Abstract. We consider a target problem for a nonlinear system under state constraints. We give a new continuous level-set approach for characterizing the optimal times and the backward-reachability sets. This approach leads to a characterization via a Hamilton–Jacobi equation, without assuming any controllability assumption. We also treat the case of time-dependent state constraints, as well as a target problem for a two-player game with state constraints. Our method gives a good framework for numerical approximations, and some numerical illustrations are included in the paper.

Key words. minimal time problem, Hamilton–Jacobi–Bellman equations, level-set method, reachability set, attainable set, state constraints, two-players game

AMS subject classifications. 93B03, 49J15, 35A18, 65N15, 49Lxx

1. Introduction. This paper studies a simple way to characterize the reachable sets and the optimal time to reach a target for a controlled nonlinear system, where the state is constrained to stay in a given domain. We are mainly interested in the case where no “controllability” assumption is made.

More precisely, we consider a control system

\[
\dot{y}(s) = f(y(s), \alpha(s)) \text{ for a.e. } s \geq 0, \quad y(0) = x, \tag{1.1}
\]

where \(\alpha : (0, +\infty) \to A\) is a measurable function, \(A\) is a given compact set in \(\mathbb{R}^m\) (set of admissible controls), and the dynamics \(f : \mathbb{R}^d \times A \to \mathbb{R}^d\). In what follows, for any initial position \(x\), we denote by \(y^\alpha_x\) the solution of (1.1) associated to the control variable \(\alpha\).

Let \(K \subset \mathbb{R}^d\) be a closed set of state constraints, and let \(C \subset K\) be a closed target. For a given time \(t \geq 0\), we consider the capture basin (backward reachable set) defined by

\[
\text{Cap}_C(t) := \{ x \in \mathbb{R}^d, \exists \alpha : (0, t) \to A \text{ measurable, } y^\alpha_x(t) \in C \text{ and } y^\alpha_x(\theta) \in K \forall \theta \in [0, t] \}. \tag{1.2}
\]

It is well known that the set \(\text{Cap}_C(\tau)\) is linked to a control problem. Indeed, consider a Lipschitz continuous function \(\vartheta_0 : \mathbb{R}^d \to \mathbb{R}\) such that

\[
\vartheta_0(x) \leq 0 \iff x \in C, \tag{1.3}
\]
and consider the control problem

\[(1.4) \quad u(x, t) := \inf \left\{ \vartheta_0(y^\alpha_x(t)) \mid \alpha \in L^\infty((0, t); \mathcal{A}), \ y^\alpha_x(\theta) \in \mathcal{K} \ \forall \theta \in [0, t] \right\}. \]

Then it is not difficult to prove that

\[
\text{Cap}_C(t) = \{ x \in \mathbb{R}^d, u(x, t) \leq 0 \}.
\]

For the unconstrained case, several works have been devoted to the characterization of the value function \( u \) as a continuous viscosity solution of a Hamilton–Jacobi equation [17, 4, 2]. In the presence of state constraints, the continuity of this value function is no longer satisfied, unless the dynamics satisfy a special controllability assumption on the boundary of the state constraints. This assumption, called “inward pointing qualification condition (IQ),” was first introduced by Soner in [31]. It asks that at each point of \( \mathcal{K} \) there exist a field of the system pointing inward \( \mathcal{K} \). Clearly this condition ensures the viability of \( \mathcal{K} \) (from any initial condition in \( \mathcal{K} \), there exist an admissible trajectory which could stay forever in \( \mathcal{K} \)). Under this assumption, the value function \( u \) is the unique continuous constrained viscosity solution of a Hamilton–Jacobi–Bellman (HJB) equation with a suitable new boundary condition [30, 31, 20, 10, 26].

Unfortunately, in many control problems, the IQ condition is not satisfied and the value function \( u \) could be discontinuous. In this framework, Frankowska introduced in [18] another controllability assumption, called “outward pointing qualification condition (OQ).” Under this assumption it is still possible to characterize the value function as the unique lower semicontinuous (lsc) solution of an HJB equation.

In the absence of any assumption of controllability, the function \( u \) is discontinuous and its characterization becomes more complicated, see, for instance, [5, 33, 9] and the references therein. In [11], the authors proved that the minimal time function for a state constrained control problem, without any controllability assumption, is the smallest nonnegative lsc supersolution of an HJB equation. This characterization leads to a numerical algorithm based on the viability approach [12, 29].

In this paper, we are interested in the case where no controllability assumption is assumed. We show that it is possible to characterize the capture basin \( \text{Cap}_C \) by means of a control problem whose value function is continuous (even Lipschitz continuous). For that, we consider a continuous function \( g : \mathbb{R}^d \to \mathbb{R} \) such that

\[(1.5) \quad g(x) \leq 0 \iff x \in \mathcal{K}. \]

Then we consider the new control problem

\[(1.6) \quad \vartheta(x, t) := \inf \left\{ \max(\vartheta_0(y^\alpha_x(t)), \max_{\theta \in [0, t]} g(y^\alpha_x(\theta))) \mid \alpha \in L^\infty((0, t); \mathcal{A}) \right\}. \]

We prove that the value function \( \vartheta \) is the unique continuous viscosity solution of the equation

\begin{align}
(1.7a) \quad & \min \left( \partial_t \vartheta(x, t) + H(x, D_x \vartheta(x, t)), \vartheta(x, t) - g(x) \right) = 0 \text{ for } t \in [0, +\infty[, \ x \in \mathbb{R}^d, \\
(1.7b) \quad & \vartheta(x, 0) = \max(\vartheta_0(x), g(x)).
\end{align}

\[1\text{By using the continuity of } g \text{ we will also have } g(x) = 0 \text{ if } x \in \partial \mathcal{K}.\]
Moreover, the capture basin is given by \( \text{Cap}_\varnothing(t) = \{ x \in \mathbb{R}^d \mid \vartheta(x, t) \leq 0 \} \), and
the minimal time for a given \( x \in \mathbb{R}^d \) to reach the target while remaining in the set of
state constraints is obtained by \( T(x) = \inf \{ t \in [0, +\infty] \mid \vartheta(x, t) \leq 0 \} \). This continuous
setting opens a large class of numerical schemes that can be used for such problems
(such as semi-Lagrangian or finite differences schemes).

Several papers in the literature deal with the link between reachability and HJB
equations. In the case when \( K = \mathbb{R}^d \), we refer to \([25, 24]\) and the references therein.
The case when \( K \) is an open set in \( \mathbb{R}^d \) is investigated in \([23]\). In the case of a constrained
reachability problem (with closed set \( K \)), a similar idea, that of introducing a control
problem in the form of (1.6), is also introduced in \([22]\). However, in that paper, the
analysis is a little bit more complicated and did not lead to a simple partial differential
equation (PDE) for characterizing \( \vartheta \) (the obstacle function is also assumed convex in
\([22]\)). Let us also refer to \([21]\) for a short discussion linking the reachability sets under
state constraints to HJB equations. The treatment in this reference assumed a \( C^1 \)
value function (see also section 2.1).

Finally, we mention that control problems with maximum costs have already
been studied by Barron \([6]\) and Barron and Ishii \([7]\), and also by Quincampoix and
Serea \([28]\) from a viability point of view. The main feature of our paper is the use
of (1.6) to deal easily with minimal time problems with state constraints and to
determine the corresponding capture basins. This idea generalizes in some sense the
known level-set approach usually used for unconstrained problems.

The paper is organized as follows. In section 2, we introduce the problem and
give the main results. In this section we also make precise the assumptions and fix
the notation that will be used in what follows. The proof of the main results is given
in section 3. We then give some extensions of the previous results. In section 4, we
shall discuss the case of time-dependent state constraints of the form

\[
y_x^\alpha(\theta) \in K_\theta \quad \forall \theta \in [0, t],
\]

where the sets \( (K_\theta)_{\theta \geq 0} \) can evolve in time (assuming some regularity of the map
\( \theta \to K_\theta \)). The case of two-player games with state constraints will also be discussed in
the appendix. Numerical approximation is studied in section 5, and an error estimate
is derived. Finally, we give some numerical illustrations in section 6.

**Notation.** Throughout the paper \( | \cdot | \) is a given norm on \( \mathbb{R}^d \) (for \( d \geq 1 \)). For
any closed set \( K \subset \mathbb{R}^d \) and any \( x \in K \), we denote by \( d(x, K) \) the distance from \( x \)
to \( K \): \( d(x, K) := \inf \{|x - y|, \ y \in K\} \). We shall also denote by \( d_K(x) \) the signed
distance function to \( K \), i.e., with \( d_K(x) := d(x, K) \geq 0 \) for \( x \notin K \), and \( d_K(x) :=
-d(x, \mathbb{R}^d \setminus K) < 0 \) for \( x \in K \).

2. **Main results.** Let \( A \) be a nonempty compact set in \( \mathbb{R}^m \) for \( m \geq 1 \). We
consider function \( f \in C(\mathbb{R}^d \times A; \mathbb{R}^d) \) satisfying the following assumptions:

(H1) There exists \( L_f > 0 \) such that for every \( (x, x', a) \in \mathbb{R}^d \times \mathbb{R}^d \times A \),

\[
|f(x, a) - f(x', a)| \leq L_f|x - x'|, \quad |f(x, a)| \leq L_f.
\]

(H2) For every \( y \in \mathbb{R}^d \), \( f(y, A) \) is a convex set of \( \mathbb{R}^d \).

