Integrability of homogeneous potentials. Application to $n$ body problems

September 27, 2013
Let us consider a system of differential equations of the form $\ddot{q} = \nabla V(q)$:

$$
\begin{align*}
\ddot{q}_1 &= \frac{\partial}{\partial q_1} V(q_1, \ldots, q_n) \\
\ddot{q}_n &= \frac{\partial}{\partial q_n} V(q_1, \ldots, q_n)
\end{align*}
$$

where $V$ is a homogeneous function, meromorphic on $\mathbb{C}^n \setminus \{0\}$ in $q_1, \ldots, q_n$.

This corresponds to the motion of a point mass under a force field with a potential $V$.

This system of equation is Hamiltonian, with $H = \frac{1}{2} \|p\|^2 - V(q)$.
Sometimes it is possible to solve such a system explicitly, and we then call this an integrable case.

**Definition**

We say that the potential $V$ is integrable in the Liouville sense if there exist $l_1, \ldots, l_n$ functions of $(p, q)$ on $\mathbb{C}^n \times (\mathbb{C}^n \setminus \{0\})$ such that

- For all $i = 1 \ldots n$, we have $\dot{l}_i = 0$
- For all $i, j = 1 \ldots n$, we have $\{l_i, l_j\} = 0$ (involution).
- The set of $(p, q)$ such that the rank of the Jacobian matrix of $(p, q) \mapsto (l_1, \ldots, l_n)$ is of maximal rank is an open dense set.

Given a potential or a family of homogeneous potentials, our aim is to prove non-integrability and to detect the integrable cases.
Available methods to prove non-integrability

- Poincaré conditions for the existence of a first integral of a perturbed integrable system

- Conditions on the multiplicators of the variational equation near a generic periodic orbit

- Ziglin conditions on the monodromy group of the variational equation near a given orbit

- Morales-Ramis conditions on the Galois group of the variational equation near a given orbit

- Morales-Ramis-Simó conditions on the Galois group of higher order variational equation near a given orbit
Theorem

Let us consider a symplectic analytic complex manifold $M$ of dimension $2n$, with the Poisson bracket defined by the symplectic form, $H$ a Hamiltonian analytic on $M$ and $\Gamma \subset M$ an orbit (not a point). If $H$ possesses a complete system of first integrals in involution, functionally independent and meromorphic on a neighbourhood of $\Gamma$, then the Galois group of variational equations is virtually abelian at any order.

In the case of homogeneous potential, we will apply this Theorem with $\Gamma$ a straight line solution associated to a Darboux point.
Definition

We say that \( c \in \mathbb{C}^2 \setminus \{0\} \) is a Darboux point if there exists \( \alpha \in \mathbb{C} \) such that \( V'(c) = \alpha c \) where \( \alpha \) is called the multiplier, and \( c \) is called non-degenerated if \( \alpha \neq 0 \). To a Darboux point, we can associate the following straight line solution

\[
q(t) = \phi(t).c \quad \frac{1}{2} \dot{\phi}^2 = \frac{\alpha}{k} \phi^k + 1
\]
Example

\[ V = \frac{1}{a_1 q_1 + a_2 q_2} + \frac{1}{a_3 q_1 + a_4 q_2} \]

This potential is homogeneous of degree \(-1\). The Darboux points correspond to critical points of \(V\) restricted to the unit circle. For \(a_2 = 0, a_3 = 0\), we have the first integrals

\[ I = \frac{1}{2} p_1^2 + \frac{1}{a_1 q_1} \quad \quad H = \frac{1}{2} (p_1^2 + p_2^2) + \frac{1}{a_1 q_1} + \frac{1}{a_4 q_2} \]

We are interested in finding all such exceptional values of the parameters \(a_1, a_2, a_3, a_4\) for which two first integral could exist.
**Theorem**

Let $V$ be a rational homogeneous potential of degree $k \neq 0$ and $c$ a Darboux point such that $V'(c) = kc$. Suppose that $\nabla^2 V(c)$ is diagonalizable. If $V$ is meromorphically integrable, then for any $\lambda \in \text{Sp}(\nabla^2 V(c))$, the couple $(k, \lambda)$ belongs to the table:

<table>
<thead>
<tr>
<th>$k$</th>
<th>$\lambda$</th>
<th>$k$</th>
<th>$\lambda$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathbb{Z}^*$</td>
<td>$\frac{1}{2} ik (ik + k - 2)$</td>
<td>$-3$</td>
<td>$-\frac{25}{8} + \frac{1}{8}(\frac{6}{5} + 6i)^2$</td>
</tr>
<tr>
<td>$\mathbb{Z}^*$</td>
<td>$\frac{1}{2} (ik + k - 1)(ik + 1)$</td>
<td>$-3$</td>
<td>$-\frac{25}{8} + \frac{1}{8}(\frac{12}{5} + 6i)^2$</td>
</tr>
<tr>
<td>2</td>
<td>$\mathbb{C}$</td>
<td>3</td>
<td>$-\frac{1}{8} + \frac{1}{8}(2 + 6i)^2$</td>
</tr>
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</tr>
<tr>
<td>$-5$</td>
<td>$-\frac{49}{8} + \frac{1}{8}(\frac{10}{3} + 10i)^2$</td>
<td>3</td>
<td>$-\frac{1}{8} + \frac{1}{8}(\frac{6}{5} + 6i)^2$</td>
</tr>
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<td>3</td>
<td>$-\frac{1}{8} + \frac{1}{8}(\frac{12}{5} + 6i)^2$</td>
</tr>
<tr>
<td>$-4$</td>
<td>$-\frac{9}{2} + \frac{1}{2}(\frac{4}{3} + 4i)^2$</td>
<td>4</td>
<td>$-\frac{1}{2} + \frac{1}{2}(\frac{4}{3} + 4i)^2$</td>
</tr>
<tr>
<td>$-3$</td>
<td>$-\frac{25}{8} + \frac{1}{8}(2 + 6i)^2$</td>
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<td>$-\frac{9}{8} + \frac{1}{8}(\frac{10}{3} + 10i)^2$</td>
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</tr>
</tbody>
</table>
**Exemple:** Consider the homogeneous potential

\[ V(a, q) = (a_1 q_1 + a_2 q_2)(q_1^2 + q_2^2) \]

The homogeneity degree is \( k = 3 \). The Darboux point equation is

\[ 3a_1 c_1^2 + a_1 c_2^2 + 2a_2 c_1 c_2 = 3c_1, \quad 3a_2 c_2^2 + a_2 c_1^2 + 2a_1 c_1 c_2 = 3c_2. \]

Its solutions \( c = (c_1, c_2) \) read

\[ c = \left( \frac{a_1}{a_1^2 + a_2^2}, \frac{a_2}{a_1^2 + a_2^2} \right), \quad c = \left( \frac{3}{2(a_1 \pm ia_2)}, \frac{\pm 3i}{2(a_1 \pm ia_2)} \right) \]

when \( a_1^2 + a_2^2 \neq 0 \). The Hessian matrices at these points are

\[
\begin{pmatrix}
\frac{2(3a_1^2 + a_2^2)}{a_1^2 + a_2^2} & \frac{4a_1 a_2}{a_1^2 + a_2^2} \\
\frac{4a_1 a_2}{a_1^2 + a_2^2} & \frac{2(a_1^2 + 3a_2^2)}{a_1^2 + a_2^2}
\end{pmatrix}, \quad
\begin{pmatrix}
\frac{3(3a_1 \pm ia_2)}{a_1 \pm ia_2} & \frac{3(a_2 \pm ia_1)}{a_1 \pm ia_2} \\
\frac{3(a_2 \pm ia_1)}{a_1 \pm ia_2} & \frac{3(a_1 \pm 3ia_2)}{a_1 \pm ia_2}
\end{pmatrix}.
\]
The second matrix is not diagonalizable since its minimal polynomial is not squarefree.

The eigenvalues of the first matrix are \{6, 2\}.

Now, Morales Ramis Theorem tells us that the allowed eigenvalues for homogeneity degree \(k = 3\) are of the form

\[
\frac{3}{2}j(3j + 1), \quad \frac{1}{2}(3j + 1)(3j + 2), \quad -\frac{1}{8} + \frac{1}{8} \left(6j + \frac{12}{5}\right)^2, \\
-\frac{1}{8} + \frac{1}{8} \left(6j + \frac{3}{2}\right)^2, \quad -\frac{1}{8} + \frac{1}{8} \left(6j + \frac{6}{5}\right)^2, \quad -\frac{1}{8} + \frac{1}{8}(6j + 2)^2.
\]

The eigenvalue 6 belongs to the table, but not 2.

\(\Rightarrow\) the potential is not integrable when \(a_1^2 + a_2^2 \neq 0\).
Let us try to apply these methods for the $n$ body problem. The Hamiltonian is

$$H = \sum_{i=1}^{n} \frac{\|p_i\|^2}{2m_i} + \sum_{1 \leq i < j \leq n} \frac{m_im_j}{\|q_i - q_j\|}$$

This is indeed a Hamiltonian of potential form, and the potential is homogeneous of degree $-1$.

