BSDEs with no driving martingale, Markov processes and associated Pseudo Partial Differential Equations. Part II: Decoupled mild solutions and Examples

Adrien BARRASSO [*] Francesco RUSSO[†]

September 2017

Abstract. Let $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times E}$ be a family of probability measures, where E is a Polish space, defined on the canonical probability space $\mathbb{D}([0,T],E)$ of E-valued cadlag functions. We suppose that a martingale problem with respect to a time-inhomogeneous generator a is well-posed. We consider also an associated semilinear *Pseudo-PDE* for which we introduce a notion of so called *decoupled mild* solution and study the equivalence with the notion of martingale solution introduced in a companion paper. We also investigate well-posedness for decoupled mild solutions and their relations with a special class of BSDEs without driving martingale. The notion of decoupled mild solution is a good candidate to replace the notion of viscosity solution which is not always suitable when the map a is not a PDE operator.

MSC 2010 Classification. 60H30; 60H10; 35S05; 60J35; 60J60; 60J75.

KEY WORDS AND PHRASES. Martingale problem; pseudo-PDE; Markov processes; backward stochastic differential equation; decoupled mild solutions.

1 Introduction

The framework of this paper is the canonical space $\Omega = \mathbb{D}([0,T],E)$ of cadlag functions defined on the interval $[0,T]$ with values in a Polish space E. This space will be equipped with a family $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times E}$ of probability measures

---

[*] ENSTA ParisTech, Unité de Mathématiques appliquées, 828, boulevard des Maréchaux, F-91120 Palaiseau, France and Ecole Polytechnique, F-91128 Palaiseau, France. E-mail: adrien.barrasso@ensta-paristech.fr

[†] ENSTA ParisTech, Unité de Mathématiques appliquées, 828, boulevard des Maréchaux, F-91120 Palaiseau, France. E-mail: francesco.russo@ensta-paristech.fr

indexed by an initial time $s \in [0,T]$ and a starting point $x \in E$. For each $(s,x) \in [0,T] \times E$, $\mathbb{P}^{s,x}$ corresponds to the law of an underlying forward Markov process with time index $[0,T]$, taking values in the Polish state space E which is characterized as the solution of a well-posed martingale problem related to a certain operator $(\mathcal{D}(a), a)$ and an increasing continuous function $V : [0,T] \rightarrow \mathbb{R}$. In the companion paper [7] we have introduced a semilinear equation generated by $(\mathcal{D}(a), a)$, called *Pseudo-PDE* of the type

$$\left\{ \begin{array}{rcl} a(u) + f\left(\cdot, \cdot, u, \sqrt{\Gamma(u,u)}\right) & = & 0 \quad \text{on } [0,T] \times E \\ u(T, \cdot) & = & g, \end{array} \right. \tag{1.1}$$

where $\Gamma(u,u) = a(u^2) - 2ua(u)$ is a potential theory operator called the *carré du champs operator*. A classical solution of (1.1) is defined as an element of $\mathcal{D}(a)$ verifying (1.1). In [7] we have also defined the notion of *martingale solution* of (1.1), see Definition 2.22. A function u is a martingale solution if (1.1) holds replacing the map a (resp. Γ) with an extended operator $\mathfrak{a}$ (resp. $\mathfrak{G}$) which is introduced in Definition 2.15 (resp. 2.17). The martingale solution extends the (analytical) notion of classical solution, however it is a probabilistic concept. The objectives of the present paper are essentially three.

1. To introduce an alternative notion of (this time analytical) solution, that we call *decoupled mild*, since it makes use of the time-dependent transition kernel associated with a. This new type of solution will be shown to be essentially equivalent to the martingale one.

2. To show existence and uniqueness of decoupled mild solutions.

3. To emphasize the link with solutions of forward BSDEs (FBSDEs) without driving martingale introduced in [7].

The aforementioned FBSDEs are of the form

$$Y_t^{s,x} = g(X_T) + \int_t^T f\left(r, X_r, Y_r^{s,x}, \sqrt{\frac{d\langle M^{s,x}\rangle}{dV}(r)}\right) dV_r - (M_T^{s,x} - M_t^{s,x}), \tag{1.2}$$

in a stochastic basis $\left(\Omega, \mathcal{F}^{s,x}, (\mathcal{F}_t^{s,x})_{t\in[0,T]}, \mathbb{P}^{s,x}\right)$ which depends on (s,x). Under suitable conditions, the solution of this FBSDES is a couple $(Y^{s,x}, M^{s,x})$ of cadlag stochastic processes where $M^{s,x}$ is a martingale. This was introduced and studied in a more general setting in [7], see [32] for a similar formulation.

We refer to the introduction and reference list of previous paper for an extensive description of contributions to non-Brownian type BSDEs.

The classical forward BSDE, which is driven by a Brownian motion is of the form

$$\left\{ \begin{array}{rcl} X_t^{s,x} & = & x + \int_s^t \mu(r, X_r^{s,x})dr + \int_s^t \sigma(r, X_r^{s,x})dB_r \\ Y_t^{s,x} & = & g(X_T^{s,x}) + \int_t^T f\left(r, X_r^{s,x}, Y_r^{s,x}, Z_r^{s,x}\right) dr - \int_t^T Z_r^{s,x} dB_r, \end{array} \right. \tag{1.3}$$

where B is a Brownian motion. Existence and uniqueness for (1.3) was established first supposing mainly Lipschitz conditions on f with respect to the third and fourth variable. μ and σ were also assumed to be Lipschitz (with respect to x) and to have linear growth. In the sequel those conditions were considerably relaxed, see [36] and references therein. This is a particular case of a more general (non-Markovian) Brownian BSDE introduced in 1990 by E. Pardoux and S. Peng in [34], after an early work of J.M. Bismut in 1973 in [8].

Equation (1.3) was a probabilistic representation of a semilinear partial differential equation of parabolic type with terminal condition:

$$\left\{\begin{array}{l} \partial_t u + \frac{1}{2}\sum\limits_{i,j\leq d}(\sigma\sigma^\intercal)_{i,j}\partial^2_{x_i x_j}u + \sum\limits_{i\leq d}\mu_i\partial_{x_i}u + f(\cdot,\cdot,u,\sigma\nabla u) = 0 \quad \text{on } [0,T[\times\mathbb{R}^d \\ u(T,\cdot) = g. \end{array}\right. \tag{1.4}$$

Given, for every (s,x), a solution $(Y^{s,x},Z^{s,x})$ of the FBSDE (1.3), under some continuity assumptions on the coefficients, see e.g. [35], it was proved that the function $u(s,x) := Y^{s,x}_s$ is a viscosity solution of (1.4), see also [37, 35, 37, 15], for related work.

We prolong this idea in a general case where the FBSDE is (1.2) with solution $(Y^{s,x},M^{s,x})$. In that case $u(s,x) := Y^{s,x}_s$ will be the decoupled mild solution of (1.1), see Theorem 3.15; in that general context the decoupled mild solution replaces the one of viscosity, for reasons that we will explain below. One celebrated problem in the case of Brownian FBSDEs is the characterization of $Z^{s,x}$ through a deterministic function v. This is what we will call the *identification problem*. In general the link between v and u is not always analytically established, excepted when u has some suitable differentiability property, see e.g. [6]: in that case v is closely related to the gradient of u. In our case, the notion of decoupled mild solution allows to identify (u,v) as the analytical solution of a deterministic problem. In the literature, the notion of mild solution of PDEs was used in finite dimension in [4], where the authors tackled diffusion operators generating symmetric Dirichlet forms and associated Markov processes thanks to the theory of Fukushima Dirichlet forms, see e.g. [21]. A partial extension to the case of non-symmetric Dirichlet forms is performed in [31]. Infinite dimensional setups were considered for example in [20] where an infinite dimensional BSDE could produce the mild solution of a PDE on a Hilbert space.

Let B be a functional Banach space $(B,\|\cdot\|)$ of real Borel functions defined on E and A be an unbounded operator on $(B,\|\cdot\|)$. In the theory of evolution equations one often considers systems of the type

$$\left\{\begin{array}{rcl} \partial_t u + Au &=& l \text{ on } [0,T]\times\mathbb{R}^d \\ u(T,\cdot) &=& g, \end{array}\right. \tag{1.5}$$

where $l : [0,T]\times\mathbb{R}^d \rightarrow \mathbb{R}$ and $g : \mathbb{R}^d \rightarrow \mathbb{R}$ are such that $l(t,\cdot)$ and g belong to B for every $t \in [0,T]$. The idea of mild solutions consists to consider $-A$ (when possible) as the infinitesimal generator of a semigroup of operators $(P_t)_{t\geq 0}$ on $(B,\|\cdot\|)$, in the following sense. There is $\mathcal{D}(A) \subset B$, a dense subset on which $-Af = \lim\limits_{t\rightarrow 0^+}\frac{1}{t}(P_t f - f)$. In particular one may think of $(P_t)_{t\geq 0}$ as the heat

kernel semi-group and A as $\frac{1}{2}\Delta$. The approach of mild solutions is also very popular in the framework of stochastic PDEs see e. g. [9].
When A is a local operator, one solution (in the sense of distributions, or in the sense of evaluation against test functions) to the linear evolution problem with terminal condition (1.5) is the so called *mild solution*

$$u(s,\cdot) = P_{T-s}[g] - \int_s^T P_{r-s}[l(r,\cdot)]dr. \tag{1.6}$$

If l is explicitly a function of u then (1.6) becomes itself an equation and a mild solution would consist in finding a fixed point of (1.6). Let us now suppose the existence of a map $S : \mathcal{D}(S) \subset B \to B$, typically S being the gradient, when (P_t) is the heat kernel semigroup. The natural question is what would be a natural replacement for a *mild solution* for

$$\left\{ \begin{array}{rcl} \partial_t u + Au & = & f(s,\cdot,u,Su) \text{ on } [0,T]\times\mathbb{R}^d \\ u(T,\cdot) & = & g. \end{array} \right. \tag{1.7}$$

If the domain of S is B, then it is not difficult to extend the notion of mild solution to this case. One novelty of our approach consists is considering the case of solutions $u : [0,T]\times\mathbb{R}^d \to \mathbb{R}$ for which $Su(t,\cdot)$ is not defined.

1. Suppose one expects a solution not to be classical, i.e. such that $u(r,\cdot)$ should not belong to the domain of $\mathcal{D}(A)$ but to be in the domain of S. In the case when Pseudo-PDEs are usual PDEs, one think to possible solutions which are not $C^{1,2}$ but admitting a gradient, typically viscosity solutions which are differentiable in x. In that case the usual idea of mild solutions theory applies to equations of type (1.7).

   In this setup, inspired by (1.6) a mild solution of the equation is naturally defined as a solution of the integral equation

   $$u(s,\cdot) = P_{T-s}[g] + \int_s^T P_{r-s}[f(r,\cdot,u(r,\cdot),Su(r,\cdot))]dr. \tag{1.8}$$

2. However, there may be reasons for which the candidate solution u is such that $u(t,\cdot)$ does not even belong to $\mathcal{D}(S)$. In the case of PDEs it is often the case for viscosity solutions of PDEs which do not admit a gradient. In that case the idea is to replace (1.8) with

   $$u(s,\cdot) = P_{T-s}[g] + \int_s^T P_{r-s}[f(r,\cdot,u(r,\cdot),v(r,\cdot))]dr. \tag{1.9}$$

   and to add a second equality which expresses in a *mild* form the equality $v(r,\cdot) = Su(r,\cdot)$.

We will work out previous methodology for the $Pseudo-PDE(f,g)$. In that case S will be given by the mapping $u \longmapsto \sqrt{\Gamma(u,u)}$. If A is the laplacian for instance one would have $\Gamma(u,u) = \|\nabla u\|^2$. For pedagogical purposes, one can first consider an operator a of type $\partial_t + A$ when $-A$ is the generator of a Markovian (time-homogeneous) semigroup. In this case,

$$\begin{array}{rcl} \Gamma(u,u) & = & \partial_t(u^2) + A(u^2) - 2u\partial_t u - 2uAu \\ & = & A(u^2) - 2uAu. \end{array}$$

Equation

$$\partial_t u + Au + f\left(\cdot,\cdot,u,\sqrt{\Gamma(u,u)}\right) = 0, \tag{1.10}$$

could therefore be decoupled into the system

$$\left\{ \begin{array}{l} \partial_t u + Au + f(\cdot,\cdot,u,v) = 0 \\ v^2 = \partial_t(u^2) + A(u^2) - 2u(\partial_t u + Au), \end{array} \right. \tag{1.11}$$

which furthermore can be expressed as

$$\left\{ \begin{array}{rcl} \partial_t u + Au & = & -f(\cdot,\cdot,u,v) \\ \partial_t(u^2) + A(u^2) & = & v^2 - 2uf(\cdot,\cdot,u,v) \end{array} \right. \tag{1.12}$$

Taking into account the existing notions of mild solution (1.6) (resp. (1.8)), for corresponding equations (1.5) (resp. (1.7)), one is naturally tempted to define a decoupled mild solution of (1.1) as a function u for which there exist $v \geq 0$ such that

$$\left\{ \begin{array}{rcl} u(s,\cdot) & = & P_{T-s}[g] + \int_s^T P_{r-s}[f(r,\cdot,u(r,\cdot),v(r,\cdot))]dr \\ u^2(s,\cdot) & = & P_{T-s}[g^2] - \int_s^T P_{r-s}[v^2(r,\cdot) - 2u(r,\cdot)f(r,\cdot,u(r,\cdot),v(r,\cdot))]dr. \end{array} \right. \tag{1.13}$$

As we mentioned before, our approach is alternative to a possible notion of viscosity solution for the $Pseudo-PDE(f,g)$. That notion will be the object of a subsequent paper, at least in the case when the driver do not depend on the last variable. In the general case the notion of viscosity solution does not fit well because of lack of suitable comparison theorems. On the other hand, even in the recent literature (see [5]) in order to show existence of viscosity solutions specific conditions exist on the driver. In our opinion our approach of decoupled mild solutions for $Pseudo-PDE(f,g)$ constitutes an interesting novelty even in the case of semilinear parabolic PDEs.

The main contributions of the paper are essentially the following. In Section 3.1, Definition 3.4 introduces our notion of decoupled mild solution of (1.1) in the general setup. In section Section 3.2, Proposition 3.7 states that under a square integrability type condition, every martingale solution is a decoupled mild solution of (1.1). Conversely, Proposition 3.8 shows that every decoupled mild solution is a martingale solution. In Theorem 3.9 we prove existence and uniqueness of a decoupled mild solution for (1.1). In Section 3.3, we show how the unique decoupled mild solution of (1.1) can be represented via the FBSDEs (1.2). In Section 4 we develop examples of Markov processes and corresponding operators a falling into our abstract setup. In Section 4.1, we work in the setup of [42], the Markov process is a diffusion with jumps and the corresponding operator is of diffusion type with an additional non-local operator. In Section 4.2

we consider Markov processes associated to pseudo-differential operators (typically the fractional Laplacian) as in [27]. In Section 4.3 we study a semilinear parabolic PDE with distributional drift, and the corresponding process is the solution an SDE with distributional drift as defined in [18]. Finally in Section 4.4 are interested with diffusions on differential manifolds and associated diffusion operators, an example being the Brownian motion in a Riemannian manifold associated to the Laplace-Beltrami operator.

2 Preliminaries

In this section we will recall the notations, notions and results of the companion paper [7], which will be used here.

Notation 2.1. *In the whole paper, concerning functional spaces we will use the following notations.*
A topological space E will always be considered as a measurable space with its Borel σ-field which shall be denoted $\mathcal{B}(E)$. Given two topological spaces, E, F, then $\mathcal{C}(E,F)$ (respectively $\mathcal{B}(E,F)$) will denote the set of functions from E to F which are continuous (respectively Borel) and if F is a metric space, $\mathcal{C}_b(E,F)$ (respectively $\mathcal{B}_b(E,F)$) will denote the set of functions from E to F which are bounded continuous (respectively bounded Borel). For any $p \in [1,\infty]$, $d \in \mathbb{N}^*$, $(L^p(\mathbb{R}^d), \|\cdot\|_p)$ will denote the usual Lebesgue space equipped with its usual norm.

On a fixed probability space $(\Omega, \mathcal{F}, \mathbb{P})$, for any $p \in \mathbb{N}^*$, L^p will denote the set of random variables with finite p-th moment.