Assumption (H2) is not needed in the entire paper, but it will be used to define
closed reachable sets (see Remark 1). Throughout the paper, we will indicate when
this assumption can be dropped. Also, the boundedness of \( f \) can be weakened, and
only a linear growth property \( "|f(x, a)| \leq L_f(1 + |x|)^\alpha" \) is needed.
Let $C$ be a nonempty closed set of $\mathbb{R}^d$ (the “target”), and also let $K$ be a nonempty closed set (of “state constraints”). Let us also denote by $A_{ad}$ the set of measurable functions on $(0, +\infty)$ and taking their values in $A$: $A_{ad} := \{ \alpha : (0, +\infty) \to \mathbb{R}^m \text{ measurable}, \alpha(t) \in A \text{ a.e.}\}$. 

Now, for $t \geq 0$, we define the capture basin as the set of all initial points $x$ from which starts a trajectory $y_x^\alpha(\cdot)$ solution of (1.1), associated to an admissible control $\alpha \in A_{ad}$, and such that $y_x^\alpha(t) \in C$, while $y_x^\alpha(\theta)$ belongs to the set of constraints $K$ for every $\theta \in [0, t]$:

$$\text{Cap}_C(t) := \left\{ x \in \mathbb{R}^d, \exists \alpha \in A_{ad}, y_x^\alpha(t) \in C, \text{ and } y_x^\alpha(\theta) \in K \forall \theta \in [0, t] \right\},$$

In this problem, the trajectory belongs to the (fixed) set of state constraints $K$.

**Remark 1.** Under (H2), for every $t \geq 0$, the capture basin $\text{Cap}_C(t)$ is a closed set.

In what follows, we will use the following definition of admissible trajectory.

**Definition 1.** Let $t$ be a fixed positive time. We will say that a solution of (1.1) $y_x^\alpha$ is admissible on $[0, t]$ if it is associated to an admissible control $\alpha \in A_{ad}$ and $y_x^\alpha(\theta)$ belongs to $K$ for every $\theta \in [0, t]$.

**Remark 2.** For every $t \geq 0$, the set $\text{Cap}_C(t)$ contains the initial positions which can be steered to the target (exactly) at time $t$. Of course, we can also define the “backward reachable set,” which is the set of points from which one can reach the target $C$ before time $t$:

$$\mathcal{R}([0, t]) := \left\{ x \in \mathbb{R}^d, \exists \alpha \in A_{ad}, y_x^\alpha(\tau) \in C, \text{ and } y_x^\alpha(\theta) \in K \forall \theta \in [0, \tau] \right\}.$$ 

$\mathcal{R}([0, t])$ is a family of increasing closed sets, with $\mathcal{R}([0, 0]) = C$. If we consider the new dynamics $F : \mathbb{R}^d \times \hat{A} \to \mathbb{R}^d$ by $F(x, (\alpha, \beta)) := \beta f(x, \alpha)$ for $(\alpha, \beta) \in \hat{A} = A \times [0, 1]$, then we can remark that $\mathcal{R}([0, t])$ is exactly the capture basin associated to the dynamics $F$ (see, for instance, [25]).

In this paper, we propose using the level-set approach in order to characterize $\text{Cap}_C(t)$ as the negative region of a continuous function $\vartheta$; i.e., we look for a continuous function such that $\text{Cap}_C(t) = \{ x, \vartheta(x, t) \leq 0 \}$.

To do so, we first consider a Lipschitz continuous function $\vartheta_0 : \mathbb{R}^d \to \mathbb{R}$ such that

$$\vartheta_0(x) \leq 0 \iff x \in C.$$

For instance, we may choose $\vartheta_0(x) := d_C(x)$; then $\vartheta_0$ is Lipschitz continuous (see, for instance, [15]). In particular, we have $\text{Cap}_C(0) = C = \{ x, \vartheta_0(x) \leq 0 \}$.

Consider the value function $u$ associated to the *Mayer problem* with final cost $\vartheta_0$:

$$u(x, t) := \inf \{ \vartheta_0(y_x^\alpha(t)), \alpha \in A_{ad}, y_x^\alpha(\theta) \in K \forall \theta \in [0, t] \}.$$

It is well known that the capture basin is characterized by

$$\text{Cap}_C(t) = \{ x, u(x, t) \leq 0 \}.$$

However, function $u$ is a value function of a state constrained problem, and we are still faced with the problem of characterizing this value function if no controllability assumption is made. To overcome this difficulty, we consider another Lipschitz continuous function $g : \mathbb{R}^d \to \mathbb{R}$ such that

$$g(x) \leq 0 \iff x \in K.$$
Note that such a function always exists since we can choose \( g(x) := d_{K}(x) \).

We then consider the control problem

\[
\vartheta(x,t) := \inf \left\{ \max \left( \vartheta_{0}(y_{x}^{\alpha}(t)), \max_{\vartheta \in [0,\alpha]} g(y_{x}^{\alpha}(\vartheta)) \right), \alpha \in \mathcal{A}_{ad} \right\}.
\]

Problem (2.4) has no “explicit” state constraint. In fact, in this new setting, the term \( \max_{\vartheta \in [0,\alpha]} g(y_{x}^{\alpha}(\vartheta)) \) plays the role of a penalization that a trajectory \( y_{x}^{\alpha} \) would pay if it violates the state constraints. We will see in Theorem 2 that the advantage of considering (2.4) is that \( \vartheta \) can now be characterized as the unique continuous solution of an HJB equation.

The central idea of the paper is that the functions \( \vartheta(\cdot,t) \) and \( u(\cdot,t) \) have the same negative regions, and so we have the following characterization of the capture basin.

**Theorem 1** (characterization of the capture basin). Assume (H1)–(H2). Let \( \vartheta_{0} \) and \( g \) be Lipschitz continuous functions defined, respectively, by (2.1) and (2.3). Let \( u \) and \( \vartheta \) be the value functions defined, respectively, by (2.2) and (2.4). Then, for every \( t \geq 0 \), we have the following:

(i) The capture basin is given by

\[
\text{Cap}_{C}(t) = \{ x, u(x,t) \leq 0 \} = \{ x, \vartheta(x,t) \leq 0 \}.
\]

(ii) If \( \vartheta(x,t) < 0 \) and \( \hat{K} = \{ x, g(x) < 0 \} \), then \( u(x,t) < 0 \), and there exists, on \( [0,t] \), an admissible trajectory \( y^{\alpha} \) that never touches the boundary \( \partial \hat{K} \).

**Remark 3.** Let us point out that the zero level sets of \( u \) and \( \vartheta \) may not coincide.

In particular, when there is an optimal trajectory \( y^{\alpha} \) that touches the boundary \( \partial \hat{K} \), we can have \( u(x,t) < 0 \) and \( \vartheta(x,t) = 0 \) (hence the converse of Theorem 1(ii) is false). This is illustrated in Example 1 of section 3.

Consider the minimal time function, which associates to any point \( x \in \mathbb{R}^{d} \) the minimal time needed to reach the target with an admissible trajectory \( y_{x}^{\alpha} \) solution of (1.1) and satisfying \( y_{x}^{\alpha}(\vartheta) \in \mathcal{K} \):

\[
\mathcal{T}(x) := \inf \{ t \geq 0, \exists \alpha \in L^{\infty}((0,t);\mathcal{A}), y_{x}^{\alpha}(t) \in C, \text{ and } y_{x}^{\alpha}(\vartheta) \in \mathcal{K} \forall \vartheta \in [0,t] \}.
\]

Many works have been devoted to the regularity of the minimum time function \( \mathcal{T} \). When \( \mathcal{K} \equiv \mathbb{R}^{d} \), and under some local metric properties around the target, the function \( \mathcal{T} \) is the unique continuous viscosity solution of an HJB equation [2].

Here, without assuming any controllability assumption at the boundary of the target, and neither at the boundary of \( \mathcal{K} \), the function \( \mathcal{T} \) may be discontinuous. Indeed, if, for \( x \in \mathbb{R}^{d} \), no trajectory \( y_{x}^{\alpha} \) reaches the target \( \mathcal{C} \), or if any trajectory leaves \( \mathcal{K} \) before reaching the target, we set \( \mathcal{T}(x) = +\infty \). Nevertheless, the next proposition states that \( \mathcal{T} \) is lsc and characterizes it by using the knowledge of the function \( \vartheta \).

**Proposition 1.** Assume (H1)–(H2). The minimal time function \( \mathcal{T} : \mathbb{R}^{d} \to \mathbb{R}^{+} \cup \{ +\infty \} \) is lsc. Moreover, we have

\[
\mathcal{T}(x) = \inf \{ t \geq 0, x \in \text{Cap}_{C}(t) \} = \inf \{ t \geq 0, \vartheta(x,t) \leq 0 \},
\]

with \( \vartheta \) the value function defined in (2.4), where \( \vartheta_{0} \) and \( g \) are any Lipschitz functions satisfying, respectively, (2.1) and (2.3).

**Remark 4.** It is known that when (H2) does not hold, the lower semicontinuity of \( \mathcal{T} \) is no longer true. In this case, it is possible to prove that \( \mathcal{T}_{\ast}(x) = \inf \{ t \geq 0, \vartheta(x,t) \leq 0 \} \), where \( \mathcal{T}_{\ast} \) is the lsc envelope of \( \mathcal{T} \).
Remark 5. The use of a level-set approach is a standard way to determine the minimal time function of unconstrained control problems [17].

In our work, we generalize this point of view to the case when the time control problem is in the presence of state constraints. Our formulation also allows us to obtain the capture basins.

As mentioned before, the function \( \vartheta \) can be characterized as the unique solution of a Hamilton–Jacobi equation. More precisely, considering the Hamiltonian

\[
H(x, p) := \max_{\alpha \in \mathcal{A}} (-f(x, \alpha) \cdot p),
\]

we have the following.

