Guided Waves by Electromagnetic Gratings and Non-uniqueness Examples for the Diffraction Problem

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Communicated by J. C. Nédélec

We consider the two-dimensional scalar problem of the diffraction of a plane wave by an infinite grating of conducting bodies immersed in a periodical dielectric medium. A Fredholm-type formulation is derived and studied. The existence of a solution is proved and some uniqueness results are established. A detailed description of the guided modes of the grating is carried out. Finally, various non-uniqueness examples for the diffraction problem are exhibited.

1. Introduction

Electromagnetic gratings are involved in many physical applications (cf. [11, 13]). They are widely used in guided optics, in particular as coupling devices. They appeared more recently in the domain of hyperfrequencies with, e.g. the phased arrays antennas or the periodical coatings which give interesting scattering properties to the bodies they cover.

We consider here the problem, often set in practice, of the diffraction of a harmonic plane wave by a planar electromagnetic grating. This grating is supposed to be periodical in one direction and invariant in the perpendicular direction. It is constituted of both conducting and dielectric materials (cf. Fig. 1).

If the direction of the incident wave is perpendicular to the invariance axis of the grating, the problem reduces classically into two uncoupled scalar two-dimensional problems corresponding to Transverse Electric (TE) or Transverse Magnetic (TM) waves (see, e.g. [4] or [6]).

In section 2, we briefly recall equations and boundary conditions for both problems. Since the grating is supposed to be infinite, the usual two-dimensional Sommerfeld radiation condition has to be replaced by a quasi-periodic radiation condition (cf. [1, 18]).

Received 12 November 1992

In section 3, following [12], we derive an equivalent formulation of the problem set in a bounded domain which is of Fredholm type. So we establish the existence of a solution. Moreover, we prove that this solution is unique except possibly for a sequence of singular frequencies without accumulation point, depending on the so-called reduced frequency. Finally, we give a condition on the conducting bodies and on the refractive index (which extends the result of [11]) so that the TM diffraction problem has no singular frequencies.

In section 4, a detailed description of the guided modes of the grating is carried out by using the min–max principle for self-adjoint operators, which gives a characterization of the eigenvalues located below the continuous spectrum. The relation between guided waves and singular frequencies for the diffraction problem is pointed out in section 5, and non-trivial examples of gratings for which singular frequencies effectively exist are constructed. These frequencies correspond to eigenvalues which are embedded in the continuous spectrum.

2. Position of the diffraction problem

2.1. Geometrical description of the grating

The grating is constituted by conducting and dielectric materials. We suppose that the conducting bodies fill a domain $\tilde{\Omega} \times \mathbb{R}$ of $\mathbb{R}^3$ which is invariant in the $x_3$-direction and periodic, with period $d$, in the $x_1$-direction. We denote by $\tilde{\Omega} \times \mathbb{R}$ the exterior domain. In $\tilde{\Omega} \times \mathbb{R}$, the medium is characterized by its refractive index $\tilde{n}(x_1, x_2, x_3)$ which is supposed to be also invariant in the $x_3$-direction, $\tilde{n}(x_1, x_2, x_3) = n(x_1, x_2)$, and periodic with period $d$ in the $x_1$-direction:

$$ \begin{cases} (x_1, x_2) \in \tilde{\Omega} \iff (x_1 + pd, x_2) \in \tilde{\Omega}, \\ n(x_1, x_2) = n(x_1 + pd, x_2), \end{cases} \quad \forall x = (x_1, x_2) \in \tilde{\Omega}, \forall p \in \mathbb{Z}. \quad (2.1) $$

Moreover, the grating is assumed to be bounded in the $x_2$-direction, so that there exists a value $H_0$ such that

$$ \begin{cases} \tilde{\Omega} = \{ x \in \mathbb{R}^2; \ |x_2| \leq H_0 \}, \\ n(x) = n_\infty \text{ if } |x_2| \geq H_0, \end{cases} \quad \forall x \in \tilde{\Omega}, \quad (2.2) $$

where $n_\infty$ is a strictly positive real.
Classically, we suppose finally that \( n \in L^\infty(\Omega) \) and that
\[
\inf n = n^- > 0.
\] (2.3)
The magnetic permeability \( \mu \) is supposed to be constant, equal to its value in the vacuum.

### 2.2. Quasi-periodicity and radiation condition

We suppose that the grating is submitted to an incident monochromatic plane wave. If the wave vector \( k \) of this plane wave is contained in the plane \((x_1, x_2)\), Maxwell’s system is classically (see [4]) equivalent to two uncoupled scalar two-dimensional equations corresponding to Transverse Electric (TE) and Transverse Magnetic (TM) waves, respectively.

In this work, we will therefore consider the following two independent systems of equations:

**TE equations:**
\[
\begin{align*}
\text{div} \left( \frac{1}{n} \nabla u \right) + k^2 u &= 0 \quad \text{in } \tilde{\Omega}, \\
\frac{\partial u}{\partial \nu} &= 0 \quad \text{on } \partial \tilde{\Omega},
\end{align*}
\] (2.4)

where \( \nu \) denotes the unit outward normal to \( \partial \tilde{\Omega} \).

**TM equations:**
\[
\begin{align*}
\Delta u + k^2 n^2 u &= 0 \quad \text{in } \tilde{\Omega}, \\
u &= 0 \quad \text{on } \partial \tilde{\Omega}.
\end{align*}
\] (2.5)

In these equations, \( u \) denotes the total field
\[
u = \nu^{\text{dif}} + \nu^{\text{inc}},
\] (2.6)
where
\[
u^{\text{inc}}(x) = \nu^{\text{inc}} e^{i(k_1 x_1 + k_2 x_2)}
\] (2.7)
and \( \nu^{\text{dif}} \) is the diffracted field. The term \( e^{-i\nu} \) has been dropped by linearity.

The incident field \( \nu^{\text{inc}} \) solves the equations in the exterior domain \( (|x_2| > H_0) \) so that the wavenumber \( k = \frac{\nu}{c} \) (where \( c \) denotes the velocity of light in vacuum) is given by
\[
k^2 = \frac{1}{n^2} (k_1^2 + k_2^2).
\] (2.8)

As usual, diffraction problems (2.4) and (2.5) are not well-posed and must be completed by a radiation condition. Since the grating is unbounded in the \( x_1 \)-direction, the classical Sommerfeld condition is not appropriate. Let us denote by \( \theta \) the unique real such that
\[
k_1 = \theta + \frac{2p_0 \pi}{d}, \quad \theta \in \left[ -\frac{\pi}{d'}, \frac{\pi}{d'} \right], \quad p_0 \in \mathbb{Z}.
\] (2.9)
We will say that \( \theta \) is the reduced wavenumber.
It is obvious that the incident field \( u^{\text{inc}} \) satisfies the following condition:

\[
\begin{align*}
  u^{\text{inc}}(x_1 + pd, x_2) &= u^{\text{inc}}(x_1, x_2) e^{ipd}, \quad \forall p \in \mathbb{Z}, \forall x \in \tilde{\Omega}.
\end{align*}
\]  

(2.10)

Suppose now that \( u \) is a solution of (2.4) or (2.5). Then, by the periodicity of \( \Omega \) and of the index profile \( n \), every field \( u^{(p)} \) of the form

\[
\begin{align*}
  u^{(p)}(x) &= u(x_1 + pd, x_2) e^{-ipd}
\end{align*}
\]

(2.11)

for \( p \in \mathbb{Z} \) is a solution of the same problem. Of course, the associated diffracted field \( (u^{(p)})^{\text{dif}} \) has the same behaviour for \( |x_2| \rightarrow + \infty \) as \( u^{\text{dif}} \), and there is no physical criterion to eliminate some of these solutions.

In order for the problem not to have infinitely many solutions, we will look only for \( \theta \)-periodic solutions, i.e. solutions \( u \) such that

\[
\begin{align*}
  u(x_1 + pd, x_2) &= u(x_1, x_2) e^{ipd}, \quad \forall p \in \mathbb{Z}, \forall x \in \tilde{\Omega}.
\end{align*}
\]

(2.12)

**Remark 2.1.** Notice that \( d \) is supposed to be a period of the grating but not necessarily the smallest one. If \( u \) satisfies (2.12) but not for the smallest period, then we can construct by (2.11), various solutions of the same problem, but only a finite number. We will see in section 5.2 that this situation can effectively arise. However, generally, the diffraction problem we consider below is well-posed (cf. section 3) and the solution satisfies (2.12) for every period of the grating and in particular for the smallest one.

The \( \theta \)-periodicity condition will allow us now to write a radiation condition in the \( x_1 \)-direction. Indeed, if the diffracted field \( u^{\text{dif}} \) is \( \theta \)-periodic, it means that the function

\[
\begin{align*}
  v(x_1, x_2) &= u^{\text{dif}}(x_1, x_2) e^{-i\theta x_1}
\end{align*}
\]

(2.13)

is periodic. Moreover, since \( u^{\text{dif}} \) is \( \mathcal{C}^\infty \) for \( |x_2| > H_0 \) by the classical regularity results for the Helmholtz equation, the Fourier decomposition of \( v \) leads us to the so-called \( \theta \)-periodic Fourier decomposition of \( u^{\text{dif}} \):

\[
\begin{align*}
  u^{\text{dif}}(x_1, x_2) = \sum_{p \in \mathbb{Z}} u_p^{\text{dif}}(x_2) e^{i(\theta + \theta_p) x_1},
\end{align*}
\]

(2.14)

where

\[
\begin{align*}
  \theta_p = \frac{2\pi p}{d}
\end{align*}
\]

(2.15)

and

\[
\begin{align*}
  u_p^{\text{dif}}(x_2) = \int_0^\infty u^{\text{dif}}(x_1, x_2) e^{-i(\theta + \theta_p) x_1} dx_1.
\end{align*}
\]

(2.16)

Since \( u^{\text{dif}} \) satisfies

\[
\begin{align*}
  \Delta u^{\text{dif}} + k^2 n_0^2 u^{\text{dif}} = 0
\end{align*}
\]

(2.17)

for \( |x_2| > H_0 \), the Fourier coefficient \( u_p^{\text{dif}} \) must solve the following differential equation:

\[
\begin{align*}
  \frac{d^2 u_p^{\text{dif}}}{dx_2^2} + (k^2 n_0^2 - (\theta + \theta_p)^2) u_p^{\text{dif}} = 0 \quad \text{if } |x_2| > H_0.
\end{align*}
\]

(2.18)
This equation has two linearly independent solutions but only one of them is physically admissible. Indeed, let us set:

\[ Z^+(k, \theta) = \{ p \in \mathbb{Z}; (\theta + \theta_p)^2 - k^2 n_{n_z}^2 > 0 \} , \]

\[ Z^0(k, \theta) = \{ p \in \mathbb{Z}; (\theta + \theta_p)^2 - k^2 n_{n_z}^2 = 0 \} , \]

\[ Z^-(k, \theta) = \{ p \in \mathbb{Z}; (\theta + \theta_p)^2 - k^2 n_{n_z}^2 < 0 \} . \]

Then

(i) for \( p \in Z^+(k, \theta) \) we select the exponentially decreasing solution

\[ u_p^{\text{diff}}(x_2) = u_p^{\text{diff}}(\pm H_0) e^{\mp \sqrt{-(\theta + \theta_p)^2 - k^2 n_{n_z}^2}}(x_2 \pm H_0) , \]

(ii) for \( p \in Z^0(k, \theta) \) we select the constant solution

\[ u_p^{\text{diff}}(x_2) = u_p^{\text{diff}}(\pm H_0) , \]

(iii) for \( p \in Z^-(k, \theta) \) we select the solution which satisfies the outgoing Sommerfeld radiation condition

\[ u_p^{\text{diff}}(x_2) = u_p^{\text{diff}}(\pm H_0) e^{\mp H_0 \sqrt{-(\theta + \theta_p)^2 - k^2 n_{n_z}^2}}(x_2 \pm H_0) . \]

Finally, the diffracted field \( u^{\text{diff}} \) has the following expression for \( |x_2| > H_0 \):

\[ u^{\text{diff}}(x) = \sum_{p \in Z^+(k, \theta)} u_p^{\text{diff}}(\pm H_0) e^{\mp \sqrt{-(\theta + \theta_p)^2 - k^2 n_{n_z}^2}}(x_2 \pm H_0) e^{i(\theta + \theta_p)x_1} \]

\[ + \sum_{p \in Z^0(k, \theta)} u_p^{\text{diff}}(\pm H_0) e^{i(\theta + \theta_p)x_1} \]

\[ + \sum_{p \in Z^-(k, \theta)} u_p^{\text{diff}}(\pm H_0) e^{\mp H_0 \sqrt{-(\theta + \theta_p)^2 - k^2 n_{n_z}^2}}(x_2 \pm H_0) e^{i(\theta + \theta_p)x_1} . \]  \hspace{1cm} (2.20)

Notice that \( Z^0(k, \theta) \) and \( Z^-(k, \theta) \) are finite sets, so that the series expansion (2.20) contains only a finite number of non-decreasing terms. However, by definition of \( \theta \) (cf. (2.9)), the cardinal of \( Z^0(k, \theta) \cup Z^-(k, \theta) \) cannot be equal to 0.