A probability space equipped with a right-continuous filtration $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{T}}, \mathbb{P})$ (where $\mathbb{T}$ is equal to $\mathbb{R}_+$ or to $[0,T]$ for some $T \in \mathbb{R}_+^*$) will be called called a ***stochastic basis*** *and will be said to* ***fulfill the usual conditions*** *if the probability space is complete and if $\mathcal{F}_0$ contains all the $\mathbb{P}$-negligible sets. When a stochastic basis is fixed, $\mathcal{P}ro$ denotes the* ***progressive σ-field*** *on $\mathbb{T} \times \Omega$.*

On a fixed stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{T}}, \mathbb{P})$, we will use the following notations and vocabulary, concerning spaces of stochastic processes, most of them being taken or adapted from [28] or [29]. $\mathcal{V}$ (resp $\mathcal{V}^+$) will denote the set of adapted, bounded variation (resp non-decreasing) processes starting at 0; $\mathcal{V}^p$ (resp $\mathcal{V}^{p,+}$) the elements of $\mathcal{V}$ (resp $\mathcal{V}^+$) which are predictable, and $\mathcal{V}^c$ (resp $\mathcal{V}^{c,+}$) the elements of $\mathcal{V}$ (resp $\mathcal{V}^+$) which are continuous.
$\mathcal{M}$ will be the space of cadlag martingales. For any $p \in [1,\infty]$ $\mathcal{H}^p$ will denote the subset of $\mathcal{M}$ of elements M such that $\underset{t\in\mathbb{T}}{sup} |M_t| \in L^p$ and in this set we identify indistinguishable elements. It is a Banach space for the norm $\|M\|_{\mathcal{H}^p} = \mathbb{E}[|\underset{t\in\mathbb{T}}{sup} M_t|^p]^{\frac{1}{p}}$, and $\mathcal{H}^p_0$ will denote the Banach subspace of $\mathcal{H}^p$ containing the elements starting at zero.
If $\mathbb{T} = [0,T]$ for some $T \in \mathbb{R}_+^*$, a stopping time will be considered as a random

variable with values in $[0,T]\cup\{+\infty\}$. We define a ***localizing sequence of stopping times*** *as an increasing sequence of stopping times $(\tau_n)_{n\geq 0}$ such that there exists $N\in\mathbb{N}$ for which $\tau_N=+\infty$. Let Y be a process and τ a stopping time, we denote Y^τ the process $t\mapsto Y_{t\wedge\tau}$ which we call* ***stopped process****. If $\mathcal{C}$ is a set of processes, we define its* ***localized class*** *$\mathcal{C}_{loc}$ as the set of processes Y such that there exist a localizing sequence $(\tau_n)_{n\geq 0}$ such that for every n, the stopped process Y^{τ_n} belongs to $\mathcal{C}$.*
For any $M\in\mathcal{M}_{loc}$, we denote $[M]$ its ***quadratic variation*** *and if moreover $M\in\mathcal{H}^2_{loc}$, $\langle M\rangle$ will denote its (predictable)* ***angular bracket****. $\mathcal{H}^2_0$ will be equipped with scalar product defined by $(M,N)_{\mathcal{H}^2}=\mathbb{E}[M_TN_T]=\mathbb{E}[\langle M,N\rangle_T]$ which makes it a Hilbert space. Two local martingales M,N will be said to be* ***strongly orthogonal*** *if MN is a local martingale starting in 0 at time 0. In $\mathcal{H}^2_{0,loc}$ this notion is equivalent to $\langle M,N\rangle=0$.*

Concerning the following definitions and results, we are given some $T\in\mathbb{R}_+^*$, and a stochastic basis $\left(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\in[0,T]},\mathbb{P}\right)$ fulfilling the usual conditions.

Definition 2.2. *Let A and B be in $\mathcal{V}^+$. We will say that dB dominates dA* **in the sense of stochastic measures** *(written $dA\ll dB$) if for almost all ω, $dA(\omega)\ll dB(\omega)$ as Borel measures on $[0,T]$.*

Let $B\in\mathcal{V}^+$. $dB\otimes d\mathbb{P}$ will denote the positive measure on $(\Omega\times[0,T],\mathcal{F}\otimes\mathcal{B}([0,T]))$ defined for any $F\in\mathcal{F}\otimes\mathcal{B}([0,T])$ by $dB\otimes d\mathbb{P}(F)=\mathbb{E}\left[\int_0^T\mathbb{1}_F(t,\omega)dB_t(\omega)\right]$. A property which holds true everywhere except on a null set for this measure will be said to be true $dB\otimes d\mathbb{P}$ almost everywhere (a.e.).

We recall that given two processes A,B in $\mathcal{V}^{p,+}$, if $dA\ll dB$, there exists a predictable process which we will denote $\frac{dA}{dB}$ and call **Radon-Nikodym derivative** of A by B, verifying $A=\int_0^\cdot\frac{dA}{dB}(r)dB_r$, see Proposition I.3.13 in [29].

As in previous paper [7] we will be interested in a Markov process which is the solution of a martingale problem which we now recall below. For definitions and results concerning Markov processes, the reader may refer to Appendix A. In particular, let E be a Polish space and $T\in\mathbb{R}_+$ be a finite value we now consider $\left(\Omega,\mathcal{F},(X_t)_{t\in[0,T]},(\mathcal{F}_t)_{t\in[0,T]}\right)$ the canonical space which was introduced in Notation A.1, and a canonical Markov class measurable in time $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times E}$, see Definitions A.6 and A.4. We will also consider the completed stochastic basis $\left(\Omega,\mathcal{F}^{s,x},(\mathcal{F}^{s,x}_t)_{t\in[0,T]},\mathbb{P}^{s,x}\right)$, see Definition A.8.

We now recall what the notion of martingale problem associated to an operator introduced in Section 4 of [7].

Definition 2.3. *Given a linear algebra $\mathcal{D}(a)\subset\mathcal{B}([0,T]\times E,\mathbb{R})$, a linear operator a mapping $\mathcal{D}(a)$ into $\mathcal{B}([0,T]\times E,\mathbb{R})$ and a non-decreasing continuous function $V:[0,T]\to\mathbb{R}_+$ starting at 0, we say that a set of probability measures $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times E}$ defined on $(\Omega,\mathcal{F})$ solves the* **Martingale Problem associated to** *$(\mathcal{D}(a),a,V)$ if, for any $(s,x)\in[0,T]\times E$, $\mathbb{P}^{s,x}$ verifies* *(a)* $\mathbb{P}^{s,x}(\forall t \in [0,s], X_t = x) = 1$*;*

(b) for every $\phi \in \mathcal{D}(a)$*, the process* $\phi(\cdot, X_\cdot) - \int_s^\cdot a(\phi)(r, X_r)dV_r$*,* $t \in [s,T]$ *is a cadlag* $(\mathbb{P}^{s,x}, (\mathcal{F}_t)_{t\in[s,T]})$*-local martingale.*

We say that the **Martingale Problem is well-posed** *if for any* $(s,x) \in [0,T]\times E$*,* $\mathbb{P}^{s,x}$ *is the only probability measure satisfying the properties (a) and (b).*

As for [7], in the sequel of the paper we will assume the following.

Hypothesis 2.4. *The Markov class* $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times E}$ *solves a well-posed Martingale Problem associated to a triplet* $(\mathcal{D}(a), a, V)$ *in the sense of Definition 2.3.*

Notation 2.5. *For every* $(s,x) \in [0,T] \times E$ *and* $\phi \in \mathcal{D}(a)$*, the process* $t \mapsto \mathbb{1}_{[s,T]}(t)\left(\phi(t,X_t) - \phi(s,x) - \int_s^t a(\phi)(r,X_r)dV_r\right)$ *will be denoted* $M[\phi]^{s,x}$*.*

$M[\phi]^{s,x}$ is a cadlag $(\mathbb{P}^{s,x}, (\mathcal{F}_t)_{t\in[0,T]})$-local martingale equal to 0 on $[0,s]$, and by Proposition A.9, it is also a $(\mathbb{P}^{s,x}, (\mathcal{F}^{s,x}_t)_{t\in[0,T]})$-local martingale.

The bilinear operator below was introduced (in the case of time-homogeneous operators) by J.P. Roth in potential analysis (see Chapter III in [38]), and popularized by P.A. Meyer and others in the study of homogeneous Markov processes (see for example Exposé II: L'opérateur carré du champs in [33] or 13.46 in [28]).

Definition 2.6. *We introduce the bilinear operator*

$$\Gamma : \begin{array}{ccc} \mathcal{D}(a)\times\mathcal{D}(a) & \rightarrow & \mathcal{B}([0,T]\times E) \\ (\phi,\psi) & \mapsto & a(\phi\psi) - \phi a(\psi) - \psi a(\phi). \end{array} \tag{2.1}$$

The operator Γ *is called the* ***carré du champs operator****.*

The angular bracket of the martingales introduced in Notation 2.5 are expressed via the operator Γ. Proposition 4.8 of [7], tells the following.

Proposition 2.7. *For any* $\phi \in \mathcal{D}(a)$ *and* $(s,x) \in [0,T]\times E$*,* $M[\phi]^{s,x}$ *is in* $\mathcal{H}^2_{0,loc}$*. Moreover, for any* $(\phi,\psi) \in \mathcal{D}(a)\times\mathcal{D}(a)$ *and* $(s,x)\in[0,T]\times E$ *we have in* $(\Omega, \mathcal{F}^{s,x}, (\mathcal{F}^{s,x}_t)_{t\in[0,T]}, \mathbb{P}^{s,x})$ *and on the interval* $[s,T]$

$$\langle M[\phi]^{s,x}, M[\psi]^{s,x}\rangle = \int_s^\cdot \Gamma(\phi,\psi)(r,X_r)dV_r. \tag{2.2}$$

We introduce some significant spaces related to V.

Notation 2.8. $\mathcal{H}^{2,V} := \{M \in \mathcal{H}^2_0 | d\langle M\rangle \ll dV\}$*.*
We will also denote $\mathcal{L}^2(dV\otimes d\mathbb{P})$ *the set of (up to indistinguishability) progressively measurable processes* ϕ *such that* $\mathbb{E}[\int_0^T \phi_r^2 dV_r] < \infty$*.*

Proposition 4.11 of [7] says the following.

Proposition 2.9. *If Hypothesis 2.4 is verified then under any* $\mathbb{P}^{s,x}$*,* $\mathcal{H}^2_0 = \mathcal{H}^{2,V}$*, where* $\mathcal{H}^{2,V}$*.*

In the sequel, several functional equations will hold up to a **zero potential** set that we recall below.

Definition 2.10. *For any* $(s,x)\in[0,T]\times E$ *we define the* ***potential measure*** $U(s,x,\cdot)$ *on* $\mathcal{B}([0,T]\times E)$ *by* $U(s,x,A):=\mathbb{E}^{s,x}\left[\int_s^T \mathbb{1}_{\{(t,X_t)\in A\}}dV_t\right]$.
A Borel set $A\in\mathcal{B}([0,T]\times E)$ *will be said to be* **of zero potential** *if, for any* $(s,x)\in[0,T]\times E$ *we have* $U(s,x,A)=0$.

Notation 2.11. *Let* $p>0$*, we define*
$\mathcal{L}^p_{s,x}:=\left\{f\in\mathcal{B}([0,T]\times E,\mathbb{R}):\,\mathbb{E}^{s,x}\left[\int_s^T|f|^p(r,X_r)dV_r\right]<\infty\right\}$ *on which we introduce the usual semi-norm* $\|\cdot\|_{p,s,x}: f\mapsto\left(\mathbb{E}^{s,x}\left[\int_s^T|f(r,X_r)|^p dV_r\right]\right)^{\frac{1}{p}}$ *We also denote* $\mathcal{L}^0_{s,x}:=\left\{f\in\mathcal{B}([0,T]\times E,\mathbb{R}):\,\int_s^T|f|(r,X_r)dV_r<\infty\,\mathbb{P}^{s,x}\text{ a.s. }\right\}$.
For any $p\geq 0$*, we then define an intersection of these spaces, i.e.*
$\mathcal{L}^p_X:=\bigcap_{(s,x)\in[0,T]\times E}\mathcal{L}^p_{s,x}$.
Finally, let $\mathcal{N}$ *the linear subspace of* $\mathcal{B}([0,T]\times E,\mathbb{R})$ *containing all functions which are equal to 0* $U(s,x,\cdot)$ *a.e. for every* (s,x)*. For any* $p\in\mathbb{N}$*, we define the quotient space* $L^p_X:=\mathcal{L}^p_X/\mathcal{N}$*. If* $p\geq 1$*,* L^p_X *can be equipped with the topology generated by the family of semi-norms* $(\|\cdot\|_{p,s,x})_{(s,x)\in[0,T]\times E}$ *which makes it into a separable locally convex topological vector space.*

The statement below was stated in Proposition 4.14 of [7].

Proposition 2.12. *Let* f *and* g *be in* $\mathcal{B}([0,T]\times E,\mathbb{R})$ *such that the processes* $\int_s^\cdot f(r,X_r)dV_r$ *and* $\int_s^\cdot g(r,X_r)dV_r$ *are finite* $\mathbb{P}^{s,x}$ *a.s. for any* $(s,x)\in[0,T]\times E$*. Then* f *and* g *are equal up a zero potential set if and only if* $\int_s^\cdot f(r,X_r)dV_r$ *and* $\int_s^\cdot g(r,X_r)dV_r$ *are indistinguishable under* $\mathbb{P}^{s,x}$ *for any* $(s,x)\in[0,T]\times E$.

We recall that if two functions f,g differ only on a zero potential set then they represent the same element of L^p_X.
We recall our notion of **extended generator**.

Definition 2.13. *We first define the* ***extended domain*** $\mathcal{D}(\mathfrak{a})$ *as the set functions* $\phi\in\mathcal{B}([0,T]\times E,\mathbb{R})$ *for which there exists* $\psi\in\mathcal{B}([0,T]\times E,\mathbb{R})$ *such that under any* $\mathbb{P}^{s,x}$ *the process*

$$\mathbb{1}_{[s,T]}\left(\phi(\cdot,X_\cdot)-\phi(s,x)-\int_s^\cdot\psi(r,X_r)dV_r\right)\tag{2.3}$$

(which is not necessarily cadlag) has a cadlag modification in $\mathcal{H}^2_0$.

Proposition 4.16 in [7] states the following.

Proposition 2.14. *Let* $\phi\in\mathcal{B}([0,T]\times E,\mathbb{R})$*. There is at most one (up to zero potential sets)* $\psi\in\mathcal{B}([0,T]\times E,\mathbb{R})$ *such that under any* $\mathbb{P}^{s,x}$*, the process defined in* (2.3) *has a modification which belongs to* $\mathcal{M}_{loc}$.
If moreover $\phi\in\mathcal{D}(a)$*, then* $a(\phi)=\psi$ *up to zero potential sets. In this case, according to Notation 2.5, for every* $(s,x)\in[0,T]\times E$*,* $M[\phi]^{s,x}$ *is the* $\mathbb{P}^{s,x}$ *cadlag modification in* $\mathcal{H}^2_0$ *of* $\mathbb{1}_{[s,T]}\left(\phi(\cdot,X_\cdot)-\phi(s,x)-\int_s^\cdot\psi(r,X_r)dV_r\right)$.

Definition 2.15. *Let $\phi \in \mathcal{D}(\mathfrak{a})$ as in Definition 2.13. We denote again by $M[\phi]^{s,x}$, the unique cadlag version of the process (2.3) in $\mathcal{H}^2_0$. Taking Proposition 2.12 into account, this will not generate any ambiguity with respect to Notation 2.5. Proposition 2.12, also permits to define without ambiguity the operator*

$$\mathfrak{a}: \begin{array}{ccc} \mathcal{D}(\mathfrak{a}) & \longrightarrow & L^0_X \\ \phi & \longmapsto & \psi. \end{array}$$

$\mathfrak{a}$ *will be called the **extended generator**.*

We also extend the carré du champs operator $\Gamma(\cdot,\cdot)$ to $\mathcal{D}(\mathfrak{a}) \times \mathcal{D}(\mathfrak{a})$.

Proposition 4.18 in [7] states the following.

Proposition 2.16. *Let ϕ and ψ be in $\mathcal{D}(\mathfrak{a})$, there exists a (unique up to zero-potential sets) function in $\mathcal{B}([0,T]\times E,\mathbb{R})$ which we will denote $\mathfrak{G}(\phi,\psi)$ such that under any $\mathbb{P}^{s,x}$, $\langle M[\phi]^{s,x}, M[\psi]^{s,x}\rangle = \int_s^{\cdot} \mathfrak{G}(\phi,\psi)(r,X_r)dV_r$ on $[s,T]$, up to indistinguishability. If moreover ϕ and ψ belong to $\mathcal{D}(a)$, then $\Gamma(\phi,\psi) = \mathfrak{G}(\phi,\psi)$ up to zero potential sets.*

Definition 2.17. *The bilinear operator $\mathfrak{G} : \mathcal{D}(\mathfrak{a}) \times \mathcal{D}(\mathfrak{a}) \longmapsto L^0_X$ will be called the **extended carré du champs operator**.*

According to Definition 2.13, we do not have necessarily $\mathcal{D}(a) \subset \mathcal{D}(\mathfrak{a})$, however we have the following.

Corollary 2.18. *If $\phi \in \mathcal{D}(a)$ and $\Gamma(\phi,\phi) \in \mathcal{L}^1_X$, then $\phi \in \mathcal{D}(\mathfrak{a})$ and $(a(\phi),\Gamma(\phi,\phi)) = (\mathfrak{a}(\phi),\mathfrak{G}(\phi,\phi))$ up to zero potential sets.*

We also recall Lemma 5.12 of [7].

Lemma 2.19. *Let $(s,x) \in [0,T]\times E$ be fixed and let ϕ,ψ be two measurable processes. If ϕ and ψ are $\mathbb{P}^{s,x}$-modifications of each other, then they are equal $dV \otimes d\mathbb{P}^{s,x}$ a.e.*

We now keep in mind the Pseudo-Partial Differential Equation (in short Pseudo-PDE), with final condition, that we have introduced in [7].
Let us consider the following data.

1. A measurable final condition $g \in \mathcal{B}(E,\mathbb{R})$;
2. a measurable nonlinear function $f \in \mathcal{B}([0,T]\times E\times\mathbb{R}\times\mathbb{R},\mathbb{R})$.

