CLUSTER DYNAMICS
IN THE TIDAL GALACTIC FIELD

Leonid P. Ossipkov

Saint Petersburg State University, Russia

GRAVASCO
$N$-body Gravitational Dynamics from $N = 2$ to infinity

Workshop 2 "Dynamical and Kinetic Theory of Self-Gravitating Systems"

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Introduction

We will discuss some aspects of dynamics of an ensemble of gravitating point masses (stars) whose barycentre is assumed to move along a circular orbit in the steady axisymmetric force field (the Galaxy).

A size of the system is assumed to be very small in comparison with the radius of its orbit.

It will be assumed for simplicity that the system can be described by the collisionless kinetic equation. But the main equations will be valid for collisional systems (under some assumptions), as well as for \( N \)-body systems.

The more or less detailed and rigorous analytical theory of tidal action can be developed for such problem.
The equations of motion and their consequences will be written in a rotating frame $x, y, z$. 
1.1. Prehistory

Earlier studies dealt with dynamics of meteor clouds, envelopes of comets, protoplanets (E. Roche, G. Schiaparelli, G. H. Darwin, C. V. L. Charlier and others).
1. Introduction
2. The Bok problem
3. Macroscopic dynamics
4. The equilibrium
5. Non-steady gross-dynamics
1. Introduction

2. The Bok problem

3. Macroscopic dynamics

4. The equilibrium

5. Non-steady gross-dynamics

1. Considérons un système formé d'une très grand nombre de particules, dont les masses respectives sont très petites, la masse totale d'elle-même étant complètement n'ignorable vis-à-vis de celle du Soleil. Supposons que ce système se soit réduit à une seule force externe, l'attraction solaire; le centre de gravité O décrit sensiblement une section conique ayant pour foyers le centre du Soleil, nous admettrons, dans ce premier chapitre, que l'excentricité de cette course est négligeable.

Proposons-nous de trouver le mouvement d'un des points de l'atome relativement à trois axes principaux passant par un point O et dirigés de la façon suivante :

- $O_1$, vers le centre du Soleil;
- $O_2$, dans le sens de la classe du point O;
- $O_3$, suivant une perpendiculaire à l'orbite de ce point.

Il nous suffit d'appliquer le théorème de Newton. La particule $A_i$ doit être considérée comme soumise à quatre forces : l'attraction $F$ du Soleil, l'attraction $f$ de l'atome, la force centrifuge $g$, la force correspondante $H$.

Si nous représentons par $x, y, z$ la quantité appelée ordinalement la coordonnée de l'atome $A_i$, d'autre part, nous désignons par $R$ la masse de la particule $A_i$, qui disparaît d'ailleurs de nos équations, et par $s$ la distance du point $A$ au centre du Soleil, le force extérieure pour autant que $A$ est dirigé de $A$ vers $S$.

Pour calculer $f$, nous faisons une dernière hypothèse sur la construction de l'atome : nous supposons limite par une sphère, et nous admettons que la distribution des masses est homogène autour du centre de cette surface, point qui se confond alors avec $O_1$. Nous négligerons donc, suivant que le corps $A$ est à l'extérieur ou à l'intérieur de l'atome. Dans le premier cas, en étant la masse totale du système, $R$ la distance OA, la force $f$ sera mesurée par $\frac{m_n}{R^3}$, on aura dans le second cas, se servir d'expression, où $m$ est remplacé par la somme des masses intérieures du rayon OA.

D'après la loi de mouvement, si nous désignons si le moment de rotation du rayon $OS$, et par $p_1$ un longueur projections de la force centrifuge $F$ sur les axes de coordonnées positivement

$$-m_2a_1x_1-m_2a_1y_1=0$$

Celles de la force centrifuge composée $F$ seront

$$z_1\frac{d^2y_1}{dt^2}-a_1z_1\frac{dz_1}{dt}=0$$

Puisque nous supposons la quantité $m$ très petite, nous aurons

$$x_1\frac{dy_1}{dt}=0$$

Nous trouvons alors pour les équations du mouvement

$$\frac{dx}{dt}=0, \frac{dy}{dt}=0, \frac{dz}{dt}=0$$

Ces équations prennent une forme encore plus simple si on compte de ces que $x, y, z$ sont, par rapport à $p_1$, des quantités extrêmement petites; on pourra alors négliger les carrés de ces quantités.