3. Resolution of the diffraction problems

Since we look now for \( \theta \)-periodic fields, the problems can be written on a cell of the grating and we introduce the following notations (cf. Fig. 2):

\[ \Omega = \{ x \in \tilde{\Omega}; 0 < x_1 < d \} , \]

\[ \Gamma = \{ x \in \partial \tilde{\Omega}; 0 \leq x_1 \leq d \} , \]

\[ \mathcal{A} = \{ x_2 \in \mathbb{R}; (0, x_2) \in \tilde{\Omega} \} . \]

Let us consider the following problems:

\[ \text{P}_{\text{TE}}\text{-problem:} \]

Find \( u \) such that

\[ \text{div} \left( \frac{1}{n^2} \nabla u \right) + k^2 u = 0 \quad \text{in} \; \Omega , \]  \hspace{1cm} (3.2)
\[ \frac{\partial u}{\partial v} = 0 \quad \text{on } \Gamma, \]  
\[ u(d, x_2) = u(0, x_2) e^{i\beta d}, \quad \forall x_2 \in \mathcal{R}, \]  
\[ \frac{\partial u}{\partial x_1} (d, x_2) = \frac{\partial u}{\partial x_1} (0, x_2) e^{i\beta d}, \quad \forall x_2 \in \mathcal{R}, \]  
\[ u^\text{ext} = u - u^\text{inc} \text{ has the form (2.20) for } |x_2| > H_0. \]  

**P_{TM} problem:**  
Find \( u \) satisfying (3.4)--(3.6) such that  
\[ \Delta u + k^2 n^2 u = 0 \quad \text{in } \Omega, \]  
\[ u = 0 \quad \text{on } \Gamma. \]  

Clearly, thanks to conditions (3.4) and (3.5), every solution of P_{TE} (resp. P_{TM}) can be extended by pseudo-periodicity to a solution of the diffraction problem (2.4) (resp. (2.5)) in the whole plane.  

For \( H > H_0 \), we set  
\[ \Omega_H = \{ x \in \Omega; \ |x_2| < H \}, \]  
\[ \Gamma_H^+ = \{ x \in \Omega; \ x_2 = \pm H \}, \]  
\[ \Gamma_H = \Gamma_H^+ \cup \Gamma_H^-, \]  
\[ \mathcal{R}_H = \{ x_2 \in \mathcal{R}; \ |x_2| < H \}. \]  

The method we use for solving P_{TE} or P_{TM} is classical and consists in writing an equivalent problem, set in the bounded domain \( \Omega_H \), which is of Fredholm type. To do that, the main point is to write a boundary condition for \( u \) on \( \Gamma_H \). This condition will be derived from the expression (2.20) of the diffracted field for \( |x_2| > H \).
Before doing that, let us recall the functional spaces which will be used in the following. They have been already studied extensively (cf. [1, 18]) and we just recall the definitions and the main Green formulas.

3.1. \( \theta \)-periodic functional spaces

We will consider the following spaces:

1. \( \mathcal{S}_0^\infty (\mathbb{R}^2) \) is the set of all functions which are \( \mathcal{C}^\infty \) on \( \mathbb{R}^2 \), satisfy (2.12) and vanish for large \( |x_2| \).
2. \( \mathcal{H}_0^\infty (\Omega) \) (resp. \( \mathcal{H}_0^\infty (\Omega_H) \)) is the set of the restrictions to \( \Omega \) (resp. \( \Omega_H \)) of all functions of \( \mathcal{S}_0^\infty (\mathbb{R}^2) \).
3. \( H_0^1 (\Omega) \) (resp. \( H_0^1 (\Omega_H) \)) is the smallest closed subspace of \( H^1 (\Omega) \) (resp. \( H^1 (\Omega_H) \)) which contains \( \mathcal{H}_0^\infty (\Omega) \) (resp. \( \mathcal{H}_0^\infty (\Omega_H) \)).
4. \( H_0^1 (\Omega) \) (resp. \( H_0^1 (\Omega_H) \)) is the space of all functions \( u \in H_0^1 (\Omega) \) (resp. \( H_0^1 (\Omega_H) \)) such that \( u = 0 \) on \( \Gamma \).

The following Green formula holds for \( i = 1 \) or 2

\[
\forall u, v \in H_0^1 (\Omega), \quad \int_\Omega \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} \, dx = - \int_\Omega u \frac{\partial \overline{v}}{\partial x_i} \, dx + \int_\Gamma u \overline{v} \, d\gamma, \quad (3.10)
\]

where \( \Omega \) and \( \Gamma \) are defined by (3.1) and where \( v = (v_1, v_2) \). Notice that formula (3.10) involves no term on the lateral boundaries \( \{x_1 = 0\} \) and \( \{x_1 = d\} \) of \( \Omega \). If moreover \( \text{div}(\rho \nabla u) \in L^2 (\Omega) \) for some periodic function \( \rho \in L^\infty (\Omega) \) and \( u \) satisfies (3.5), then

\[
\int_{\Omega_n} \text{div}(\rho \nabla u) \overline{v} \, dx = \int_{\Omega_n} \rho \nabla u \cdot \nabla \overline{v} \, dx - \int_{\Gamma \cup \Gamma_n} \rho \frac{\partial u}{\partial \nu} \overline{v} \, d\gamma. \quad (3.11)
\]

Likewise, we have

\[
\forall u, v \in H_0^1 (\Omega_H), \quad \int_{\Omega_n} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} \, dx = - \int_{\Omega_n} u \frac{\partial \overline{v}}{\partial x_i} \, dx + \int_{\Gamma \cup \Gamma_n} u \overline{v} \, d\gamma \quad (3.12)
\]

and if \( \text{div}(\rho \nabla u) \in L^2 (\Omega_H) \) and \( u \) satisfies (3.18),

\[
\int_{\Omega_n} - \text{div}(\rho \nabla u) \overline{v} \, dx = \int_{\Omega_n} \rho \nabla u \cdot \nabla \overline{v} \, dx - \int_{\Gamma \cup \Gamma_n} \rho \frac{\partial u}{\partial \nu} \overline{v} \, d\gamma. \quad (3.13)
\]

We also introduce the following space:

\[
H^{1/2}_0 (\Gamma_H^\pm) = \left\{ v = \sum_{p \in \mathbb{Z}} v_p (\pm H) e^{i(\theta_p + \theta_p^*) x_1}; \sum_{p \in \mathbb{Z}} (1 + (\theta + \theta_p))^{1/2} |v_p|^2 < + \infty \right\},
\]

\( H^{1/2}_0 (\Gamma_H^\pm) \) is a closed subspace of the usual Sobolev space \( H^{1/2} (\Gamma_H^\pm) \) and the norm

\[
|v|_{1/2, \partial}^2 = \sum_{p \in \mathbb{Z}} (1 + (\theta + \theta_p))^{1/2} |v_p|^2
\]

is equivalent on \( H^{1/2}_0 (\Gamma_H^\pm) \) to the classical \( H^{1/2} \)-norm. The dual space of \( H^{1/2}_0 (\Gamma_H^\pm) \) is

\[
H^{-1/2}_0 (\Gamma_H^\pm) = \left\{ v = \sum_{p \in \mathbb{Z}} v_p (\pm H) e^{i(\theta_p + \theta_p^*) x_1}; \sum_{p \in \mathbb{Z}} (1 + (\theta + \theta_p))^{-1/2} |v_p|^2 < + \infty \right\}
\]
and the associated norm is

$$|v|^2_{-1/2, \delta} = d \sum_{p \in Z} (1 + (\theta + \theta_p))^{-1/2} |v_p|^2.$$  

Of course, $H^{1/2}(\Gamma^H)$ is exactly the space of the traces on $\Gamma^H$ of all functions of $H^{1}(\Omega_H)$. The duality product between $H^{1/2}(\Gamma^H)$ and $H^{-1/2}(\Gamma^H)$ is given by

$$\langle u, v \rangle_{H^{1/2}(\Gamma^H)} = d \sum_{p \in Z} u_p (\pm H) \delta_p (\pm H).$$

### 3.2. Truncation of the domain

If $u$ is a solution of $P_{\text{T}E}$ or $P_{\text{T}M}$, then, by (2.20), $u^\text{diff}$ satisfies on $\Gamma^H$ and $\Gamma^{H-}$, the following boundary conditions:

$$\frac{\partial u^\text{diff}}{\partial n} = - T^\pm (k, \theta) u^\text{diff} \quad \text{on} \quad \Gamma^H,$$

where the operators $T^+ (k, \theta)$ and $T^- (k, \theta)$ are defined as follows:

$$T^\pm (k, \theta) \begin{cases} H^{1/2}(\Gamma^H) \\ \sum_{p \in Z} v_p (\pm H) e^{i(k \cdot x + \theta_p)} x_1 \end{cases} \rightarrow \begin{cases} H^{-1/2}(\Gamma^H) \\ \sum_{p \in Z} \mu_p (k, \theta) v_p (\pm H) e^{i(k \cdot x + \theta_p)x_1}, \end{cases}$$

where

$$\mu_p (k, \theta) = \begin{cases} \sqrt{\left(- k^2 n_x^2 + (\theta + \theta_p)^2\right)} & \text{if} \ p \in Z^+ (k, \theta), \\ 0 & \text{if} \ p \in Z^0 (k, \theta), \\ i \sqrt{\left(k^2 n_x^2 - (\theta + \theta_p)^2\right)} & \text{if} \ p \in Z^- (k, \theta). \end{cases}$$

One can easily check that $T^\pm (k, \theta)$ is continuous.

Consider now the following problems:

**P_{\text{T}E}-problem:**

Find $u \in H^{1}(\Omega_H)$ such that

$$\text{div} \left( \frac{1}{n^2} \nabla u \right) + k^2 u = 0 \quad \text{in} \quad \Omega_H,$$  

$$\frac{\partial u}{\partial n} = 0 \quad \text{on} \quad \Gamma,$$  

$$\frac{\partial u}{\partial x_1} (d, x_2) = \frac{\partial u}{\partial x_1} (0, x_2) e^{i\theta_d} \quad \forall x_2 \in \partial \Omega_H,$$  

$$\frac{\partial u}{\partial n} = \frac{\partial u^{\text{inc}}}{\partial n} - T^\pm (k, \theta) (u - u^{\text{inc}}) \quad \text{on} \quad \Gamma^H.$$  

**P_{\text{T}M}-problem:**

Find $u \in H^{1}_{0, \mathcal{O}}(\Omega_H)$ satisfying (3.18) and (3.19) such that

$$\Delta u + k^2 n^2 u = 0 \quad \text{in} \quad \Omega_H.$$  

(3.20)
Problems $P_{TE}$ (resp. $P_{TM}$) and $P_{TE}^H$ (resp. $P_{TM}^H$) are clearly equivalent in the sense of the following proposition.

**Proposition 3.1.** If $u$ is a solution of $P_{TE}$ (resp. $P_{TM}$) such that $\tilde{u} = u|_{\Gamma_1} \in H^4_\delta(\Omega_H)$, then $\tilde{u}$ is a solution of $P_{TE}^H$ (resp. $P_{TM}^H$). Conversely, if $\tilde{u}$ is a solution of $P_{TE}^H$ (resp. $P_{TM}^H$), it can be extended to a solution $u$ of $P_{TE}$ (resp. $P_{TM}$).

**Proof.** The first part of the proposition is obvious since (3.19) is equivalent to (3.14). Conversely, if $\tilde{u}$ is a solution of $P_{TE}^H$ (resp. $P_{TM}^H$) set

$$
\begin{align*}
    u = \begin{cases} 
        \tilde{u} & \text{if } |x_2| < H, \\
        \sum_{\ell \in \mathbb{Z}} \tilde{a}_\ell (\pm H) e^{j \mu \rho_\ell \theta \xi_2} (x_2 H) + i (\theta + \xi_1) & \text{if } |x_2| > H. 
    \end{cases}
\end{align*}
$$

(3.21)

This function is clearly continuous through $|x_2| = H$. Moreover, the continuity of the normal derivative is obtained thanks to condition (3.19), and one can check easily that $u$ is a solution of $P_{TE}$ (resp. $P_{TM}$). \hfill \Box

**Remark 3.1.** Notice that problems $P_{TE}^H$ and $P_{TM}^H$ can be solved numerically using finite elements after truncation of the series expansion. This method for the resolution of diffraction problems is known as the 'localized finite elements method' (see [10]).

### 3.3. Fredholm decomposition and existence result

For the sake of simplicity, we will consider now in detail the case of a TE diffraction problem. The TM diffraction problem can be studied exactly in the same way and we only state the main results.

Problem $P_{TE}^H$ has the following variational formulation:

$$
\begin{align*}
    \begin{cases} 
        \text{Find } u \in H^4_\delta(\Omega_H) \text{ such that } \\
        a_{TE}^H(u, v) = l_{TE}^H(v), \\
        \forall v \in H^4_\delta(\Omega_H),
    \end{cases}
\end{align*}
$$

(3.22)

where the forms $a_{TE}^H(u, v)$ and $l_{TE}^H(v)$ are defined by

$$
\begin{align*}
    a_{TE}^H(u, v) &= \int_{\Omega_H} \left( \frac{1}{n^2} \nabla u \cdot \nabla \bar{v} - k^2 u \bar{v} \right) dx \\
    &\quad + \frac{1}{n^2} \left( \langle T^+(k, \theta) u, v \rangle_{\theta} - \langle T^-(k, \theta) u, v \rangle_{\theta} \right),
\end{align*}
$$

$$
\begin{align*}
    l_{TE}^H(v) &= \frac{1}{n^2} \left( \left\langle \frac{\partial u^{inc}}{\partial v} + T^+(k, \theta) u^{inc}, v \right\rangle_{\theta, r_z^+} + \left\langle \frac{\partial u^{inc}}{\partial v} + T^-(k, \theta) u^{inc}, v \right\rangle_{\theta, r_z^-} \right).
\end{align*}
$$

We now establish a Fredholm decomposition for formulation (3.22). The following identity is obvious:

$$
\begin{align*}
    a_{TE}^H(u, v) = b(u, v) + c(u, v),
\end{align*}
$$

(3.23)

where

$$
\begin{align*}
    b(u, v) &= \int_{\Omega_H} \left( \frac{1}{n^2} \nabla u \cdot \nabla \bar{v} + u \bar{v} \right) dx + \frac{1}{n^2} \left( \langle T^+(k, \theta) u, v \rangle_{\theta} + \langle T^-(k, \theta) u, v \rangle_{\theta} \right),
    \end{align*}
$$

$$
\begin{align*}
    c(u, v) &= \frac{1}{n^2} \left( \langle T^+(k, \theta) u, v \rangle_{\theta} - \langle T^-(k, \theta) u, v \rangle_{\theta} \right).
\end{align*}
$$
and
\[ c(u, v) = -(k^2 + 1) \int_{\Omega_H} u \bar{v} \, dx. \]

Since \( b(u, v) \) and \( c(u, v) \) are continuous sesquilinear forms on \( H^1_0(\Omega_H) \), we can define two bounded operators \( B \) and \( C \) of \( H^1_0(\Omega_H) \) by the identities
\[ (Bu, v)_{H^1_0(\Omega_H)} = b(u, v) \quad \text{and} \quad (Cu, v)_{H^1_0(\Omega_H)} = c(u, v) \quad \text{for all} \ u, v \in H^1_0(\Omega_H). \]

We have the following theorem.