The equation is

$$\left\{ \begin{array}{rcll} a(u)(t,x) + f\left(t,x,u(t,x),\sqrt{\Gamma(u,u)(t,x)}\right) & = & 0 & \text{on } [0,T]\times E \\ u(T,\cdot) & = & g. & \end{array} \right. \tag{2.4}$$

Notation 2.20. *Equation (2.4) will be denoted $Pseudo-PDE(f,g)$.*

Definition 2.21. *We will say that u is a **classical solution** of $Pseudo-PDE(f,g)$ if it belongs to $\mathcal{D}(a)$ and verifies (2.4).*

Definition 2.22. *A function $u : [0,T] \times E \to \mathbb{R}$ will be said to be a* **martingale solution** *of $Pseudo - PDE(f,g)$ if $u \in \mathcal{D}(\mathfrak{a})$ and*

$$\left\{ \begin{array}{rcl} \mathfrak{a}(u) & = & -f(\cdot,\cdot,u,\sqrt{\mathfrak{G}(u,u)}) \\ u(T,\cdot) & = & g. \end{array} \right. \tag{2.5}$$

Until the end of these preliminaries, we will assume some generalized moments conditions on X, and some growth conditions on the functions (f,g). Those will be related to two functions $\zeta, \eta \in \mathcal{B}(E, \mathbb{R}_+)$.

Hypothesis 2.23. *The Markov class will be said* ***to verify*** $H^{mom}(\zeta,\eta)$ *if*

1. *for any $(s,x) \in [0,T] \times E$, $\mathbb{E}^{s,x}[\zeta^2(X_T)]$ is finite;*
2. *for any $(s,x) \in [0,T] \times E$, $\mathbb{E}^{s,x}\left[\int_0^T \eta^2(X_r)dV_r\right]$ is finite.*

Hypothesis 2.24. *A couple of functions $f \in \mathcal{B}([0,T] \times E \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ and $g \in \mathcal{B}(E,\mathbb{R})$ will be said* ***to verify*** $H^{lip}(\zeta,\eta)$ *if there exist positive constants K^Y, K^Z, C, C' such that*

1. $\forall x : \quad |g(x)| \leq C(1 + \zeta(x))$;
2. $\forall (t,x) : \quad |f(t,x,0,0)| \leq C'(1 + \eta(x))$;
3. $\forall (t,x,y,y',z,z') : \quad |f(t,x,y,z) - f(t,x,y',z)| \leq K^Y|y-y'| + K^Z|z-z'|$.

Finally, if we assume that g and $f(\cdot,\cdot,0,0)$ are bounded, we will say that **they verify** H^{lip}_b.

Remark 2.25. *Even if the underlying process X admits no generalized moments, given a couple (f,g) verifying H^{lip}_b, the considerations of this paper still apply.*

We conclude these preliminaries by stating the Theorem of existence and uniqueness of a martingale solution for $Pseudo - PDE(f,g)$. It was the object of Theorem 5.21 of [7].

Theorem 2.26. *Let $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times E}$ be a Markov class associated to a transition function measurable in time (see Definitions A.6 and A.4) which fulfills Hypothesis 2.4, i.e. it is a solution of a well-posed Martingale Problem associated with the triplet $(\mathcal{D}(a), a, V)$. Moreover we suppose Hypothesis $H^{mom}(\zeta,\eta)$ for some positive ζ,η. Let (f,g) be a couple verifying $H^{lip}(\zeta,\eta)$, cf. Hypothesis 2.24.*

Then $Pseudo - PDE(f,g)$ has a unique martingale solution.

We also had shown (see Proposition 5.19 in [7]) that the unique martingale solution is the only possible classical solution if there is one, as stated below.

Proposition 2.27. *Under the conditions of previous Theorem 2.26, a classical solution u of $Pseudo - PDE(f,g)$ such that $\Gamma(u,u) \in \mathcal{L}^1_X$, is also a martingale solution.*
Conversely, if u is a martingale solution of $Pseudo - PDE(f,g)$ belonging to $\mathcal{D}(a)$, then u is a classical solution of $Pseudo - PDE(f,g)$ up to a zero-potential set, meaning that the first equality of (2.4) holds up to a set of zero potential.

3 Decoupled mild solutions of Pseudo-PDEs

All along this section we will consider a canonical Markov class $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times E}$ associated to a transition function p measurable in time (see Definitions A.6, A.4) verifying Hypothesis 2.4 for a certain $(\mathcal{D}(a), a, V)$. Positive functions ζ, η are fixed and we will assume that the Markov class verifies $H^{mom}(\zeta,\eta)$, see Hypothesis 2.23. We are also given a couple of functions $f \in \mathcal{B}([0,T]\times E\times\mathbb{R}\times\mathbb{R},\mathbb{R})$ and $g \in \mathcal{B}(E,\mathbb{R})$.

3.1 Definition

As mentioned in the introduction, in this section we introduce a notion of solution of our $Pseudo-PDE(f,g)$ that we will denominate *decoupled mild*, which is a generalization of the mild solution concept for partial differential equation. We will show that such solution exists and is unique. Indeed, that function will be the one appearing in Theorem 3.13.

A function u will be a *decoupled mild* solution of $Pseudo-PDE(f,g)$ if there is a function v such that the couple (u,v) is a (decoupled mild) solution of the identification problem $IP(f,g)$. In this section we first go through a notion of *decoupled mild solution* for the identification problem, which has particular interest in itself.

We will be interested in functions (f,g) which satisfy weaker conditions than those of type $H^{lip}(\zeta,\eta)$ (see Hypothesis 2.24) namely the following ones.

Hypothesis 3.1. *A couple of functions* $f \in \mathcal{B}([0,T]\times E\times\mathbb{R}\times\mathbb{R},\mathbb{R})$ *and* $g \in \mathcal{B}(E,\mathbb{R})$ *will be said* ***to verify*** $H^{growth}(\zeta,\eta)$ *if there exist positive constants* C, C' *such that*

1. $\forall x: \quad |g(x)| \leq C(1+\zeta(x))$;
2. $\forall (t,x,y,z): \quad |f(t,x,y,z)| \leq C'(1+\eta(x)+|y|+|z|)$.

Notation 3.2. *Let* s,t *in* $[0,T]$ *with* $s \leq t$, $x \in E$ *and* $\phi \in \mathcal{B}(E,\mathbb{R})$, *if the expectation* $\mathbb{E}^{s,x}[|\phi(X_t)|]$ *is finite, then* $P_{s,t}[\phi](x)$ *will denote* $\mathbb{E}^{s,x}[\phi(X_t)]$.

We recall two important measurability properties.

Remark 3.3. *Let* $\phi \in \mathcal{B}(E,\mathbb{R})$.

- *Suppose that for any* (s,x,t), $\mathbb{E}^{s,x}[|\phi(X_t)|] < \infty$ *then by Proposition A.12,* $(s,x,t) \longmapsto P_{s,t}[\phi](x)$ *is Borel.*
- *Suppose that for every* (s,x), $\mathbb{E}^{s,x}[\int_s^T |\phi(X_r)|dV_r] < \infty$. *Then by Lemma A.11,* $(s,x) \longmapsto \int_s^T P_{s,r}[\phi](x)dVr$ *is Borel.*

In our general setup, considering some operator a, the equation

$$a(u) + f\left(\cdot,\cdot,u,\sqrt{\Gamma(u,u)}\right) = 0, \tag{3.1}$$

can be naturally decoupled into

$$\left\{\begin{array}{rcl} a(u) & = & -f(\cdot,\cdot,u,v) \\ \Gamma(u,u) & = & v^2. \end{array}\right. \tag{3.2}$$

Since $\Gamma(u,u) = a(u^2) - 2ua(u)$, this system of equation will be rewritten as

$$\left\{\begin{array}{rcl} a(u) & = & -f(\cdot,\cdot,u,v) \\ a(u^2) & = & v^2 - 2uf(\cdot,\cdot,u,v). \end{array}\right. \tag{3.3}$$

On the other hand our Markov process X is time non-homogeneous and V_t can be more general than t, which leads us to the following definition of a decoupled mild solution.

Definition 3.4. *Let (f,g) be a couple verifying $H^{growth}(\zeta,\eta)$.*
Let $u, v \in \mathcal{B}([0,T] \times E, \mathbb{R})$ be two Borel functions with $v \geq 0$.

1. *The couple (u,v) will be called* **solution of the identification problem determined by** *(f,g) or simply* **solution of** *$IP(f,g)$ if u and v belong to $\mathcal{L}^2_X$ and if for every $(s,x) \in [0,T] \times E$,*

$$\left\{\begin{array}{rcl} u(s,x) & = & P_{s,T}[g](x) + \int_s^T P_{s,r}\left[f\left(r,\cdot,u(r,\cdot),v(r,\cdot)\right)\right](x)dV_r \\ u^2(s,x) & = & P_{s,T}[g^2](x) - \int_s^T P_{s,r}\left[v^2(r,\cdot) - 2uf\left(r,\cdot,u(r,\cdot),v(r,\cdot)\right)\right](x)dV_r. \end{array}\right. \tag{3.4}$$

2. *The function u will be called* **decoupled mild solution** *of $Pseudo-PDE(f,g)$ if there is a function v such that the couple (u,v) is a solution of $IP(f,g)$.*

Lemma 3.5. *Let $u, v \in \mathcal{L}^2_X$, and let f be a Borel function satisfying the second item of $H^{growth}(\zeta,\eta)$, then $f(\cdot,\cdot,u,v)$ belongs to $\mathcal{L}^2_X$ and $uf(\cdot,\cdot,u,v)$ to $\mathcal{L}^1_X$.*

Proof. Thanks to the growth condition on f in $H^{growth}(\zeta,\eta)$, there exists a constant $C > 0$ such that for any $(s,x) \in [0,T] \times E$,

$$\begin{array}{rl} & \mathbb{E}^{s,x}\left[\int_t^T f^2(r,X_r,u(r,X_r),v(r,X_r))dV_r\right] \\ \leq & C\mathbb{E}^{s,x}\left[\int_t^T (f^2(r,X_r,0,0) + u^2(r,X_r) + v^2(r,X_r))dV_r\right] < \infty, \end{array} \tag{3.5}$$

since we have assumed that u^2, v^2 belong to $\mathcal{L}^1_X$, and since we have made Hypothesis 2.23 and $H^{growth}(\zeta,\eta)$. This means that $f^2(\cdot,\cdot,u,v)$ belongs to $\mathcal{L}^1_X$. Since $2\left|uf(\cdot,\cdot,u,v)\right| \leq u^2 + f^2(\cdot,\cdot,u,v)$ then $uf(\cdot,\cdot,u,v)$ also belongs to $\mathcal{L}^1_X$. ☐

Remark 3.6. *Consequently, under the assumptions of Lemma 3.5 all the terms in (3.4) make sense.*

3.2 Existence and uniqueness of a solution

Proposition 3.7. *Assume that* (f,g) *verifies* $H^{growth}(\zeta,\eta)$ *(see Hypothesis 3.1) and let* $u \in \mathcal{L}^2_X$ *be a martingale solution of* $Pseudo-PDE(f,g)$*. Then* $(u,\sqrt{\mathfrak{G}(u,u)})$ *is a solution of* $IP(f,g)$ *and in particular,* u *is a decoupled mild solution of* $Pseudo-PDE(f,g)$*.*

Proof. Let $u \in \mathcal{L}^2_X$ be a martingale solution of $Pseudo-PDE(f,g)$. We emphasize that, taking Definition 2.13 and Proposition 2.16 into account, $\mathfrak{G}(u,u)$ belongs to $\mathcal{L}^1_X$, or equivalently that $\sqrt{\mathfrak{G}(u,u)}$ belongs to $\mathcal{L}^2_X$. By Lemma 3.5, it follows that $f\left(\cdot,\cdot,u,\sqrt{\mathfrak{G}(u,u)}\right) \in \mathcal{L}^2_X$ and $uf\left(\cdot,\cdot,u,\sqrt{\mathfrak{G}(u,u)}\right) \in \mathcal{L}^1_X$.
We fix some $(s,x) \in [0,T]\times E$ and the corresponding probability $\mathbb{P}^{s,x}$. We are going to show that

$$\left\{\begin{array}{rcl} u(s,x) &=& P_{s,T}[g](x) + \int_s^T P_{s,r}\left[f\left(r,\cdot,u(r,\cdot),\sqrt{\mathfrak{G}(u,u)}(r,\cdot)\right)\right](x)dV_r \\ u^2(s,x) &=& P_{s,T}[g^2](x) - \int_s^T P_{s,r}\left[\mathfrak{G}(u,u)(r,\cdot) - 2uf\left(r,\cdot,u(r,\cdot),\sqrt{\mathfrak{G}(u,u)}(r,\cdot)\right)\right](x)dV_r. \end{array}\right. \tag{3.6}$$

Combining Definitions 2.13, 2.15, 2.22, we know that on $[s,T]$, the process $u(\cdot,X_\cdot)$ has a cadlag modification which we denote $U^{s,x}$ which is a special semimartingale with decomposition

$$U^{s,x} = u(s,x) - \int_s^\cdot f\left(\cdot,\cdot,u,\sqrt{\mathfrak{G}(u,u)}\right)(r,X_r)dV_r + M[u]^{s,x}, \tag{3.7}$$

where $M[u]^{s,x} \in \mathcal{H}^2_0$. Definition 2.22 also states that $u(T,\cdot) = g$, implying that

$$u(s,x) = g(X_T) + \int_s^T f\left(\cdot,\cdot,u,\sqrt{\mathfrak{G}(u,u)}\right)(r,X_r)dV_r - M[u]^{s,x}_T \text{ a.s.} \tag{3.8}$$

Taking the expectation, by Fubini's theorem we get

$$\begin{array}{rcl} u(s,x) &=& \mathbb{E}^{s,x}\left[g(X_T) + \int_s^T f\left(\cdot,\cdot,u,\sqrt{\mathfrak{G}(u,u)}\right)(r,X_r)dV_r\right] \\ &=& P_{s,T}[g](x) + \int_s^T P_{s,r}\left[f\left(r,\cdot,u(r,\cdot),\sqrt{\mathfrak{G}(u,u)}(r,\cdot)\right)\right](x)dV_r. \end{array} \tag{3.9}$$

By integration by parts, we obtain

$$d(U^{s,x})^2_t = -2U^{s,x}_t f\left(\cdot,\cdot,u,\sqrt{\mathfrak{G}(u,u)}\right)(t,X_t)dV_t + 2U^{s,x}_{t^-}dM[u]^{s,x}_t + d[M[u]^{s,x}]_t, \tag{3.10}$$

so integrating from s to T, we get

$$\begin{array}{rl} & u^2(s,x) \\ = & g^2(X_T) + 2\int_s^T U^{s,x}_r f\left(\cdot,\cdot,u,\sqrt{\mathfrak{G}(u,u)}\right)(r,X_r)dV_r - 2\int_s^T U^{s,x}_{r^-}dM[u]^{s,x}_r - [M[u]^{s,x}]_T \\ = & g^2(X_T) + 2\int_s^T uf\left(\cdot,\cdot,u,\sqrt{\mathfrak{G}(u,u)}\right)(r,X_r)dV_r - 2\int_s^T U^{s,x}_{r^-}dM[u]^{s,x}_r - [M[u]^{s,x}]_T, \end{array} \tag{3.11}$$

where the latter line is a consequence of Lemma 2.19. The next step will consist in taking the expectation in equation (3.11), but before, we will check that $\int_s^{\cdot} U^{s,x}_{r^-} dM[u]^{s,x}_r$ is a martingale. Thanks to (3.7) and Jensen's inequality, there exists a constant $C > 0$ such that

$$\sup_{t\in[s,T]} (U^{s,x}_t)^2 \leq C\left(\int_s^T f^2\left(\cdot,\cdot,u,\sqrt{\mathfrak{G}(u,u)}\right)(r,X_r)dV_r + \sup_{t\in[s,T]}(M[u]^{s,x}_t)^2\right). \tag{3.12}$$

Since $M[u]^{s,x}\in\mathcal{H}^2_0$ and $f\left(\cdot,\cdot,u,\sqrt{\mathfrak{G}(u,u)}\right)\in\mathcal{L}^2_X$, it follows that $\sup\limits_{t\in[s,T]}(U^{s,x}_t)^2\in L^1$ and Lemma 3.15 in [7] states that $\int_s^{\cdot} U^{s,x}_{r^-} dM[u]^{s,x}_r$ is a martingale. Taking the expectation in (3.11), we now obtain

$$\begin{array}{rcl} u^2(s,x) & = & \mathbb{E}^{s,x}\left[g^2(X_T) + \int_s^T 2uf\left(\cdot,\cdot,u,\sqrt{\mathfrak{G}(u,u)}\right)(r,X_r)dV_r - [M[u]^{s,x}]_T\right] \\ & = & \mathbb{E}^{s,x}\left[g^2(X_T) + \int_s^T 2uf\left(\cdot,\cdot,u,\sqrt{\mathfrak{G}(u,u)}\right)(r,X_r)dV_r - \langle M[u]^{s,x}\rangle_T\right] \\ & = & \mathbb{E}^{s,x}\left[g^2(X_T)\right] - \mathbb{E}^{s,x}\left[\int_s^T\left(\mathfrak{G}(u,u) - 2uf\left(\cdot,\cdot,u,\sqrt{\mathfrak{G}(u,u)}\right)\right)(r,X_r)dV_r\right] \\ & = & P_{s,T}[g^2](x) - \int_s^T P_{s,r}\left[\mathfrak{G}(u,u)(r,\cdot) - 2u(r,\cdot)f\left(r,\cdot,u(r,\cdot),\sqrt{\mathfrak{G}(u,u)}(r,\cdot)\right)\right](x)dV_r, \end{array} \tag{3.13}$$

where the third equality derives from Proposition 2.16 and the fourth from Fubini's theorem. This concludes the proof. □

We now show the converse result of Proposition 3.7.