All seems good to apply the previous integrability conditions, except one:

The potential $V$ is not meromorphic if the dimension $d \geq 2$.
Example of an integrable case: \( V = (q_1^2 + q_2^2)^{-1/2} \).

This potential has an additional first integral

\[
C = p_1 q_2 - p_2 q_1
\]

Problem: \( V \) is not meromorphic on \( \mathbb{C}^2 \setminus \{0\} \), but is defined on

\[
C = \{(q_1, q_2, r) \in \mathbb{C}^3, \ r^2 = q_1^2 + q_2^2, \ r \neq 0\}
\]

⇒ It would be reasonable to study meromorphic potentials on complex algebraic manifolds.
Let $G \in \mathbb{C}[q, w]^s$ be weight homogeneous polynomials with weights $(1, \ldots, 1, k_1, \ldots, k_s)$, and the manifold

$$S = \{(q, w) \in \mathbb{C}^{n+s}, G(q, w) = 0\}$$

Let $J$ be the Jacobian matrix of the application $w \mapsto G(q, w)$ and the derivation of $V$ on an open set $U \subset S$ by

$$\frac{\partial}{\partial q_i} V = \partial_i V + [J^{-1}(\partial_{n+1} V, \ldots \partial_{n+s} V)^\top]_i$$

The equation $\ddot{q} = \nabla V(q)$ is well defined on $U$ outside of

$$\Sigma(V) = \{(q, w) \in U, V(q, w) \notin \mathbb{C} \text{ ou } \det(J)(q, w) = 0\}$$
Introduction

Algebraic potentials

Application to the planar $n$ body problem

Application to the colinear 3 body problem

The 4 body case

Integrability of homogeneous potentials. Application to $n$ body prob...
Theorem

Let $V$ be a homogeneous potential of degree $-1$, meromorphic on an open set $U$ of a complex algebraic surface, and $\Sigma(V)$ the associated singular set. Let $\Gamma \subset \mathbb{C}^n \times U$ be a non-stationary orbit of $V$. Assume that $\Gamma \not\subset \mathbb{C}^n \times \Sigma(V)$. If $V$ is integrable on the Liouville sense with first integrals meromorphic on $\mathbb{C}^n \times (U \setminus \Sigma(V))$, then the identity component of the Galois group of variational equations near $\Gamma$ is abelian over the field of meromorphic functions on $\Gamma \setminus (\mathbb{C}^n \times \Sigma(V))$. 
Let $V$ be a homogeneous potential of degree $-1$, meromorphic on a complex algebraic surface $S$ and $\Sigma(V)$ the associated singular set. Let $c \in S \setminus (\{0\} \cup \Sigma(V))$ be a Darboux point. If $V$ is integrable on the Liouville sense with first integrals meromorphic on $\mathbb{C}^n \times (S \setminus \Sigma(V))$, then

$$\text{Sp}(\nabla^2 V(c)) \subset \left\{ \frac{1}{2}(i - 1)(i + 2), \ i \in \mathbb{N} \right\}$$
Let first remark that the colinear $n$ body problem can be seen as an invariant manifold of the $d$-dimensional $n$ body problem.

This suggest that the Hessian matrix at a colinear central configuration will have a lot of structure.

**Proposition**

*Let $c$ be a colinear central configuration. Then the Hessian matrix of $V$ is of the form*

$$
\nabla^2 V(c) = \begin{pmatrix}
A & 0 & \ldots & 0 \\
0 & \frac{1}{2}A & 0 & \ldots \\
0 & \ldots & 0 & \frac{1}{2}A
\end{pmatrix}
$$
Theorem (Pacella)

For positive masses, all the eigenvalues of $A$ are strictly superior than 2.

Corollary

The $n$ body problem with positive masses in dimension $d \geq 2$ is not meromorphically integrable.
Proof: The possible eigenvalues from the Morales-Ramis table are of the form

$$\frac{1}{2}(k - 1)(k + 2), \quad k \in \mathbb{N}$$

If $\lambda \in \text{Sp}(\nabla^2 V(c))$, then $-\lambda/2 \in \text{Sp}(\nabla^2 V(c))$. The only possible non-positive eigenvalue are 0, $-1$, and thus the only possible $\lambda = 0, 2$.

But $\lambda > 2$, and thus all eigenvalues cannot belong to the Morales-Ramis table.
It would be enough to check for any positive masses that the Morales-Ramis integrability condition

\[ \lambda \in \frac{1}{2}(k - 1)(k + 2), \quad k \in \mathbb{N} \]

is not satisfied. But this is not so easy...