**Lemma 3.1.** The operator \( B \) is an automorphism of \( H^1_0(\Omega_H) \) and the operator \( C \) is compact on \( H^1_0(\Omega_H) \).

**Proof.** By definition of \( T^\pm(k, \theta) \) (cf. (3.15)), we have
\[ \forall v \in H^1_0(\Omega_H), \quad \text{Re} \langle T^\pm(k, \theta)v, v \rangle_{H^1_0(\Omega_H)} = d \sum_{p \in Z^2} \sqrt{((\theta + \theta_p)^2 - k^2 n_x^2)} |v_p(\pm H)|^2 \geq 0 \]
and consequently, \( b(\ldots) \) is coercive on \( H^1_0(\Omega_H) \),
\[ \forall u \in H^1_0(\Omega_H), \quad \text{Re} \ b(u, u) \geq \frac{1}{n^2} \int_{\Omega_H} |\nabla u|^2 + |u|^2 \, dx. \]
This proves that \( B \) is an automorphism of \( H^1_0(\Omega_H) \). The compactness of \( C \) is a direct consequence of the compactness of the injection of \( H^1_0(\Omega_H) \) into \( L^2(\Omega_H) \). \( \square \)

Let us denote by \( w^{\text{inc}} \) the unique element of \( H^1_0(\Omega_H) \) such that
\[ (w^{\text{inc}}, v)_{H^1_0(\Omega_H)} = l^T_{\text{TE}}(v), \quad \forall v \in H^1_0(\Omega_H). \]
Then problem \( P^H_{\text{TE}} \) can be formulated as follows:

Find \( u \in H^1_0(\Omega_H) \) such that \( Bu + Cu = w^{\text{inc}} \). \( \quad (3.24) \)

By Lemma 3.1 and by the Fredholm alternative, the existence of a solution will follow from the study of the homogeneous problem

Find \( u \in H^1_0(\Omega_H) \) such that \( Bu + Cu = 0 \) \( \quad (3.25) \)

which has the equivalent following form

Find \( u \in H^1_0(\Omega_H) \) such that \( a^H_{\text{TE}}(u, v) = 0, \quad \forall v \in H^1_0(\Omega_H). \) \( \quad (3.26) \)

**Lemma 3.2.** Let \( u \) be a solution of (3.26). Then \( u_p(\pm H) = 0, \forall p \in Z^+(k, \theta). \)

**Proof.** If follows directly from the identity
\[ \text{Im}(a^H_{\text{TE}}(u, u)) = 0. \]
\( \square \)

Lemma 3.2 means that a solution of the homogeneous problem is either exponentially decreasing or tends to a constant function for large \( |x_z| \).

From Lemmas 3.1 and 3.2, we deduce the following theorem.

**Theorem 3.1.** Problem \( P^H_{\text{TE}} \) has at least one solution and the set of solutions is at most a finite-dimensional affine space.
Proof. This result is based on the Fredholm alternative. If (3.26) has no non-trivial solution, then (3.22) has one and only one solution.

Suppose now that (3.26) has non-trivial solutions and let us denote by \( V_{\text{TE}} \) the set of these solutions. Then \( \dim V_{\text{TE}} < + \infty \) and problem (3.22) has a solution if and only if

\[
I_{\text{TE}}^H(v) = 0, \quad \forall v \in V_{\text{TE}}. \tag{3.27}
\]

Let us prove (3.27). By (2.7) and (2.9),

\[
u_\text{inc}^\text{te}(x) = i_0^\text{te} e^{i k x} e^{i(\phi + \theta_\text{p}) x},
\]

and, by (2.8), \( p_0 \in Z^- \) or \( p_0 \in Z^0 \). Then an easy calculation gives

\[
I_{\text{TE}}^H(v) = \frac{i}{n_\text{io}} (\langle |k_2| + k_2 \rangle v_{\theta, r_{\text{p}}}^\text{te} + (|k_2| - k_2 \rangle v_{\theta, r_{\text{p}}}^\text{te}).
\]

If \( p_0 \in Z^-(k, \theta) \), then, by Lemma 3.2, \( \langle v_{\text{inc}}^\text{te}, v \rangle_{\theta, r_{\text{p}}} = 0 \) for every \( v \in V_{\text{TE}} \). If \( p_0 \in Z^0(k, \theta) \), then \( k_2 = 0 \) and therefore,

\[
I_{\text{TE}}^H(v) = 0, \quad \forall v \in H_0^1(\Omega_\text{H}).
\]

An analogous result stands for problem \( P_{\text{TM}}^H \).

\[\Box\]

Theorem 3.2. Problem \( P_{\text{TM}}^H \) has at least one solution and the set of solutions is at most a finite-dimensional affine space.

3.4. Characterization of the singular frequencies

In the following, we will call singular frequency for problem \( P_{\text{TE}} \), every value of \( k \) such that (3.26) has non-trivial solutions. In this section, we will prove that, for a given reduced frequency \( \theta \), the singular frequencies form at most a countable sequence without accumulation point. The idea of the proof is due to Nedelec and Starling [12].

Let us denote by \( T_{\theta}^\pm(k, \theta) \) the 'real part' of \( T^\pm(k, \theta) \):

\[
T_{\theta}^\pm : \sum_{p \in Z} v_p(\pm H) e^{i(\theta + \theta_\text{p}) x} \to \sum_{p \in Z^+ (k, \theta)} \mu_p(k, \theta) v_p(\pm H) e^{i(\theta + \theta_\text{p}) x},
\]

and set

\[
a_{\theta}(k; u, v) = \int_{\Omega_\text{H}} \frac{1}{i n_\text{io}} \nabla u \cdot \nabla \bar{v} + \int_{\Omega_\text{H}} \left( \langle T_{\theta}^+(k, \theta) u, v \rangle_{\theta, r_{\text{p}}} + \langle T_{\theta}^-(k, \theta) u, v \rangle_{\theta, r_{\text{p}}} \right).
\]

A direct consequence of Lemma 3.2 is the following lemma.

Lemma 3.3. If \( u \) is a non-trivial solution of (3.26), then it is a solution of

\[
u \in H_0^1(\Omega_\text{H}), \quad u \neq 0,
\]

\[
a_{\theta}(k; u, v) = k^2 \int_{\Omega_\text{H}} u \bar{v} \, dx \quad \forall v \in H_0^1(\Omega_\text{H}). \tag{3.28}
\]

Notice that the converse statement is false. Indeed, a solution \( u \) of (3.28) is a solution of (3.26) if and only if \( u_p(\pm H) = 0 \) for \( p \in Z^-(k, \theta) \).
The reason why we introduced problem (3.28) is that the bilinear form $a_{\theta}(k; \ldots)$ is hermitian, while $a_{\theta}(k; \ldots)$ was not. Let us first consider the following eigenvalue problem:

$$\begin{align*}
\text{Find } \lambda \text{ such that there exists } u \in H_0^1(\Omega_H), & \ u \neq 0, \text{ satisfying} \\
\int_{\Omega_H} a_{\theta}^H(k; u, u) & = \lambda \int_{\Omega_H} u \overline{v} \ dx \quad \forall v \in H_0^1(\Omega_H). 
\end{align*}$$

(3.29)

Since $a_{\theta}^H(k; u, v) + \int_{\Omega_H} u \overline{v} \ dx$ is coercive on $H_0^1(\Omega_H)$, the study of problem (3.29) is very classical (cf. [9]) and we can state the following proposition.

**Proposition 3.2.** The eigenvalues of problem (3.29) form a sequence $(\lambda_m(k))_{k \geq 1}$ which tends to $+\infty$ and one has the following characterization:

$$\lambda_m(k) = \min_{\mathcal{V}_m(H_0^1(\Omega_H))} \sup_{u \neq 0} \frac{a_{\theta}^H(k; u, u)}{\int_{\Omega_H} |u|^2 \ dx},$$

(3.30)

where $\mathcal{V}_m(H_0^1(\Omega_H))$ denotes the set of all $m$-dimensional subspaces of $H_0^1(\Omega_H)$.

A direct consequence of the previous proposition is the following corollary.

**Corollary 3.4.** Problem (3.28) has solutions if and only if $k$ solves for some $m \geq 1$ the equation

$$\lambda_m(k) = k^2.$$  

(3.31)

We can now establish the main result of this subsection.

**Theorem 3.3.** Problem $P_{TE}$ is well-posed for every value of $k$ except maybe for the values $k = k_{m}^{TE}(\theta, H)$, where $k_{m}^{TE}(\theta, H)$ denotes the unique solution of equation (3.31). Moreover, the sequence $(k_{m}^{TE}(\theta, H))_{m \geq 1}$ tends to infinity as $m \to +\infty$.

**Proof.** To prove that equation (3.31) has one and only one solution, it suffices to check that the function $k \mapsto \lambda_m(k)$ is continuous and non-increasing (cf. Fig. 3). The monotonicity of $\lambda_m$ results directly from the monotonicity of the functions $k \mapsto a_{\theta}(k; u, u) / \int_{\Omega_H} |u|^2 \ dx$. Let us prove now the continuity. Suppose $k < k'$ and $k$ near $k'$. We will consider two cases:

![Fig. 3. The function $k \mapsto \lambda_m(k)$](image-url)
(1) If $Z^0(k', \theta) = \emptyset$, then, $Z^+(k, \theta) = Z^+(k', \theta)$ and for $u \in H^1_\Gamma(\Omega_\mu)$, $u \neq 0$, we have

$$0 \leq a^H_{\mu}(k; u, u) - a^H_{\mu}(k'; u, u)$$

$$\leq \frac{1}{n_{\mu}} \sum_{p \in Z^+(k', \theta)} (\mu_p(k, \theta) - \mu_p(k', \theta))(|u_p(H)|^2 + |u_p(-H)|^2)$$

$$\leq \frac{1}{n_{\mu}} M(k', \theta) \left( \|u\|^2_{L^2(\Gamma_\mu)} + \|u\|^2_{L^2(\Gamma_{\partial \mu})} \right),$$

where

$$M(k', \theta) = \min_{p \in Z^+(k, \theta)} \sqrt{(\theta + \theta_p)^2 - (k'n_{\mu})^2}.$$

(2) If $Z^0(k', \theta) \neq \emptyset$, then, $Z^+(k, \theta) = Z^+(k', \theta) \cup Z^0(k', \theta)$ and for $u \in H^1_\Gamma(\Omega_\mu)$, $u \neq 0$, we have

$$0 \leq a^H_{\mu}(k; u, u) - a^H_{\mu}(k'; u, u)$$

$$\leq \frac{1}{n_{\mu}} \sum_{p \in Z^+(k', \theta)} (\mu_p(k, \theta) - \mu_p(k', \theta))(|u_p(H)|^2 + |u_p(-H)|^2)$$

$$+ \frac{1}{n_{\mu}} \sum_{p \in Z^0(k', \theta)} (|u_p(H)|^2 + |u_p(-H)|^2)$$

$$\leq \frac{1}{n_{\mu}} \left( \sqrt{k'^2 - k^2} n_{\mu} + \frac{k^2 - k'^2}{M(k', \theta)} \right) \|u\|^2_{L^2(\Gamma_\mu)}.$$

We deduce from the previous estimates, from the continuity of the trace application, from $H_\Gamma(\Omega_\mu)$ on $L^2(\Gamma_\mu \cup \Gamma_{\partial \mu})$ and from the coercivity of $a^H_{\mu}(k', \ldots)$ that

$$\frac{a^H_{\mu}(k'; u, u)}{\|u\|^2_{L^2(\Gamma_\mu)}} \leq \frac{a^H_{\mu}(k; u, u)}{\|u\|^2_{L^2(\Gamma_\mu)}} \leq (1 + \varepsilon(k, k')) \frac{a^H_{\mu}(k'; u, u)}{\|u\|^2_{L^2(\Gamma_\mu)}},$$

where $\lambda_m(k') \leq \lambda_m(k) \leq (1 + \varepsilon(k, k')) \lambda_m(k')$,

which proves the continuity of $\lambda_m$. $\square$

Likewise we can establish the following theorem.

**Theorem 3.4.** Problem $P_{TM}$ is well-posed for every value of $k$ except maybe for values $k = k_{TM}^{\mu}(\theta, H)$, where $(k_{TM}^{\mu}(\theta, H))_{m \geq 1}$ is an increasing sequence which tends to infinity as $m \to + \infty$.

### 3.5. Unicity result

To conclude section 3, we will give a condition on the refractive index $\mu$ and on the boundary of the conducting bodies $\Gamma$ such that problem $P_{TM}$ is well-posed for every value of $k$. In other words, under that condition, there are no singular frequencies for problem $P_{TM}$. Our result is an extension of the result established by Alber (cf. [1]), who considered the case of an homogeneous medium and supposed $k^2 n_{\infty}^2 \neq (\theta + \theta_p)^2$ for all $p \in \mathbb{Z}$.