Proposition 3.8. *Assume that (f,g) verifies $H^{growth}(\zeta,\eta)$, see Hypothesis 3.1. Every decoupled mild solution of $Pseudo-PDE(f,g)$ is a also a martingale solution. Moreover, if (u,v) solves $IP(f,g)$, then $v^2=\mathfrak{G}(u,u)$ (up to zero potential sets).*

Proof. Let u and $v\geq 0$ be a couple of functions in $\mathcal{L}^2_X$ verifying (3.4). We first note that, the first line of (3.4) with $s=T$, gives $u(T,\cdot)=g$.
We fix $(s,x)\in[0,T]\times E$ and the associated probability $\mathbb{P}^{s,x}$, and on $[s,T]$, we set $U_t := u(t,X_t)$ and $N_t := u(t,X_t) - u(s,x) + \int_s^t f(r,X_r,u(r,X_r),v(r,X_r))dV_r$.

Combining the first line of (3.4) applied in $(s,x)=(t,X_t)$ and the Markov property (A.3), and since $f(\cdot,\cdot,u,v)$ belongs to $\mathcal{L}^2_X$ (see Lemma 3.5) we get the a.s. equalities

$$\begin{array}{rcl} U_t & = & u(t,X_t) \\ & = & P_{t,T}[g](X_t) + \int_t^T P_{t,r}\left[f\left(r,\cdot,u(r,\cdot),v(r,\cdot)\right)\right](X_t)dV_r \\ & = & \mathbb{E}^{t,X_t}\left[g(X_T) + \int_t^T f(r,X_r,u(r,X_r),v(r,X_r))dV_r\right] \\ & = & \mathbb{E}^{s,x}\left[g(X_T) + \int_t^T f(r,X_r,u(r,X_r),v(r,X_r))dV_r|\mathcal{F}_t\right], \end{array} \tag{3.14}$$

from which we deduce that $N_t = \mathbb{E}^{s,x}\left[g(X_T) + \int_s^T f(r,X_r,u(r,X_r),v(r,X_r))dV_r|\mathcal{F}_t\right] - u(s,x)$ a.s. So N is a martingale. We can therefore consider on $[s,T]$ and under

$\mathbb{P}^{s,x}$, $N^{s,x}$ the cadlag version of N, and the special semi-martingale $U^{s,x} := u(s,x) - \int_s^{\cdot} f(r, X_r, u(r,X_r), v(r,X_r))dV_r + N^{s,x}$ which is a cadlag version of U. By Jensen's inequality for both expectation and conditional expectation, we have

$$\begin{array}{rcl} \mathbb{E}^{s,x}[(N^{s,x})^2_t] & = & \mathbb{E}^{s,x}\left[\left(\mathbb{E}^{s,x}\left[g(X_T) + \int_s^T f(r,X_r,u(r,X_r),v(r,X_r))dV_r|\mathcal{F}_t\right] - u(s,x)\right)^2\right] \\ & \leq & 3u^2(s,x) + 3\mathbb{E}^{s,x}[g^2(X_T)] + 3\mathbb{E}^{s,x}\left[\int_s^T f^2(r,X_r,u(r,X_r),v(r,X_r))dV_r\right] \\ & < & \infty, \end{array} \tag{3.15}$$

where the second term is finite because of $H^{mom}(\zeta,\eta)$ and $H^{growth}(\zeta,\eta)$, and the same also holds for the third one because $f(\cdot,\cdot,u,v)$ belongs to $\mathcal{L}^2_X$, see Lemma 3.5. So $N^{s,x}$ is square integrable. We have therefore shown that under any $\mathbb{P}^{s,x}$, the process $u(\cdot,X_\cdot) - u(s,x) + \int_s^{\cdot} f(r,X_r,u(r,X_r),v(r,X_r))dV_r$ has on $[s,T]$ a modification in $\mathcal{H}^2_0$. Definitions 2.13 and 2.15, justify that $u \in \mathcal{D}(\mathfrak{a})$, $\mathfrak{a}(u) = -f(\cdot,\cdot,u,v)$ and that for any $(s,x) \in [0,T]\times E$, $M[u]^{s,x} = N^{s,x}$.

To conclude that u is a martingale solution of $Pseudo-PDE(f,g)$, there is left to show that $\mathfrak{G}(u,u) = v^2$, up to zero potential sets. By Proposition 2.16, this is equivalent to show that for every $(s,x)\in[0,T]\times E$,
$\langle N^{s,x}\rangle = \int_s^{\cdot} v^2(r,X_r)dV_r$, in the sense of indistinguishability.

We fix again $(s,x)\in[0,T]\times E$ and the associated probability, and now set $N'_t := u^2(t,X_t) - u^2(s,x) - \int_s^t (v^2 - 2uf(\cdot,\cdot,u,v))(r,X_r)dV_r$. Combining the second line of (3.4) applied in $(s,x) = (t,X_t)$ and the Markov property (A.3), and since $v^2, uf(\cdot,\cdot,u,v)$ belong to $\mathcal{L}^1_X$ (see Lemma 3.5) we get the a.s. equalities

$$\begin{array}{rcl} u^2(t,X_t) & = & P_{t,T}[g^2](X_t) - \int_t^T P_{t,r}\left[(v^2(r,\cdot) - 2u(r,\cdot)f(r,\cdot,u(r,\cdot),v(r,\cdot)))\right](X_t)dV_r \\ & = & \mathbb{E}^{t,X_t}\left[g^2(X_T) - \int_t^T (v^2 - 2uf(\cdot,\cdot,u,v))(r,X_r)dV_r\right] \\ & = & \mathbb{E}^{s,x}\left[g^2(X_T) - \int_t^T (v^2 - 2uf(\cdot,\cdot,u,v))(r,X_r)dV_r|\mathcal{F}_t\right], \end{array} \tag{3.16}$$

from which we deduce that for any $t\in[s,T]$,
$N'_t = \mathbb{E}^{s,x}\left[g^2(X_T) - \int_s^T (v^2 - uf(\cdot,\cdot,u,v))(r,X_r)dV_r|\mathcal{F}_t\right] - u^2(s,x)$ a.s. So N' is a martingale. We can therefore consider on $[s,T]$ and under $\mathbb{P}^{s,x}$, $N'^{s,x}$ the cadlag version of N'.

The process $u^2(s,x) + \int_s^{\cdot}(v^2 - uf(\cdot,\cdot,u,v))(r,X_r)dV_r + N'^{s,x}$ is therefore a cadlag special semi-martingale which is a $\mathbb{P}^{s,x}$-version of $u^2(\cdot,X)$ on $[s,T]$. But we also had shown that $U^{s,x} = u(s,x) - \int_s^{\cdot} f(r,X_r,u(r,X_r),v(r,X_r))dV_r + N^{s,x}$ is a version of $u(\cdot,X)$, which by integration by part implies that
$u^2(s,x) - 2\int_s^{\cdot} U^{s,x}_r f(\cdot,\cdot,u,v)(r,X_r)dV_r + 2\int_s^{\cdot} U^{s,x}_{r-}dN^{s,x}_r + [N^{s,x}]$ is another cadlag semi-martingale which is a $\mathbb{P}^{s,x}$-version of $u^2(\cdot,X)$ on $[s,T]$.

$\int_s^{\cdot}(v^2 - 2uf(\cdot,\cdot,u,v))(r,X_r)dV_r + N'^{s,x}$ is therefore indistinguishable from $-2\int_s^{\cdot} U^{s,x}_r f(\cdot,\cdot,u,v)(r,X_r)dV_r + 2\int_s^{\cdot} U^{s,x}_{r-}dN^{s,x}_r + [N^{s,x}]$ which can be written $\langle N^{s,x}\rangle - 2\int_s^{\cdot} U^{s,x}_r f(\cdot,\cdot,u,v)(r,X_r)dV_r + 2\int_s^{\cdot} U^{s,x}_{r-}dN^{s,x}_r + ([N^{s,x}] - \langle N^{s,x}\rangle)$ where $\langle N^{s,x}\rangle - 2\int_s^{\cdot} U^{s,x}_r f(\cdot,\cdot,u,v)(r,X_r)dV_r$ is predictable with bounded variation and

$2\int_s^{\cdot} U^{s,x}_{r^-} dN^{s,x}_r + ([N^{s,x}] - \langle N^{s,x}\rangle)$ is a local martingale. By uniqueness of the decomposition of a special semi-martingale, we have
$\int_s^{\cdot}(v^2 - 2uf(\cdot,\cdot,u,v))(r,X_r)dV_r = \langle N^{s,x}\rangle - 2\int_s^{\cdot} U^{s,x}_r f(\cdot,\cdot,u,v)(r,X_r)dV_r$, and by Lemma 2.19, $\int_s^{\cdot}(v^2 - 2uf(\cdot,\cdot,u,v))(r,X_r)dV_r = \langle N^{s,x}\rangle - 2\int_s^{\cdot} uf(\cdot,\cdot,u,v)(r,X_r)dV_r$, which finally yields $\langle N^{s,x}\rangle = \int_s^{\cdot} v^2(r,X_r)dV_r$ as desired. □

We recall that $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times E}$ is a Markov class associated to a transition function measurable in time (see Definitions A.6 and A.4) which fulfills Hypothesis 2.4, i.e. it is a solution of a well-posed Martingale Problem associated with the triplet $(\mathcal{D}(a), a, V)$. Moreover we suppose Hypothesis $H^{mom}(\zeta,\eta)$ for some positive ζ,η, see Hypothesis 2.23.

Theorem 3.9. *Let (f,g) be a couple verifying $H^{lip}(\zeta,\eta)$, see Hypothesis 2.24. Then $Pseudo-PDE(f,g)$ has a unique decoupled mild solution.*

Proof. This derives from Theorem 2.26 and Propositions 3.7, 3.8. □

Corollary 3.10. *Assume that (f,g) verifies $H^{lip}(\zeta,\eta)$, see Hypothesis 2.24. A classical solution u of $Pseudo-PDE(f,g)$ such that $\Gamma(u,u)\in\mathcal{L}^1_X$, is also a decoupled mild solution.*
Conversely, if u is a decoupled mild solution of $Pseudo-PDE(f,g)$ belonging to $\mathcal{D}(a)$, then u is a classical solution of $Pseudo-PDE(f,g)$ up to a zero-potential set, meaning that the first equality of (2.4) *holds up to a set of zero potential.*

Proof. The statement holds by Proposition 3.8 and Proposition 2.27. □

3.3 Representation of the solution via FBSDEs with no driving martingale

In the companion paper [7], the following family of FBSDEs with no driving martingale indexed by $(s,x)\in[0,T]\times E$ was introduced.

Definition 3.11. *Let $(s,x)\in[0,T]\times E$ and the associated stochastic basis $\left(\Omega,\mathcal{F}^{s,x},(\mathcal{F}^{s,x}_t)_{t\in[0,T]},\mathbb{P}^{s,x}\right)$ be fixed. A couple $(Y^{s,x},M^{s,x})\in\mathcal{L}^2(dV\otimes d\mathbb{P}^{s,x})\times\mathcal{H}^2_0$ will be said to solve $FBSDE^{s,x}(f,g)$ if it verifies on $[0,T]$, in the sense of indistinguishability*

$$Y^{s,x} = g(X_T) + \int_{\cdot}^{T} f\left(r, X_r, Y^{s,x}_r, \sqrt{\frac{d\langle M^{s,x}\rangle}{dV}(r)}\right)dV_r - (M^{s,x}_T - M^{s,x}_{\cdot}). \quad (3.17)$$

If (3.17) *is only satisfied on a smaller interval $[t_0,T]$, with $0<t_0<T$, we say that $(Y^{s,x},M^{s,x})$ solves $FBSDE^{s,x}(f,g)$ on $[t_0,T]$.*

The following result follows from Theorem 3.22 in [7].

Theorem 3.12. *Assume that (f,g) verifies $H^{lip}(\zeta,\eta)$, see Hypothesis 2.24. Then for any $(s,x)\in[0,T]\times E$, $FBSDE^{s,x}(f,g)$ has a unique solution.*

In the following theorem, we summarize the links between the $FBSDE^{s,x}(f,g)$ and the notion of martingale solution of $Pseudo-PDE(f,g)$. These are shown in Theorem 5.14, Remark 5.15, Theorem 5.20 and Theorem 5.21 of [7].

Theorem 3.13. *Assume that (f,g) verifies $H^{lip}(\zeta,\eta)$ (see Hypothesis 2.24) and let $(Y^{s,x},M^{s,x})$ denote the (unique) solution of $FBSDE^{s,x}(f,g)$ for fixed (s,x). Let u be the unique martingale solution of $Pseudo-PDE(f,g)$. For every $(s,x)\in[0,T]\times E$, on the interval $[s,T]$,*

- *$Y^{s,x}$ and $u(\cdot,X_\cdot)$ are $\mathbb{P}^{s,x}$-modifications, and equal $dV\otimes d\mathbb{P}^{s,x}$ a.e.;*
- *$M^{s,x}$ and $M[u]^{s,x}$ are $\mathbb{P}^{s,x}$-indistinguishable.*

Moreover u belongs to $\mathcal{L}^2_X$ and for any $(s,x)\in[0,T]\times E$, we have $\frac{d\langle M^{s,x}\rangle}{dV}=\mathfrak{G}(u,u)(\cdot,X_\cdot)\ dV\otimes d\mathbb{P}^{s,x}$ a.e.

Remark 3.14. *The martingale solution u of $Pseudo-PDE$ exists and is unique by Theorem 2.26.*

We can therefore represent the unique decoupled mild solution of $Pseudo-PDE(f,g)$ via the stochastic equations $FBSDE^{s,x}(f,g)$ as follows.

Theorem 3.15. *Assume that (f,g) verifies $H^{lip}(\zeta,\eta)$ (see Hypothesis 2.24) and let $(Y^{s,x},M^{s,x})$ denote the (unique) solution of $FBSDE^{s,x}(f,g)$ for fixed (s,x).*
Then for any $(s,x)\in[0,T]\times E$, the random variable $Y^{s,x}_s$ is $\mathbb{P}^{s,x}$ a.s. equal to a constant (which we still denote $Y^{s,x}_s$), and the function

$$u:(s,x)\longmapsto Y^{s,x}_s \tag{3.18}$$

is the unique decoupled mild solution of $Pseudo-PDE(f,g)$.

Proof. By Theorem 3.13, there exists a Borel function u such that for every $(s,x)\in[0,T]\times E$, $Y^{s,x}_s=u(s,X_s)=u(s,x)$ $\mathbb{P}^{s,x}$ a.s. and u is the unique martingale solution of $Pseudo-PDE(f,g)$. By Proposition 3.7, it is also its unique decoupled mild solution. $\square$

Remark 3.16. *The function v such that (u,v) is the unique solution of the identification problem $IP(f,g)$ also has a stochastic representation since it verifies for every $(s,x)\in[0,T]\times E$, on the interval $[s,T]$,*
$\frac{d\langle M^{s,x}\rangle}{dV}=v^2(\cdot,X_\cdot)\ dV\otimes d\mathbb{P}^{s,x}$ a.e. where $M^{s,x}$ is the martingale part of the solution of $FBSDE^{s,x}$.

Conversely, under the weaker condition $H^{growth}(\zeta,\eta)$ if one knows the solution of $IP(f,g)$, one can (for every (s,x)) produce a version of a solution of $FBSDE^{s,x}(f,g)$ as follows. This is only possible with the notion of decoupled mild solution: even in the case of Brownian BSDEs the knowledge of the viscosity solution of the related PDE would (in general) not be sufficient to reconstruct the family of solutions of the BSDEs.

Proposition 3.17. *Assume that* (f,g) *verifies* $H^{growth}(\zeta,\eta)$, *see Hypothesis 3.1. Suppose the existence of a solution* (u,v) *to* $IP(f,g)$, *and let* $(s,x)\in[0,T]\times E$ *be fixed. Then*

$$\left(u(\cdot,X),\quad u(\cdot,X)-u(s,x)+\int_s^{\cdot} f(\cdot,\cdot,u,v)(r,X_r)dV_r\right) \tag{3.19}$$

admits on $[s,T]$ *a* $\mathbb{P}^{s,x}$*-version* $(Y^{s,x},M^{s,x})$ *which solves* $FBSDE^{s,x}$ *on* $[s,T]$.