On a, dans le triangle $AOS$,

$$p_1^2=p_1^2+2p_1^2x+2p_1^2y$$

ou en déduit

$$\frac{L^2}{L} = \frac{1}{L^2} (1 + \frac{3}{L^2} - \ldots)$$

les termes non écrites pouvant être négligés.
1936 — the critical review by N. F. Rein.
1951 — the paper by V. G. Fessenkov "The condition of tidal stability and its application in cosmogony."

APÉCU CIDITE DE LA THÉORIE DE LA STABILITE DES ESSAIS
MÉTEORIQUES ET DES AMAS STELLAIRES,

Par Natalie Ralu

Resumé

Le présent travail est consacré à l'examen des théories concernant le problème de la stabilité des condensations gravitationnelles. Nous sommes persuadés que c'est un problème majeur et des plus délicats de la cosmogonie. Il nous semble nécessaire de comprendre ce problème en tenant compte des conclusions de diverses investigations concernant ce sujet. Nous avons donc décidé de faire un aperçu succinct de l'évolution des connaissances sur ce sujet.

1. Introduction

2. The Bok problem

3. Macroscopic dynamics

4. The equilibrium

5. Non-steady gross-dynamics

Dans la plupart des investigations, l'effet du mouvement circulaire de la condensation est supposé être négligeable. En ce sens, il est impossible de déterminer si la condensation est stable ou non. C'est pourquoi nous avons décidé de faire un aperçu de l'évolution des connaissances sur ce sujet.

6. The critical review by N. F. Rein.


8. Conclusion

Dans la plupart des investigations, l'effet du mouvement circulaire de la condensation est supposé être négligeable. En ce sens, il est impossible de déterminer si la condensation est stable ou non. C'est pourquoi nous avons décidé de faire un aperçu de l'évolution des connaissances sur ce sujet.
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Saint Petersburg State University, Russia

CLUSTER DYNAMICS IN THE TIDAL GALACTIC FIELD
1.2. First works on cluster dynamics

J. H. Jeans (1922) — ignored Coriolis and centrifugal forces.
B. Bok (1934) — the pioneering work, in which

- the Hill critical surface was constructed,
- the equations of star motion were solved for the model of a homogeneous cluster.

→ two conditions for stability of a cluster that were coinciding (that was a mistake!)

\[ \rho > \frac{\kappa^2}{G \pi \beta}, \]

where \( G \) is the gravitational constant, \( \kappa^2 = 4A(A - B) \) (\( A, B \) being Oort’s dynamical coefficients). \( \beta \) is a dimensionless factor depending on the shape of a cluster (1.3 for spheres, 0.2 for discs).

H. Mineur (1939) — the self-consistent problem (the equilibrium shape of the fluid ellipsoid in the tidal field)
U. van Wijk (1949), G. G. Kuzmin (1963) — the tensor virial theorem and generalizing these results for non-homogeneous systems.
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Leonid P. Ossipkov

Saint Petersburg State University, Russia

CLUSTER DYNAMICS IN THE TIDAL GALACTIC FIELD
Last decades: simulations (will not be discussed)
analytical and semi-analytical works by D. Heggie, V. M. Danilov and their teams.
2. The Bok problem.
2.1. The equations of motion

The equations of motion in the rotating frame:

\[
\begin{align*}
\ddot{x} - 2\Omega \dot{y} &= -\frac{\partial \Phi_c}{\partial x} + \kappa_R^2 x, \\
\ddot{y} + 2\Omega \dot{x} &= -\frac{\partial \Phi_c}{\partial y}, \\
\ddot{z} &= -\frac{\partial \Phi_c}{\partial z} - \kappa_z^2 z.
\end{align*}
\]