We did not succeed in establishing a similar result for problem $P_{RE}$, except in the case of a pure dielectrical grating without conducting bodies ($\Gamma = \emptyset$).
The variational form of problem $P_{TM}$ is given by

\[
\begin{align*}
\text{find } & u \in H^1_{\delta,0}(\Omega_H) \text{ such that } \\
& a^H_{TM}(u,v) = l^H_{TM}(v), \quad \forall v \in H^1_{\delta}(\Omega_H),
\end{align*}
\]

where the forms $a^H_{TM}(u,v)$ and $l^H_{TM}(v)$ are defined by

\[
\begin{align*}
a^H_{TM}(u,v) &= \int_{\Omega_H} (\nabla u \cdot \nabla v - k^2 n^2 u \bar{v}) \, dx \\
& \quad + \left( \langle T^+(k,\theta) u, v \rangle_{\partial \Gamma^+_H} + \langle T^-(k,\theta) u, v \rangle_{\partial \Gamma^-_H} \right), \\
l^H_{TM}(v) &= \left( \left\langle \frac{\delta u^{inc}}{\delta n} + T^+(k,\theta) u^{inc}, v \right\rangle_{\partial \Gamma^+_H} + \left\langle \frac{\delta u^{inc}}{\delta n} + T^-(k,\theta) u^{inc}, v \right\rangle_{\partial \Gamma^-_H} \right).
\end{align*}
\]

A value $k$ is a singular frequency for problem $P_{TM}$ if the homogeneous problem

\[
\begin{align*}
\text{find } & u \in H^1_{\delta,0}(\Omega_H) \text{ such that } \\
& a^H_{TM}(u,v) = 0, \quad \forall v \in H^1_{\delta}(\Omega_H)
\end{align*}
\]

has non-trivial solution.

Let us now prove the following theorem.

**Theorem 3.5.** If

\[
\begin{align*}
& v_1 x_2 \leq 0 \quad \text{on } \Gamma \\
& \frac{\partial}{\partial x_2} (n^2) x_2 \geq 0 \quad \text{in the sense of } \mathcal{D}'(\Omega),
\end{align*}
\]

where $v = (v_1, v_2)$ denotes the unit outward normal to $\Gamma$, then $P_{TM}$ is well-posed for every value of $k$.

**Proof.** Following [5], one can check by using Greens formula (3.13) that

\[
\forall u \in H^1_{\delta,0}(\Omega_H) \cap H^2(\Omega_H),
\]

\[
2 \text{Re} \left( \int_{\Omega_H} x_2 \frac{\partial u}{\partial x_2} \Delta \bar{u} \, dx \right) = \int_{\Omega_H} \left( \left| \frac{\partial u}{\partial x_1} \right|^2 - \left| \frac{\partial u}{\partial x_2} \right|^2 \right) \, dx \\
- \int_{\Gamma^+_H} x_2 \left( |\nabla u|^2 v_2 - 2 \text{Re} \left( \frac{\partial u}{\partial x_2} \frac{\partial \bar{u}}{\partial x_2} \right) \right) \, dy. \quad (3.35)
\]

Let $u$ be a solution of (3.33). Then

\[
\Delta u + k^2 n^2 u = 0 \quad \text{in } \Omega_H \\
u = 0 \quad \text{on } \Gamma, \\
\frac{\partial u}{\partial n} = -T^\pm(k,\theta) u \quad \text{on } \Gamma^\pm_H.
\]

Consequently,

\[
2 \text{Re} \left( \int_{\Omega_H} x_2 \frac{\partial u}{\partial x_2} \Delta \bar{u} \, dx \right) = k^2 \int_{\Omega_H} n^2 |u|^2 \, dx + k^2 \left( x_2 \frac{\partial}{\partial x_2} (n^2), |u|^2 \right) \\
- k^2 n^2 \int_{\Gamma_H} |u|^2 \, dy. \quad (3.36)
\]
(where \( \langle \ldots \rangle \) denotes the duality product between \( H^1(\Omega_H) \) and its dual) and

\[
\int_{\Omega_H} \left( \frac{\partial u}{\partial x_1}^2 - \left| \frac{\partial u}{\partial x_2} \right|^2 \right) \, dx = k^2 \int_{\Omega_H} n^2 |u|^2 \, dx - 2 \int_{\Omega_H} \frac{\partial u}{\partial x_2} \, dx.
\]  

(3.37)

Moreover, since \( u = 0 \) on \( \Gamma \), \( \nabla u = (\partial u/\partial v)v \) on \( \Gamma \) and therefore,

\[
\int_{\Gamma} x_2 \left( |\nabla u|^2 v_2 - 2 \text{Re} \left( \frac{\partial u}{\partial x_2} \frac{\partial u}{\partial v} \right) \right) \, dy = -\int_{\Gamma} x_2 v_2 \left| \frac{\partial u}{\partial v} \right|^2 \, dy.
\]  

(3.38)

From (3.35)–(3.38), we deduce

\[
2 \int_{\Omega_H} \left| \frac{\partial u}{\partial x_2} \right|^2 \, dx + k^2 \left( \int_{\Omega_H} x_2 \frac{\partial}{\partial x_2} (n^2) \, dx \right) + \langle T^+(k, \theta) u, u \rangle_{\theta, \Gamma_H} + \langle T^-(k, \theta) u, u \rangle_{\theta, \Gamma_H} = \int_{\Gamma} x_2 v_2 \left| \frac{\partial u}{\partial v} \right|^2 \, dy + F_H(u),
\]  

(3.39)

where

\[
F_H(u) = -H \int_{\Gamma_H} \left( \frac{\partial u}{\partial x_1}^2 - \left| \frac{\partial u}{\partial v} \right|^2 - k^2 n^2_x |u|^2 \right) \, dy.
\]

But from \( \text{Im}(a_{\theta_H}(u, u)) = 0 \), we deduce that \( u_p(\pm H) = 0 \) if \( p \in Z^- (k, \theta) \) and consequently,

\[
F_H(u) = -H \sum_{p \in Z} \left( \theta + \theta_p \right)^2 - k^2 n^2_x - \mu^2_p(k, \theta) \left( |u_p(H)|^2 + |u_p(-H)|^2 \right) = 0
\]

and

\[
\langle T^+(k, \theta) u, u \rangle_{\theta, \Gamma_H} + \langle T^-(k, \theta) u, u \rangle_{\theta, \Gamma_H} \in \mathbb{R}^+.
\]

The result then follows from the identity (3.39) and from hypothesis (3.34).  

\[\square\]

Likewise, one can establish the following theorem.

**Theorem 3.6.** If

\[
\Gamma = \emptyset,
\]

(3.40)

\[
\frac{\partial}{\partial x_2} (n^2) x_2 \geq 0 \text{ in the sense of } \mathcal{D}'(\Omega),
\]

then \( P_{RE} \) is well-posed for every value of \( k \).

4. Study of the guided modes of the grating

4.1. Definition of the guided modes

For a given \( \theta \in ] - \pi/d, \pi/d [ \), we now consider the following problems:

\[
\begin{cases}
\text{Find } k \in \mathbb{R}^+ \text{ such that there exists } u \in H^1_{b}(\Omega), \ u \neq 0, \\
satisfying (3.2)–(3.5). 
\end{cases}
\]

(4.1)

\[
\begin{cases}
\text{Find } k \in \mathbb{R}^+ \text{ such that there exists } u \in H^1_{b,0}(\Omega), \ u \neq 0, \\
satisfying (3.7), (3.4) \text{ and (3.5).}
\end{cases}
\]

(4.2)
A solution \((k, u)\) of (4.1) (resp. (4.2)) is called a TE (resp. TM) guided mode.

Let \((k, u)\) be a solution of (4.1) and set

\[ v(x_1, x_2, t) = u(x) e^{-i k x_1} e^{i n x_2}, \quad \forall x \in \tilde{\Omega}, \quad pd \leq x_1 \leq (p + 1)d, \]

then \(v\) is a solution of

\[ \frac{1}{c^2} \frac{\partial^2 v}{\partial t^2} - \text{div} \left( \frac{1}{n^2} \nabla v \right) = 0 \quad \text{in} \quad \tilde{\Omega} \]

\[ \frac{\partial v}{\partial n} = 0 \quad \text{on} \quad \tilde{\Gamma} \]

and moreover

\[ v(x_1 + d, x_2, t) = v(x_1, x_2, t - \frac{\theta d}{\omega}), \quad \forall x \in \tilde{\Omega}, \quad \forall t \in \mathbb{R}. \]

Consequently, the function \(v\) describes a wave which propagates in the direction \(x_1\) without attenuation with the velocity \((k/\theta)c\). The condition \(u \in H^1_\Omega(\Omega)\) expresses the fact that the energy of \(v\) is confined near the grating. For that reason, we say that \(v\) is a guided or a surface wave.

The link between problems (4.1) and (4.2) and the diffraction problems studied in sections 2 and 3 is given by the following proposition.

**Proposition 4.1.** Suppose

\[ \theta^2 \leq k^2 n_\infty^2. \]

Then:

1. If \((k, u)\) is a solution of (4.1) (resp. (4.2)), then \(k\) is a singular frequency for problem \(P_{\text{TE}}\) (resp. \(P_{\text{TM}}\)) and \(u\) is a non-trivial solution of the associated homogeneous problem.

2. If \(k\) is a singular frequency for problem \(P_{\text{TE}}\) (resp. \(P_{\text{TM}}\)) such that \(k^2 n_\infty^2 \neq (\theta + \theta_p)^2\) for every \(p \in \mathbb{Z}\), and if \(u\) is a non-trivial solution of the associated homogeneous problem, then \((k, u)\) is a solution of (4.1) (resp. (4.2)).

**Remark 4.1.** Notice that (4.4) is a direct consequence of (2.8) and (2.9). Therefore, this condition is automatically satisfied if we consider the diffraction problems. However, it does not play any role in the proofs of the previous section, except for the existence results of Theorems 3.1 and 3.2. In particular, Theorems 3.3–3.6 remain valid for every \(\theta \in ] - \pi/d, \pi/d]\) and \(k \in \mathbb{R}^+\).

**Proof.** The first statement is trivial. Conversely, suppose that \(u\) is a solution of \(P_{\text{TE}}\) with \(u^{\text{ext}} = 0\). Then \(u|_{\Omega^H}\) is a solution of (3.26) and by Lemma 3.2, \(u_p(\pm H) = 0\) for every \(p \in Z^-(k, \theta)\). If we suppose moreover that \(k^2 n_\infty^2 \neq (\theta + \theta_p)^2, \forall p \in \mathbb{Z}\), then \(Z^0(k, \theta) = \emptyset\) and therefore, by (2.20), \(u\) decreases exponentially as \(|x_2| \to +\infty\). Consequently, \(u \in H^1_\Omega(\Omega)\) and \((k, u)\) is a solution of (4.1). A similar result stands for problem \(P_{\text{TM}}\).

The analysis which is carried out in this section has to be compared to mathematical studies of guided modes in various other applications (cf. [2, 8, 3]).
4.2. Position of the eigenvalue problems and general estimates for the spectrum

Using the Green formula (3.13), one can derive the variational formulations of problems (4.1) and (4.2):

(1) \((k, u)\) is a solution of (4.1) if and only if
\[
\begin{align*}
&k \in \mathbb{R}^+, \quad u \in H^1_0(\Omega), \quad u \neq 0, \\
&a_{TE}(u, v) = k^2(u, v)_{TE}, \quad \forall v \in H^1_0(\Omega),
\end{align*}
\]
(4.5)

(2) \((k, u)\) is a solution of (4.2) if and only if
\[
\begin{align*}
&k \in \mathbb{R}^+, \quad u \in H^1_{k,0}(\Omega), \quad u \neq 0, \\
&a_{TM}(u, v) = k^2(u, v)_{TM}, \quad \forall v \in H^1_{k,0}(\Omega),
\end{align*}
\]
(4.6)

where
\[
\begin{align*}
a_{TE}(u, v) &= \int_{\Omega} \frac{1}{n^2} \nabla u \cdot \nabla \bar{v} \, dx, \quad (u, v)_{TE} = \int_{\Omega} u \bar{v} \, dx, \\
a_{TM}(u, v) &= \int_{\Omega} \nabla u \cdot \nabla \bar{v} \, dx, \quad (u, v)_{TM} = \int_{\Omega} n^2 u \bar{v} \, dx.
\end{align*}
\]
(4.7)

We set moreover,
\[
\|u\|_{TE} = \sqrt{(u, u)_{TE}} \quad \text{and} \quad \|u\|_{TM} = \sqrt{(u, u)_{TM}}
\]
(4.8)
and we denote by \(L^2_T(\Omega)\) (resp. \(L^2_M(\Omega)\)) the Hilbert space isomorphic to the usual space \(L^2(\Omega)\) and equipped with the norm \(\| \cdot \|_{TE}\) (resp. \(\| \cdot \|_{TM}\)).