Proof. By Proposition 3.8, u is a martingale solution of $Pseudo-PDE(f,g)$ and $v^2=\mathfrak{G}(u,u)$. We now fix $(s,x)\in[0,T]\times E$. Combining Definitions 2.15, 2.17 and 2.22, we know that $u(T,\cdot)=g$ and that on $[s,T]$, $u(\cdot,X)$ has a $\mathbb{P}^{s,x}$-version $U^{s,x}$ with decomposition $U^{s,x}=u(s,x)-\int_s^{\cdot} f(\cdot,\cdot,u,v)(r,X_r)dV_r+M[u]^{s,x}$, where $M[u]^{s,x}$ is an element of $\mathcal{H}^2_0$ of angular bracket $\int_s^{\cdot} v^2(r,X_r)dV_r$ and is a version of $u(\cdot,X)-u(s,x)+\int_s^{\cdot} f(\cdot,\cdot,u,v)(r,X_r)dV_r$. By Lemma 2.19, taking into account $u(T,\cdot)=g$, the couple $(U^{s,x},M[u]^{s,x})$ verifies on $[s,T]$, in the sense of indistinguishability

$$U^{s,x}=g(X_T)+\int_{\cdot}^{T} f\left(r,X_r,U^{s,x}_r,\sqrt{\frac{d\langle M[u]^{s,x}\rangle}{dV}(r)}\right)dV_r-(M[u]^{s,x}_T-M[u]^{s,x}_{\cdot}) \tag{3.20}$$

with $M[u]^{s,x}\in\mathcal{H}^2_0$ verifying $M[u]^{s,x}_s=0$ (see Definition 2.15) and $U^{s,x}_s$ is deterministic so in particular is a square integrable r.v. Following a slight adaptation of the proof of Lemma 3.25 in [7] (see Remark 3.18 below), this implies that $U^{s,x}\in\mathcal{L}^2(dV\otimes d\mathbb{P}^{s,x})$ and therefore that that $(U^{s,x},M[u]^{s,x})$ is a solution of $FBSDE^{s,x}(f,g)$ on $[s,T]$. □

Remark 3.18. *Indeed Lemma 3.25 in [7], taking into account Notation 5.5 ibidem, can be applied rigorously only under* $H^{lip}(\zeta,\eta)$ *for* (f,g). *However, the same proof easily allows an extension to our framework* $H^{growth}(\zeta,\eta)$.

4 Examples of applications

We now develop some examples. Some of the applications that we are interested in involve operators which only act on the space variable, and we will extend them to time-dependent functions. The reader may consult Appendix B, concerning details about such extensions. In all the items below there will be a canonical Markov class with transition function measurable in time which is solution of a well-posed Martingale Problem associated to some triplet $(\mathcal{D}(a),a,V)$ as introduced in Definition 2.3. Therefore all the results of this paper will apply to all the examples below, namely Theorem 2.26, Propositions 2.27, 3.7 and 3.8, Theorem 3.9, Corollaries 3.10 and 3.10, Theorems 3.12, 3.13 and 3.15 and Proposition 3.17. In particular, Theorem 3.9 states in all the cases, under suitable Lipschitz type conditions for the driver f, that the corresponding Pseudo-PDE admits a unique decoupled mild solution. In all the examples $T\in\mathbb{R}^*_+$ will be fixed.

4.1 Markovian jump diffusions

In this subsection, the state space will be $E := \mathbb{R}^d$ for some $d \in \mathbb{N}^*$. We are given $\mu \in \mathcal{B}([0,T] \times \mathbb{R}^d, \mathbb{R}^d)$, $\alpha \in \mathcal{B}([0,T] \times \mathbb{R}^d, S^*_+(\mathbb{R}^d))$ (where $S^*_+(\mathbb{R}^d)$ is the space of symmetric strictly positive definite matrices of size d) and K a Lévy kernel: this means that for every $(t,x) \in [0,T] \times \mathbb{R}^d$, $K(t,x,\cdot)$ is a σ-finite measure on $\mathbb{R}^d \backslash \{0\}$, $\sup\limits_{t,x} \int \frac{\|y\|^2}{1+\|y\|^2} K(t,x,dy) < \infty$ and for every Borel set $A \in \mathcal{B}(\mathbb{R}^d \backslash \{0\})$, $(t,x) \longmapsto \int_A \frac{\|y\|^2}{1+\|y\|^2} K(t,x,dy)$ is Borel. We will consider the operator a defined by

$$\partial_t \phi + \frac{1}{2} Tr(\alpha \nabla^2 \phi) + (\mu, \nabla \phi) + \int \left(\phi(\cdot, \cdot + y) - \phi(\cdot, y) - \frac{(y, \nabla \phi)}{1 + \|y\|^2} \right) K(\cdot, \cdot, dy), \tag{4.1}$$

on the domain $\mathcal{D}(a)$ which is here the linear algebra $\mathcal{C}^{1,2}_b([0,T] \times \mathbb{R}^d, \mathbb{R})$ of real continuous bounded functions on $[0,T] \times \mathbb{R}^d$ which are continuously differentiable in the first variable with bounded derivative, and twice continuously differentiable in the second variable with bounded derivatives.

Concerning martingale problems associated to parabolic PDE operators, one may consult [42]. Since we want to include integral operators, we will adopt the formalism of D.W. Stroock in [41]. Its Theorem 4.3 and the penultimate sentence of its proof states the following.

Theorem 4.1. *Suppose that μ is bounded, that α is bounded continuous and that for any $A \in \mathcal{B}(\mathbb{R}^d \backslash \{0\})$, $(t,x) \longmapsto \int_A \frac{y}{1+\|y\|^2} K(t,x,dy)$ is bounded continuous. Then, for every (s,x), there exists a unique probability $\mathbb{P}^{s,x}$ on the canonical space (see Definition A.1) such that $\phi(\cdot, X_\cdot) - \int_s^\cdot a(\phi)(r, X_r)dr$ is a local martingale for any $\phi \in \mathcal{D}(a)$ and $\mathbb{P}^{s,x}(X_s = x) = 1$. Moreover $(\mathbb{P}^{s,x})_{(s,x) \in [0,T] \times \mathbb{R}^d}$ defines a Markov class and its transition function is measurable in time.*

The Martingale Problem associated to $(\mathcal{D}(a), a, V_t \equiv t)$ in the sense of Definition 2.3 is therefore well-posed and solved by $(\mathbb{P}^{s,x})_{(s,x) \in [0,T] \times \mathbb{R}^d}$.

In this context, $\mathcal{D}(a)$ is an algebra and for ϕ, ψ in $\mathcal{D}(a)$, the carré du champs operator is given by

$$\Gamma(\phi, \psi) = \sum_{i,j \leq d} \alpha_{i,j} \partial_{x_i} \phi \partial_{x_j} \psi + \int_{\mathbb{R}^d \backslash \{0\}} (\phi(\cdot, \cdot + y) - \phi)(\psi(\cdot, \cdot + y) - \psi) K(\cdot, \cdot, dy).$$

In order to obtain a general result without having to make considerations on the integrability of X, we consider a couple (f,g) satisfying H^{lip}_b meaning that g and $f(\cdot, \cdot, 0, 0)$ are bounded.

Proposition 4.2. *Under the assumptions of Theorem 4.1, and if (f,g) verify H^{lip}_b (see Hypothesis 2.24), $Pseudo-PDE(f,g)$ admits a unique decoupled mild solution in the sense of Definition 3.4.*

Proof. $\mathcal{D}(a)$ is an algebra. Moreover $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times\mathbb{R}^d}$ is a Markov class which is measurable in time, and it solves the well-posed Martingale Problem associated to $(\mathcal{D}(a), a, V_t \equiv t)$. Therefore our Theorem 3.9 applies. □

4.2 Pseudo-Differential operators and Fractional Laplacian

This section concerns pseudo-differential operators with negative definite symbol, see [26] for an extensive description. A typical example of such operators will be the fractional Laplacian $\Delta^{\frac{\alpha}{2}}$ with $\alpha \in]0,2[$, see Chapter 3 in [13] for a detailed study of this operator. We will mainly use the notations and vocabulary of N. Jacob in [25], [26] and [27], some results being attributed to W. Hoh [22]. We fix $d \in \mathbb{N}^*$. $\mathcal{C}_c^\infty(\mathbb{R}^d)$ will denote the space of real functions defined on $\mathbb{R}^d$ which are infinitely continuously differentiable with compact support and $\mathcal{S}(\mathbb{R}^d)$ the Schwartz space of fast decreasing real smooth functions also defined on $\mathbb{R}^d$. $\mathcal{F}u$ will denote the Fourier transform of a function u whenever it is well-defined. For $u \in L^1(\mathbb{R}^d)$ we use the convention $\mathcal{F}u(\xi) = \frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d} e^{-i(x,\xi)}u(x)dx$.

Definition 4.3. *A function $\psi \in \mathcal{C}(\mathbb{R}^d,\mathbb{R})$ will be said* ***negative definite*** *if for any $k \in \mathbb{N}$, $\xi_1, \cdots, \xi_k \in \mathbb{R}^d$, the matrix $(\psi(\xi^j) + \psi(\xi^l) - \psi(\xi^j - \xi^l))_{j,l=1,\cdots,k}$ is symmetric positive definite.*

A function $q \in \mathcal{C}(\mathbb{R}^d \times \mathbb{R}^d, \mathbb{R})$ will be called a ***continuous negative definite symbol*** *if for any $x \in \mathbb{R}^d$, $q(x,\cdot)$ is continuous negative definite*
In this case we introduce the pseudo-differential operator $q(\cdot, D)$ defined by

$$q(\cdot,D)(u)(x) = \frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d} e^{i(x,\xi)}q(x,\xi)\mathcal{F}u(\xi)d\xi. \tag{4.2}$$

Remark 4.4. *By Theorem 4.5.7 in [25], $q(\cdot,D)$ maps the space $\mathcal{C}_c^\infty(\mathbb{R}^d)$ of smooth functions with compact support into itself. In particular $q(\cdot,D)$ will be defined on $\mathcal{C}_c^\infty(\mathbb{R}^d)$. However, the proof of this Theorem 4.5.7 only uses the fact that if $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^d)$ then $\mathcal{F}\phi \in \mathcal{S}(\mathbb{R}^d)$ and this still holds for every $\phi \in \mathcal{S}(\mathbb{R}^d)$. Therefore $q(\cdot,D)$ is well-defined on $\mathcal{S}(\mathbb{R}^d)$ and maps it into $\mathcal{C}(\mathbb{R}^d,\mathbb{R})$.*

A typical example of such pseudo-differential operators is the fractional Laplacian defined for some fixed $\alpha \in]0,2[$ on $\mathcal{S}(\mathbb{R}^d)$ by

$$(-\Delta)^{\frac{\alpha}{2}}(u)(x) = \frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d} e^{i(x,\xi)}\|\xi\|^\alpha\mathcal{F}u(\xi)d\xi. \tag{4.3}$$

Its symbol has no dependence in x and is the continuous negative definite function $\xi \mapsto \|\xi\|^\alpha$. Combining Theorem 4.5.12 and 4.6.6 in [27], one can state the following.

Theorem 4.5. *Let ψ be a continuous negative definite function satisfying for some $r_0, c_0 > 0$: $\psi(\xi) \geq c_0\|\xi\|^{r_0}$ if $\|\xi\| \geq 1$. Let M be the smallest integer strictly superior to $(\frac{d}{r_0} \vee 2) + d$. Let q be a continuous negative symbol verifying, for some $c, c' > 0$ and $\gamma : \mathbb{R}^d \to \mathbb{R}_+^*$, the following items.*

- *$q(\cdot,0)=0$ and $\underset{x\in\mathbb{R}^d}{sup}|q(x,\xi)|\underset{\xi\to 0}{\longrightarrow}0$;*

- *q is $\mathcal{C}^{2M+1-d}$ in the first variable and for any $\beta\in\mathbb{N}^d$ with $\|\beta\|\leq 2M+1-d$, $\|\partial_x^\beta q\|\leq c(1+\psi)$;*

- *$q(x,\xi)\geq\gamma(x)(1+\psi(x))$ if $x\in\mathbb{R}^d$, $\|\xi\|\geq 1$;*

- *$q(x,\xi)\leq c'(1+\|\xi\|^2)$ for every (x,ξ).*

Then the homogeneous Martingale Problem associated to $(-q(\cdot,D),\mathcal{S}(\mathbb{R}^d))$ is well-posed (see Definition B.3) and its solution $(\mathbb{P}^x)_{x\in\mathbb{R}^d}$ defines a homogeneous Markov class, see Notation B.1.

We will now introduce the time-inhomogeneous domain which will be used to extend $\mathcal{D}(-q(\cdot,D))=\mathcal{S}(\mathbb{R}^d)$.

Definition 4.6. *Let τ be a Hausdorff topological linear space. We will denote by $\mathcal{C}^1([0,T],\tau)$ the set of functions $\phi\in\mathcal{C}([0,T],\tau)$ such that there exists a function $\partial_t\phi\in\mathcal{C}([0,T],\tau)$ verifying the following. For every $t_0\in[0,T]$ we have $\frac{1}{(t-t_0)}(\phi(t)-\phi(t_0))\underset{t\to t_0}{\xrightarrow{\tau}}\partial_t\phi(t_0)$.*

We recall that a topological algebra is a topological space equipped with a structure of linear algebra such that addition, multiplication and multiplication by a scalar are continuous.

Lemma 4.7. *Let $\mathcal{A}$ be a (Hausdorff) topological algebra, then $\mathcal{C}^1([0,T],\mathcal{A})$ is a linear algebra, and for any $\phi,\psi\in\mathcal{C}^1([0,T],\mathcal{A})$, we have $\partial_t(\phi\psi)=\psi\partial_t\phi+\phi\partial_t\psi$.*

Proof. The proof is very close to the one of $\mathbb{R}$. □

Remark 4.8. *Classical examples of topological algebras are $\mathcal{C}_c^\infty(\mathbb{R}^d)$, $\mathcal{S}(\mathbb{R}^d)$, $\mathcal{C}^\infty(\mathbb{R}^d)$, $\mathcal{C}^m(\mathbb{R}^d)$ for some $m\in\mathbb{N}$ (equipped with their usual topologies), or $W^{k,p}(\mathbb{R}^d)\bigcap W^{k,\infty}(\mathbb{R}^d)$ for any $k\in\mathbb{N}^*,p\geq 1$, where $W^{k,p}(\mathbb{R}^d)$ denotes the usual Sobolev space of parameters k,p. Those are all Fréchet algebras except for $\mathcal{C}_c^\infty(\mathbb{R}^d)$ which is only locally convex one.*

Notation 4.9. *We set $\mathcal{D}(\partial_t-q(\cdot,D)):=\mathcal{C}^1([0,T],\mathcal{S}(\mathbb{R}^d))$.*

Elements in $\mathcal{C}([0,T],\mathcal{S}(\mathbb{R}^d))$ will also be seen as functions of two variables, and since convergence in $\mathcal{S}(\mathbb{R}^d)$ implies pointwise convergence, the usual notion of partial derivative coincides with the notation ∂_t introduced in Definition 4.6. Any $\phi\in\mathcal{D}(\partial_t-q(\cdot,D))$ clearly verifies

- $\forall t\in[0,T]$, $\phi(t,\cdot)\in\mathcal{S}(\mathbb{R}^d)$ and $\forall x\in\mathbb{R}^d$, $\phi(\cdot,x)\in\mathcal{C}^1([0,T],\mathbb{R})$;

- $\forall t\in[0,T]$, $\partial_t\phi(t,\cdot)\in\mathcal{S}(\mathbb{R}^d)$.

Our goal now is to show that $\mathcal{D}(\partial_t-q(\cdot,D)$ also verifies the other items needed to be included in $\mathcal{D}^{max}(\partial_t-q(\cdot,D))$ (see Notation B.5) and therefore that Corollary B.8 applies with this domain.

Notation 4.10. *Let* $\alpha, \beta \in \mathbb{N}^d$ *be multi-indices, we introduce the semi-norm*

$$\| \cdot \|_{\alpha,\beta} : \begin{array}{rcl} \mathcal{S}(\mathbb{R}^d) & \longrightarrow & \mathbb{R} \\ \phi & \longmapsto & \underset{x\in\mathbb{R}^d}{sup} |x^\alpha \partial_x^\beta \phi(x)|. \end{array} \tag{4.4}$$

$\mathcal{S}(\mathbb{R}^d)$ is a Fréchet space whose topology is determined by the family of seminorms $\| \cdot \|_{\alpha,\beta}$. In particular those seminorms are continuous.
In what follows, $\mathcal{F}_x$ will denote the Fourier transform taken in the space variable.

Proposition 4.11. *Let* $\phi \in \mathcal{C}([0,T], \mathcal{S}(\mathbb{R}^d))$. *Then* $\mathcal{F}_x\phi \in \mathcal{C}([0,T], \mathcal{S}(\mathbb{R}^d))$. *Moreover if* $\phi \in \mathcal{C}^1([0,T], \mathcal{S}(\mathbb{R}^d))$, *then* $\mathcal{F}_x\phi \in \mathcal{C}^1([0,T], \mathcal{S}(\mathbb{R}^d))$ *and*
$\partial_t \mathcal{F}_x\phi = \mathcal{F}_x\partial_t\phi$.