Here \(\Phi_c(x, y, z)\) is the gravitational potential of the cluster, \(\Omega\), the circular frequency of the cluster. \(\kappa_R, \kappa_z\) are local parameters of Galaxy’s gravitational field. For the Solar neighbourhood
\[
\kappa_R^2 = 4A(A - B), \quad A, B \text{ being Oort’s dynamical parameters},
\]
\[
\Omega \approx 27 \text{ km} \cdot \text{s}^{-1} \cdot \text{kpc}^{-1}, \quad \kappa_R \approx 42 \text{ km} \cdot \text{s}^{-1} \cdot \text{kpc}^{-1}, \quad \kappa_z \approx 85 \text{ km} \cdot \text{s}^{-1} \cdot \text{kpc}^{-1}.
\]
\(\kappa_z\) (Kuzmin’s parameter) is the frequency of small vertical star oscillations. \(\kappa_R\) was suggested to call the \textit{tidal increment}. 
The dimensionless form of these equations:
The unit of length will be $r_0 = \left(\frac{GM}{\kappa_R^2}\right)^{1/3}$ ($M$ is the mass of the cluster).
The dimensionless coordinates $\xi = x/r_0$, $\eta = y/r_0$, $\zeta = z/r_0$.
The unit of time will be $t_0 = \kappa_R^{-1}$. Denote $\tau = t/t_0$.

Then

$$
\frac{d^2}{d\tau^2} \xi - \gamma \frac{d}{d\tau} \eta = -\frac{\partial \varphi}{\partial \xi} + \xi,
$$

$$
\frac{d^2}{d\tau^2} \eta + \gamma \frac{d}{d\tau} \xi = -\frac{\partial \varphi}{\partial \eta},
$$

$$
\frac{d^2}{d\tau^2} \zeta = -\frac{\partial \varphi}{\partial \zeta} - k\zeta.
$$

Here $\varphi(\xi, \eta, \zeta) = \Phi_c(\xi r_0, \eta r_0, \zeta r_0)/(GMr_0^{-1})$ is the dimensionless potential of the cluster, $\gamma = 2\Omega \kappa_R^{-1}$ and $k = \left(\kappa_z/\kappa_R\right)^2$. For the Solar neighbourhood $\gamma \approx 1.25$, $k \approx 4$. 
When both the Galaxy and a cluster are point masses → the classical Hill problem of Celestial Mechanics. Our problem is more general and is suggested to be called as the *Bok problem*. If the potential of a cluster is steady, the Jacobi integral can be written:

\[ H = \frac{1}{2} \left[ \left( \frac{d\xi}{d\tau} \right)^2 + \left( \frac{d\eta}{d\tau} \right)^2 + \left( \frac{d\zeta}{d\tau} \right)^2 \right] + \varphi - \frac{1}{2}(\xi^2 - k\zeta^2). \]

If a cluster is a homogeneous ellipsoid → the equations of motion are linear ones, and two other integrals of motion can be written down (Mineur, 1939). Then the general solution of equations of motion can be written explicitly (Bok, 1934).

But it seems very probable that no more global integrals of motion exist in general, and orbits can be chaotic, as calculations by Jefferys (1970) and Carpintero et al. (1999) show. Danilov & Leskov (2005) calculated Lyapounoff’s exponents. Search for conditions of orbit chaotization can be called the *internal Bok problem*. As the first step to its analysis the author tried to investigate libration points (LO, 2007).
2.2. Libration points

It was found that all libration points lie on the $\xi$-axis. If a cluster is spherical, $\varphi(\xi, \eta, \zeta) = -f(r)$, $r = (\xi^2 + \eta^2 + \zeta^2)^{1/2}$, then

$$\xi(r + f') = \eta f' = \zeta(-kr + f') = 0.$$ 

If $f' \neq 0$, then $\eta = \zeta = 0$ and positions of libration points can be found from the equation

$$f'(|\xi|) + |\xi| = 0.$$ 

At least one pair of libration points exists for plausible models of cluster. A condition of their stability was found from equations in variations. If $a > 0$ is a coordinate of a libration point, then a necessary condition for its stability is as follows:

$$\left.\frac{f'(r)}{r}\right|_{r=a} < 0.$$ 

It is not fulfilled for the Hill problem and reasonable cluster models.
It is evident that the centre of cluster $\xi = \eta = \zeta = 0$ is a libration point. To check its stability the equations in variations are to be studied:

\[
\begin{align*}
\frac{d^2}{d\tau^2} \xi_1 - \gamma \frac{d}{d\tau} \eta_1 &= (1 - \beta)\xi_1, \\
\frac{d^2}{d\tau^2} \eta_1 + \gamma \frac{d}{d\tau} \xi_1 &= -\beta\eta_1, \\
\frac{d^2}{d\tau^2} \zeta_1 &= -(k + \beta)\zeta_1,
\end{align*}
\]

with $\beta = -f''(0)$, $\xi_1$, $\eta_1$, $\zeta_1$ as variations. The problem is reduced to Bok’s analysis of equations of motion for a homogeneous cluster. Then the necessary condition of its stability has the same form as Bok’s one, $\varrho$ being now a central density of cluster. Fessenkov (1951) proved the stability for the case when the Galaxy is a point mass. Hence, one can expect from the KAM-theory that stellar orbits nearby the centre are ordered. As we can judge, transition to chaos for the internal Bok problem was not studied yet.

Leonid P. Ossipkov
Saint Petersburg State University, Russia

CLUSTER DYNAMICS IN THE TIDAL GALACTIC FIELD
2.3. The Hill surface and escape

Let $L_a = (a, 0, 0)$ be a libration point, $F = \varphi - (\xi^2 - k\zeta^2)/2$, the effective potential, $H_a = F(a, 0, 0)$, the value of the Jacobi integral. We consider the zero velocity surface

$$S_a = \{(\xi, \eta, \zeta) | F(\xi, \eta, \zeta) = H_a\}.$$

For simplicity let suppose that only one pair of libration pints, $L_a, L_{-a}$, exists. $S_a$ is the critical Hill surface for the Bok problem. If the Galaxy is a point mass and a cluster is a sphere, $S_s$ will be a closed surface (Cimino, 1956). Generally, it is not clear, how to find a shape of $S_a$.

The case of cluster as a point mass (or a sphere of a small finite radius): $S_a$ intersects axis at $a = 1$, $\eta_a = \pm 2/3$, $|\zeta_a|$ that is a root of the equation $k\zeta^3 + 3\zeta = 2$. If $k = 4$, then $\zeta_a = \pm 1/2$ (that was ignored by Bok).
The maximal size of a stable spherical cluster $r_* = r_0/2$. If a cluster is a homogeneous sphere then its density

$$\rho_* = \frac{6\kappa_R^2}{\pi G}.$$ 

Hence, it is higher than Bok’s critical density in 8 times.

If $S_a$ is closed and the Jacobi constant $H < H_a$, the orbits are finite. What will be if $H > H_a$? That is the external Bok problem. Usually it assumed that these stars escape. But it cannot be proved.


Orbit calculations were done by A. Davydenko for the Schuster–Plummer sphere

$$\Phi_c = -\Phi_0/(1 + r^2/r_c^2)^{1/2}.$$ 

There were calculated about $10^6$ orbits for various initial conditions that were chosen using the Lindblad diagram. The time interval of integration was 10. An orbit considered as escaper if its distance from the centre of a cluster was larger than 10.
Examples of orbits: finite ones (an initial angular momentum was zero)
Examples of orbits: tails (retrograde orbits)
Examples of orbits: escapes (direct orbits)
Escapers at Lindblad’s diagram for $r_c/r_0 = 0.1$ (left figure) and $r_c/r_0 = 0.3$ (right figure)
Escapers at Lindblad’s diagram for $r_c/r_0 = 0.5$
Escapers at Lindblad’s diagram for $r_c/r_0 = 0.7$
Escapers at Lindblad’s diagram for $r_c/r_0 = 0.9$