Let us now consider the two operators \(A_{TE}(\theta)\) and \(A_{TM}(\theta)\) defined as follows: \(A_{TE}(\theta)\) (resp. \(A_{TM}(\theta)\)) is the unbounded operator of \(L^2_T(\Omega)\) (resp. \(L^2_M(\Omega)\)) associated with the sesquilinear form \(a_{TE}(\ldots, \ldots)\) (resp. \(a_{TM}(\ldots, \ldots)\)), i.e.
\[
\begin{align*}
&\{ D(A_{TE}(\theta)) = \{ u \in H^1_0(\Omega); \exists C/ |a_{TE}(u, v)| \leq C \|u\|_{TE} \|v\|_{TE} \forall v \in H^1_0(\Omega) \} \\
&\forall u \in D(A_{TE}(\theta)) \quad \forall v \in H^1_0(\Omega), \quad (A_{TE}(\theta)u, v)_{TE} = a_{TE}(u, v),
\end{align*}
\]
(4.9)

\[
\begin{align*}
&\{ D(A_{TM}(\theta)) = \{ u \in H^1_{k,0}(\Omega); \exists C/ |a_{TM}(u, v)| \leq C \|u\|_{TM} \|v\|_{TM} \forall v \in H^1_{k,0}(\Omega) \} \\
&\forall u \in D(A_{TM}(\theta)) \quad \forall v \in H^1_{k,0}(\Omega), \quad (A_{TM}(\theta)u, v)_{TM} = a_{TM}(u, v).
\end{align*}
\]
(4.10)

First we have the following lemma.

**Lemma 4.1.** (1) \((k, u)\) is a solution of (4.1) (resp. (4.2)) if and only if \(k^2\) is an eigenvalue of \(A_{TE}(\theta)\) (resp. \(A_{TM}(\theta)\)) and \(u\) is an associated eigenvector.

(2) \(A_{TE}(\theta)\) and \(A_{TM}(\theta)\) are positive self-adjoint operators.

**Proof.** The self-adjointness results classically from the coercivity of \(a_{TE}(u, v) + (u, v)_{TE}\) and \(a_{TM}(u, v) + (u, v)_{TM}\), and the positivity from the positivity of \(a_{TE}\) and \(a_{TM}\). \(\square\)

Let us denote by \(\sigma_{TE}\) (resp. \(\sigma_{TM}\)) the spectrum of \(A_{TE}(\theta)\) (resp. \(A_{TM}(\theta)\)) and by \(\sigma_{TE}^0\) (resp. \(\sigma_{TM}^0\)) its essential spectrum. Since \(A_{TE}(\theta)\) and \(A_{TM}(\theta)\) are self-adjoint operators bounded from below, the eigenvalues located below the essential spectrum can be characterized by the Min–Max principle which is stated in the next subsection.

Therefore, the first step consists in the determination of \(\sigma_{TE}^0\) and \(\sigma_{TM}^0\).

Let us establish a preliminary lemma.
**Lemma 4.2.** Let \( K = \{(x_1, x_2); \ 0 < x_1 < d \quad \text{and} \quad a < x_2 < b\} \) with \(-\infty \leq a < b \leq +\infty\). Then
\[
\forall u \in H^1_b(K), \quad \int_K \left| \frac{\partial u}{\partial x_1} \right|^2 \, dx \geq \theta^2 \int_K |u|^2 \, dx.
\]

**Proof.** Let \( u \in \mathcal{C}_b^\infty(K) \). Then
\[
u(x_1, x_2) = \sum_{p \in Z} u_p(x_2) e^{i(\theta + \theta_p)x_1}, \quad \text{where} \quad u_p(x_2) = \int_0^d u(x) e^{-i(\theta + \theta_p)x_1} \, dx_1.
\]
This series expansion and its derivatives converge uniformly. Consequently,
\[
\int_K |u|^2 \, dx = d \sum_{p \in Z} \int_0^b |u_p(x_2)|^2 \, dx_2,
\]
\[
\int_K \left| \frac{\partial u}{\partial x_1} \right|^2 \, dx = d \sum_{p \in Z} (\theta + \theta_p)^2 \int_0^b |u_p(x_2)|^2 \, dx_2
\]
and the lemma follows since \( \inf_{p \in Z} (\theta + \theta_p)^2 = \theta^2 \).

We can now establish the following theorem.

**Theorem 4.1.** (1) \( \sigma_{TE} \subset \mathbb{R}^+ \),

(2) \( \sigma_{TM} = \left[ \frac{\theta^2}{n_*^2} + \infty \right] \), where \( n_* = \sup_{x \in \Omega} n(x) \),

(3) If \( \Gamma = \emptyset \), \( \sigma_{TE} = \left[ \frac{\theta^2}{n_*^2} + \infty \right] \),

(4) \( \sigma^*_T = \sigma_{TM} = \left[ \frac{\theta^2}{n_*^2} + \infty \right] \).

**Proof.** (1) The inclusion \( \sigma_{TE} \subset \mathbb{R}^+ \) follows immediately from the positivity of \( A_{TE} \).

(2) Every function of \( u \in H^1_{\theta, 0}(\Omega) \) can be extended by 0 to a function \( \tilde{u} \in H^1_b(K) \) with \( K = ]0, d[ \times \mathbb{R} \). Consequently, by Lemma 4.2,
\[
\int_\Omega \left| \frac{\partial u}{\partial x_1} \right|^2 \, dx \geq \frac{\theta^2}{n_*^2} \int_\Omega |u|^2 \, dx, \quad \forall u \in H^1_{\theta, 0}(\Omega).
\]
This proves the second inclusion.

(3) Assertion (3) can be established in the same way.

(4) Let us consider e.g. \( \sigma_{TE} \) (the determination of \( \sigma_{TM} \) can be handled in the same way). Let \( \lambda \geq \theta^2/n_*^2 \) and consider the sequence \( (u^{(m)}_{\lambda})_{m \geq 1} \) of \( D(A_{TE}) \) defined by
\[
u(x_1, x_2) = \frac{1}{\sqrt{m}} \psi \left( \frac{x_2}{m} \right) e^{\sqrt{2\lambda m^2 - \theta^2} x_1},
\]
where \( \psi \) is a \( \mathcal{C}^\infty \) compactly supported function such that \( \psi(x_2) = 0 \) for \( |x_2| < H_0 \) and \( \int_0^\infty (\psi(x_2))^2 \, dx_2 = 1 \). One can easily check that \( (u^{(m)}) \) is a singular sequence associated to \( \lambda \), i.e. (cf. [16]):
\( (1) \quad \| u^{(m)} \|_{TE} = 1, \quad \forall m \)

\( (2) \quad \| A_{TE}(\theta) u^{(m)} - \lambda u^{(m)} \|_{TE} \to +\infty \)

\( (3) \quad u^{(m)} \to 0 \quad \text{in} \quad L^2(TE) \).

Consequently, \( \lambda \in \sigma_{TE}^+ \). This proves the inclusion \( [\theta^2/n_\infty^2, +\infty[ \subset \sigma_{TE}^+ \).

To prove the converse inclusion, suppose \( \lambda \in \sigma_{TE}^+ \). Then there exists a singular sequence \((u^{(m)})\) of \( D(A_{TE}(\theta)) \) associated to \( \lambda \) satisfying 1–3. From 1 and 2, we deduce that

\[ (A_{TE}(\theta) u^{(m)}, u^{(m)})_{TE} \to \lambda. \quad \text{(4.11)} \]

Moreover, if \( K = \Omega \setminus \Omega_{Ho} \), Lemma 4.2 provides the following inequality:

\[ (A_{TE}(\theta) u^{(m)}, u^{(m)})_{TE} \geq \frac{1}{n_\infty^2} \int_K \left| \frac{\partial u^{(m)}}{\partial x_1} \right|^2 \, dx \geq \frac{\theta^2}{n_\infty^2} \int_K |u^{(m)}|^2 \, dx. \quad \text{(4.12)} \]

By 1 and (4.11), \( u^{(m)} \) is bounded in \( H^1_0(\Omega) \) and therefore, by 3, \( u^{(m)} \to 0 \) in \( L^2(\Omega_{Ho}) \). Consequently,

\[ \int_K |u^{(m)}|^2 \, dx \to 1, \]

which, joined with (4.11) and (4.12), gives

\[ \lambda \geq \frac{\theta^2}{n_\infty^2}. \]

\[ \square \]

4.3. Min–max principle and dispersion curves

Let us set for \( u \in H_0^1(\Omega), u \neq 0, \)

\[ R_{TE}(u) = \frac{d_{TE}(u, u)}{\| u \|_{TE}^2}, \]

\[ R_{TM}(u) = \frac{d_{TM}(u, u)}{\| u \|_{TM}^2} \]

and for \( m \in \mathbb{N}, \)

\[ \lambda_{m}^{TE}(\theta) = \inf_{\gamma_m \in \gamma_m(H_0^1(\Omega))} \sup_{u \in \gamma_m, u \neq 0} R_{TE}(u), \]

\[ \lambda_{m}^{TM}(\theta) = \inf_{\gamma_m \in \gamma_m(H_0^1(\Omega))} \sup_{u \in \gamma_m, u \neq 0} R_{TM}(u), \quad \text{(4.13)} \]

where \( \gamma_m(X) \) denotes the set of all \( m \)-dimensional subspaces of \( X \).

We denote by \( \mathcal{N}_{TE}(\theta) \) (resp. \( \mathcal{N}_{TM}(\theta) \)) the number of eigenvalues \( \lambda \) of \( A_{TE}(\theta) \) (resp. \( A_{TM}(\theta) \)) located strictly below the essential spectrum, i.e. \( \lambda < \theta^2/n_\infty^2 \). Each eigenvalue is counted a number of times equal to its multiplicity.

Then we have the min–max principle (cf. [14]):

1. \( (\lambda_{m}^{TE}(\theta))_{m \geq 1} \) is an increasing sequence which converges to \( \theta^2/n_\infty^2 \).
2. If \( \lambda_{m}^{TE}(\theta) < \theta^2/n_\infty^2 \) then \( \mathcal{N}_{TE}(\theta) \geq m \) and \( \lambda_{m}^{TE}(\theta), \lambda_{m}^{TE}(\theta), \ldots, \lambda_{m}^{TE}(\theta) \) are the \( m \) first eigenvalues of \( A_{TE}(\theta) \).
3. If \( \lambda_{m}^{TE}(\theta) = \theta^2/n_\infty^2 \) then \( \mathcal{N}_{TE}(\theta) < m. \)
The same principle can be stated for the TM problem. It will be used now to establish existence results for guided modes such that \( k^2 < \theta^2/n_2^2 \). The existence of guided modes such that \( k^2 \geq \theta^2/n_2^2 \) will be discussed in section 5.

To make explicit the dependence of \( \lambda_m^{TE}(\theta) \) and \( \lambda_m^{TM}(\theta) \) with respect to \( \theta \), we will establish equivalent formulas such that \( \theta \) appears in the expression of the Rayleigh quotients and not in that of the functional spaces. Indeed, using the change of test function,

\[
v(x_1, x_2) = u(x_1, x_2) e^{-i\theta x_1},
\]

we derive from (4.13),

\[
\lambda_m^{TE}(\theta) = \inf_{v \neq 0} \sup_{\mathcal{V}_{m \in \mathcal{V}_m(\Omega)_0} v \neq 0} \mathcal{R}_{TE}(\theta; v), \quad \lambda_m^{TM}(\theta) = \inf_{v \neq 0} \sup_{\mathcal{V}_{m \in \mathcal{V}_m(\Omega)_0} v \neq 0} \mathcal{R}_{TM}(\theta; v), \tag{4.14}
\]

where

\[
H^1_\omega(\Omega) = H^1_\omega(\Omega) \quad \text{and} \quad H^1_{\omega,0}(\Omega) = H^1_{\omega,0}(\Theta) \quad \text{for} \quad \theta = 0
\]

\[
\mathcal{R}_{TE}(\theta; v) = \frac{a_{TE}(\theta; v, v)}{v^{\frac{1}{2}}}, \quad \mathcal{R}_{TM}(\theta; v) = \frac{a_{TM}(\theta; v, v)}{v^{\frac{1}{2}}},
\]

\[
a_{TE}(\theta; v, v) = \int_\Omega \frac{1}{\Omega} \left( |v|^2 - 2\theta \text{Im} \left( v \frac{\partial v}{\partial x_1} \right) + \theta^2 |v|^2 \right) dx, \tag{4.15}
\]

\[
a_{TM}(\theta; v, v) = \int_\Omega \left( |v|^2 - 2\theta \text{Im} \left( v \frac{\partial v}{\partial x_1} \right) + \theta^2 |v|^2 \right) dx.
\]

(The notations \( H^1_\omega(\Omega) \) and \( H^1_{\omega,0}(\theta) \) are introduced to avoid confusion with the standard space \( H^1(\Omega) \).

\textbf{Remark 4.2:} Since \( a_{TE}(-\theta; v, v) = a_{TE}(-\theta; \bar{v}, \bar{v}) \), it is clear that \( \lambda_m^{TE}(\theta) = \lambda_m^{TE}(-\theta) \) and likewise \( \lambda_m^{TM}(\theta) = \lambda_m^{TM}(-\theta) \). Consequently, in the following, we suppose that \( \theta \in [0, \pi/d] \).

![Fig. 4. The functions \( \lambda_m^{TE}(\theta) \)](image-url)
Thanks to formulas (4.14), we can establish some properties of the dispersion curves.

**Theorem 4.2.** (1) The functions $\theta \to \lambda_{m}^{\text{TE}}(\theta)$ and $\theta \to \lambda_{m}^{\text{TM}}(\theta)$ are continuous on $[0, \pi/d]$. 
(2) If $n(x) \geq n_0$ a.e. $x \in \Omega$, then the function $\theta \to \lambda_{m}^{\text{TE}}(\theta) - (\theta^2/n_0^2)$ is non-increasing on $[0, \pi/d]$ and consequently, the function $\theta \to \mathcal{N}_{\text{TE}}(\theta)$ is non-decreasing on $[0, \pi/d]$. 
(3) The function $\theta \to \lambda_{m}^{\text{TM}}(\theta) - (\theta^2/n_0^2)$ is non-increasing on $[0, \pi/d]$ and consequently, the function $\theta \to \mathcal{N}_{\text{TM}}(\theta)$ is non-decreasing on $[0, \pi/d]$.