**Theorem 2.** Assume (H1) and that \( \vartheta_0 \) and \( g \) are Lipschitz continuous. Then \( \vartheta \) is the unique continuous viscosity solution (see Definition 2 in section 3) of the variational inequality (obstacle problem)

\[
\begin{align*}
\min(\partial_t \vartheta + H(x, \nabla \vartheta), \vartheta - g(x)) &= 0, \quad t > 0, \ x \in \mathbb{R}^d, \\
\vartheta(x, 0) &= \max(\vartheta_0(x), g(x)), \quad x \in \mathbb{R}^d.
\end{align*}
\]

**Remark 6.** In practice, since the target \( C \) is a subset of \( \mathcal{K} \), it is always possible to choose \( \vartheta_0 \) and \( g \) in such a way that \( \vartheta_0 \geq g \).

Inequalities such as (2.7) appear also in the framework of exit time problems, where the obstacle \( g \) represents the exit cost that should be paid for exit. Here, \( g \) is a “fictitious cost” that a target would pay if it leaves \( \mathcal{K} \).

**Remark 7.** From a theoretical point of view, the choice of \( g \) is not important, and \( g \) can be any Lipschitz function satisfying (2.3). Of course, the value function \( \vartheta \) is dependent on \( g \), while the set \( \{ x \in \mathbb{R}^d, \vartheta(x, t) \leq 0 \} \) does not depend on \( g \).

Let us also point out that the obstacle term in (2.7) comes from the presence of the sup-norm \( \max_{\theta \in [0, t]} g(y^{\alpha}_x(\theta)) \) in the cost function which defined \( \vartheta \) (see (2.4)). We refer to the works of Barron and Ishii [7] and the references therein for optimal control problems with sup-norm cost functions.

**2.1. Some comments.** Here we quote some other techniques introduced in the literature and that link the reachability sets to PDEs.

**Approach 1.** Following Kurzhanski and Varaiya [21], for any \( \eta > 0 \), we can define a value function defined by

\[
\vartheta^\eta(x, t) := \inf \left\{ d(y^\alpha_x(t), \mathcal{C}) + \eta \int_0^t d(y^\alpha_x(s), \mathcal{K}) ds \mid \alpha \in L^\infty((0, t); \mathcal{A}) \right\}.
\]

It is easy to prove that, for any \( \eta > 0 \) and any \( t \geq 0 \), the capture basin \( \text{Cap}_\mathcal{C}(t) \) is given by

\[
\text{Cap}_\mathcal{C}(t) = \{ x \in \mathbb{R}^m, \ \vartheta^\eta(x, t) \leq 0 \} = \{ x \in \mathbb{R}^m, \ \vartheta^\eta(x, t) = 0 \},
\]

and \( \vartheta^\eta \) is the unique continuous viscosity solution of the PDE

\[
\begin{align*}
\partial_t \vartheta + H(x, D_x \vartheta) - \eta d(x, \mathcal{K}) &= 0, \quad t > 0, \ x \in \mathbb{R}^d, \\
\vartheta(x, 0) &= d(x, \mathcal{C}).
\end{align*}
\]

In [21], the characterization of \( \vartheta^\eta \) (when \( \eta = 1 \)) is given under the assumption that \( \vartheta^\eta \) is very smooth. However, it is possible to derive (2.9) by using standard arguments.
of the viscosity framework for any $\eta > 0$. The drawback of approach (2.9) is that for every $t \geq 0$, $\vartheta^\eta(.,t)$ is nonsmooth in the neighborhood of the capture basin. In section 6, we will compare this approach (for several choices of $\eta$) to our method based on (2.7).

**Approach 2.** This consists of a classical idea of penalizing the state constrained control problem (2.2). Thus, for every $\epsilon > 0$, we consider for $t \geq 0$ and $x \in \mathbb{R}^d$ the problem

$$u^\epsilon(x,t) := \inf \left\{ \vartheta_0(y_\alpha^\epsilon(x,t)) + \int_0^t \frac{1}{\epsilon} d(y_\alpha^\epsilon(s), K) \, ds \mid \alpha \in L^\infty((0,t); A) \right\}.$$  

When $\epsilon$ tends to 0, $u^\epsilon(x,t)$ converges locally uniformly to $u(x,t)$. Moreover, $u^\epsilon$ is the unique continuous viscosity solution to the following PDE:

$$\begin{align*}
\partial_t v + H(x, D_x v) - \frac{1}{\epsilon} d(x, K) &= 0, & t > 0, & x \in \mathbb{R}^d, \\
v(x,0) &= \vartheta_0(x).
\end{align*}$$

We point out that (2.9) is very close to (2.11), but the initial conditions can be different. Indeed, in (2.11), $\vartheta_0$ can be chosen smoother than $d(\cdot, K)$. While, for any $\eta > 0$, (2.9) gives an exact characterization to the capture basin sets, the functions $u^\epsilon$ give only “an approximation” to these sets. Moreover, it is not clear how to choose the parameter $\epsilon$ in order to obtain a good approximation (see section 6).

**Approach 3.** Another formal way consists of considering that the dynamics vanishes on the obstacle. The capture basin at time $t$ is (formally) identified to the set

$$\{ \tilde{u}(x,t) \leq 0 \},$$

where $\tilde{u}$ solves the HJB equation

$$\begin{align*}
u_t + &-f(x, \alpha)1_K(x) \cdot \nabla u = 0, & t > 0, & x \in \mathbb{R}^d, \\
v(x,0) &= \vartheta_0(x),
\end{align*}$$

where the dynamics $f$ is replaced by $f1_K$, with $1_K(x) = 1$ if $x \in K$, and 0 if $x \not\in K$. This approach is usually used for numerical purposes. However, from the theoretical point of view, (2.12) cannot be handled with the continuous viscosity framework since the Hamiltonian $H(\cdot, p)$ is discontinuous on the boundary of $K$. In section 6, we study this approach in a simple numerical example.

### 3. Proofs of the main results.

**Proof of Theorem 1.** (i) Assume that $u(x,t) \leq 0$. Then by the definition of $u$ and assumptions (H1)–(H2), there exists an admissible trajectory $y_\alpha^\epsilon$ such that

$$\vartheta_0(y_\alpha^\epsilon(t)) \leq 0, \quad y_\alpha^\epsilon(\theta) \in K \quad \text{for every } \theta \in [0,t].$$

Hence, $\max_{\theta \in [0,t]} g(y_\alpha^\epsilon(\theta)) \leq 0$, and we have

$$\vartheta(x,t) \leq \max_{\theta \in [0,t]} \left( \vartheta_0(y_\alpha^\epsilon(t)), \max_{\theta \in [0,t]} g(y_\alpha^\epsilon(\theta)) \right) \leq 0.$$  

Conversely, assume that $\vartheta(x,t) \leq 0$. Then there exists a trajectory $y_\alpha^\epsilon$ such that

$$0 \geq \vartheta(x,t) = \max_{\theta \in [0,t]} \left( \vartheta_0(y_\alpha^\epsilon(t)), \max_{\theta \in [0,t]} g(y_\alpha^\epsilon(\theta)) \right).$$
Thus, for all \( \theta \in [0, t] \), \( g(y^\alpha_x(\theta)) \leq 0 \), i.e., \( y^\alpha_x(\theta) \in \mathcal{K} \), and so \( y^\alpha_x \) is an admissible trajectory. Moreover, we have \( \vartheta_0(y^\alpha_x(t)) \leq 0 \); hence
\[
 u(x, t) \leq \vartheta_0(y^\alpha_x(t)) \leq 0.
\]

(ii) The proof uses arguments similar to those in (i).

The following example shows that the converse of Theorem 1(ii) is false in general; i.e., we can have \( \vartheta(x, t) = 0 \) and \( u(x, t) < 0 \).

**Example 1.** Consider \( f = (1, 1)^T \), the target \( \mathcal{C} = [1, 2]^2 \), the constraint set \( \mathcal{K} := \mathbb{R}^2 \setminus [1, 0] \times [0, 1] \), and \( x := (-1, -1) \). We assume that \( \vartheta_0 < 0 \) on the interior \( \overset{\circ}{\mathcal{C}} \) of \( \mathcal{C} \).

In this example, we do not have any control variable, and the only possible trajectory starting from \( x \) is the one defined by \( y_x(t) = x + (1)^T t \). At time \( t = 2.5 \) we have \( y_x(t) = (1.5, 1.5)^T \in \overset{\circ}{\mathcal{C}} \), and then \( u(x, t) = \vartheta_0(y_x(t)) < 0 \). On the other hand, since \( y_x(1) = (0, 0) \in \partial \mathcal{K} \), we have \( \max_{\theta \in [0, t]} g(y_x(\theta)) = 0 \); thus \( \vartheta(x, t) = 0 \) (see Figure 3.1).

**Fig. 3.1.** The target \( \mathcal{C} \), the state constraints set \( \mathcal{K} \), and the trajectory \( y_x \).

We now give, for the sake of completeness, the proof of Proposition 1.

**Proof of Proposition 1.** The lower semicontinuity of \( \mathcal{T} \) has already been proved in [11].

Let \( \overline{T}(x) := \inf \{ t \geq 0, \ x \in \text{Cap}_x(t) \} \). The fact that \( \overline{T}(x) = \inf \{ t, \vartheta(x, t) \leq 0 \} = \inf \{ t, u(x, t) \leq 0 \} \) is a consequence of Theorem 1(i) and of the definition of \( \text{Cap}_x(t) \).

It remains just to prove that \( T(x) = \overline{T}(x) \).

Let \( t := T(x) \). Since \( t \) is the minimal time, by using assumptions (H1)–(H2) and compactness arguments as in [29], there exists an admissible trajectory \( y^\alpha_x \), such that \( y^\alpha_x(0) \in \mathcal{K} \). Hence \( \vartheta(x, t) \leq \vartheta_0(y^\alpha_x(t)) \leq 0 \), and thus \( \overline{T}(x) \leq t = T(x) \).