Proof. $\mathcal{F}_x : \mathcal{S}(\mathbb{R}^d) \longrightarrow \mathcal{S}(\mathbb{R}^d)$ is continuous, so $\phi \in \mathcal{C}([0,T], \mathcal{S}(\mathbb{R}^d)$ implies $\mathcal{F}_x\phi \in \mathcal{C}([0,T], \mathcal{S}(\mathbb{R}^d))$. If $\phi \in \mathcal{C}^1([0,T], \mathcal{S}(\mathbb{R}^d))$ then $\partial_t\phi \in \mathcal{C}([0,T], \mathcal{S}(\mathbb{R}^d)$ so $\mathcal{F}_x\partial_t\phi \in \mathcal{C}([0,T], \mathcal{S}(\mathbb{R}^d)$. Then for any $t_0 \in [0,T]$, the convergence
$\frac{1}{t-t_0}(\phi(t,\cdot) - \phi(t_0,\cdot)) \underset{t\to t_0}{\overset{\mathcal{S}(\mathbb{R}^d)}{\longrightarrow}} \partial_t\phi(t_0,\cdot)$ is preserved by the continuous mapping $\mathcal{F}_x$ meaning that (by linearity)
$\frac{1}{t-t_0}(\mathcal{F}_x\phi(t,\cdot) - \mathcal{F}_x\phi(t_0,\cdot)) \underset{t\to t_0}{\overset{\mathcal{S}(\mathbb{R}^d)}{\longrightarrow}} \mathcal{F}_x\partial_t\phi(t_0,\cdot)$. Since $\mathcal{F}_x\partial_t\phi \in \mathcal{C}([0,T], \mathcal{S}(\mathbb{R}^d))$, we have shown that $\mathcal{F}_x\phi \in \mathcal{C}^1([0,T], \mathcal{S}(\mathbb{R}^d))$ and $\partial_t\mathcal{F}_x\phi = \mathcal{F}_x\partial_t\phi$. ☐

Proposition 4.12. *If* $\phi \in \mathcal{C}([0,T], \mathcal{S}(\mathbb{R}^d))$, *then for any* $\alpha, \beta \in \mathbb{N}^d$,
$(t,x) \longmapsto x^\alpha \partial_x^\beta \phi(t,x)$ *is bounded.*

Proof. Let α, β be fixed. Since the maps $\| \cdot \|_{\alpha,\beta} : \mathcal{S}(\mathbb{R}^d) \to \mathbb{R}$ are continuous, for every $\phi \in \mathcal{C}([0,T], \mathcal{S}(\mathbb{R}^d))$, the application $t \mapsto \|\phi(t,\cdot)\|_{\alpha,\beta}$ is continuous on the compact interval $[0,T]$ and therefore bounded, which yields the result.
☐

Proposition 4.13. *If* $\phi \in \mathcal{C}([0,T], \mathcal{S}(\mathbb{R}^d))$ *and* $\alpha, \beta \in \mathbb{N}^d$, *then there exist non-negative functions* $\psi_{\alpha,\beta} \in L^1(\mathbb{R}^d)$ *such that for every* $(t,x) \in [0,T] \times \mathbb{R}^d$, $|x^\alpha \partial_x^\beta \phi(t,x)| \leq \psi_{\alpha,\beta}(x)$.

Proof. We decompose

$$\begin{aligned} |x^\alpha \partial_x^\beta \phi(t,x)| &= |x^\alpha \partial_x^\beta \phi(t,x)| \mathbb{1}_{[-1,1]^d}(x) + |x^{\alpha+(2,\cdots,2)} \partial_x^\beta \phi(t,x)| \frac{1}{\prod_{i\leq d} x_i^2} \mathbb{1}_{\mathbb{R}^d \setminus [-1,1]^d}(x) \\ &\leq C(\mathbb{1}_{[-1,1]^d}(x) + \frac{1}{\prod_{i\leq d} x_i^2} \mathbb{1}_{\mathbb{R}^d \setminus [-1,1]^d}(x)), \end{aligned} \tag{4.5}$$

where C is some constant which exists thanks to Proposition 4.12. ☐

Proposition 4.14. *Let* q *be a continuous negative definite symbol verifying the assumptions of Theorem 4.5 and let* $\phi \in \mathcal{C}^1([0,T], \mathcal{S}(\mathbb{R}^d))$. *Then for any* $x \in \mathbb{R}^d$, $t \mapsto q(\cdot, D)\phi(t,x) \in \mathcal{C}^1([0,T], \mathbb{R})$ *and* $\partial_t q(\cdot,D)\phi = q(\cdot,D)\partial_t\phi$.

Proof. We fix $\phi \in \mathcal{C}^1([0,T], \mathcal{S}(\mathbb{R}^d))$, and $x \in \mathbb{R}^d$. We wish to show that for any $\xi \in \mathbb{R}^d$, $t \longmapsto \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{i(x,\xi)} q(x,\xi) \mathcal{F}_x \phi(t,\xi) d\xi$ is $\mathcal{C}^1$ with derivative $t \longmapsto \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{i(x,\xi)} q(x,\xi) \mathcal{F}_x \partial_t \phi(t,\xi) d\xi$.
Since $\phi \in \mathcal{C}^1([0,T], \mathcal{S}(\mathbb{R}^d))$, then $\partial_t \phi \in \mathcal{C}([0,T], \mathcal{S}(\mathbb{R}^d))$ and by Proposition 4.11, $\mathcal{F}_x \partial_t \phi \in \mathcal{C}([0,T], \mathcal{S}(\mathbb{R}^d))$. Moreover since q verifies the assumptions of Theorem 4.5, then $|q(x,\xi)|$ is bounded by $c'(1 + \|\xi\|^2)$ for some constant c'. Therefore by Proposition 4.13, there exists a non-negative $\psi \in L^1(\mathbb{R}^d)$ such that for every t, ξ, $|q(x,\xi)\mathcal{F}_x \partial_t \phi(t,\xi)| \leq \psi(\xi)$. Since by Proposition 4.11, $\mathcal{F}_x \partial_t \phi = \partial_t \mathcal{F}_x \phi$, this implies that for any (t,ξ), $|\partial_t e^{i(x,\xi)} q(x,\xi) \mathcal{F}_x \phi(t,\xi)| \leq \psi(\xi)$. So by the theorem about the differentiation of integrals depending on a parameter, for any $\xi \in \mathbb{R}^d$, $t \longmapsto \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{i(x,\xi)} q(x,\xi) \mathcal{F}_x \phi(t,\xi) d\xi$ is of class $\mathcal{C}^1$ with derivative $t \longmapsto \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{i(x,\xi)} q(x,\xi) \mathcal{F}_x \partial_t \phi(t,\xi) d\xi$.

□

Proposition 4.15. *Let q be a continuous negative definite symbol verifying the assumptions of Theorem 4.5 and let $\phi \in \mathcal{C}^1([0,T], \mathcal{S}(\mathbb{R}^d))$. Then ϕ, $\partial_t \phi$, $q(\cdot, D)\phi$ and $q(\cdot, D)\partial_t \phi$ are bounded.*

Proof. Proposition 4.12 implies that any element of $\mathcal{C}([0,T], \mathcal{S}(\mathbb{R}^d))$ is bounded, so we immediately deduce that ϕ and $\partial_t \phi$ are bounded.
Since q verifies the assumptions of Theorem 4.5, for any fixed $(t,x) \in [0,T] \times \mathbb{R}^d$, we have

$$\begin{array}{rcl} |q(\cdot, D)\phi(t,x)| & = & \left| \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{i(x,\xi)} q(x,\xi) \mathcal{F}_x \phi(t,\xi) d\xi \right| \\ & \leq & C \int_{\mathbb{R}^d} (1 + \|\xi\|^2) |\mathcal{F}_x \phi(t,\xi)| d\xi, \end{array} \tag{4.6}$$

for some constant C. Since $\phi \in \mathcal{C}([0,T], \mathcal{S}(\mathbb{R}^d))$ then, by Proposition 4.11, $\mathcal{F}_x \phi$ also belongs to $\mathcal{C}([0,T], \mathcal{S}(\mathbb{R}^d))$, and by Proposition 4.12, there exists a positive $\psi \in L^1(\mathbb{R}^d)$ such that for any (t,ξ), $(1 + \|\xi\|^2)|\mathcal{F}_x \phi(t,\xi)| \leq \psi(\xi)$, so for any (t,x), $|q(\cdot, D)\phi(t,x)| \leq \|\psi\|_1$.

Similar arguments hold replacing ϕ with $\partial_t \phi$ since it also belongs to $\mathcal{C}([0,T], \mathcal{S}(\mathbb{R}^d))$.

□

Remark 4.16. *$\mathcal{C}^1([0,T], \mathcal{S}(\mathbb{R}^d))$ seems to be a domain which is particularly appropriate for time-dependent Fourier analysis and it fits well for our framework. On the other hand it is not so fundamental to require such regularity for classical solutions for Pseudo-PDEs, so that we could consider a larger domain. For example the Fréchet algebra $\mathcal{S}(\mathbb{R}^d)$ could be replaced with the Banach algebra $W^{d+3,1}(\mathbb{R}^d) \bigcap W^{d+3,\infty}(\mathbb{R}^d)$ in all the previous proofs.*
Even bigger domains are certainly possible, we will however not insist on such refinements.

Corollary 4.17. *Let q be a continuous negative definite symbol verifying the hypotheses of Theorem 4.5. Then $\mathcal{D}(\partial_t - q(\cdot, D))$ is a linear algebra included in $\mathcal{D}^{max}(\partial_t - q(\cdot, D))$ as defined in Notation B.5.*

Proof. We recall that, according to Notation 4.9 $\mathcal{D}(\partial_t - q(\cdot, D)) = \mathcal{C}^1([0,T], \mathcal{S}(\mathbb{R}^d))$. The proof follows from Lemma 4.7, Propositions 4.14 and 4.15, and the comments under Notation 4.9. $\square$

Corollary 4.18. *Let q be a continuous negative definite symbol verifying the hypotheses of Theorem 4.5, let $(\mathbb{P}^x)_{x\in\mathbb{R}^d}$ be the corresponding homogeneous Markov class exhibited in Theorem 4.5, let $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times\mathbb{R}^d}$ be the corresponding Markov class (see Notation B.1), let $(\mathcal{D}(\partial_t - q(\cdot, D)), \partial_t - q(\cdot, D))$ be as in Notation 4.9. Then*

- $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times\mathbb{R}^d}$ *solves the well-posed Martingale Problem associated to* $(\mathcal{D}(\partial_t - q(\cdot, D)), \partial_t - q(\cdot, D), V_t \equiv t)$;
- *its transition function is measurable in time.*

Proof. The first statement directly comes from Theorem 4.5 and Corollaries 4.17 B.8, and the second from Proposition B.2. $\square$

Remark 4.19. *The symbol of the fractional Laplacian $q : (x, \xi) \mapsto \|\xi\|^\alpha$ trivially verifies the assumptions of Theorem 4.5. Indeed, it has no dependence in x, so it is enough to set $\psi : \xi \mapsto \|\xi\|^\alpha$, $c_0 = c = c' = 1$, $r_0 = \alpha$ and $\gamma = \frac{1}{2}$.*

The Pseudo-PDE that we focus on is the following.

$$\left\{ \begin{array}{r} \partial_t u - q(\cdot, D)u = f(\cdot,\cdot,u,\sqrt{\Gamma(u,u)}) \text{ on } [0,T]\times\mathbb{R}^d \\ u(T,\cdot) = g, \end{array} \right. \tag{4.7}$$

where q is a continuous negative definite symbol verifying the assumptions of Theorem 4.5 and Γ is the associated carré du champs operator, see Definition 2.6.

Remark 4.20. *By Proposition 3.3 in [13], for any $\alpha \in]0,2[$, there exists a constant c_α such that for any $\phi \in \mathcal{S}(\mathbb{R}^d)$,*

$$(-\Delta)^{\frac{\alpha}{2}}\phi = c_\alpha PV \int_{\mathbb{R}^d} \frac{(\phi(\cdot + y) - \phi)}{\|y\|^{d+\alpha}} dy, \tag{4.8}$$

where PV is a notation for principal value, see (3.1) in [13]. Therefore in the particular case of the fractional Laplace operator, the carré du champs operator Γ^α associated to $(-\Delta)^{\frac{\alpha}{2}}$ is given by

$$\begin{array}{rl} & \Gamma^\alpha(\phi,\phi) \\ = & c_\alpha PV \int_{\mathbb{R}^d} \frac{(\phi^2(\cdot,\cdot+y)-\phi^2)}{\|y\|^{d+\alpha}} dy - 2\phi c_\alpha PV \int_{\mathbb{R}^d} \frac{(\phi(\cdot,\cdot+y)-\phi)}{\|y\|^{d+\alpha}} dy \\ = & c_\alpha PV \int_{\mathbb{R}^d} \frac{(\phi(\cdot,\cdot+y)-\phi)^2}{\|y\|^{d+\alpha}} dy. \end{array} \tag{4.9}$$

Proposition 4.21. *Let q be a continuous negative symbol verifying the assumptions of Theorem 4.5, let $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times\mathbb{R}^d}$ be the Markov class which by Corollary 4.18 solves the well-posed Martingale Problem associated to $(\mathcal{D}(\partial_t - q(\cdot, D)), \partial_t - q(\cdot, D), V_t \equiv t)$.*
For any (f,g) verifying H^{lip}_b (see Hypothesis 2.24), $Pseudo-PDE(f,g)$ admits a unique decoupled mild solution in the sense of Definition 3.4.

Proof. The assertion comes from Corollary 4.18 and Theorem 3.9. $\square$

4.3 Parabolic semi-linear PDEs with distributional drift

In this section we will use the formalism and results obtained in [18] and [19], see also [39], [10] for more recent developments. In particular the latter paper treats interesting applications to polymers. Those papers introduced a suitable framework of Martingale Problem related to a PDE operator containing a distributional drift b' which is the derivative of a continuous function. [17] established a first work in the n-dimensional setting.

Let $b, \sigma \in \mathcal{C}^0(\mathbb{R})$ such that $\sigma > 0$. By mollifier, we intend a function $\Phi \in \mathcal{S}(\mathbb{R})$ with $\int \Phi(x)dx = 1$. We denote $\Phi_n(x) = n\Phi(nx)$, $\sigma_n^2 = \sigma^2 * \Phi_n$, $b_n = b * \Phi_n$.
We then define $L_n g = \frac{\sigma_n^2}{2} g'' + b_n' g'$. $f \in \mathcal{C}^1(\mathbb{R})$ is said to be a solution to $Lf = \dot{l}$ where $\dot{l} \in \mathcal{C}^0$, if for any mollifier Φ, there are sequences (f_n) in $\mathcal{C}^2$, $(\dot{l}_n)$ in $\mathcal{C}^0$ such that $L_n f_n = (\dot{l}_n)$, $f_n \xrightarrow{\mathcal{C}^1} f$, $\dot{l}_n \xrightarrow{\mathcal{C}^0} \dot{l}$. We will assume that $\Sigma(x) = \lim\limits_{n\to\infty} 2\int_0^x \frac{b_n'}{\sigma_n^2}(y)dy$ exists in $\mathcal{C}^0$ independently from the mollifier.

By Proposition 2.3 in [18] there exists a solution $h \in \mathcal{C}^1$ to $Lh = 0$, $h(0) = 0$, $h'(0) = 1$. Moreover it verifies $h' = e^{-\Sigma}$. Moreover by Remark 2.4 in [18], for any $\dot{l} \in \mathcal{C}^0$, $x_0, x_1 \in \mathbb{R}$, there exists a unique solution of

$$Lf(x) = \dot{l},\ f \in \mathcal{C}^1,\ f(0) = x_0,\ f'(0) = x_1. \tag{4.10}$$

$\mathcal{D}_L$ is defined as the set of $f \in \mathcal{C}^1$ such that there exists some $\dot{l} \in \mathcal{C}^0$ with $Lf = \dot{l}$. And by Lemma 2.9 in [18] it is equal to the set of $f \in \mathcal{C}^1$ such that $\frac{f'}{h'} \in \mathcal{C}^1$. So it is clearly an algebra.
h is strictly increasing, I will denote its image. Let L^0 be the classical differential operator defined by $L^0\phi = \frac{\sigma_0^2}{2}\phi''$, where

$$\sigma_0(y) = \begin{cases} (\sigma h')(h^{-1}(y)) & : \ y \in I \\ 0 & : \ y \in I^c. \end{cases} \tag{4.11}$$

Let v be the unique solution to $Lv = 1$, $v(0) = v'(0) = 0$, we will assume that

$$v(-\infty) = v(+\infty) = +\infty, \tag{4.12}$$

which represents a non-explosion condition. In this case, Proposition 3.13 in [18] states that the Martingale Problem associated to $(\mathcal{D}_L, L, V_t \equiv t)$ is well-posed. Its solution will be denoted $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times\mathbb{R}^d}$. By Proposition 2.13, $\mathcal{D}_{L^0} = C^2(I)$. and by Proposition 3.2 in [18], the Martingale Problem associated to $(\mathcal{D}_{L^0}, L^0, V_t \equiv t)$ is also well-posed, we will call $(\mathbb{Q}^{s,x})_{(s,x)\in[0,T]\times\mathbb{R}^d}$ its solution. Moreover under any $\mathbb{P}^{s,x}$ the canonical process is a Dirichlet process, and $h^{-1}(X)$ is a semi-martingale that we call Y solving the SDE $Y_t = h(x) + \int_s^t \sigma_0(Y_s)dW_s$ in law, where the law of Y is $\mathbb{Q}^{s,x}$. X_t is a $\mathbb{P}^{s,x}$-Dirichlet process whose martingale component is $\int_s^\cdot \sigma(X_r)dW_r$. $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times\mathbb{R}^d}$ and $(\mathbb{Q}^{s,x})_{(s,x)\in[0,T]\times\mathbb{R}^d}$ both define Markov classes.
We introduce now the domain that we will indeed use.