**Proof.** (1) To prove the continuity, we proceed as in the proof of Theorem 3.3. Consider, e.g. the case of the TE modes. Let $v \in H_{d}^{1}(\Omega)$ such that $\|v\|_{i}^{2} = 1$ and $\theta_1 \neq \theta_2$. Then

$$a_{\text{TE}}(\theta_1; v, v) - a_{\text{TE}}(\theta_2; v, v) = 2(\theta_2 - \theta_1) \int_{\Omega} \frac{1}{n^2} \text{Im} \left( v \frac{\partial \tilde{\phi}}{\partial x_1} \right) dx + (\theta_1^2 - \theta_2^2) \int_{\Omega} \frac{1}{n^2} |v|^2 dx,$$

and consequently

$$|a_{\text{TE}}(\theta_1; v, v) - a_{\text{TE}}(\theta_2; v, v)| \leq |\theta_1 - \theta_2| \frac{2}{n^2} \left( \left\| \frac{\partial v}{\partial x_1} \right\|_{\text{TE}}^2 + \max(\theta_1 - \theta_2) \right). \tag{4.16}$$

Moreover, from the Young inequality,

$$2\theta \text{Im} \left( v \frac{\partial \tilde{\phi}}{\partial x_1} \right) \leq \frac{1}{2} \left| \frac{\partial v}{\partial x_1} \right|^2 + 2\theta^2 |v|^2,$$

we deduce the coercivity estimate

$$a_{\text{TE}}(\theta_1; v, v) + \theta^2 \int_{\Omega} \frac{1}{n^2} |v|^2 dx \geq \int_{\Omega} \frac{1}{2} \left| \frac{\partial v}{\partial x_1} \right|^2 + \left| \frac{\partial v}{\partial x_2} \right|^2 \right) dx. \tag{4.17}$$

From (4.16) and (4.17), we deduce finally

$$a_{\text{TE}}(\theta_1; v, v) \leq a_{\text{TE}}(\theta_2; v, v)$$

$$+ \frac{2}{n^2} |\theta_1 - \theta_2| \left( \sqrt{2n^2 \left( a_{\text{TE}}(\theta_2; v, v) + \frac{\theta_2^2}{n^2} \right) + \max(\theta_1, \theta_2) \right),$$

which implies

$$\lambda_{m}^{\text{TE}}(\theta_1) \leq \lambda_{m}^{\text{TE}}(\theta_2)$$

$$+ \frac{2}{n^2} |\theta_1 - \theta_2| \left( \sqrt{2n^2 \left( \lambda_{m}^{\text{TE}}(\theta_2) + \frac{\theta_2^2}{n^2} \right) + \max(\theta_1, \theta_2) \right).$$

The result follows by interchanging $\theta_1$ and $\theta_2$.

(2) Suppose now that $n(x) \geq n_0$ a.e. By (4.14) and since $\lambda_{m}^{\text{TE}}(\theta) \leq (\theta^2/n_0^2)$, we have

$$\lambda_{m}^{\text{TE}}(\theta) - \frac{\theta^2}{n_0^2} = \inf_{V_m \neq V_0 \in H_{d}^{1}(\Omega) \setminus \{0\}} \sup \{ \frac{a_{\text{TE}}(\theta; v, v) - a_0^2 \|v\|^2_{\text{TE}}}{\|v\|^2_{\text{TE}}}, 0 \}.$$
Moreover, for every given \( v \), the function

\[
\theta \to \min(a_{\text{TE}}(\theta; v, v) - \frac{\theta^2}{n_\infty^2} \| v \|_{\text{TE}}^2, 0)
\]

is decreasing since

\[
\theta \to \int_\Omega \frac{1}{n^2} \left( |\nabla v|^2 - 2\theta \text{Im} \left( v \frac{\partial \varphi}{\partial x_1} \right) + \theta^2 \left( 1 - \frac{n^2}{n_\infty^2} \right) |v|^2 \right) dx
\]

is a polynomial function of the form \( a\theta^2 + b\theta + c \) with \( a \geq 0 \) (since \( n(x) \geq n_\infty \) a.e.) and \( c > 0 \). This proves assertion 2.

(3) The case of the TM modes can be handled in the same way. Indeed, for any given \( v \in H^{1,0}_0(\Omega) \), the function

\[
\theta \to \min(a_{\text{TM}}(\theta; v, v) - \frac{\theta^2}{n_\infty^2} \| v \|_{\text{TM}}^2, 0)
\]

is decreasing. To prove this, consider a function \( v \) such that

\[
a_{\text{TM}}(\theta; v, v) < \frac{\theta^2}{n_\infty^2} \| v \|_{\text{TM}}^2.
\]  \hspace{1cm} (4.18)

Then the polynomial function

\[
\theta \to \int_\Omega \left( |\nabla v|^2 - 2\theta \text{Im} \left( v \frac{\partial \varphi}{\partial x_1} \right) + \theta^2 \left( 1 - \frac{n^2}{n_\infty^2} \right) |v|^2 \right) dx
\]

is again of the form \( a\theta^2 + b\theta + c \) with \( a \) and \( c \) positive. The positivity of \( a \) is not obvious since we did not make any hypothesis on the refractive profile. However, by plugging in (4.18) the Fourier expansion of \( v \),

\[
v(x) = \sum_{\rho \in \mathbb{Z}} v_\rho(x_2) e^{i\rho x_1},
\]

we get

\[
\sum_{\rho \in \mathbb{Z}} \left( (\theta + \rho)^2 \int_\Omega |v_\rho(x_2)|^2 dx_2 + \int_\Omega \frac{\partial v_\rho}{\partial x_2}^2 dx_2 \right) < \frac{\theta^2}{n_\infty^2} \int_\Omega n^2 |v|^2 dx,
\]

and since \( \theta^2 = \text{Inf}(\theta + \rho)^2 \), this proves the following inequality:

\[
\theta^2 \int_\Omega \left( \frac{n^2}{n_\infty^2} - 1 \right) |v|^2 dx \geq 0.
\]  \hspace{1cm} (4.19)

**Remark 4.3.** (1) We do not know if the condition \( n(x) \geq n_\infty \) is necessary to establish the monotonicity of \( A_{\text{TE}}(\theta) \) but we did not succeed to establish directly an estimate analogous to (4.19) for the TE modes, even for a dielectric grating without conducting bodies (\( \Gamma = \emptyset \)).

(2) If \( \lambda \) is an eigenvalue of \( A_{\text{TE}}(\theta) \) (or likewise \( A_{\text{TM}}(\theta) \)) located strictly below \( \theta^2/n_\infty^2 \), the associated eigenfunction \( u \) has the following expression for \( |x_2| > H_0 \):

\[
u(x) = \sum_{\rho \in \mathbb{Z} \setminus \{0\}} u_\rho(\pm H_0) e^{\mp \sqrt{(\theta + \rho)^2 - \lambda} x_2} \chi_{\{|x_2| > H_0\}} e^{\pm (\theta + \rho) x_1}.
\]  \hspace{1cm} (4.20)

Consequently, \( u \) decreases exponentially with \( x_2 \) as \( e^{-\sqrt{\theta^2 - \lambda} x_2} \chi_{\{|x_2| \}} \).
Assertions 2 and 3 of the previous theorem show that the decreasing of the guided mode in the $x_2$-direction is maximal for the greatest value of $\theta$, $\theta = \pi/d$.

4.4. Existence and non-existence results

In this section, we will prove, thanks to the min--max principle, that guided modes can effectively exist. The existence of TE guided waves can result either from a variation of the refractive index profile $n$ or from a specific geometry of the boundary $\Gamma$ of the conducting bodies. For the TM modes, if the index profile is constant equal to $n_\infty$, the spectrum is equal to the essential spectrum and therefore, guided waves corresponding to eigenvalues below $\theta^2/n_\infty^2$ cannot exist. However, we prove that TM guided modes can exist if $n_+ > n_\infty$.

First we prove that, for every integer $m \geq 1$, one can construct examples of index profiles such that the numbers $\mathcal{N}_{TE}(\theta)$ and $\mathcal{N}_{TM}(\theta)$ of guided modes (below the essential spectrum) are greater than $m$.

**Theorem 4.3.** Let $K \subset \Omega$ and suppose that $n = n_+$ on $K$. Then

\[ \forall \theta > 0, \forall m \geq 1, \exists \eta_m(\theta) \text{ such that if } n_+ \geq \eta_m(\theta), \mathcal{N}_{TE}(\theta) \geq m \text{ and } \mathcal{N}_{TM}(\theta) \geq m. \]

**Proof.** Let $\mu_m$ denote the $m$th eigenvalue of $-\Delta$ in $K$ with a Dirichlet boundary condition. Classical min--max formulas give

\[ \mu_m = \min_{\mu_m \in \sigma_{-\Delta}(H_0^1(K))} \max_{\|u\|_{H_0^1(K)} = 1} \frac{\int_K |\nabla u|^2 \, dx}{\int_K |u|^2 \, dx}. \]

But every function of $H_0^1(K)$ can be extended by 0 to a function of $H^1_0(\Omega)$ and therefore, thanks to formula (4.13), we get

\[ \lambda_m^{TE}(\theta) \leq \frac{\mu_m}{n_+^2} \text{ and } \lambda_m^{TM}(\theta) \leq \frac{\mu_m}{n_+^2}. \]

The result follows since $\mu_m/n_+^2 < \theta^2/n_\infty^2$ for $n_+$ great enough. \hfill \Box

**Remark 4.4.** Suppose, e.g. that $K = ]l_1, l_2[ \times ]h_1, h_2[$ and set $L = l_2 - l_1$ and $H = h_2 - h_1$. Then a simple calculation shows that, for any strictly positive integers $p$ and $q$,

\[ \mathcal{N}_{TE}(\theta) \geq pq \text{ if } n_+^2 > \left( \frac{p^2}{L^2} + \frac{q^2}{H^2} \right) \frac{\pi^2 n_\infty^2}{\theta^2}. \]

The following theorem gives a sufficient condition on the refractive index profile for the first TE guided mode to exist for every value of the reduced frequency $\theta > 0$. If we consider a dielectrically grating, without conducting bodies ($\Gamma = \emptyset$), a similar result can be proved for the first TM guided mode. But if $\Gamma \neq \emptyset$, we show that the first TM mode does not exist for low reduced frequencies ($\theta$ near 0).

**Theorem 4.4.** (1) If $n_+ > n_\infty$ and

\[ \int_{\Omega} \left( \frac{1}{n^2(x)} - \frac{1}{n_\infty^2} \right) \, dx \leq 0, \]

then
then
\[ N_{TE}(\theta) \geq 1 \]
for every \( \theta \in \left[ 0, \frac{\pi}{d} \right] \).

(2) If \( \Gamma = \emptyset \), then
\[ N_{TM}(\theta) \geq 1 \quad \forall \theta \in \left[ 0, \frac{\pi}{d} \right] \text{ if and only if } n_+ > n_\infty \]
and \( \int_{\Omega} (n^2(x) - n_\infty^2) \, dx \geq 0 \).

(3) If \( \Gamma \neq \emptyset \), there exists \( \theta^* > 0 \) such that \( N_{TM}(\theta) = 0 \) if \( \theta \leq \theta^* \).

**Remark 4.5.** Notice that conditions
\[ \int_{\Omega} \left( \frac{1}{n^2(x)} - \frac{1}{n_\infty^2} \right) \, dx \leq 0, \quad \text{and} \quad \int_{\Omega} (n^2(x) - n_\infty^2) \, dx \geq 0 \]
are not equivalent.

**Proof.** (1) Consider the function \( v^H \) defined by
\[ v^H(x) = \begin{cases} \frac{1}{H} & \text{if } |x_2| < H, \\ \frac{2H - |x_2|}{H} & \text{if } H < |x_2| < 2H, \\ 0 & \text{if } |x_2| > 2H. \end{cases} \]

By (4.14),
\[ \lambda_{TE}^T(\theta) \leq a_{TE}(\theta; v^H, v^H) = \frac{2d}{n_\infty^2} \frac{\int_{\Omega} \left( \frac{1}{n^2} - \frac{1}{n_\infty^2} \right) \, dx}{\| v^H \|^2_{L_2} \| v^H \|^2_{L_2}} + \frac{\theta^2}{n_\infty^2}. \]

If \( \int_{\Omega} (1/n^2 - 1/n_\infty^2) \, dx < 0 \), this proves by taking \( H \) great enough that \( \lambda_{TE}^T(\theta) < \theta^2/n_\infty^2 \). If \( \int_{\Omega} (1/n^2 - 1/n_\infty^2) \, dx = 0 \), consider a function \( w \in H^1_2(\Omega) \) such that
\[ \int_{\Omega} (1/n^2 - 1/n_\infty^2) w \, dx < 0 \] and set \( w_\alpha^H = v^H + \alpha w \) for \( \alpha \in \mathbb{R}^+ \). Then

\[
\alpha_{TE}(\theta; w_\alpha^H, w_\alpha^H) - \theta^2 \left\| \frac{\alpha^2}{n_\infty^2} \right\|_{TE}^2 = \int_{\Omega} \frac{1}{n^2} |\nabla w_\alpha^H|^2 \, dx + \theta^2 \int_{\Omega} \left( \frac{1}{n^2} - \frac{1}{n_\infty^2} \right) |w_\alpha^H|^2 \, dx
\]

\[
\leq \int_{\Omega} \frac{1}{n^2} |\nabla v^H|^2 \, dx + 2\alpha \theta^2 \int_{\Omega} \left( \frac{1}{n^2} - \frac{1}{n_\infty^2} \right) w \, dx
\]

\[
+ \alpha^2 \alpha_{TE}(\theta; w, w)
\]

The term on the right-hand side of the previous inequality is strictly negative if \( \alpha \) is taken small enough and then \( H \) large enough.