BUT THE BOK PROBLEM IS NOT SELF-CONSISTENT!
3. Macroscopic dynamics

3.1. The distribution function

DF: \( f = f(r, v, t) = f(x, y, z, u, v, w) \)

The kinetic equation:

\[
\frac{\partial f}{\partial t} + u \frac{\partial f}{\partial x} + v \frac{\partial f}{\partial y} + w \frac{\partial f}{\partial z} + \left(- \frac{\partial \Phi_c}{\partial x} + \nu_\infty^2 x + 2\Omega v\right) \frac{\partial f}{\partial u} \\
+ \left(- \frac{\partial \Phi_c}{\partial y} - 2\Omega u\right) \frac{\partial f}{\partial v} + \left(- \frac{\partial \Phi_c}{\partial z} - \nu_z^2 z\right) \frac{\partial f}{\partial w} = \text{St}f.
\]

The Jeans theorem \( \rightarrow \) steady DF is a function of isolating integrals of motion. They are known for a model of a homogeneous ellipsoid only \( \rightarrow \) the model by M. Fellhauer & D. Heggie (2005).

V. Danilov (2006, 2008) found an approximate three-integral DF with an ellipsoidal velocity distribution.

We will apply the method of moments to study the cluster dynamics, neglecting evaporation and dynamical friction.
3.2. Generalized Jeans equations

Denote:

\[ \varrho \mathbf{V}(\mathbf{r}, t) = \varrho(V_x, V_y, V_z) = \int \mathbf{v} f(\mathbf{r}, \mathbf{v}, t) \, d^3\mathbf{v} \]

– the mean velocity,

\[ \varrho S_{ij}(\mathbf{r}, t) = \int v_i v_j f(\mathbf{r}, \mathbf{v}, t) \, d^3\mathbf{v} \]

– the velocity dispersion tensor.

Multiplying the equation for DF by velocities and integrating yields moment equations.

The continuity equation:

\[ \frac{\partial \varrho}{\partial t} + \frac{\partial}{\partial x} \varrho V_x + \frac{\partial}{\partial y} \varrho V_y + \frac{\partial}{\partial z} \varrho V_z = 0 \]
The equations of mean motion (the generalized Jeans equations):

\[
\frac{\partial}{\partial t} \varrho V_x + \frac{\partial}{\partial x} \varrho S_{xx} + \frac{\partial}{\partial y} \varrho S_{xy} + \frac{\partial}{\partial z} \varrho S_{xz} + \varrho \left( \frac{\partial \Phi_c}{\partial x} + \kappa^2_R x \right) - 2\Omega \varrho V_y = 0,
\]

\[
\frac{\partial}{\partial t} \varrho V_y + \frac{\partial}{\partial x} \varrho S_{xy} + \frac{\partial}{\partial y} \varrho S_{yy} + \frac{\partial}{\partial z} \varrho S_{yz} + \varrho \frac{\partial \Phi_c}{\partial y} + 2\Omega \varrho V_x = 0,
\]

\[
\frac{\partial}{\partial t} \varrho V_z + \frac{\partial}{\partial x} \varrho S_{xz} + \frac{\partial}{\partial y} \varrho S_{yz} + \frac{\partial}{\partial z} \varrho S_{zz} + \varrho \left( \frac{\partial \Phi_c}{\partial z} - \kappa^2_z z \right) = 0.
\]

Also, the equations for \(dS_{ij}/dt\) (the equations of micromotion according to the terminology by Kuzmin (1965)) can be found.

Problems: to close these equations (e.g. to assume that the cluster is steady, the velocity distribution is isotropic etc), to find boundary conditions (at the Hill surface?), to solve them.
3.3. Gross-dynamics equations

The continuity equation \( \frac{d}{dt} I_{ij} = L_{ij} + L_{ji} \),

where \( I_{ij} = M \langle x_i x_j \rangle \), \( L_{ij} = M \langle x_i V_j \rangle \).