Definition 4.22. *We set*

$$\mathcal{D}(a) = \left\{ \phi \in \mathcal{C}^{1,1}([0,T] \times \mathbb{R}) : \frac{\partial_x \phi}{h'} \in \mathcal{C}^{1,1}([0,T] \times \mathbb{R}) \right\}, \tag{4.13}$$

which clearly is a linear algebra.
On $\mathcal{D}(a)$, we set $L\phi := \frac{\sigma^2 h'}{2} \partial_x (\frac{\partial_x \phi}{h'})$ and $a(\phi) := \partial_t \phi + L\phi$.

Proposition 4.23. *Let Γ denote the carré du champ operator associated to a, let ϕ, ψ be in $\mathcal{D}(a)$, then $\Gamma(\phi, \psi) = \sigma^2 \partial_x \phi \partial_x \psi$.*

Proof. We fix ϕ, ψ in $\mathcal{D}(a)$. We write

$$\begin{array}{rcl} \Gamma(\phi,\psi) & = & (\partial_t + L)(\phi\psi) - \phi(\partial_t + L)(\psi) - \psi(\partial_t + L)(\phi) \\ & = & \frac{\sigma^2 h'}{2} \left(\partial_x \left(\frac{\partial_x \phi\psi}{h'} \right) - \phi \partial_x \left(\frac{\partial_x \psi}{h'} \right) - \psi \partial_x \left(\frac{\partial_x \phi}{h'} \right) \right) \\ & = & \sigma^2 \partial_x \phi \partial_x \psi. \end{array} \tag{4.14}$$

□

Emphasizing that b' is a distribution, the equation that we will study in this section is therefore given by

$$\left\{ \begin{array}{l} \partial_t u + \frac{1}{2}\sigma^2 \partial_x^2 u + b' \partial_x u + f(\cdot, \cdot, u, \sigma|\partial_x u|) = 0 \quad \text{on } [0,T] \times \mathbb{R} \\ u(T, \cdot) = g. \end{array} \right. \tag{4.15}$$

Proposition 4.24. $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times\mathbb{R}^d}$ *solves the Martingale Problem associated to $(a, \mathcal{D}(a), V_t \equiv t)$.*

Proof. $(t,y) \mapsto \phi(t, h^{-1}(y))$ is of class $\mathcal{C}^{1,2}$; moreover $\partial_x \left(\phi(r,\cdot) \circ h^{-1} \right) = \frac{\partial_x \phi}{h'} \circ h^{-1}$ and $\partial_x^2 \left(\phi(r,\cdot) \circ h^{-1} \right) = \frac{2L\phi}{\sigma^2 h'^2} \circ h^{-1} = \frac{2L\phi}{\sigma_0^2} \circ h^{-1}$. By Itô formula we have

$$\begin{array}{rcl} \phi(t, X_t) & = & \phi(t, h^{-1}(Y_t)) \\ & = & \phi(s,x) + \int_s^t \left(\partial_t \phi(r, h^{-1}(Y_r)) + \frac{1}{2}\sigma_0^2(Y_r) \partial_x^2 \left(\phi(r,\cdot) \circ h^{-1} \right)(Y_r) \right) dr \\ & & + \int_s^t \sigma_0(r, h^{-1}(Y_r)) \partial_x \left(\phi(r,\cdot) \circ h^{-1} \right)(Y_r) dW_r \\ & = & \phi(s,x) + \int_s^t \left(\partial_t \phi(r, h^{-1}(Y_r)) + L\phi(r, h^{-1}(Y_r)) \right) dr \\ & & + \int_s^t \sigma_0(r, h^{-1}(Y_r)) \frac{\partial_x \phi(r, h^{-1}(Y_r))}{h'(Y_r)} dW_r \\ & = & \phi(s,x) + \int_s^t \left(\partial_t \phi(r, X_r) + l(r, X_r) \right) dr + \int_s^t \sigma(r, X_r) \partial_x \phi(r, X_r) dW_r. \end{array} \tag{4.16}$$

Therefore $\phi(t, X_t) - \phi(s,x) - \int_s^t a(\phi)(r, X_r) dr = \int_s^t \sigma(r, X_r) \partial_x \phi(r, X_r) dW_r$ is a local martingale. □

In order to consider the $FBSDE^{s,x}(f,g)$ for functions (f,g) having polynomial growth in x we will show the following result. We formulate here the supplementary assumption, called (TA) in [18]. This means the existence of strictly positive constants c_1, C_1 such that

$$c_1 \leq \frac{e^{\Sigma}}{\sigma} \leq C_1. \tag{4.17}$$

Proposition 4.25. *We suppose that (TA) is fulfilled and σ has linear growth. Then, for any $p > 0$ and $(s,x) \in [0,T]\times\mathbb{R}$, $\mathbb{E}^{s,x}[|X_T|^p] < \infty$ and $\mathbb{E}^{s,x}[\int_s^T |X_r|^p dr] < \infty$. In other words, the Markov class $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times\mathbb{R}^d}$ verifies $H^{mom}(\zeta,\eta)$ (see Hypothesis 2.23), for $\zeta : x \longmapsto |x|^p$ and $\eta : x \longmapsto |x|^q$, for every $p, q > 0$.*

Proof. We start by proving the proposition in the divergence form case, meaning that $b = \frac{\sigma^2}{2}$.
Let (s,x) and $t \in [s,T]$ be fixed. Thanks to the Aronson estimates, see e.g. [2] and also Section 5. of [18], there is a constant $M > 0$ such that

$$\begin{array}{rcl} \mathbb{E}^{s,x}[|X_t|^p] & = & \int_{\mathbb{R}} |y|^p p_{t-s}(x,y)dy \\ & \leq & \frac{M}{\sqrt{t-s}} \int_{\mathbb{R}} |y|^p e^{-\frac{|x-y|^2}{M(t-s)}} dz \\ & = & M^{\frac{3}{2}} \int_{\mathbb{R}} |x + z\sqrt{M(t-s)}|^p e^{-z^2} dz \\ & \leq & \sum_{k=0}^p M^{\frac{3+k}{2}} \binom{p}{k} |x|^k |t-s|^{\frac{p-k}{2}} \int_{\mathbb{R}} |z|^{p-k} e^{-z^2} dz, \end{array} \tag{4.18}$$

which (for fixed (s,x)) is bounded in $t \in [s,T]$ and therefore Lebesgue integrable in t on $[s,T]$. This in particular shows that $\mathbb{E}^{s,x}[|X_T|^p]$ and $\mathbb{E}^{s,x}[\int_s^T |X_r|^p dr](= \int_s^T \mathbb{E}^{s,x}[|X_r|^p]dr)$ are finite.
Now we will consider the case in which X only verifies (4.17) and we will add the hypothesis that σ has linear growth.
Then there exists a process Z (see Lemma 5.6 in [18]) solving an SDE with distributional drift of divergence form generator, and a function k of class $\mathcal{C}^1$ such that $X = k^{-1}(Z)$. The (4.17) condition implies that there exist two constants such that $0 < c \leq k'\sigma \leq C$ implying that for any x, $(k^{-1})'(x) = \frac{1}{k'\circ k^{-1}(x)} \leq \frac{\sigma\circ k^{-1}(x)}{c} \leq C_2(1+|k^{-1}(x)|)$, for a positive constant C_2. So by Gronwall Lemma there exists $C_3 > 0$ such that $k^{-1}(x) \leq C_3 e^{C_2|x|}$, $\forall x \in \mathbb{R}$.
Now thank to the Aronson estimates on the transition function p^Z of Z, for every $p > 0$, we have

$$\begin{array}{rcl} \mathbb{E}^{s,x}[|X|_t^p] & \leq & C_3 \int e^{C_2 p|z|} p_{t-s}^Z(k(x),z)dz \\ & \leq & \int e^{C_2 p|z|} \frac{M}{\sqrt{t}} e^{-\frac{|k(x)-z|^2}{Mt}} dz \\ & \leq & M^{\frac{3}{2}} \int e^{C_2 p(\sqrt{Mt}|y|+k(x))} e^{-y^2} dy \\ & \leq & Ae^{Bk(x)}, \end{array} \tag{4.19}$$

where A, B are two constants depending on p and M. This implies that $\mathbb{E}^{s,x}[|X_T|^p] < \infty$ and $\mathbb{E}^{s,x}[\int_s^T |X_r|^p dr] < \infty$. ☐

We can now state the main result of this section.

Proposition 4.26. *Assume that the non-explosion condition* (4.12) *is verified and the validity of the two following items.*

- *the (TA) condition* (4.17) *is fulfilled, σ has linear growth and (f,g) verifies $H^{lip}(\zeta,\eta)$ (see Hypothesis 2.24) with $\zeta : x \mapsto |x|^p$, $\eta : x \mapsto |x|^q$ for some $p, q \in \mathbb{R}_+$;*

- *(f, g) verifies H_b^{lip}, see Hypothesis 2.24.*

Then (4.15) *has a unique decoupled mild solution u in the sense of Definition 3.4.*

Proof. The assertion comes from Theorem 3.9 which applies thanks to Propositions 4.24, 4.25 and B.2. ☐

Remark 4.27.

1. *A first analysis linking PDEs (in fact second order elliptic differential equations) with distributional drift and BSDEs was performed by [40]. In those BSDEs the final horizon was a stopping time.*
2. *In [24], the authors have considered a class of BSDEs involving particular distributions.*

4.4 Diffusion equations on differential manifolds

In this section, we will provide an example of application in a non Euclidean space. We consider a compact connected smooth differential manifold M of dimension n. We denote by $\mathcal{C}^\infty(M)$ the linear algebra of smooth functions from M to $\mathbb{R}$, and $(U_i, \phi_i)_{i\in I}$ its atlas. The reader may consult [30] for an extensive introduction to the study of differential manifolds, and [23] concerning diffusions on differential manifolds.

Lemma 4.28. *M is Polish.*

Proof. By Theorem 1.4.1 in [30] M may be equipped with a Riemannian metric, that we denote by g and its topology may be metricized by the associated distance which we denote by d. As any compact metric space, (M, d) is separable and complete so that M is a Polish space. ☐

We denote by $(\Omega, \mathcal{F}, (X_t)_{t\in[0,T]}(\mathcal{F})_{t\in[0,T]})$ the canonical space associated to M and T, see Definition A.1.

Definition 4.29. *An operator $L : \mathcal{C}^\infty(M) \longrightarrow \mathcal{C}^\infty(M)$ will be called a* ***smooth second order elliptic non degenerate differential operator on*** *M if for any chart $\phi : U \longrightarrow \mathbb{R}^n$ there exist smooth $\mu : \phi(U) \longrightarrow \mathbb{R}^n$ and $\alpha : \phi(U) \longrightarrow S_+^*(\mathbb{R}^n)$ such that on $\phi(U)$ for any $f \in \mathcal{C}^\infty(M)$ we have*

$$Lf(\phi^{-1}(x)) = \frac{1}{2}\sum_{i,j=1}^{n} \alpha^{i,j}(x)\partial_{x_i x_j}(f\circ\phi^{-1})(x) + \sum_{i=1}^{n}\mu^i(x)\partial_{x_i}(f\circ\phi^{-1})(x). \quad (4.20)$$

α and μ depend on the chart ϕ but this dependence will be sometimes omitted and we will say that for some given local coordinates,
$Lf = \frac{1}{2}\sum\limits_{i,j=1}^{n}\alpha^{i,j}\partial_{x_i x_j} f + \sum\limits_{i=1}^{n}\mu^i\partial_{x_i} f$.

The following definition comes from [23], see Definition 1.3.1.

Definition 4.30. *Let L denote a smooth second order elliptic non degenerate differential operator on M. Let $x \in M$. A probability measure $\mathbb{P}^x$ on $(\Omega, \mathcal{F})$ will be called an* ***L-diffusion starting in*** *x if*

- $\mathbb{P}^x(X_0 = x) = 1$;
- *for every $f \in \mathcal{C}^\infty(M)$, $f(X) - \int_0^\cdot Lf(X_r)dr$ is a $(\mathbb{P}^x, (\mathcal{F})_{t\in[0,T]})$ local martingale.*

Remark 4.31. *No explosion can occur for continuous stochastic processes with values in a compact space, so there is no need to consider paths in the compactification of M as in Definition 1.1.4 in [23].*
Theorems 1.3.4 and 1.3.5 in [23] state that for any $x \in M$, there exists a unique L-diffusion starting in x. Theorem 1.3.7 in [23] implies that those probability measures $(\mathbb{P}^x)_{x\in M}$ define a homogeneous Markov class.

For a given operator L, the carré du champs operator Γ is given (in local coordinates) by $\Gamma(\phi, \psi) = \sum\limits_{i,j=1}^{n} \alpha^{i,j}\partial_{x_i}\phi\partial_{x_j}\phi$, see equation (1.3.3) in [23]. We wish to emphasize here that the carré du champs operator has recently become a powerful tool in the study of geometrical properties of Riemannian manifolds. The reader may refer e.g. to [3].

Definition 4.32. *$(\mathbb{P}^x)_{x\in M}$ will be called the* ***L-diffusion****. If M is equipped with a specific Riemannian metric g and L is chosen to be equal to $\frac{1}{2}\Delta_g$ where Δ_g the Laplace-Beltrami operator associated to g, then $(\mathbb{P}^x)_{x\in M}$ will be called the* ***Brownian motion associated to*** *g, see [23] Chapter 3 for details.*

We now fix some smooth second order elliptic non degenerate differential operator L and the L-diffusion $(\mathbb{P}^x)_{x\in M}$. We introduce the associated Markov class $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times M}$ as described in Notation B.1, which by Proposition B.2 is measurable in time.

Notation 4.33. *We define $\mathcal{D}(\partial_t + L)$ the set of functions $u : [0,T] \times M \longrightarrow \mathbb{R}$ such that, for any chart $\phi : U \longrightarrow \mathbb{R}^n$, the mapping*

$$\begin{array}{rcl} [0,T] \times \phi(U) & \longrightarrow & \mathbb{R} \\ (t,x) & \longmapsto & u(t, \phi^{-1}(x)) \end{array} \tag{4.21}$$

belongs to $\mathcal{C}^\infty([0,T] \times \phi(U), \mathbb{R})$, the set of infinitely continuously differentiable functions in the usual Euclidean setup.

Lemma 4.34. *$\mathcal{D}(\partial_t + L)$ is a linear algebra included in $\mathcal{D}^{max}(\partial_t + L)$ as defined in Notation B.5.*

Proof. For some fixed chart $\phi : U \longrightarrow \mathbb{R}^n$, $\mathcal{C}^\infty([0,T] \times \phi(U), \mathbb{R})$ is an algebra, so it is immediate that $\mathcal{D}(\partial_t + L)$ is an algebra.
Moreover, if $u \in \mathcal{D}(\partial_t + L)$, it is clear that

- $\forall x \in M$, $u(\cdot, x) \in \mathcal{C}^1([0,T], \mathbb{R})$ and $\forall t \in [0,T]$, $u(t, \cdot) \in \mathcal{C}^\infty(M)$;

- $\forall t\in[0,T]$, $\partial_t u(t,\cdot)\in\mathcal{C}^\infty(M)$ and $\forall x\in M$, $Lu(\cdot,x)\in\mathcal{C}^1([0,T],\mathbb{R})$.

Given a chart $\phi : U \longrightarrow \mathbb{R}^n$, by the Schwarz Theorem allowing the commutation of partial derivatives (in the classical Euclidean setup), we have for $x\in\phi(U)$

$$\begin{array}{rcl}\partial_t\circ L(u)(t,\phi^{-1}(x)) & = & \frac{1}{2}\sum\limits_{i,j=1}^{n}\alpha^{i,j}(x)\partial_t\partial_{x_ix_j}(u(\cdot,\phi^{-1}(\cdot))(t,x)+\sum\limits_{i=1}^{n}\mu^i(x)\partial_t\partial_{x_i}(u(\cdot,\phi^{-1}(\cdot))(t,x)\\ & = & \frac{1}{2}\sum\limits_{i,j=1}^{n}\alpha^{i,j}(x)\partial_{x_ix_j}\partial_t(u(\cdot,\phi^{-1}(\cdot))(t,x)+\sum\limits_{i=1}^{n}\mu^i(x)\partial_{x_i}\partial_t(u(\cdot,\phi^{-1}(\cdot))(t,x)\\ & = & L\circ\partial_t(u)(t,\phi^{-1}(x)).\end{array}\tag{4.22}$$

So $\partial_t\circ Lu = L\circ\partial_t u$. Finally $\partial_t u$, Lu and $\partial_t\circ Lu$ are continuous (since they are continuous on all the sets $[0,T]\times U$ where U is the domain of a chart) and are therefore bounded as continuous functions on the compact set $[0,T]\times M$. This concludes the proof. ☐

Corollary 4.35. *$(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times M}$ solves the well-posed Martingale Problem associated to $(\partial_t+L,\mathcal{D}(\partial_t+L),V_t\equiv t)$ in the sense of Definition 2.3.*

Proof. The corollary derives from Lemma 4.34 and Corollary B.8. ☐

We fix a couple (f,g) verifying H_b^{lip} (see Hypothesis 2.24) and consider the PDE

$$\left\{\begin{array}{l}\partial_t u+Lu+f(\cdot,\cdot,u,\sqrt{\Gamma(u,u)})=0 \quad \text{on } [0,T]\times M\\ u(T,\cdot)=g.\end{array}\right.\tag{4.23}$$

Since Theorem 3.9 applies, we have the following result.

