Approximate Models for the Scattering by a Thin Periodic Layer
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Context
- Thickness of the ring δ. Angular periodicity \( \approx \delta \).
- \( \delta \approx \varepsilon \approx \delta < R_0 \).
- Difficulty: two different scales \( \delta, \varepsilon \).

Goal: replacing the periodic ring by an approximate transmission condition across \( \Gamma \).
Method:
- Asymptotic expansion of the solution with respect to the small parameter \( \delta \): matched asymptotic expansion / homogenisation (see [1], [2],[3]).
- Construction of stable approximate models using this expansion.

Two Dimensional Model Problem

**Description**

Far field:

(1) \( \mathbf{u}^\ast = \sum_{n=1}^{N} \mathbf{u}_n \mathbf{e}_n (r, \theta) \)

Near field:

(2) \( \mathbf{u} = \sum_{n=1}^{N} \mathbf{u}_n (V, \theta) \)

\( \mathbf{u}_n \Delta \mathbf{u} + \omega^2 \mathbf{u}_n = f_{1n} \) in \( \Omega \)

\( \mathbf{u}_n \Delta \mathbf{u} + \omega^2 \mathbf{u}_n = f_{2n} \) in \( \Omega \)

\( \mathbf{u}_n \Delta \mathbf{u} + \omega^2 \mathbf{u}_n = f_{3n} \) in \( \Omega \)

\( \mathbf{u}_n \Delta \mathbf{u} + \omega^2 \mathbf{u}_n = f_{4n} \) in \( \Omega \)

\( \mathbf{u}_n \Delta \mathbf{u} + \omega^2 \mathbf{u}_n = f_{5n} \) in \( \Omega \)

+ matching conditions: (1) and (2) coincide in an overlapping area.

Approximate Model
- Main idea: we want to find an approximate well-posed problem whose solution \( \mathbf{u}^\ast \) is closed to the two first terms of the far field asymptotic expansion \( u_1 + u_2 \).
- Method: we use the fact that \( |u_3| \approx |u_1 + u_2| \).

**Asymptotic Expansion**

- First terms of the far field asymptotic expansion

**Numerical Results**

Figure 1: Scattering of a plane wave: 'exact' solution, approximate solution, convergence rate

3D Maxwell Problem

**Description**

\[ \text{curl} \left( \frac{1}{\varepsilon} \text{curl} \mathbf{E} \right) - \omega^2 \mathbf{E} = F \text{ in } \Omega \]

\( \mathbf{E} \) \( L_2 \)-periodic in \( x \)

\( \mathbf{E} \) \( L_2 \)-periodic in \( y \)

\[ \frac{\partial \mathbf{E}}{\partial n} = 0 \text{ on } \Sigma^\pm \]

For an \( \mathbf{C} \) large enough,
- \( D_1^\mathbf{C} \) and \( D_2^\mathbf{C} \) are positive diagonal matrices.
- \( a^\mathbf{C}, b^\mathbf{C} \) are positive constants.

**Existence, Uniqueness, and Stability**

Operator \( \mathbf{G} : \mathbf{H}^{1/2} (\partial \Gamma) \rightarrow \mathbf{H} (\partial \Gamma) \)

\[ F = \lambda \mathbf{a}^\mathbf{C} \mathbf{u} + \mathbf{b}^\mathbf{C} \mathbf{v} = \int_\Omega \mathbf{F} \cdot \mathbf{v} \quad \forall \mathbf{v} \in \mathbf{X} \]

Assumption: \( \lambda \) is such that \( \mathbf{a}^\mathbf{C} \mathbf{u} + \mathbf{b}^\mathbf{C} \mathbf{v} \) is well defined.

Variational formulation:

\[ \mathbf{X} = \left\{ \mathbf{E} \in \mathbf{H} \text{per}(\partial \Omega) \bigg| \left( \mathbf{E}, \mathbf{G} \right) = 0, \mathbf{E} \perp \mathbf{F} \right\} \]

Proposition: \( \mathbf{P}_1 \) is well posed. Moreover, there exists \( C>0 \) and \( \delta_0>0 \) such that, for any \( \delta \leq \delta_0 \),

**Main ideas of the proof** (based on the ideas of [41])

- Helmholtz decomposition:

\[ \mathbf{X} = \mathbf{U} \oplus \mathbf{W} \]

\[ S = \left\{ \mathbf{E} \in \mathbf{H} \text{per}(\partial \Omega) \bigg| \left( \mathbf{E}, \mathbf{G} \right) = 0 \right\} \]

- Uniform estimate: proof by contradiction \( \Rightarrow \) uniqueness.

- Existence: compactness of \( \mathbf{U} \) on \( \mathbf{X}_0 \), Fredholm Alternative.

**References**