The equations of motion (the tensor generalisations of the Lagrange-Jacobi equation)

\[
\frac{d}{dt} L_{ix} = K_{ix} + W_{ix} + \kappa_R^2 l_{ix} + 2\Omega L_{iy}, \\
\frac{d}{dt} L_{iy} = K_{iy} + W_{iy} - 2\Omega L_{ix}, \\
\frac{d}{dt} L_{iz} = K_{iz} + W_{iz} + \kappa_z^2 l_{iz}.
\]

Here \( K_{ij} = M \langle S_{ij} \rangle \), \( W_{ij} = M \langle x_i \partial \Phi_c / \partial x_j \rangle \).
Equations of micromotion →

\[
\begin{align*}
\frac{d}{dt} K_{xx} & = 2H_{xx} + 2\kappa^2 R L_{xx} + 4\Omega K_{xy} + Mc_x, \\
\frac{d}{dt} K_{yy} & = 2H_{yy} - 4\Omega K_{xy} + Mc_y, \\
\frac{d}{dt} K_{zz} & = 2H_{zz} - 2\kappa^2 z L_{zz} + Mc_z, \\
\frac{d}{dt} K_{xy} & = H_{xy} + H_{yx} + 2\kappa^2 R L_{xy} + 2\Omega(K_{yy} - K_{xx}), \\
\frac{d}{dt} K_{xz} & = H_{xz} + H_{zx} + \kappa^2 R L_{xz} - \kappa^2 z L_{zx} + 2\Omega K_{yz}, \\
\frac{d}{dt} K_{yz} & = H_{yz} + H_{zy} - \kappa^2 z L_{zy} - 2\Omega K_{xy},
\end{align*}
\]

where \( H_{ij} = -M \langle V_i \partial \Phi_c / \partial x_j \rangle \) and \( c_i \) are results of averaging the collisional term. Their trace yields the Jacobi integral for the cluster.
4. The equilibrium

Assume that
- the cluster is steady in the rotating frame
- there are no mean motions.

Denote

\[ \alpha_{xy} = \sigma_x^2 / \sigma_y^2, \quad \alpha_{zy} = \sigma_z^2 / \sigma_y^2 \]

(anisotropy parameters, \( \sigma_i^2 \) are mean velocity dispersions),

\[ \varepsilon_{xy}^2 = W_{xx} / W_{yy}, \quad \varepsilon_{zy}^2 = W_{zz} / W_{yy} \]

(parameters of a shape).

Denote also

\[ \tau_x^2 = I_{xx} / (-W_{xx}), \quad \tau_z^2 = I_{zz} / (-W_{zz}) \]

(these quantities have an order of the crossing time).
The tensor virial equations will be written as follows (L0, 2007):

\[ \alpha_{xy} = \varepsilon_{xy}^2 (1 - \kappa_R^2 \tau_x^2), \]

\[ \alpha_{zy} = \varepsilon_{zy}^2 (1 + \kappa_z^2 \tau_z^2). \]

Then the **general necessary condition of existing a steady cluster in the field of tidal forces** can be written in the following form:

\[ \kappa_R^2 \tau_x^2 \leq 1. \]

For the homogeneous cluster it coincides with the Bok condition:

\[ \rho \geq \frac{3}{4\pi G} \kappa_R^2; \]
Let us write the tensor virial equations in the following form:

\[
1 = \frac{\varepsilon_{xy}^2}{\alpha_{xy}} (1 - \kappa_R^2 \tau_x^2) = \frac{\varepsilon_{zy}^2}{\alpha_{zy}} (1 + \kappa_z^2 \tau_z^2).
\]

and study connection between shape and anisotropy.

If \(\varepsilon_{xy} = 1\) then \(\alpha_{xy} < 1\), that is \(\sigma_x^2 < \sigma_y^2\).

If \(\varepsilon_{zy} = 1\) then \(\sigma_z^2 > \sigma_y^2\).

The latter means that **steady spherical clusters with isotropic velocity distribution cannot exist**.

In the case of spherical velocity distribution \(\varepsilon_{xy}^2 > 1, \varepsilon_{zy} < 1\). Hence, such cluster is elongated along \(x\)-axis and flattened along \(z\)-axis (that coincides with Kuzmin’s result).
5. Non-steady gross-dynamics

5.1. Some theorems

Some general theorem can be drawn from gross-dynamics equations.