On the other hand, let \( \hat{t} := \overline{T}(x) \). For any \( n \geq 1 \), there exists some \( t_n \in [\hat{t}, \hat{t} + \frac{1}{n}] \) such that \( \vartheta(x, t_n) \leq 0 \). We can consider an associated optimal trajectory \( y_n := y^\alpha_x \) such that \( y^\alpha_x \) is admissible and \( y^\alpha_x(t_n) \in \mathcal{K} \). By again using a compactness argument, and since \( \mathcal{K} \) and \( \mathcal{C} \) are closed subsets, we can extract a convergent subsequence and an admissible trajectory \( y \), such that \( y_n \to y \) uniformly on \([0, \hat{t}]\), and \( y(\hat{t}) \in \mathcal{C} \). Hence \( T(x) \leq \hat{t} \), which concludes the proof.

Before giving the proof of Theorem 2, we need the following dynamic programming principle (DPP) for \( \vartheta \).
Lemma 1 (DPP). The function $\vartheta$ is characterized by the following:

(i) For all $t \geq 0$ and $\tau \geq 0$, for all $x \in \mathbb{R}^d$,

$$
\vartheta(x, t + \tau) = \inf \left\{ \max \left( \vartheta(y_x^\alpha(\tau), t), \max_{\theta \in [0, \tau]} g(y_x^\alpha(\theta)) \right), \alpha \in L^\infty((0, \tau); A) \right\};
$$

(ii) $\vartheta(x, 0) = \max(\vartheta_0(x), g(x))$.

Proof. One can refer, for instance, to Barron and Ishii [7, Proposition 3.1].

The first consequence of the above lemma is the Lipschitz continuity of the value function $\vartheta$.

Proposition 2. Assume (H1). Let $\vartheta_0$ and $g$ be Lipschitz continuous functions satisfying (2.1) and (2.3). Let $\vartheta$ be the value function defined as in (2.4). For every $T > 0$, $\vartheta$ is Lipschitz continuous on $\mathbb{R}^d \times [0, T]$.

Proof. Let $T > 0$, and let $x, x' \in \mathbb{R}^d$ and $t \in [0, +\infty]$. By using the definition of $\vartheta$ and the simple inequalities

$$
\max(A, B) - \max(C, D) \leq \max(A - C, B - D),
$$

and

$$
\inf A_\alpha - \inf B_\alpha \leq \sup(A_\alpha - B_\alpha),
$$

we get

$$
|h(x, t) - h(x', t)| \\
\leq \sup_{\alpha(\cdot) \in A} \max \left( \vartheta_0(y_x^\alpha(t)) - \vartheta_0(y_{x'}^\alpha(t)), \max_{\theta \in [0, t]} g(y_x^\alpha(\theta)) - g(y_{x'}^\alpha(\theta)) \right),
$$

where $L_0$ and $L_g$ denote, respectively, the Lipschitz constant of $\vartheta_0$ and $g$. By assumption (H1), we know that $|y_x^\alpha(\theta) - y_{x'}^\alpha(\theta)| \leq e^{L_f|\theta|}|x - x'|$. Then we conclude that

$$
|h(x, t) - h(x', t)| \leq \max(L_0, L_g)e^{L_{f,t}|x - x'|}
$$

for any $x, x' \in \mathbb{R}^d$ and any $t \in [0, T]$ for $T \geq 0$. Now, let $x \in \mathbb{R}^d$ and $t, h \geq 0$. Remarking that $\vartheta(x, t) \geq g(x)$, we deduce from Lemma 1 that

$$
|h(x, t + h) - h(x, t)| = \inf_{\alpha} \max \left( \vartheta(y_x^\alpha(h), t), \max_{\theta \in [0, h]} g(y_x^\alpha(\theta)) \right) - \max \left( \vartheta(x, t), g(x) \right)
$$

$$
\leq \sup_{\alpha} \max \left( \left| \vartheta(y_x^\alpha(h), t) - \vartheta(x, t) \right|, \max_{\theta \in [0, h]} g(y_x^\alpha(\theta)) - g(x) \right)
$$

$$
\leq L_f \max(\max(L_0, L_g)e^{L_{f,t}|x - x'|}, L_g)h,
$$

where we have used (3.3) and assumption (H1). This completes the proof.

Now, we recall the definition of the viscosity solution for (2.7).

Definition 2 (viscosity solution). An upper semicontinuous (resp., lsc) function $\vartheta : \mathbb{R}^d \times \mathbb{R}_+ \to \mathbb{R}$ is a viscosity subsolution (resp., supersolution) of (2.7) if $\vartheta(x, 0) \leq \vartheta_0(x)$ in $\mathbb{R}^d$ (resp., $\vartheta(x, 0) \geq \vartheta_0(x)$) and for any $(x, t) \in \mathbb{R}^d \times (0, \infty)$ and any test function $\phi \in C^1(\mathbb{R}^d \times \mathbb{R}_+)$ such that $\vartheta - \phi$ attains a maximum (resp., a minimum) at the point $(x, t) \in \mathbb{R}^d \times (0, +\infty)$, then we have

$$
\min(\partial_t \phi + H(x, \nabla \phi), \vartheta - g(x)) \leq 0,
$$

where $H(x, \nabla \phi)$ is the Hamiltonian associated with (2.7).
A continuous function $\vartheta$ is a viscosity solution of (2.7) if $\vartheta$ is a viscosity subsolution and a viscosity supersolution of (2.7).

We now give the proof of Theorem 2.

Proof of Theorem 2. The proof can be deduced from [7, Proposition 2.6]. Here, we give the main lines of a direct proof for completeness. We first show that $\vartheta$ is a solution of (2.7). The fact that $\vartheta$ satisfies the initial condition is a direct consequence of Lemma 1(ii).

Let us check the supersolution property of $\vartheta$. By Lemma 1(i), we get that for any $\tau \geq 0$,

$$\vartheta(x, t + \tau) \geq \inf_{\alpha} \vartheta(y^\alpha_x(\tau), t).$$

Hence, with classical arguments, we can obtain

$$\partial_t \vartheta + H(x, \nabla \vartheta) \geq 0$$

in the viscosity sense. Moreover, by the definition of $\vartheta$, for every $(x, t) \in \mathbb{R}^d \times \mathbb{R}^+$, we have

$$\vartheta(x, t) \geq \inf_{\alpha} \max_{\theta \in [0, t]} g(y^\alpha_x(\theta)) \geq g(x).$$

Combining these two inequalities, we get

$$\min(\partial_t \vartheta + H(x, \nabla \vartheta), \vartheta(x, t) - g(x)) \geq 0$$

in the viscosity sense; i.e., $\vartheta$ is a supersolution of (2.7).

Let us now prove that $\vartheta$ is a subsolution. Let $x \in \mathbb{R}^d$ and $t > 0$. If $\vartheta(x, t) \leq g(x)$, it is obvious that $\vartheta$ satisfies

$$\min(\partial_t \vartheta + H(x, \nabla \vartheta), \vartheta(x, t) - g(x)) \leq 0.$$

Now, assume that $\vartheta(x, t) > g(x)$. By continuity of $g$ and $\vartheta$, there exists some $\tau > 0$ such that $\vartheta(y^\alpha_x(\theta), t) > g(y^\alpha_x(\theta))$ for all $\theta \in [0, \tau]$ (since $y^\alpha_x(\theta)$ will stay in a neighborhood of $x$, which is controlled uniformly with respect to $\alpha$). Hence, by using Lemma 1(i), we get that

$$\vartheta(x, t + h) = \inf_{\alpha} \vartheta(y^\alpha_x(h), t) \quad \text{for any } 0 \leq h \leq \tau.$$

We then deduce by classical arguments [2] that $\partial_t \vartheta(x, t) + H(x, \nabla \vartheta(x, t)) \leq 0$ in the viscosity sense. Therefore, $\vartheta$ is a viscosity subsolution of (2.7).

The fact that $\vartheta$ is the unique solution of (2.7) follows from the comparison principle for (2.7) (which is classical; see, for instance, [4]) and the fact that the Hamiltonian function $H$ satisfies

\begin{align}
|H(x_2, p) - H(x_1, p)| &\leq C(1 + |p|) |x_2 - x_1|, \\
|H(x, p_2) - H(x, p_1)| &\leq C|p_2 - p_1|
\end{align}

for some constant $C \geq 0$ and for all $x_i, p_i, x$, and $p$ in $\mathbb{R}^d$. \[\square\]
4. **Time-dependent state constraints.** Let \((K_{\theta})_{\theta \geq 0}\) be a family of closed subsets of \(\mathbb{R}^d\). We assume the following:

(H3) The set-valued application \(\theta \mapsto K_{\theta}\) is Lipschitz continuous\(^2\) on \([0, +\infty[\).

For \(x \in K_0\), we consider the trajectories solution of (1.1) and satisfying the time-dependent constraints:

\[
y_\alpha^x(t) \in K_{\theta} \quad \forall \theta \in [0, t].
\]

An example of such a situation is the case where we want to avoid a mobile obstacle located at every \(\theta \geq 0\) at the open subset \(O_{\theta}\) while remaining in a given closed set \(K \subset \mathbb{R}^d\). In this case, we should set \(K_{\theta} := K \setminus O_{\theta}\).

Now, let us define the minimal time function:

\[
T^t(x) = \inf \{ t \geq 0 \mid \exists \alpha \in L^\infty((0, t); A), y_\alpha^x(t) \in C \text{ and } y_\alpha^x(\theta) \in K_{\theta} \forall \theta \in [0, t]\}.\]

In this setting, the function \(T^t\) cannot be characterized by an HJB equation. The reason for that comes from the time-dependency of the state constraints. Actually, function \(T^t\) does not even satisfy the DPP (this is also the case of the minimal time function of nonautonomous systems; see [8]). Also, here we cannot use the ideas developed in the previous sections to determine the capture basins. Nevertheless, we shall see that an equation similar to (2.4) can be used to determine the reachable sets.