2. Likewise, if \( \Gamma = \emptyset \), \( H_{\infty,0}^1(\Omega) = H^1_2(\Omega) \) and we have

\[
\lambda_{TM}(\theta) \leq \frac{\alpha_{TM}(\theta; v^H, v^H)}{\|v^H\|_{TM}^2} = \frac{2d}{H} + \theta^2 \int_{\Omega} \left( 1 - \frac{n^2}{n_\infty^2} \right) \, dx + \theta^2 \int_{\Omega} \left( \frac{1}{n^2} - \frac{1}{n_\infty^2} \right) \, dx
\]

which proves the existence of the first guided mode if \( \int_{\Omega} (n^2 - n_\infty^2) \, dx > 0 \). In the case of equality, we proceed as above.

Conversely, suppose there exist a sequence \( \theta_i \to 0 \) and a sequence \( v_i \in H^1_2(\Omega) \) such that

\[
a_{TM}(\theta_i; v_i, v_i) < \frac{\theta_i^2}{n_\infty^2} \|v_i\|_{TM}^2.
\]

By Lemma 4.2, since \( u_i = v_i e^{i\theta_1} \in H^1_2(\Omega) \),

\[
\int_{\Omega \cap \Omega_{\infty}} \left| \frac{\partial v_i}{\partial x_1} + i\theta_1 v_i \right|^2 \, dx \geq \theta_i^2 \int_{\Omega \cap \Omega_{\infty}} |v_i|^2 \, dx
\]

and consequently, by (4.21) and (4.22),

\[
\int_{\Omega \cap \Omega_{\infty}} \left( |\nabla v_i|^2 - 2\theta_1 \Im \left( v_i \frac{\partial v_i}{\partial x_1} \right) + \theta_i^2 |v_i|^2 \right) \, dx < \frac{\theta_i^2}{n_\infty^2} \int_{\Omega \cap \Omega_{\infty}} n^2 |v_i|^2 \, dx.
\]

If the sequence \( (\theta_i) \) is normalized by

\[
\int_{\Omega \cap \Omega_{\infty}} |v_i|^2 \, dx = 1,
\]

we deduce from (4.23) that \( \nabla v_i \to 0 \) in \( L^2(\Omega_{\infty}) \). Moreover, by the compactness of the injection of \( H^1(\Omega_{\infty}) \) into \( L^2(\Omega_{\infty}) \), there is a subsequence still denoted by \( (v_i) \) which converges strongly in \( H^1(\Omega_{\infty}) \) to a constant function \( v \). By (4.24), \( v \) is not equal to 0 and consequently, we deduce from (4.19) that \( \int_{\Omega} (n^2 - n_\infty^2) \, dx \geq 0 \) holds.

3. We assume now that \( \Gamma \neq \emptyset \). As in the previous part of the proof, suppose by contradiction the existence of a sequence \( \theta_i \to 0 \) and \( v_i \in H^1_{\infty,0}(\Omega) \) satisfying (4.21) and (4.24). Then, as above, there exists a subsequence converging strongly in \( H^1_{\infty,0}(\Omega_{\infty}) \) to a non-trivial constant function \( v \). But \( H^1_{\infty,0}(\Omega_{\infty}) \) does not contain any constant function except the function \( v \equiv 0 \). That achieves the proof.

If \( n_+ = n_\infty \) (in particular if the dielectric medium is homogeneous), by Theorem 4.1, \( \sigma_{TM} = \sigma_{TE} \) and consequently, \( \mathcal{N}_{TM}(\theta) = 0 \) for every value of \( \theta \). Likewise, if \( n_+ = n_\infty \) and \( \Gamma = \emptyset \), \( \sigma_{TE} = \sigma_{TE} \) and \( \mathcal{N}_{TE}(\theta) = 0 \) for every \( \theta \).
The TE problem of a grating with conducting bodies is more complicated and, to conclude this paragraph, we will show that TE guided waves may exist even if \( n_+ = n_- \).

Let \( K = \cup_{l_1, l_2 \in \mathbb{R}} h_1, h_2 \) and set \( L = l_2 - l_1 \) and \( H = h_2 - h_1 \). We decompose the boundary \( \partial K \) of \( K \) into two parts:

\[
\partial K = \Gamma_1 \cup \Gamma_h,
\]

where \( \Gamma_1 = \{ x \in \partial K ; x_1 = l_1 \text{ or } x_1 = l_2 \} \) and \( \Gamma_h = \{ x \in \partial K ; x_2 = h_1 \text{ or } x_2 = h_2 \} \). Then we have the following theorem.

**Theorem 4.5.** Suppose \( K \subset \Omega, \Gamma_1 \subset \Gamma \) (cf. Fig. 6) and \( n(x) = n_\infty \) a.e. in \( \Omega \). Then

\[
m^2 \leq \frac{\theta^2 H^2}{\pi^2} \Rightarrow \mathcal{N}_{TE}(\theta) \geq m.
\]

**Proof.** Consider the following eigenvalue problem

\[
\begin{cases}
- \frac{1}{n_\infty^2} \Delta v = \lambda v & \text{in } K, \\
v = 0 & \text{on } \Gamma_h, \\
\frac{\partial v}{\partial n} = 0 & \text{on } \Gamma_1.
\end{cases}
\]

The eigenvalues form a sequence \( (\lambda_{p,q}^K) \) tending to \( +\infty \) which can be calculated explicitly as

\[
\lambda_{p,q}^K = \left( \frac{p^2}{L^2} + \frac{q^2}{H^2} \right) \frac{\pi^2}{n_\infty^2}, \quad p \in \mathbb{N} \quad \text{and} \quad q \in \mathbb{N}^*. \]

But every function of \( H^1(K) \) which vanishes on \( \Gamma_h \) can be extended by 0 to a function of \( H^1(\Omega) \) and therefore, by the same argument as in the proof of Theorem 4.3, we have

\[
\lambda_{m,\infty}^{TE}(\theta) \leq \lambda_{p,q}^K \text{ if } m \leq q(p + 1).
\]

In particular, \( \lambda_{m,\infty}^{TE}(\theta) < \theta^2 / n_\infty^2 \) if \( \lambda_{0,m}^K < \theta^2 / n_\infty^2 \), which proves the theorem. \( \square \)

---

**Fig. 6. Illustration of Theorem 4.5**
5. Examples of singular frequencies for the diffraction problem

In section 3.4, we proved that, for a given $\theta$, diffraction problems $P_{TE}$ and $P_{TM}$ are well-posed for every $k$, except some values of $k$ (the so-called singular frequencies) which form at most a countable set without accumulation point.

We will prove now that this result is optimal by constructing examples of gratings which admit singular frequencies for some values of $\theta$. By Proposition 4.1, it is sufficient to find gratings such that problem (4.1) (or (4.2)) has solutions satisfying $k^2 \geq \theta^2/n_a^2$, or equivalently (cf. Theorem 4.1) such that the operator $A_{TE}(\theta)$ (or $A_{TM}(\theta)$) has eigenvalues embedded in its essential spectrum.

Notice that more generally, the question of the existence of eigenvalues embedded in the essential spectrum of a self-adjoint operator is a difficult problem which has been studied by various authors. Some non-existence results have been established in acoustics (see [15, 14, 17]). On the other hand, some examples of embedded eigenvalues have been exhibited in various applications (cf. [19, 7]).

A simple technique to find such examples is the following. Consider a self-adjoint operator $A : D(A) \subset E \to E$ and suppose that $F$ is a closed subspace of $E$ such that $D(A) \cap F$ is dense in $F$ and $A(D(A) \cap F) \subset F$. Then the operator $A_F : D(A) \cap F \to F$ is a self-adjoint operator on $F$. Suppose moreover that the essential spectrum of $A_F$ is strictly included in the essential spectrum of $A$ and that

$$\inf \sigma_{es}(A) < \inf \sigma_{es}(A_F).$$

Then there can exist eigenvalues of $A_F$ located below $\sigma_{es}(A_F)$ and embedded in $\sigma_{es}(A)$. Since an eigenvalue of $A_F$ is a fortiori an eigenvalue of $A$, these values are in fact eigenvalues of $A$ embedded in $\sigma_{es}(A)$ (cf. Fig. 7). Notice that these eigenvalues may be studied by applying the Min–Max principle to the operator $A_F$.

Let us now present two examples of gratings for which this technique works.

5.1. The symmetric grating

Suppose that, in addition to (2.1), the grating is symmetric:

$$\begin{align*}
(x_1, x_2) \in \Omega \iff (d - x_1, x_2) &\in \Omega, \\
n(x_1, x_2) = n(d - x_1, x_2) &\forall (x_1, x_2) \in \Omega
\end{align*}$$

and that the incident field is normal to the grating

$$u^{inc}(x) = u^{inc}_0 e^{ik \cdot x},$$

so that $\theta = k_y = 0$.

![Diagram](https://example.com/diagram.png)

Fig. 7. Spectra of $A$ and $A_F$
Consider, e.g. the operator \( A = A_{\text{TKE}}(0) \) and denote by
\[
F = \{ u \in L^2_{\text{TKE}}(\Omega); u(d -x_1, x_2) = -u(x_1, x_2) \text{ a.e. } x \in \Omega \},
\]
the space of antisymmetric functions. Then \( D(A) \cap F \) is dense in \( F \) (consider regular functions) and, by (5.1), it is obvious that \( A_{\text{TKE}}(0)(D(A) \cap F) \subset F \). Consequently, \( A_F = A_{\text{TKE}}(\theta) \cap F \) is self-adjoint and we have the following Lemma.

**Lemma 5.1.**

\[ \sigma_{\text{ess}}(A_F) = \left[ \frac{4 \pi^2}{d^2 n_m^2}, + \infty \right), \quad \sigma(A_F) \subset \mathbb{R}^+. \]

**Proof.** We use the same technique as in the proof of Theorem 4.1.

First, we establish the inclusion \( [4 \pi^2/d^2 n_m^2, + \infty) \subset \sigma_{\text{ess}}(A_F) \) by considering the following singular sequence:
\[
u^{\lambda}(x) = \frac{1}{\sqrt{m}} \psi \left( \frac{x_2}{m} \right) e^{i \lambda x_2} \sin \left( \frac{2 \pi x_1}{d} \right)
\]
where \( \lambda > 0 \) and \( \psi \) is a compactly supported function which vanishes for \( |x_2| < H_0 \).

Conversely, let \( u \in F \) and consider its Fourier decomposition
\[
u(x) = \sum_{p \in \mathbb{Z}} u_p(x_2) e^{i \theta_p x_1},
\]
which holds for \( |x_2| > H_0 \). By definition of \( F \), the coefficients \( u_p \) necessarily satisfy
\[
u_p(x_2) = -u_{-p}(x_2), \quad \forall p \in \mathbb{Z},
\]
and consequently
\[
u(x) = 2i \sum_{p > 1} u_p(x_2) \sin(\theta_p x_1).
\]

From this decomposition, we deduce easily the following estimate:
\[
u u \in F, \quad \int_{\Omega \cap \Omega_{1/2}} \left| \frac{\partial u}{\partial x_1} \right|^2 dx \geq \frac{4 \pi^2}{d^2} \int_{\Omega \cap \Omega_{1/2}} |u|^2 dx.
\]

The lemma follows as in the proof of Theorem 4.1. \(\square\)

**Remark 5.1.** Set
\[
\Omega_{1/2} = \left\{ x \in \Omega; x_1 < \frac{d}{2} \right\},
\]
(5.3)

The operator \( A_F \) can be identified to the operator \(-\div((1/n^2) \nabla u)\) with a Neumann homogeneous boundary condition on \( \Gamma \cap \Omega_{1/2} \) and Dirichlet homogeneous conditions on the lateral boundaries of \( \Omega_{1/2} \).

By Lemma 5.1, it is clear that every eigenvalue of \( A_F \) is an embedded eigenvalue for \( A_{\text{TKE}}(0) \) (since \( \sigma_{\text{ess}}(A_{\text{TKE}}(0)) = \mathbb{R}^+ \)). And one can exhibit various examples of gratings such that \( A_F \) has eigenvalues. The same technique can be applied to \( A_{\text{TM}}(0) \).

For example, let \( K = [l]_1 l_2 [\times] h_1, h_2 [\text{ and set } L = l_2 - l_1 \text{ and } H = h_2 - h_1. \)
Theorem 5.1. Suppose that (5.1) is satisfied and that $K \subset \Omega_{1/2}$. Then

(i) If $\Gamma_1 = \{x \in \partial K; x_1 = l_1 \text{ or } x_1 = l_2\} \subset \Gamma$ (cf. Fig. 8), if $n(x) = n_\infty$ in $K$ and if

$$m^2 < \frac{4H^2}{d^2},$$

(5.4)

then, for $\theta = 0$, problem $P_{TE}$ has at least $m$ singular frequencies.

(2) If $n(x) = n_+$ in $K$ and if

$$m^2 < \frac{4H^2n_+^2}{d^2n_\infty^2} - \frac{H^2}{L^2},$$

(5.5)

then, for $\theta = 0$, problems $P_{TE}$ and $P_{TM}$ have at least $m$ singular frequencies.

Remark 5.2. Condition (5.5) may be satisfied only if $n_+ > n_\infty$ since by hypothesis

$K \subset \Omega_{1/2}$ and thus $L < d/2$. For the first example, the existence of singular frequencies for $P_{TE}$ problem results from particular geometries of the conducting bodies. In the second example, it results from particular variations of the refractive index.

Proof. If (5.4) holds, then, with the notations of Theorem 4.5

$$\lambda_{\theta, m}^K < \frac{4\pi^2}{d^2n_\infty^2}.$$

Moreover, every function of $H^1(K)$ which vanishes on $\Gamma_\theta$ can be identified to a function of $H^1(\Omega) \cap F$. Consequently, as in the proof of Theorem 4.5, the previous inequality implies that $A_{TE}(0)_F$ has at least $m$ eigenvalues below the essential spectrum.