**Theorem 1.** If \( l_{ij} = 0, i \neq j \), then \( L_{ij} = -L_{ji} \).

**Theorem 2.** If \( (I_{xx} - I_{yy}) = 0 \) (e.g. in the case of rotational symmetry), then \( L_{xx} = L_{yy} \).

**Theorem 3.** If a system rotates around the z-axis as a rigid body, then \( I_{xy} = \text{const} \).

**Theorem 4.** If \( I_{xy} = 0 \) then

\[
H_z = (L_{xy} - L_{yx}) + 2\Omega(I_{xx} + I_{yy}) = \text{const}.
\]

It is evident that the integral \( H_z \) is a generalization of the angular momentum integral for a non-isolated system. Earlier it was found by Chandrasekhar (1942) for \( N \)-body system.
Theorem 5. If $I_{yz} = 0$, $L_{xz} = 0$ then $H_x = L_{yz} - L_{zy} = \text{const}$.

We conclude also that if $H_x = \text{const}$ then, in general, $I_{yz} = 0$, $L_{xz} = 0$, for $\kappa_z$, $\Omega$ are given external parameters of the problem.

Theorem 6. If $I_{xz} = 0$, $L_{zy} = 0$, then $H_y = L_{xz} - L_{zx} = \text{const}$.

Theorems 5, 6 were also proved by Chandrasekhar (1942) for $N$-body systems. $H_x$, $H_y$, $H_z$ can be called Chandrasekhar integrals.

We can conclude again that if $H_y = \text{const}$ then, in general, $I_{xz} = 0$, $L_{zy} = 0$.

Theorem 7. If conditions of theorems 5 and 6 are fulfilled, then $K_{yz} + W_{yz} = 0$ and $K_{xz} + W_{xz} = 0$.

Corollary 1. In this case $H_x = H_y = 0$, that is these Chandrasekhar integrals will be invariant relations.

Corollary 2. If $W_{xz} = W_{yz} = 0$, then the total ellipsoid for stars of cluster will not be inclined to the galactic plane.
Theorem 8. If \( I_{xy} = 0 \), then \( \dot{L}_{xy} = 2\Omega(L_{xx} - L_{yy}) \). If \( W_{xy} = 0 \), then \( K_{xy} = 2\Omega(I_{xx} - I_{yy}) \).

Corollary. If \( I_{xx} - I_{yy} = \text{const} \), then \( K_{xy} = 0 \).

Lemma 1. If \( K_{xy} = 0 \), \( W_{xy} = 0 \), then \( L_{xy} = \Omega I_{xx} = C_1 = \text{const} \).

Lemma 2. If \( K_{xy} = 0 \), \( W_{xy} = 0 \), \( I_{xy} = 0 \), then \( L_{xy} - \Omega I_{yy} = C_2 = \text{const} \).

Theorem 9. If \( I_{xy} = 0 \), \( W_{xy} = 0 \), \( K_{xy} = 0 \), then \( I_{xx} - I_{yy} = \text{const} \).

It is difficult to image a cluster with \( I_{xx}(t) - I_{yy}(t) = \text{const} \neq 0 \). Then one can conclude about rotational symmetry of non-stationary clusters.

Combining Theorem 8 and Corollary of Theorem 7 leads to the following conclusion:

Theorem 10. If \( I_{xy} = 0 \), \( W_{xy} = 0 \), then \( I_{xx} - I_{yy} = \text{const} \) iff \( K_{xy} = 0 \).
5.2. Small virial oscillations

To solve non-stationary problems, it is necessary

- to close the system of gross-dynamics equations,
- to linearize these equations and to find eigen-frequencies,
- try to analyze non-linear equations.