Let \(D\) be a given nonempty closed set of \(\mathbb{R}^d\) (\(D\) can be a singleton). For \(t \geq 0\), we consider the attainable set (or, reachability region) starting from \(D\), defined as the set of points that can be reached at time \(t\) by a trajectory starting from \(D\) and satisfying the time-dependent state constraint (4.1), i.e.,

\[
\text{Att}_D(t) := \left\{ y_\alpha^x(t) \in \mathbb{R}^d \mid x \in D, \exists \alpha \in L^\infty((0, t); A), y_\alpha^x(\theta) \in K_{\theta} \forall \theta \in [0, t]\right\}.
\]

As in section 2, we consider a Lipschitz continuous function \(g^t : [0, +\infty[ \times \mathbb{R}^d\) such that, for all \(\theta \geq 0\),

\[
g^t(x, \theta) \leq 0 \iff x \in K_{\theta}.
\]

Such a function always exists since we can choose \(g^t(x, \theta) := d_{K_{\theta}}(x)\) (note that by assumption (H3), the sign distance function to \(K_{\theta}\) is also Lipschitz continuous in both variables \((x, \theta))\). We also consider \(\vartheta_0^t : \mathbb{R}^d \to \mathbb{R}\) such that

\[
\vartheta_0^t(x) \leq 0 \iff x \in D.
\]

We then consider the following control problem:

\[
\vartheta^t(x, t) := \inf \left\{ \max \left( \vartheta_0^t(y_\alpha^x(t)), \max_{\theta \in [0, t]} g^t(y_\alpha^x(\theta), \theta) \right), \alpha \in L^\infty((0, t); A) \right\}.
\]

Similar arguments, as in section 2, lead to the following.

**Theorem 3.** Assume (H1)–(H3) hold. For every \(t \geq 0\), the attainable set is characterized by

\[
\text{Att}_D(t) = \{ x \mid \vartheta^t(x, t) \leq 0 \}.
\]

\(^2\)That is, \(\exists C \geq 0 \forall \theta, \theta' \in [0, +\infty[ \mid d_H(K_{\theta}, K_{\theta'}) \leq C|\theta - \theta'|\), where \(d_H\) is the Hausdorff distance.
Theorem 4. We assume (H1) and (H3). Let \( \vartheta_0^d \) and \( g^t \) be Lipschitz continuous functions satisfying, respectively, (4.3) and (4.2). Then \( \vartheta^2 \) is the unique continuous viscosity solution of

\begin{align}
(4.5a) \quad & \min(\partial_t \vartheta^t + H(x, D_x \vartheta^t), \vartheta^2 - g^t(x, t)) = 0, \quad t \geq 0, \ x \in \mathbb{R}^d, \\
(4.5b) \quad & \vartheta^2(x, 0) = \max(\vartheta_0^d(x), g^0(x, 0)), \quad x \in \mathbb{R}^d.
\end{align}

Remark 8. It is worth remarking that the HJB inequality (4.5) (with a time-dependent obstacle function \( g^t \)) allows us to determine the reachable sets but not the capture basins.

Remark 9. For \( x \in \mathcal{K}_0 \), if we want to know the minimal time needed to reach the target \( \mathcal{C} \), starting from \( x \) and satisfying the state constraints (4.1), we should consider \( \mathcal{D} = \{x\} \) and set \( \vartheta_0^d(y) := d(x, y) \) (this is the signed distance to the set \( \mathcal{D} := \{x\} \)). Then let \( \vartheta^t \) be the solution of (4.5) and where \( g^t \) represents the time-dependent state constraints. In that case, the set of points that can be reached at time \( t \) and starting from \( x \) is

\[
\mathrm{Att}_{\{x\}}(t) := \{x, \ \vartheta^2(x, t) \leq 0\}
\]

(which is also identical to \( \{x, \ \vartheta^t(x, t) = 0\} \) in this specific case). Finally, we can recover the minimal time to reach \( \mathcal{C} \) as

\[
T(x) := \inf \{t \geq 0, \ \mathrm{Att}_{\{x\}}(t) \cap \mathcal{C} \neq \emptyset\}.
\]

5. Numerical scheme and error estimates. In this section we propose a finite difference scheme to approximate the solution \( u \) of (2.7) or (4.5). In this section \( H : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R} \) is assumed to be only a given continuous function satisfying (3.4).

For given mesh sizes \( \Delta x > 0, \ \Delta t > 0 \), we define

\[
\mathcal{G} := \{I \Delta x, \ I \in \mathbb{Z}^d\}
\]

where \( N_T \) is the integer part of \( T/\Delta t \). The discrete running point is \((x_I, t_n)\) with \( x_I = I \Delta x, \ t_n = n \Delta t \). The approximation of the solution \( \vartheta \) at the node \((x_I, t_n)\) is written indifferently as \( v(x_I, t_n) \) or \( v^n_I \) according to whether we view it as a function defined on the lattice or as a sequence.

Now, given a numerical Hamiltonian \( \mathcal{H} : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R} \) (which will be a consistent approximation of the Hamiltonian \( H \)), we consider the scheme

\[
(5.1) \quad \begin{cases}
\min \left( \frac{v^{n+1}_I - v^n_I}{\Delta t} + \mathcal{H}(x_I, D^+ v^n(x_I), D^- v^n(x_I)) \right) = 0, \\
v^n_I = \tilde{u}_0(x_I),
\end{cases}
\]

where \( \tilde{u}_0 \) is an approximation of \( u_0 \) and

\[
D^+ v^n(x_I) = (D^+ v^n(x_I), \ldots, D^+ v^n(x_I)), \\
D^- v^n(x_I) = (D^- v^n(x_I), \ldots, D^- v^n(x_I))
\]

are the discrete space gradients of the function \( v^n \) at point \( x_I \) defined for a general function \( w \) by

\[
D^\pm v(x_I) = \pm \frac{w(x_I + \pm) - w(x_I)}{\Delta x}.
\]
with the notation \( I^{k,\pm} = (i_1, \ldots, i_{k-1}, i_k \pm 1, i_{k+1}, \ldots, i_d) \).

We make the following assumptions on the numerical Hamiltonian \( H \):

(H4) There exists \( C_1 > 0 \) such that for all \( x_I \in \mathcal{G}, \ (P^+, P^-) \in \mathbb{R}^d \times \mathbb{R}^d \),
\[
|H(x_I, P^+, P^-)| \leq C_1(|P^+|_\infty + |P^-|_\infty).
\]

(H5) There exists \( C_2 > 0 \) such that for all \( x_I \in \mathcal{G}, \ P^+, P^-, Q^+, Q^- \in \mathbb{R}^d \),
\[
|H(x_I, P^+, P^-) - H(x_I, Q^+, Q^-)| \leq C_2(|P^+ - Q^+| + |P^- - Q^-|).
\]

(H6) \( H = H(x_I, P^+_1, \ldots, P^+_d, P^-_1, \ldots, P^-_d) \) satisfies the following monotonicity condition, a.e. \( (x, P^+, P^-) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \):
\[
\frac{\partial H}{\partial P^+_i}(x, P^+, P^-) \leq 0 \quad \text{and} \quad \frac{\partial H}{\partial P^-_i}(x, P^+, P^-) \geq 0.
\]

Remark 10. This assumptions holds in the a.e. sense and makes sense by the Lipschitz assumption.

(H7) (consistency). There exists \( C_3 > 0 \) such that for all \( x_I \in \mathcal{G}, \ x \in \mathbb{R}^d \), and \( P \in \mathbb{R}^d \),
\[
|H(x_I, P, P) - H(x, P)| \leq C_3|x_I - x|.
\]

In the next section we will give some examples of numerical schemes satisfying (H4)–(H7).

Remark 11. It is well known that the monotonicity assumption (H6), together with the Courant–Friedrichs–Lewy (CFL) condition
\[
\frac{\Delta t}{\Delta x} \sum_{i=1}^d \left( \left| \frac{\partial H}{\partial P^+_i}(x, P^+, P^-) \right| + \left| \frac{\partial H}{\partial P^-_i}(x, P^+, P^-) \right| \right) \leq 1,
\]
a.e. \( (x, P^+, P^-) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \),

ensures that the scheme
\[
\begin{align*}
v^{n+1}_I &= v^n_I + \Delta t H(x_I, D^+ v^n(x_I), D^- v^n(x_I)), \quad v^0_I = \tilde{u}_0(x_I) \\
v^n_I &= \max \left( v^n_I - \Delta t H(x_I, D^+ v^n(x_I), D^- v^n(x_I)), \ g(x_I, t_{n+1}) \right)
\end{align*}
\]

is monotone (i.e., if \((v^n)_n\) is a subsolution of (5.4) and if \((w^n)_n\) is a supersolution of (5.4) and such that \(v^n \leq w^n\), then \(v^{n+1} \leq w^{n+1}\)).

Furthermore, by (H5) we have \( |\frac{\partial H}{\partial P^+_i}| \leq C_2 \), and thus the CFL condition is satisfied as soon as \( \frac{\Delta t}{\Delta x} \leq 1/(2dC_2) \).

Remark 12. Equation (5.1) implies also that
\[
v^{n+1}_I = \max \left( v^n_I - \Delta t H(x_I, D^+ v^n(x_I), D^- v^n(x_I)), \ g(x_I, t_{n+1}) \right)
\]

We deduce, assuming (H6) and the CFL condition (5.3), that the scheme (5.1) is monotone.