In the 3 body problem, we can always suppose \( m_1 + m_2 + m_3 = 1 \).

After reduction, the Darboux points and Hessian eigenvalues are \( c = (-1, 0, \rho) \) with

\[
(m_2+m_3)+(2m_2+3m_3)\rho+(3m_3+m_2)\rho^2-(3m_1+m_2)\rho^3-(3m_1+2m_2)\rho^4-(m_1+m_2)\rho^5 = 0
\]

\[
\left\{ 0, 2, -\frac{4(1+\rho)\rho^3(2\rho^2+3\rho+2)}{(m_1\rho^4+2m_1\rho^3-\rho^2+\rho^2m_1-2\rho+2\rho m_1-1+m_1)(1+2\rho+\rho^2+2\rho^3+\rho^4)} \right\}
\]
Grobner basis: Elimination of variable $\rho$, and we get the equation

$$[\lambda^j] \ P_{m_1,m_3}(\lambda) - \beta \prod_{i=1}^{5} (\lambda - \frac{1}{2}(k_i - 1)(k_i + 2)) = 0 \quad j = 0 \ldots 5$$

Elimination on $m_1$, $m_3$, $\beta$ is impossible in practice, but we have the following relation

$$\sum_{i=1}^{5} \frac{1}{\lambda_i + 1} = 1$$

Admissible solutions: $[2, 2, 5, 9, 14], [2, 5, 5, 5, 5]$
⇒ No solutions.

But for $m_i \in \mathbb{C}$, there are integrable cases! The flaw is that there could be degenerated cases, with fewer Darboux points...

Solution and Problem: Comprehensive Groebner basis, but high computational cost, doubly exponential in the number of Darboux points, out of reach for $n = 4$.

We need to use somehow the positivity of the masses ⇒ One should consider only real Darboux points.
Theorem (Moulton)

The colinear $n$ body problem possesses exactly one real Darboux point.

Theorem

If the colinear 3 body problem with positive masses $m_1 + m_2 + m_3 = 1$ is meromorphically integrable, then it exists $\rho \in \mathbb{R}^*_+$ such that

$$
m_1 = \frac{(\rho + 1)(-8\rho^5 + k\rho^5 - 12\rho^4 + 3k\rho^4 - 8\rho^3 + 3k\rho^3 + 3k\rho^2 + 3k\rho + k)}{k(1 + 2\rho^3 + \rho^4 + 2\rho + \rho^2)^2},$$

$$
m_2 = -\frac{(-8\rho^4 + k\rho^4 - 28\rho^3 + 2k\rho^3 + k\rho^2 - 40\rho^2 - 28\rho + 2k\rho - 8 + k)\rho^2}{k(1 + 2\rho^3 + \rho^4 + 2\rho + \rho^2)^2} (E_k),$$

$$
m_3 = \frac{(\rho + 1)(k\rho^5 + 3k\rho^4 + 3k\rho^3 - 8\rho^2 + 3k\rho^2 - 12\rho + 3k\rho - 8 + k)\rho^2}{k(1 + 2\rho^3 + \rho^4 + 2\rho + \rho^2)^2},$$

with $k \in \{5, 9, 14\}$. 
Curves \((E_k)\) in barycentric coordinates.

We need to study higher variational equations to conclude using only this Darboux point.
The first order variational equation is of the form

\[
\dot{X} = \begin{pmatrix}
A_1(t) & 0 & 0 \\
0 & A_2(t) & 0 \\
0 & \ldots & 0 \\
0 & \ldots & A_n(t)
\end{pmatrix} X = A(t)X
\]

with \(A_i(t) \in M_2(\mathbb{C}(t))\).

The matrices \(A_i\) involve the eigenvalues of the Hessian matrix.

The first order equation is of hypergeometric type. Classification of solvable equations by Kimura \(\Rightarrow\) Morales-Ramis criterion.
Higher variational equations are of the form

\[ \dot{X} = \left( \begin{array}{cccc} Sym^k(A(t)) & 0 & \ldots & 0 \\ * & Sym^{k-1}(A(t)) & \ldots & 0 \\ * & \ldots & * & A(t) \\ * & \ldots & * & A(t) \end{array} \right) X \]

The * are rational fraction depending linearly on some parameters: the higher order derivatives of the potential.

⇒ It is always possible to solve this equation through the method of variation of parameters.