Likewise, let us denote by $\gamma_{\theta \theta}^K$ the eigenvalues of the Dirichlet problem

$$\begin{cases}
\frac{1}{n_+^2} \Delta \psi = \gamma \psi & \text{in } K, \\
\psi = 0 & \text{on } \partial K,
\end{cases}$$

which are given by

$$\gamma_{\theta \theta}^K = \left( \frac{p^2}{L^2} + \frac{q^2}{H^2} \right) \frac{\pi^2}{n_+^2}, \quad p, q \in \mathbb{N}^*.$$

![Fig 8. Illustration of Theorem 5.1](image-url)
Then by (5.5),

\[ \gamma_{1,m}^K < \frac{4\pi^2}{d^2 n_m^2}. \]

That proves that \( A_{TM}(0) \) has at least \( m \) eigenvalues below the essential spectrum. Notice that this second example differs from the first one by the fact that we did not suppose that \( \Gamma_1 \subset \Gamma. \)

Since every function of \( H^1_0(K) \) can be identified to a function of \( H^1_{1,0}(\Omega) \), the same argument holds for \( A_{TM}(0) \). \( \square \)

Remark 5.3. (1) In fact, the symmetry of the grating and of the incident field (cf. (5.1) and (5.2)) would lead to look for symmetric solutions of the diffraction problems \( P_{TE} \) and \( P_{TM} \). But the property of symmetry is not a consequence of the equations of the problems.

The examples of singular frequencies we have exhibited correspond to antisymmetric solutions of the homogeneous problem.

(2) The singular frequencies constructed above are such that \( k^2 < 4\pi^2/d^2 n_m^2. \) They correspond to the non-uniqueness for the diffraction problem only if the incident field is of the form (5.2) (i.e. \( k_1 = 0 \)). Indeed, if we have \( k_1 = 2\pi p_0/d \) with \( p_0 \neq 0 \), then \( \theta = 0 \) but \( k^2 = (1/n_m^2) (k_1^2 + k_2^2) \geq 4\pi^2/d^2 n_m^2. \)

5.2. Grating with periodical cells

In the previous examples, we found singular frequencies associated to the reduced frequency \( \theta = 0 \). In the examples below, \( \theta \) can be chosen arbitrary.

Suppose that the smallest period of the grating is not \( d \) but \( d/p \) where \( p \geq 2 \). The cell \( \Omega \) therefore satisfies the following property:

\[ x \in \Omega \text{ and } x_1 < \frac{p-1}{p} d \iff (x_1 + \frac{d}{p}, x_2) \in \Omega \text{ and } n(x) = n(x_1 + \frac{d}{p}, x_2). \] (5.6)

We suppose however that the diffraction problems are set in the cell \( \Omega \) of width \( d \) and not in the cell of width \( d/p \) denoted in the following by \( \Omega_{1/p} \):

\[ \Omega_{1/p} = \left\{ x \in \Omega; x_1 < \frac{d}{p} \right\}. \]

Then we denote by \( F \) the subspace of all functions of \( L^2_{TE}(\Omega) \) such that

\[ u(x_1 + \frac{d}{p}, x_2) = u(x_1, x_2) e^{i\hat{\theta} x_1} \text{ for } 0 < x_1 < \frac{p-1}{p} d, \] (5.7)

where \( \hat{\theta} \) is defined as follows:

\[ \hat{\theta} = \begin{cases} \theta + (p-1) \frac{\pi}{d} & \text{if } p \text{ is odd,} \\ \theta - \frac{pn}{d} & \text{if } p \text{ is even,} \end{cases} \]

so that, for \( 0 \leq \theta \leq \pi/d \)

\[ \left( \frac{(p-1)\pi}{d} \right)^2 \leq \hat{\theta}^2 \leq \left( \frac{pn}{d} \right)^2. \]
By (5.5) and by definition of $\hat{\theta}$, for $A = A_{TE}(\theta)$ (or $A_{TM}(\theta)$), $F \cap D(A)$ is dense in $F$ and $A(D(A) \cap F) \subset F$. We can therefore consider the self-adjoint operator $A_{\hat{\theta}}$ defined as above. Moreover, we have the following lemma:

**Lemma 5.2.**

$$\sigma_{ess}(A_{\hat{\theta}}) = \left[ \frac{\hat{\theta}^2}{n_\infty^2}, + \infty \right]$$

**Proof.** This lemma is a direct consequence of Theorem 4.1 if $\theta$ is replaced by $\hat{\theta}$ and $d$ by $d/p$. \qed

Note that, except if $p = 2$ and $\theta = \pi/d$, $\theta^2 < \hat{\theta}^2$. Consequently, to find examples of embedded eigenvalues for $A_{TE}(\theta)$ (or $A_{TM}(\theta)$), it suffices to find examples of eigenvalues of $A_{\hat{\theta}}$ located in the interval

$$\left[ \frac{\hat{\theta}^2}{n_\infty^2}, \frac{\hat{\theta}^2}{n_\infty^2} \right].$$

If $\hat{\theta} \neq 0$, the problem is more complicated than the previous one since we need lower bounds for the eigenvalues of $A_{\hat{\theta}}$. Let $K = ]l_1, l_2[ \times ]h_1, h_2[, K_\infty = \{ x; 0 < x < d/p; x_2 < h_1 \text{ or } x_2 > h_2 \}$ and set $L = l_2 - l_1$ and $H = h_2 - h_1$.

**Theorem 5.2.** Suppose that (5.6) is satisfied and that $K \subset \Omega_{1,p}$. Then

1. If $\Omega_{1,p} = K \cup K_\infty$ (cf. Fig. 10), if $n(x) = n_\infty$ a.e. in $\Omega$ and if

$$\frac{(m + 1)^2}{\hat{\theta}^2} < \frac{H^2}{n_\infty^2} < \frac{1}{\hat{\theta}^2},$$

then problem $P_{TE}$ has at least $m$ singular frequencies.

2. If $n(x) = n_+$ in $K$ and if

$$\frac{\hat{\theta}^2}{n_+^2} \geq \frac{\hat{\theta}^2}{n_\infty^2},$$

and

$$m^2 < \frac{\hat{\theta}^2 H^2 n_+^2}{n_\infty^2 n_\infty^2} - \frac{H^2}{L^2},$$

then problem $P_{TM}$ has at least $m$ singular frequencies.
then problem $P_{TM}$ has at least $m$ singular frequencies. Under the same conditions and if $\Gamma = \emptyset$ (no conductor bodies), problem $P_{TE}$ has at least $m$ singular frequencies.

Remark 5.4. A simple calculation shows that the function $Z: \theta \to \frac{\theta^2}{\theta^2}$ is strictly decreasing on $[0, \pi/d]$. Moreover, $\lim_{\theta \to 0} Z(\theta) = +\infty$ and $Z(\pi/d) = p^2$ if $p$ is odd and $(p - 1)^2$ if $p$ is even.

Consequently, for any given $m$, condition

$$\frac{\theta^2}{\theta^2} > (m + 1)^2$$

is automatically satisfied for small values of $\theta$, or, if $p \geq m + 2$, for all values of $\theta$. Then $H$ can be chosen such that (5.8) holds.

Likewise, for any given index values $n_+ \geq n_\infty$, (5.9) holds for small values of $\theta$, or if $p > (n_+ / n_\infty) + 1$, for all values of $\theta$. Then, if $n_+ > n_\infty$, (5.10) holds for $H$ great enough.

Proof. By Theorem 4.5, if (5.8) holds, $A_{TE}(\theta)|_{\Gamma}$ has at least $m + 1$ eigenvalues below $\frac{\theta^2}{\theta^2}$. Let us denote these eigenvalues by $\lambda_1, \ldots, \lambda_{m+1}$. We will now prove that, under condition (5.8), $\lambda_2, \ldots, \lambda_{m+1}$ are embedded eigenvalues of $A_{TE}(\theta)$, or equivalently that

$$\lambda_2 \geq \frac{\theta^2}{n_\infty^2}. \quad (5.11)$$

To do that, consider the following eigenvalue problem:

$$\begin{cases}
-\frac{1}{n_\infty^2} \Delta v = \gamma v & \text{in } K, \\
\frac{\partial v}{\partial \nu} = 0 & \text{on } \partial K,
\end{cases} \quad (5.12)$$

and denote by $\gamma_1, \gamma_2, \ldots$ the increasing sequence of its eigenvalues. The min–max characterizations of $\lambda_j$ and $\gamma_j$ are the following:

$$\lambda_j = \min_{\gamma_j \in \mathcal{R}_j (\Omega_j, \iota_j)} \frac{\int_{\Omega_j} |\nabla u|^2 \, dx}{\int_{\Omega_j} |u|^2 \, dx}$$

and

$$\gamma_j = \max_{u \neq 0} \frac{\int_{\Omega_j} |\nabla u|^2 \, dx}{\int_{\Omega_j} |u|^2 \, dx}. $$
and

\[
\gamma_j = \min_{V \in [\mathcal{H}(\Gamma)]} \max_{u \neq 0} \frac{1}{\int_{\Omega} |\nabla u|^2 \, dx} \frac{\int_{\Omega} |\nabla u|^2 \, dx}{\int_{\Omega} |u|^2 \, dx}.
\]

Now, if \( u \in H^1(\Omega_{1/p}) \), \( u \neq 0 \), then by Lemma 4.2,

\[
\frac{1}{n_\gamma^2} \int_{\Omega_{1/p}} |\nabla u|^2 \, dx \geq \frac{\theta^2}{n_\gamma^2} \int_{\Omega_{1/p}} |u|^2 \, dx
\]

and therefore, since \( \Omega_{1/p} = K \cup K_w \),

\[
\lambda_j < \frac{\theta^2}{n_\gamma^2} \Rightarrow \gamma_j \leq \lambda_j.
\]

In particular, \( \lambda_2 \geq \min(\gamma_2, \theta^2/n_\gamma^2) \). Moreover, the values of the first solutions of (5.12) are \( \gamma_1 = 0 \) and \( \gamma_2 = \pi^2/n_\gamma^2 \). To conclude, it suffices to notice that, by (5.8), \( \gamma_2 \geq \theta^2/n_\gamma^2 \).

The second part of the proof is easier. One proceeds exactly like in the proof of Theorem 5.1 to prove that, if (5.11) holds, \( A_{TE}(\theta)|_F \) and \( A_{TM}(\theta)|_F \) have at least \( m \) eigenvalues. By Theorem 4.1, these eigenvalues are located in the interval \( [\theta^2/n_\gamma^2, \theta^2/n_\gamma^2] \). Consequently, these are embedded eigenvalues for \( A_{TE}(\theta) \) or \( A_{TM}(\theta) \) if \( \theta^2/n_\gamma^2 \geq \theta^2/n_\gamma^2 \).

**Remark 5.5.** Let \( k \) be a singular frequency constructed by the previous theorem. Then by construction, \( k^2 < \frac{\theta^2}{n_\gamma^2} \). This inequality implies that \( p_0 \) cannot be chosen arbitrarily in (2.9). More precisely, we must have

\[
\left( \frac{\theta + 2p_0 \pi}{d} \right)^2 < \frac{p^2 \pi^2}{d^2}.
\]

There are only a finite number of possible values of \( p_0 \).

5.3. Concluding remarks

To conclude, we want to point out some essential features of the examples we have constructed.

First, notice that the existence of singular frequencies for the TE problem can occur for gratings with or without dielectric heterogeneity. We cannot construct such examples for the TM case if the medium around the conducting bodies is homogeneous. In that case we have even no examples of guided waves. However, except under conditions (3.39), we have not proved that singular frequencies cannot exist.

By the examples of section 5.2, it is clear that no general upper bound for the singular frequencies can be derived. In the examples above, there are only a finite number of singular frequencies for a given \( \theta \). But there is another example, very much simpler than the previous ones (although less interesting), which admits for any \( \theta > 0 \) infinitely many singular frequencies.

Indeed, consider a ‘degenerate’ grating, invariant in the \( x_1 \)-direction, and set

\[
F_p = \{ u = v(x_2) e^{i(\theta \theta + \theta_\delta)} x_1 \}.
\]
Again for $A = A_{TE}(\theta)$ or $A = A_{TM}(\theta)$, $D(A) \cap F_\theta$ is dense in $F_\theta$ and $A(D(A) \cap F_\theta) \subset F_\theta$. The operator $A_\theta = A|_{F_\theta}$ is a self-adjoint differential operator. If, e.g. $\Omega = \mathbb{R}^2$ and

$$n(x) = \begin{cases} n_+ & \text{if } x_2 < H_0, \\ n_- & \text{if } x_2 > H_0, \end{cases}$$

the spectrum of $A_{\theta}$ can be determined analytically (i.e. the classical problem of the slab waveguide (cf. [6, 11, 16])). In particular, $\sigma_{\text{ess}}(A_{\theta}) = \left[ \frac{\left(\theta + \theta_0\right)^2}{n_\pm^2}, + \infty \right)$ and $A_{\theta}$ has eigenvalues (if $\theta + \theta_0 \neq 0$) located in the interval $\left[\left(\theta + \theta_0\right)^2/n_+^2, \left(\theta + \theta_0\right)^2/n_-^2\right]$. Consequently, in that case, $A_{TE}(\theta)$ and $A_{TM}(\theta)$ admit an infinity of embedded eigenvalues without accumulation point.

**Remark 5.6.** In this degenerate case, the pseudo-Fourier transform provides a complete spectral representation of the operators $A_{TE}(\theta)$ and $A_{TM}(\theta)$ by a Rayleigh–Bloch wave expansion (see [18]).

**References**