We then have the following error estimate.
Theorem 5 (discrete-continuous error estimate). Assume (H4)–(H7) and that
\( \vartheta_0 \) and \( g \) are Lipschitz continuous and bounded. Let \( T > 0 \). There exists a constant \( K \geq 0 \) (depending only on \( d, C_1, C_2, C_3 \|D\vartheta_0\|_{\infty}, \|Dg\|_{\infty}, \|\vartheta_0\|_{\infty}, \|g\|_{\infty}, \) and \( \|\partial H/\partial p\|_{\infty} \)) such that if we choose \( \Delta x \) and \( \Delta t \) sufficiently small, such that the CFL condition (5.3) holds and
\[
\left( \sqrt{T} (\Delta x + \Delta t)^{1/2} + \sup_{\vartheta} |\vartheta_0 - \tilde{u}_0| \right) \leq \frac{1}{K},
\]
then the error between the solution \( \vartheta \) of (2.7) (with a Hamiltonian satisfying (3.4)) and the discrete solution \( v \) of the finite difference scheme (5.1) satisfy
\[
\sup_{0 \leq n \leq N_T} \sup_{\vartheta} |\vartheta(\cdot, t_n) - v^n| \leq K \left( \max(T, \sqrt{T}) (\Delta x + \Delta t)^{1/2} + \sup_{\vartheta} |\vartheta_0 - \tilde{u}_0| \right).
\]

Remark 13. The fact that \( \vartheta_0 \) and \( g \) are bounded is not a restriction since we can truncate them, and this will not change the set \( \{x, \vartheta(x, t) \leq 0\} \).

Proof. The proof is an adaptation of that of Crandall and Lions \[14\], revisited by Alvarez et al. \[1\]. Nevertheless, for the reader’s convenience, we give the main steps to show how to take into account the obstacle. The main idea of the proof is the same as that of comparison principles, i.e., to consider the maximum of \( u - v \), to duplicate the variable, and to use the viscosity inequalities to get the result. We consider that \( \Delta x + \Delta t \leq 1 \).

We first assume that
\[
\vartheta_0(x_I) \geq \tilde{u}_0(x_I) \quad \forall I \in \mathbb{Z}^d,
\]
and we set
\[
\mu^0 := \sup_{\vartheta} (\vartheta_0 - \tilde{u}_0) \geq 0.
\]
We denote throughout by \( K \) various constants depending only on \( d, C_1, C_2, C_3, \|D\vartheta_0\|_{\infty}, \|Dg\|_{\infty}, \|\vartheta_0\|_{\infty}, \|g\|_{\infty}, \) and \( \|\partial H/\partial p\|_{\infty} \). Since \( \vartheta_0 \) and \( g \) are bounded, we deduce that \( \vartheta \) is bounded.

The proof is split into three steps.

Step 1. Estimate on \( v \). We have the following estimate for the discrete solution:
\[
-Kt_n - \mu^0 \leq v(x_I, t_n) - \vartheta_0(x_I) \leq Kt_n + \mu^0.
\]
To show this, it suffice to consider \( w^\pm(x_I, t_n) = \vartheta_0(x_I) \pm Kt_n \pm \mu^0 \) and to show that \( w^- \) (resp., \( w^+ \)) is a subsolution (resp., a supersolution) of the scheme for \( K \) large enough. The result will then follow by the monotonicity of the scheme. Let us prove that \( w^+ \) is a supersolution. On one hand, we have
\[
\frac{w_I^{x,n+1} - w_I^{x,n}}{\Delta t} + \mathcal{H}(x_I, D^+ w^{x,n}(x_I), D^- w^{x,n}(x_I)) = K + \mathcal{H}(x_I, D^+ \vartheta_0(x_I), D^- \vartheta_0(x_I)) \geq K - 2C_1 \|D\vartheta_0\|_{\infty} \geq 0
\]
for \( K \geq 2C_1 \|D\vartheta_0\|_{\infty} \) and where we have used assumption (H4) for the second line.
On the other hand, we also have
\[
\begin{align*}
    w^+(x_I, t_{n+1}) - g(x_I, t_{n+1}) &= \vartheta_0(x_I) + Kt_{n+1} + \mu^0 - g(x_I, t_{n+1}) \\
    &\geq \vartheta_0(x_I) - g(x_I, 0) + Kt_{n+1} + g(0, x_I) - g(x_I, t_{n+1}) \\
    &\geq t_{n+1}(K - \|Dg\|_\infty) \\
    &\geq 0
\end{align*}
\]
for \( K \geq \|Dg\|_\infty \).

From the two previous inequalities, we deduce that \( w^+ \) is a supersolution of the scheme. Remarking, moreover, that
\[
    w^+ + (x_I, 0) = \vartheta_0(x_I) + \mu^0 \geq v_0(x_I)
\]
i.e.,
\[
    v(x_I, t_n) \leq \vartheta_0(x_I) + Kt_n + \mu_0.
\]
To obtain the reverse inequality, we show in a similar way that
\[
    \frac{w^+_{n+1} - w^+_n}{\Delta t} + \mathcal{H}(x_I, D^-w^+_{n+1}(x_I), D^-w^+_{n}(x_I)) \leq 0,
\]
which implies that \( w^- \) is a subsolution of the scheme, and obtain the desired result.

Before continuing the proof, we need some notation. We put
\[
    \mu := \sup_{\mathcal{G}} (\vartheta - v).
\]
We want to bound from above \( \mu \) by \( \mu_0 \) plus a constant. We assume that \( \mu > 0 \) (otherwise the estimate is trivial). For every \( 0 < \alpha \leq 1, 0 < \varepsilon \leq 1, \) and \( 0 < \eta \leq 1, \) we set
\[
    M_{\eta, \varepsilon}^{\alpha} := \sup_{\mathbb{R}^N \times \mathcal{G} \times (0, T) \times \{0, \ldots, T_N\}} \Psi_{\eta, \varepsilon}^{\alpha, \varepsilon}(x, x_I, t, t_n)
\]
with
\[
    \Psi_{\eta, \varepsilon}^{\alpha, \varepsilon}(x, x_I, t, t_n) := \vartheta(x, t) - v(x_I, t_n) - \frac{|x - x_I|^2}{2\varepsilon} - \frac{|t - t_n|^2}{2\varepsilon} - \eta t - \alpha(|x|^2 + |x_I|^2).
\]
We shall drop the superscripts and subscripts on \( \Psi \) if there is no ambiguity. We remark that for \( \eta \) and \( \alpha \) small enough, we have \( M_{\eta, \varepsilon}^{\alpha} \geq \frac{\mu}{4} \).

Since \( \vartheta \) and \( v \) are bounded (using Step 1 for \( v \)), we then deduce that \( \Psi \) achieves its maximum at some point that we denote by \( (x, x_I, t, t_n) \).

**Step 2.** Estimates for the maximum point of \( \Psi \). Here we show that there exists a constant \( K > 0 \) such that the following estimates hold:
\[
    \alpha(|x|^2 + |x_I|^2) \leq K
\]
and

\[(5.9) \quad |x - x_I| \leq K \varepsilon \quad \text{and} \quad |t - t_n| \leq K \varepsilon.\]

To prove (5.8), it suffices to use the inequality \(\Psi(x, x_I, t, t_n) \geq \Psi(0, 0, 0, 0) \geq 0.\) Indeed, this implies

\[\alpha(|x|^2 + |x_I|^2) \leq \vartheta(x, t) - v(x_I, t_n) \leq K.\]

To prove the first estimate of (5.9), we use the inequality \(\Psi(x, x_I, t, t_n) \geq \Psi(x_I, x_I, t, t_n)\) to get

\[\frac{|x - x_I|^2}{2\varepsilon} \leq \vartheta(x_I, t) - \vartheta(x, t) + \alpha(|x_I|^2 - |x|^2) \leq |x - x_I| \left(\alpha(|x_I| + |x|) + K\right) \leq K|x - x_I|\]

which implies the result.

The last inequality is obtained in the same way by using the inequality \(\Psi(x, x_I, t, t_n) \geq \Psi(x_I, x_I, t_n, t_n)\).

**Step 3. Upper bound of \(\mu\).** First, we claim that for \(\eta\) large enough, we have either \(t = 0\), or \(t_n = 0\), or

\[\mu \leq K \sqrt{T \varepsilon + \Delta x + \Delta t}.\]

We argue by contradiction. We suppose that the function \((y, s) \mapsto \Psi(y, x_I, s, t_n)\) achieves its maximum at a point \((x, t)\) of \(\mathbb{R}^N \times (0, T)\). Then, using the fact that \(\vartheta\) is a subsolution of (2.7), we deduce that

\[\min \left( p_t + \eta + H(x, p_x + 2ax), \vartheta(x, t) - g(x, t) \right) \leq 0,\]

where

\[p_t = \frac{t - t_n}{\varepsilon} \quad \text{and} \quad p_x = \frac{x - x_I}{\varepsilon}.\]

We now distinguish two cases.

**Case 1.** \(\vartheta(x, t) \leq g(x, t)\). Since \(v\) is a solution of the scheme, we also have

\[v(x_I, t_n) \geq g(x_I, t_n).\]

We then deduce that

\[\frac{\mu}{2} \leq \vartheta(x, t) - v(x_I, t_n) \leq K(|x - x_I| + |t - t_n|) \leq K\varepsilon.\]

Choosing \(\varepsilon \leq \sqrt{T \varepsilon + \Delta x + \Delta t}\), we get a contradiction.