But the virtual abelianity condition imposes here that one should not use iterated integrals.
Example

\[(t^2 - 1)\ddot{y} + 2t\dot{y} - 2y = a\arctanh(1/t) + bt\]

Is it possible to solve this equation without iterated integrals? ⇔
Solving this equation in \(\mathbb{C}(t)[\arctanh(1/t)]\)

Answer: if and only if \(b = 0\) and then solutions are

\[y(t) = c_1 t \arctanh\left(\frac{1}{t}\right) + c_2 t + c_1 \frac{2/3-t^2}{t^2-1} + \frac{at \arctanh\left(\frac{1}{t}\right)}{6(t^2-1)} - \frac{a}{2} \arctanh\left(\frac{1}{t}\right)^2\]
Idea: an element of $\mathbb{C}(t)[\text{arctanh}(1/t)]$ has a zero residue at infinity.

$\Rightarrow$ search solutions in $\mathbb{C}(t)[\text{arctanh}(1/t)]$ block by block and eventually compute residues at infinity

Example: For $V$ a homogeneous potential in dimension 2 of degree $-1$ with $c = (1, 0)$ Darboux point, $V(c) = 1$ and $\lambda = 9$

Order 1: $\frac{\partial^2 V(c)}{\partial q_2^2} = 9$

Order 2: $\frac{\partial^3 V(c)}{\partial q_2^3} = a \in \mathbb{C}$

Order 3: $\frac{\partial^4 V(c)}{\partial q_2^4} = \frac{39933}{67} + \frac{3289}{18090}a^2$

Order 4: $\frac{\partial^5 V(c)}{\partial q_2^5} = \frac{495855}{2144}a + \frac{363467}{8683200}a^3$

Order 5: $\frac{\partial^6 V(c)}{\partial q_2^6} = -\frac{13085295305625}{233350414} - \frac{46044432837}{2333504140}a^2$ with

$$\frac{45927}{2729312} + \frac{41381067}{3473868313600}a^2 + \frac{158469311}{126622500030720000}a^4 = 0$$
No real solution at order 5 $\Rightarrow$ There exists no real integrable homogeneous potential of degree $-1$ in the plane with this eigenvalue.

Doing the same with eigenvalues 5,14 proves

**Theorem**

*The colinear 3-body problem with positive masses is never meromorphically integrable.*
For \( n = 4 \), we first need to find the possible eigenvalues. The Darboux point can be written

\[
c = (-\rho_1, -1, 1, \rho_2) \quad \rho_1 \geq \rho_2 > 1
\]

We just need to bound some real algebraic functions (eigenvalues in function of \( \rho_1, \rho_2 \)): Real algebraic geometry programs deal with this problem.

The possible specturms are \( Sp = \{0, 2, \lambda_1, \lambda_2\} \) with

\[
\{\lambda_1, \lambda_2\} = \{5, 5\}, \{5, 9\}, \{5, 14\}, \{5, 20\}, \ldots, \{27, 35\}
\]
26 cases: we study them one by one.

In principle, one just need to compute some additional integrability conditions and test them.

⇒ to prove that some real bivariate polynomial has no real roots.

But the degree is often too high: Conditions on 4-th derivatives lead to polynomial of degree 220.
Example: \( \{ \lambda_1, \lambda_2 \} = \{ 9, 54 \} \).

We have \( V(q_1 - 1, q_2, q_3) = 1 - q_1 + \)

Order 1: \( q_1^2 + 9/2q_2^2 + 27q_3^2 + O(q^3) \)

Order 2: \( \cdots + u_4q_3^3 - 27/2q_1q_2^2 + u_3q_2q_3^2 - 81q_1q_3^2 - q_1^3 + u_1q_2^3 + O(q^4) \)

Order 3: \( \cdots + q_1^4 - 4q_1u_4q_3^3 - 4q_1u_1q_2^3 + \)

\[
\left( \frac{663499046129}{14245925733270} u_4^2 + \frac{3843346549}{105721155720} u_3^2 + \frac{93293112339}{97889959} \right) q_3^4 + \cdots + O(q^5)
\]

At order 4, we prove that

\[
\partial_{q_2}^2 \partial_{q_3} V(c) = \partial_{q_2}^3 \partial_{q_3} V(c) = \partial_{q_2}^4 \partial_{q_3} V(c) = 0
\]
$\Rightarrow$ at order 5, it exists an invariant plane. The eigenvalue on this plane is 9, and so impossible for integrability.

**Theorem**

*If $V$ is a meromorphic real homogeneous potential of degree $-1$ in dimension 3, $c = (1, 0, 0)$ a Darboux point, $V(c) = 1$, and the eigenvalues correspond to one of the 26 cases, then $V$ is not meromorphically integrable.*

**Corollary**

*The colinear 4-body problem with positive masses is never meromorphically integrable.*