We suppose that the cluster is axisymmetric (later Danilov considered a more general case) and

\[
-W_{ii} = -\frac{1}{2} s^2 G \frac{M^{5/2} I_{ii}}{(I_{xx} + I_{yy} + I_{zz})^{3/2}}, \quad s = \text{const}
\]

\[
H_{ii} = -\frac{1}{4} s^2 G \frac{M^{5/2} \dot{I}_{ii}}{(I_{xx} + I_{yy} + I_{zz})^{3/2}}
\]

(the quasi-homologousity assumption). Transform our equations into a dimensionless form. Let \( E \) denotes a Jacobi constant for a cluster,

\[
l_0 = (1/4s^2)^2 G^2 M^5/(-E)^2 \quad \text{— a unit of inertia moments},
\]

\[
t_0 = (1/2)l_0^{1/2}(-E)^{-1/2} \quad \text{— a unit of time (the crossing time)}.\]
Put $\tau = t/t_0$, $'= \frac{d}{d\tau}$, $\iota = \iota_{ii}/l_0$, $k_i = (1/2)K_{ii}/(-E)$. Then

$$
\iota''_x - k_x = -\frac{\iota_x}{(2\iota_x + \iota_z)^{3/2}} + \gamma_1 \iota_x \\
\iota''_z - k_z = -\frac{\iota_z}{(2\iota_x + \iota_z)^{3/2}} - \gamma_2 \iota_z \\
k'_x = -\frac{\iota'_x}{(2\iota_x + \iota_z)^{3/2}} + 2\gamma_3 \iota'_x \\
k'_z = -\frac{\iota'_z}{(2\iota_x + \iota_z)^{3/2}} - 2\gamma_2 \iota'_x
$$

Here $\gamma_1 = t_0^2(\chi_r^2 - 2\Omega^2) = 2t_0^2(A^2 - B^2)$, $\gamma_2 = t_0^2 \chi_z^2$, $\gamma_3 = t_0^2 \chi_R^2 = 4t_0^2 A(A - B)$.

Denote equilibrium values of $\iota_x$, $\iota_z$, $k_x$, $k_z$ by $a^2$, $b^2$, $\kappa_x$, $\kappa_z$. Then

$$2a^2 + b^2 = 1, \quad \kappa_x = 2a^2 - \gamma_1, \quad \kappa_z = b^2 + \gamma_2.$$
Put $2\nu_x = 2a^2 + x$, $\nu_z = b^2 + z$, $\nu_x' = X$, $\nu_z' = Z$, $k_x = \kappa_x + \xi$, $k_z = \kappa_y + \zeta$. Write down linearized equations:

\[
\begin{align*}
X' &= -\alpha_x x - \alpha_z z + \xi, \\
Z' &= -\beta_x x - \beta_z z + \zeta, \\
\xi' &= -(1 - 2\gamma_3)X, \\
\zeta' &= -(1 + 2\gamma_2)Z, \\
x' &= X, \\
z' &= Z,
\end{align*}
\]

where $\alpha_x = -3a^2 + 1 - \gamma_1$, $\alpha_z = -3a^2$, $\beta_x = -3b^2$, $\beta_z = -3b^2 + 1 + \gamma_2$. Now one can seek for eigenfrequencies $\lambda_i$. Denoting $\lambda^2 = \sigma$, we find

\[
\sigma_{1,2} = -\frac{5}{4} - \frac{p_1}{2} \pm \frac{1}{4} \sqrt{(5 + 2p_1)^2 - 16(1 + p_2)}.
\]

with $p_1 = \gamma_1 - 2\gamma_2 + 2\gamma_3$, $p_2 = b^2(5\gamma_1 + 2\gamma_3) - 4\gamma_3 - 2\gamma_2(\gamma_1 + 2\gamma_3)$. 
For isolated clusters $p_1 = p_2 = 0$, and $\sigma_1 = -2$, $\sigma_2 = -\frac{1}{2}$. Corrections due to $\gamma_i$ are small.

For realistic values of parameters $\sigma_1 < 0$, $\sigma_2 < 0$, i.e. $\lambda^2 < 0$ and the cluster oscillate under joint action of its gravity and external action of the Galaxy.

Danilov confirmed this conclusion.

Danilov also studied oscillations of a system consisting of a core and envelope.
THANK YOU FOR YOUR PATIENCE!