**Case 2.** \(p_t + \eta + H(x, p_x + 2ax) \leq 0\). In this case, using the fact that

\[\frac{v^{n+1} - v^n}{\Delta t} + H(x_I, D^+ v^n(x_I), D^- v^n(x_I)) \geq 0,\]

we deduce, using the classical arguments of the proof of Crandall and Lions, that

\[(5.10) \quad p_t + \frac{\Delta t}{2\varepsilon} \geq -H \left( x_I, p_x - \frac{\Delta x}{2\varepsilon} - \alpha(2x_I + \Delta x), p_x + \frac{\Delta x}{2\varepsilon} - \alpha(2x_I - \Delta x) \right).\]
Subtracting (5.10) to the inequation satisfied by \( \vartheta \), we get

\[
\eta \leq \frac{\Delta t}{2\varepsilon} + \mathcal{H}(x_I, p_x - \frac{\Delta x}{2\varepsilon} - \alpha(2x_I + \Delta x), p_x + \frac{\Delta x}{2\varepsilon} - \alpha(2x_I - \Delta x)) \\
- H(x, p_x + 2\alpha x)
\]

\[
\leq \frac{\Delta t}{2\varepsilon} + 2K\alpha|x| + \mathcal{H}(x_I, p_x, p_x) - H(x, p_x) + 2C_2 \left( \frac{\Delta x}{2\varepsilon} + 2\alpha|x_I| + \alpha|\Delta x| \right)
\]

\[
\leq K\frac{\Delta x + \Delta t}{2\varepsilon} + K\sqrt{\alpha} + C_3|x_I - x|
\]

\[
\leq K\frac{\Delta x + \Delta t}{2\varepsilon} + K\sqrt{\alpha} + K\varepsilon,
\]

where we have used the Lipschitz continuity of \( H \) in \( p \), assumption (H5) for the second line, assumption (H7) and (5.8) for the third, and (5.9) for the last.

We then deduce that for \( 1 \geq \eta \geq \eta^* := K\frac{\Delta x + \Delta t}{2\varepsilon} + K\sqrt{\alpha} + K\varepsilon \), we have either \( t = 0 \) or \( t_n = 0 \).

If \( t = 0 \), then we have

\[
M^{0,\varepsilon}_\eta(x, x_I, 0, t_n) \leq \vartheta_0(x) - v(x_I, t_n)
\]

\[
\leq Kt_n + \mu^0 + K|x - x_I|
\]

\[
\leq K\varepsilon + \mu^0,
\]

where we have used Step 1, the Lipschitz continuity of \( \vartheta \), and (5.9). In the same way, if \( t_n = 0 \), we get

\[
M^{0,\varepsilon}_\eta(x, x_I, t, 0) \leq \vartheta(x, t) - v(x_I, 0)
\]

\[
\leq K(|x - x_I| + |t|) + \mu^0
\]

\[
\leq K\varepsilon + \mu^0.
\]

We obtain that for all \((s_n, y_t) \in \{0, \ldots, N_T\Delta t\} \times \mathcal{G}\), we have

\[
\vartheta(y_t, s_n) - v(y_t, s_n) - K\left( \frac{\Delta x + \Delta t}{2\varepsilon} + \sqrt{\alpha} + \varepsilon \right)T - 2\alpha|y_t|^2 \leq M^{0,\varepsilon}_\eta \leq K\varepsilon + \mu^0.
\]

Sending \( \alpha \) to 0, taking the supremum over \((y_t, s_n) \in \mathcal{G} \times \{0, \ldots, N_T\Delta t\}\), and choosing \( \varepsilon = \sqrt{T} \sqrt{\Delta x + \Delta t} \), we finally get

\[
\sup_{\mathcal{G} \times \{0, \ldots, N_T\Delta t\}} \vartheta(y_t, s_n) - v(y_t, s_n) = \mu \leq K\sqrt{T} \sqrt{\Delta x + \Delta t} + \mu^0
\]

provided that \( \Delta x + \Delta t \leq \frac{1}{K^2} \) and \( 0 \leq \mu_0 \leq 1 \). Using the same arguments as in Alvarez et al. [1, Theorem 2], we easily deduce the result in the general case when \(-1 \leq \mu_0 \leq 1\).

This ends the proof of Theorem 5. \( \square \)

6. Numerical simulations. We keep the notation of the previous section. We now apply finite difference schemes for solving (2.7). We consider here the case of dimension \( d = 2 \), and the Hamiltonian defined by (2.6). We denote by \( f = (f_1, f_2) \) the two components of the dynamics \( f \).

In order to ensure convergence of the scheme in the viscosity framework, we need monotonicity properties (assumption (H6)). A basic standard finite difference scheme
is obtained with
\[
\mathcal{H}(x, P^+, P^-) := \max_{\alpha \in A} \left( \max(0, f_1(x, \alpha)) P_1^- + \min(0, f_1(x, \alpha)) P_1^+ \\
+ \max(0, f_2(x, \alpha)) P_2^- + \min(0, f_2(x, \alpha)) P_2^+ \right).
\]

Then the scheme
\[
v^{n+1}_i = \max \left( v^n_i - \Delta t \mathcal{H}(x_j, D^+ v_n(x_i), D^- v_n(x_i)), g(x_i) \right)
\]
is consistent with (2.7) and satisfies assumptions (H4)–(H7). The CFL condition, which ensures the monotonicity of the scheme, is then given by
\[
\frac{\Delta t}{\Delta x} \max_x \max_\alpha (|f_1(x, \alpha)| + |f_2(x, \alpha)|) \leq 1.
\]

Another standard scheme is obtained by
\[
v^{n+1}_i = \max \left( v^n_i - \Delta t \mathcal{H}^{LF}(x_j, D^+ v_n(x_i), D^- v_n(x_i)), g(x_i) \right),
\]
where \( \mathcal{H}^{LF} \) is the Lax–Friedrich (LF) Hamiltonian
\[
\mathcal{H}^{LF}(x, P^+, P^-) := H \left( x, \frac{P^+ + P^-}{2} \right) - \frac{C_1(x)}{2} (P_1^+ - P_1^-) - \frac{C_2(x)}{2} (P_2^+ - P_2^-),
\]
and where \( C_i(x) \) are chosen such that \( \max_P |\partial \mathcal{H} / \partial P(x, P)| \leq C_i(x) \). Then, under the CFL condition \( \frac{\Delta t}{\Delta x} \max_x (C_1(x) + C_2(x)) \leq 1 \), the scheme is monotone and satisfies (H4)–(H7).

Although monotone schemes ensure convergence properties as well as error estimates, they are at most of first order [19]. This can lead to numerical diffusion problems as time increases. One way to diminish this diffusion problem is to use higher order essentially nonoscillatory (ENO) schemes as proposed by Osher and Shu [27]. Instead of (6.2), the scheme can be formulated as follows:
\[
v^{n+1}_i = \max \left( v^n_i - \Delta t \mathcal{H}^{LF}(x_j, \tilde{D}^+ v_n(x_i), \tilde{D}^- v_n(x_i)), g(x_i) \right),
\]
where \( \tilde{D}^\pm v_n(x_i) \) correspond to higher order numerical approximations of the derivatives \( \partial v / \partial x_i \) (this can also be coupled with a Runge–Kutta time discretization scheme). The scheme (6.3) is not necessarily monotone, and its convergence is not proved. Its relevance is proved in many numerical experiments (see [27] and Example 2 below).

In our illustrations, except when otherwise noted, we shall use a second-order ENO scheme [27], denoted “ENO2.”

Example 2 (backward reachable set with obstacle). In this example we compute the backward reachable set for the target \( C \) which is the ball centered at \((1,1)\) of radius \(0.5\), and with a rotation-type dynamics: \( f(x_1, x_2) = 2\pi(-x_2, x_1) \). We also consider an obstacle which is the square centered in \((-0.5, 0)\) and of length 0.8.

In Figure 6.1, we use the first-order LF scheme, and the number of mesh points \((M_{x_1} \times M_{x_2})\) is either 100\(^2\) or 200\(^2\). We observe a numerical convergence towards the


\[ M_x = M_y = 100 \]

\[ M_x = M_y = 200 \]

**Fig. 6.1.** Example 2. Backward reachable set, \( t = 0.75 \), with first-order LF scheme.

\[ M_x = M_y = 100 \]

**Fig. 6.2.** Example 2. Backward reachable set, \( t = 0.75 \), with second-order ENO2 scheme (left), and isovalues (right).

\[ (a) \text{ Solving Eq. (2.12)} \quad (b) \text{ Solving Eq. (2.7)} \]

**Fig. 6.3.** Example 2. Backward reachable set, \( t = 0.75 \), with second-order ENO2 scheme (a) solving (2.12) and (b) solving (2.7), \( M_{x_1} = M_{x_2} = 100 \).

exact front, but at a slow rate. In Figure 6.2, we use the second-order ENO2 scheme, with \( 100^2 \) mesh points. We see that the result is greatly improved.

When we enlarge the size of the obstacle, the backward reachable set becomes narrow. And still in this case the numerical solution of (2.7) gives a good approximation results. In Figure 6.3, we compare our approach to the formal one based on (2.12). The obstacle is now the square centered in \((-0.5, 0.3)\) and of length 1.0. We observe that the numerical results based on (2.12) are less accurate than those
Example 2. Hausdorff distance between numerical front and the exact one: comparison of (2.12) and (2.7) (same data as in Figure 6.3)

<table>
<thead>
<tr>
<th>$M_{x_1} = M_{x_2}$</th>
<th>Approach (2.12)</th>
<th>Approach (2.7)</th>
</tr>
</thead>
<tbody>
<tr>
<td>75</td>
<td>0.4590</td>
<td>0.2810</td>
</tr>
<tr>
<td>150</td>
<td>0.2540</td>
<td>0.1244</td>
</tr>
<tr>
<td>300</td>
<td>0.1646</td>
<td>0.0762</td>
</tr>
</tbody>
</table>

Obtained by solving (2.7). A numerical comparison is given in Table 6.1, where we are given the Hausdorff distances between the numerical front and the exact one.

Example 3. In this example we consider a simplified Zermelo problem: a swimmer wants to reach the target $C := B(0, r)$, which is the ball centered at the origin and of radius $r = 0.25$. The dynamics depends on a control $u = (u_1, u_2)$, where $u_1^2 + u_2^2 \leq 1$ is given by

$$f(x, u) = (c + u_1, u_2)$$

for $x := (x_1, x_2) \in \mathbb{R}^2$,

where $c := 2$ is the speed of the current and $u$ is the speed of the swimmer. We consider two fixed obstacles as represented in Figure 6.4. In order to obtain the set of points that can reach the target up to time $t$, the Hamiltonian function considered here is (see Remark 2)

$$H(x, \nabla v) := \max_{\lambda \in [0, 1], u_1^2 + u_2^2 \leq 1} \lambda (-c \partial x_1 v - u_1 \partial x_1 v - u_2 \partial x_2 v) = \max (0, -c \partial x_1 v + ||\nabla v||).$$

(Note that the set of points that can reach the target at time $t$ exactly would be obtained by using simply $H(x, \nabla v) := -c \partial x_1 v + ||\nabla v||$.)

Numerical results are given in Figure 6.4. We have used the ENO2 scheme with 100$^2$ grid points. Computations are performed up to time $t = 2$ and on the domain $[-2, 2] \times [-1.5, 1.5]$. For a given $x = (x_1, x_2)$, the obstacle function is defined by

$$(6.4) g(x) := \max \left( g_{\min}, C_1 (r_a - ||x - a||_\infty), C_1 (r_b - \max \left( x_1 - b_1, \frac{1}{5} (x_2 - b_2) \right)) \right)$$

where $r_a = 0.2, a = (0.3, 0.4); r_b = 0.2, b = (-1, -1.5); C_1 := 20; \text{ and } g_{\min} := -0.2$.

The initial data is defined by $\vartheta_0(x) := C_1 \min (r_0, ||x|| - r_0)$, where $r_0 = 0.25$.

We observe a small gap between the exact front and the approximated one.
Fig. 6.5. Example 3. Approach (2.9) with various $\eta$ parameters and $M_{x_1} = M_{x_2} = 100$.

Fig. 6.6. Example 3. Penalization approach (2.11) with various $\epsilon$ parameters and $M_{x_1} = M_{x_2} = 100$.

Remark 14. As noted in Remark 7, the theoretical results hold for any choice of $\vartheta_0$ and $g$ such satisfying (2.3) and (2.1). However, the choice of $\vartheta_0$ and $g$ seems important for numerical purposes. Indeed when we consider $\vartheta_0(x) := \min(r_0, \|x\|-r_0)$ and

$$g(x) := \max \left( r_a - \|x-a\|_\infty, \ r_b - \max \left( |x_1-b_1|, \frac{1}{3} |x_2-b_2| \right) \right)$$

instead of (6.4), the numerical results are less accurate.

In Figure 6.5, we give also some numerical results obtained by using the approach based on (2.9). As noted in section 2.1, this approach characterizes the backward reachable sets for every $\eta > 0$. However, when we set $\eta = 1$, the obstacle is almost not taken into account, while for bigger parameters of $\eta$, a numerical diffusion is observed.

In Figure 6.6 we give the results obtained with the penalization approach (2.11) and with different parameters $\epsilon$. For $\epsilon \geq 10^{-2}$, the penalization approach does not take into account the obstacle. On the other hand, for small parameters $\epsilon \leq 10^{-3}$, the obstacle is well taken into account. However, the Lipschitz constant of function $u^\epsilon$ becomes very high, leading to less accuracy in the numerical computations (the error estimate depends on this Lipschitz constant).

Example 4. In this last example we consider the Zermelo problem with a nonlinear dynamics. The target $C := B(0, r)$ is the ball centered at the origin and of radius
Approach (2.9), $\eta = 1$  
Approach (2.9), $\eta = 10^3$  
Approach (2.9), $\eta = 10^5$

Fig. 6.8. Example 4. Approach (2.9) with various $\eta$ parameters, $M_{x_1} = M_{x_2} = 100$.

Approach (2.11), $\epsilon = 10^{-3}$  
Approach (2.11), $\epsilon = 10^{-5}$  
Approach (2.11), $\epsilon = 10^{-7}$

Fig. 6.9. Example 4. Penalization approach (2.11) with various $\epsilon$ parameters, $M_{x_1} = M_{x_2} = 100$.

$r = 0.25$. The dynamics is now given by

\[ f(x, u) = (c - ax_2^2 + u_1, u_2) \quad \text{for} \quad x = (x_1, x_2) \in \mathbb{R}^2, \]

where $a = 0.5$, $c := 2$ is the speed of the current, and $u = (u_1, u_2)$ is the speed of the swimmer, with $||u|| := (u_1^2 + u_2^2)^{1/2} \leq 1$ (on the boundary $x_2 = \pm 2$, the current speed vanishes). We consider the same two fixed obstacles as before. The Hamiltonian function considered here is thus

\[ H(x, \nabla v) := \max \left( 0, -(c - ax_2^2) \partial_{x_1} v + \|\nabla v\| \right). \]
In this example, the computational domain is $[-2, 2] \times [-2, 2]$. We show in Figures 6.7, 6.8, and 6.9 the numerical results obtained, respectively, with the obstacle approach (2.7), approach (2.9), and the penalization method (2.11).

These results are computed at time $T = 3$. Comparison is made with the numerical solution computed by solving (2.7) in the refined grid with $M_{x_1} = M_{x_2} = 400$ (black curve). On this nonlinear example, we see that the obstacle approach (2.7) is more accurate than the other two.

**Appendix. Two-player games with state constraints.** We present a generalization of the previous approach to the case of two-player deterministic games with state constraints, without assuming any controllability assumption. We refer to [34, 16, 2, 32] and references therein for an introduction and some results for deterministic two-player games with infinite horizon.

In the literature, a controllability assumption or continuity of the value function is in general assumed [3] in order to deal with state constrained problems. Note that in the work of Cardaliaguet, Quincampoix, and Saint-Pierre [13], no controllability assumption is made, and a characterization is obtained involving nonsmooth analysis.

Let $A$ and $B$ be two nonempty compact subset of $\mathbb{R}^m$ and $\mathbb{R}^p$, respectively. For $t \geq 0$, let $\mathbb{A}_t$ be the set of measurable functions $\alpha : (0, t) \to A$, and let $\mathbb{B}_t$ be the set of measurable functions $\beta : (0, t) \to B$. We consider a continuous dynamics $\mathbb{f} : \mathbb{R}^d \times A \times B \to \mathbb{R}^d$ and, for every $x \in \mathbb{R}^d$ and $(\alpha, \beta) \in \mathbb{A}_t \times \mathbb{B}_t$, its associated trajectory $y = y_{x, \alpha, \beta}$ solution of

$$\hat{y}(s) = \mathbb{f}(y(s), \alpha(s), \beta(s)) \text{ for a.e. } s \in [0, t], \quad y(0) = x. \tag{A.1}$$

We consider a game involving two players. The first player wants to steer the system (initially at point $x$) toward the target $C$ in minimal time by staying in $K$ (and using input $\alpha$), while the second player tries to steer the system away from $C$ or from $K$ (with input $\beta$). We define the set of nonanticipative strategies for the first player as follows:

$$\Gamma_t := \left\{ \alpha : \mathbb{B}_t \to \mathbb{A}_t \right\} \forall (\beta, \tilde{\beta}) \in \mathbb{B}_t, \text{ and } \forall s \in [0, t],$$

$$\left( \beta(\theta) = \tilde{\beta}(\theta) \text{ a.e. } \theta \in [0, s] \right) \Rightarrow \left( \alpha[\beta](\theta) = \alpha[\tilde{\beta}](\theta) \text{ a.e. on } [0, s] \right).$$

Then we are interested in characterizing the following capture basin for the first player:

$$\text{Cap}_C(t) := \left\{ x, \exists \alpha \in \Gamma_t \forall \beta \in \mathbb{B}_t, \left( y_{x, \alpha, \beta}[t] \in C, \text{ and } y_{x, \alpha, \beta}[\theta] \in K \forall \theta \in [0, t] \right) \right\}. \tag{A.2}$$

Now we consider a function $g$ satisfying (2.3), and we define the following value function for the first player:

$$\overline{\mathcal{V}}(x, t) := \inf_{\alpha \in \Gamma_t} \max_{\beta \in \mathbb{B}_t} \left\{ \max \left( \hat{g}_0(y_{x, \alpha, \beta}[t]), \max_{\theta \in [0, t]} g(y_{x, \alpha, \beta}[\theta]) \right) \right\}. \tag{A.2}$$

By using similar arguments as before, we have the following.

**Theorem 6.** (i) For any $t \geq 0$, the capture basin for the first player is characterized by

$$\text{Cap}_C(t) = \{ x, \ \vartheta(x, t) \leq 0 \}. \tag{I}$$
(ii) If $g$ and $\vartheta_0$ are Lipschitz continuous, $\vartheta$ is the unique continuous viscosity solution of

\[
\begin{align*}
(A.3a) \quad & \min(\partial_t \vartheta + \mathcal{H}(x, \nabla \vartheta), \vartheta - g(x)) = 0, \quad t \geq 0, \ x \in \mathbb{R}^d, \\
(A.3b) \quad & \vartheta(x, 0) = \max(\vartheta_0(x), g(x)), \quad x \in \mathbb{R}^d,
\end{align*}
\]

where $\mathcal{H}(x, p) := \max_{\alpha \in A} \min_{\beta \in B} -f(x, \alpha, \beta) \cdot p$.

This gives again a characterization of the capture basin with state constraints by using a continuous viscosity approach. The corresponding minimal time function can then be recovered as in Proposition 1.

REFERENCES