Proposition 4.36. *Equation* (4.23) *admits a unique decoupled mild solution u in the sense of Definition 3.4.*

Remark 4.37. *Since M is compact, we emphasize that if g is continuous and f is continuous in t,x Lipschitz in y,z then (f,g) verifies H_b^{lip}.*

Appendices

A Markov classes

In this Appendix we recall some basic definitions and results concerning Markov processes. For a complete study of homogeneous Markov processes, one may consult [12], concerning non-homogeneous Markov classes, our reference was chapter VI of [14]. The first definition refers to the canonical space that one can find in [28], see paragraph 12.63.

Notation A.1. *In the whole section E will be a fixed Polish space (a separable completely metrizable topological space). E will be called the* ***state space****.*

We consider $T \in \mathbb{R}_+^*$. We denote $\Omega := \mathbb{D}([0,T],E)$ the space of functions from $[0,T]$ to E right-continuous with left limits and continuous at time T, e.g. cadlag. For any $t \in [0,T]$ we denote the coordinate mapping $X_t : \omega \mapsto \omega(t)$, and we introduce on Ω the σ-field $\mathcal{F} := \sigma(X_r|r \in [0,T])$.
On the measurable space $(\Omega, \mathcal{F})$, we introduce the measurable ***canonical process***

$$X : \begin{array}{ccc} (t,\omega) & \longmapsto & \omega(t) \\ ([0,T]\times\Omega, \mathcal{B}([0,T])\otimes\mathcal{F}) & \longrightarrow & (E,\mathcal{B}(E)), \end{array}$$

and the right-continuous filtration $(\mathcal{F}_t)_{t\in[0,T]}$ where $\mathcal{F}_t := \bigcap_{s\in]t,T]} \sigma(X_r|r \le s)$ if $t < T$, and $\mathcal{F}_T := \sigma(X_r|r \in [0,T]) = \mathcal{F}$.
$\big(\Omega, \mathcal{F}, (X_t)_{t\in[0,T]}, (\mathcal{F}_t)_{t\in[0,T]}\big)$ will be called the ***canonical space*** *(associated to T and E).*
For any $t \in [0,T]$ we denote $\mathcal{F}_{t,T} := \sigma(X_r|r \ge t)$, and for any $0 \le t \le u < T$ we will denote $\mathcal{F}_{t,u} := \bigcap_{n\ge 0} \sigma(X_r|r \in [t, u + \frac{1}{n}])$.

Since E is Polish, we recall that $\mathbb{D}([0,T],E)$ can be equipped with a Skorokhod distance which makes it a Polish metric space (see Theorem 5.6 in chapter 3 of [16], and for which the Borel σ-field is $\mathcal{F}$, see Proposition 7.1 in Chapter 3 of [16]. This in particular implies that $\mathcal{F}$ is separable, as the Borel σ-field of a separable metric space.

Remark A.2. *Previous definitions and all the notions of this Appendix, extend to a time interval equal to $\mathbb{R}_+$ or replacing the Skorokhod space with the Wiener space of continuous functions from $[0,T]$ (or $\mathbb{R}_+$) to E.*

Definition A.3. *The function*

$$p : \begin{array}{ccc} (s,x,t,A) & \longmapsto & p(s,x,t,A) \\ [0,T]\times E\times[0,T]\times\mathcal{B}(E) & \longrightarrow & [0,1], \end{array}$$

will be called ***transition function*** *if, for any s,t in $[0,T]$, $x_0 \in E$, $A \in \mathcal{B}(E)$, it verifies*

1. *$x \mapsto p(s,x,t,A)$ is Borel,*
2. *$B \mapsto p(s,x_0,t,B)$ is a probability measure on $(E,\mathcal{B}(E))$,*
3. *if $t \le s$ then $p(s,x_0,t,A) = \mathbb{1}_A(x_0)$,*
4. *if $s < t$, for any $u > t$, $\int_E p(s,x_0,t,dy)p(t,y,u,A) = p(s,x_0,u,A)$.*

The latter statement is the well-known **Chapman-Kolmogorov equation**.

Definition A.4. *A transition function p for which the first item is reinforced supposing that $(s,x) \longmapsto p(s,x,t,A)$ is Borel for any t, A, will be said* ***measurable in time****.*

Remark A.5. *Let p be a transition function which is measurable in time. By approximation by step functions, one can easily show that, for any Borel function ϕ from E to $\bar{\mathbb{R}}$ then $(s,x) \mapsto \int \phi(y)p(s,x,t,dy)$ is Borel, provided ϕ is quasi integrable for every (s,x).*

Definition A.6. *A* ***canonical Markov class*** *associated to a transition function p is a set of probability measures $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times E}$ defined on the measurable space $(\Omega, \mathcal{F})$ and verifying for any $t \in [0,T]$ and $A \in \mathcal{B}(E)$*

$$\mathbb{P}^{s,x}(X_t \in A) = p(s,x,t,A), \tag{A.1}$$

and for any $s \leq t \leq u$

$$\mathbb{P}^{s,x}(X_u \in A|\mathcal{F}_t) = p(t, X_t, u, A) \quad \mathbb{P}^{s,x} \text{ a.s.} \tag{A.2}$$

Remark A.7. *Formula 1.7 in Chapter 6 of [14] states that for any $F \in \mathcal{F}_{t,T}$ yields*

$$\mathbb{P}^{s,x}(F|\mathcal{F}_t) = \mathbb{P}^{t,X_t}(F) = \mathbb{P}^{s,x}(F|X_t) \ \ \mathbb{P}^{s,x} \text{a.s.} \tag{A.3}$$

Property (A.3) *will be called* ***Markov property***.

For the rest of this section, we are given a canonical Markov class $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times E}$ with transition function p.
We will complete the σ-fields $\mathcal{F}_t$ of the canonical filtration by $\mathbb{P}^{s,x}$ as follows.

Definition A.8. *For any $(s,x) \in [0,T]\times E$ we will consider the (s,x)-**completion** $\left(\Omega, \mathcal{F}^{s,x}, (\mathcal{F}_t^{s,x})_{t\in[0,T]}, \mathbb{P}^{s,x}\right)$ of the stochastic basis $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P}^{s,x}\right)$ by defining $\mathcal{F}^{s,x}$ as the $\mathbb{P}^{s,x}$-completion of $\mathcal{F}$, by extending $\mathbb{P}^{s,x}$ to $\mathcal{F}^{s,x}$ and finally by defining $\mathcal{F}_t^{s,x}$ as the $\mathbb{P}^{s,x}$-closure of $\mathcal{F}_t$ (meaning $\mathcal{F}_t$ augmented with the $\mathbb{P}^{s,x}$-negligible sets) for every $t \in [0,T]$.*

We remark that, for any $(s,x) \in [0,T] \times E$, $\left(\Omega, \mathcal{F}^{s,x}, (\mathcal{F}_t^{s,x})_{t\in[0,T]}, \mathbb{P}^{s,x}\right)$ is a stochastic basis fulfilling the usual conditions, see (1.4) in chapter I of [29]). We recall that considering a conditional expectation with respect to a σ-field augmented with the negligible sets or not, does not change the result. In particular we have the following.

Proposition A.9. *Let $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times E}$ be a canonical Markov class. Let $(s,x) \in [0,T] \times E$ be fixed, Z be a random variable and $t \in [s,T]$, then*
$\mathbb{E}^{s,x}[Z|\mathcal{F}_t] = \mathbb{E}^{s,x}[Z|\mathcal{F}_t^{s,x}]$ $\mathbb{P}^{s,x}$ a.s.

Proposition A.10. *Let Z be a random variable. If the function $(s,x) \longmapsto \mathbb{E}^{s,x}[Z]$ is well-defined (with possible values in $[-\infty, \infty]$), then at fixed $s \in [0,T]$, $x \longmapsto \mathbb{E}^{s,x}[Z]$ is Borel. If moreover the transition function p is measurable in time then, $(s,x) \longmapsto \mathbb{E}^{s,x}[Z]$ is Borel.*
In particular if $F \in \mathcal{F}$ be fixed, then at fixed $s \in [0,T]$, $x \longmapsto \mathbb{P}^{s,x}(F)$ is Borel.
If the transition function p is measurable in time then, $(s,x) \longmapsto \mathbb{P}^{s,x}(F)$ is Borel.

Proof. We will only deal with the case of a measurable in time transition function since the other case is proven in a very similar way.
We consider first the case $Z = 1_F$ where $F \in \mathcal{F}$. We start by assuming that F is of the form $\bigcap\limits_{i\leq n}\{X_{t_i} \in A_i\}$, where $n \in \mathbb{N}^*$, $0 = t_0 \leq t_1 < \cdots < t_n \leq T$ and $A_1, \cdots, A_n$ are Borel sets of E, and we denote by Π the set of such events.
In this proof we will make use of monotone class arguments, see for instance Section 4.3 in [1] for the definitions of π-systems and λ-systems and for the presently used version of the monotone class theorem, also called the Dynkin's lemma.
We remark that Π is a π-system (see Definition 4.9 in [1]) generating $\mathcal{F}$. For such events we can explicitly compute $\mathbb{P}^{s,x}(F)$. We compute this when (s,x) belongs to $[t_{i^*-1}-1, t_{i^*}[\times E$ for some $0 < i^* \leq n$. On $[t_n, T] \times E$, a similar computation can be performed. We will show below that those restricted functions are Borel, the general result will follow by concatenation. We have

$$\begin{array}{rl}
& \mathbb{P}^{s,x}(F) \\
= & \prod\limits_{i=1}^{i^*-1} \mathbb{1}_{A_i}(x)\mathbb{E}^{s,x}\left[\prod\limits_{j=i^*}^{n} \mathbb{1}_{A_i}(X_{t_i})\right] \\
= & \prod\limits_{i=1}^{i^*-1} \mathbb{1}_{A_i}(x)\mathbb{E}^{s,x}\left[\prod\limits_{j=i^*}^{n-1} \mathbb{1}_{A_i}(X_{t_i})\mathbb{E}^{s,x}[\mathbb{1}_{A_n}(X_{t_n})|\mathcal{F}_{t_{n-1}}]\right] \\
= & \prod\limits_{i=1}^{i^*-1} \mathbb{1}_{A_i}(x)\mathbb{E}^{s,x}\left[\prod\limits_{j=i^*}^{n-1} \mathbb{1}_{A_i}(X_{t_i})p(t_{n-1}, X_{t_{n-1}}, t_n, A_n)\right] \\
= & \cdots \\
= & \prod\limits_{i=1}^{i^*-1} \mathbb{1}_{A_i}(x)\int\left(\prod\limits_{j=i^*+1}^{n} \mathbb{1}_{A_j}(x_j)p(t_{j-1}, x_{j-1}, t_j, dx_j)\right)\mathbb{1}_{A_{i^*}}(x_{i^*})p(s, x, t_{i^*}, dx_{i^*}),
\end{array}$$

which indeed is Borel in (s,x) thank to Definition A.4 and Remark A.5.
We can extend this result to any event F by the monotone class theorem. Indeed, let Λ be the set of elements F of $\mathcal{F}$ such that $(s,x) \mapsto \mathbb{P}^{s,x}(F)$ is Borel. For any two events F^1, F^2, in Λ with $F^1 \subset F^2$, since for any (s,x),
$\mathbb{P}^{s,x}(F^2\backslash F^1) = \mathbb{P}^{s,x}(F^2) - \mathbb{P}^{s,x}(F^1)$, $(s,x) \mapsto \mathbb{P}^{s,x}(F^2\backslash F^1)$ is still Borel. For any increasing sequence $(F^n)_{n\geq 0}$ of elements of Λ, $\mathbb{P}^{s,x}(\bigcup\limits_{n\in\mathbb{N}} F^n) = \lim\limits_{n\to\infty} \mathbb{P}^{s,x}(F^n)$ so $(s,x) \mapsto \mathbb{P}^{s,x}(\bigcup\limits_{n\in\mathbb{N}} F^n)$ is still Borel, therefore Λ is a λ-system containing the π-system Π which generates $\mathcal{F}$. So by the monotone class theorem, $\Lambda = \mathcal{F}$, which shows the case $Z = 1_F$.
We go on with the proof when Z is a general r.v. If $Z \geq 0$, there exists an increasing sequence $(Z_n)_{n\geq 0}$ of simple functions on Ω converging pointwise to Z, and thank to the first statement of the Proposition, for each of these functions, $(s,x) \mapsto \mathbb{E}^{s,x}[Z_n]$ is Borel. Therefore since by monotonic convergence, $\mathbb{E}^{s,x}[Z_n] \underset{n\to\infty}{\longrightarrow} \mathbb{E}^{s,x}[Z]$, then $(s,x) \mapsto \mathbb{E}^{s,x}[Z]$ is Borel as the pointwise limit of Borel functions. For a general Z one just has to consider its decomposition $Z = Z^+ - Z^-$ where Z^+ and Z^- are positive. □

Lemma A.11. *Assume that the transition function of the canonical Markov class is measurable in time.*
Let V be a continuous non-decreasing function on $[0,T]$ and $f \in \mathcal{B}([0,T]\times E,\mathbb{R})$ be such that for every $(s,x) \in [0,T] \times E$, $\mathbb{E}^{s,x}[\int_s^T |f(r,X_r)|dV_r] < \infty$. Then $(s,x) \longmapsto \mathbb{E}^{s,x}[\int_s^T f(r,X_r)dV_r]$ is Borel.

Proof. We will start by showing that on $([0,T] \times E) \times [0,T]$, the function $k^n : (s,x,t) \mapsto \mathbb{E}^{s,x}[\int_t^T ((-n) \vee f(r,X_r) \wedge n)dV_r]$ is Borel, where $n \in \mathbb{N}$.
Let $t \in [0,T]$ be fixed. Then by Proposition A.10, $(s,x) \mapsto \mathbb{E}^{s,x}[\int_t^T ((-n) \vee f(r,X_r) \wedge n)dV_r]$ is Borel. Let $(s,x) \in [0,T] \times E$ be fixed and $t_m \underset{m\to\infty}{\longrightarrow} t$ be a converging sequence in $[0,T]$. Since V is continuous, $\int_{t_m}^T ((-n) \vee f(r,X_r) \wedge n)dV_r \underset{m\to\infty}{\longrightarrow} \int_t^T ((-n) \vee f(r,X_r) \wedge n)dV_r$ a.s. Since this sequence is uniformly bounded, by dominated convergence theorem, the same convergence holds under the expectation. This implies that $t \mapsto \mathbb{E}^{s,x}[\int_t^T ((-n) \vee f(r,X_r) \wedge n)dV_r]$ is continuous. By Lemma 4.51 in [1], k^n is therefore jointly Borel.
By composing with $(s,x,t) \mapsto (s,x,s)$, we also have that for any $n \geq 0$, $\tilde{k}^n : (s,x) \longmapsto \mathbb{E}^{s,x}[\int_s^T ((-n) \vee f(r,X_r) \wedge n)dV_r]$ is Borel. Then by letting n tend to infinity, $(-n) \vee f(r,X_r) \wedge n$ tends $dV \otimes d\mathbb{P}^{s,x}$ a.e. to $f(r,X_r)$ and since we assumed $\mathbb{E}^{s,x}[\int_s^T |f(r,X_r)|dV_r] < \infty$, by dominated convergence, $\mathbb{E}^{s,x}[\int_s^T ((-n) \vee f(r,X_r) \wedge n)dV_r]$ tends to $\mathbb{E}^{s,x}[\int_s^T f(r,X_r)dV_r]$.
$(s,x) \longmapsto \mathbb{E}^{s,x}[\int_s^T f(r,X_r)dV_r]$ is therefore Borel as the pointwise limit of the $\tilde{k}^n$ which are Borel. □

Proposition A.12. *Let $f \in \mathcal{B}([0,T] \times E,\mathbb{R})$ be such that for any (s,x,t), $\mathbb{E}^{s,x}[|f(t,X_t)|] < \infty$ then at fixed $s \in [0,T]$, $(x,t) \longmapsto \mathbb{E}^{s,x}[f(t,X_t)]$ is Borel. If moreover the transition function p is measurable in time, then $(s,x,t) \longmapsto \mathbb{E}^{s,x}[f(t,X_t)]$ is Borel.*

Proof. We will only show the case in which p is measurable in time since the other case is proven very similarly.
We start by showing the statement for $f \in \mathcal{C}_b([0,T] \times E,\mathbb{R})$. X is cadlag so $t \longmapsto f(t,X_t)$ also is. So for any fixed $(s,x) \in [0,T] \times E$ if we take a converging sequence $t_n \underset{n\to\infty}{\longrightarrow} t^+$(resp. t^-), an easy application of the Lebesgue dominated convergence theorem shows that $t \longmapsto \mathbb{E}^{s,x}[f(t,X_t)]$ is cadlag. On the other hand, by Proposition A.10, for a fixed t, $(s,x) \longmapsto \mathbb{E}^{s,x}[f(t,X_t)]$ is Borel. Therefore by Theorem 15 Chapter IV of [11], $(s,x,t) \longmapsto \mathbb{E}^{s,x}[f(t,X_t)]$ is jointly Borel.
In order to extend the result to any $f \in \mathcal{B}_b([0,T] \times E,\mathbb{R})$, we consider the subset $\mathcal{I}$ of functions $f \in \mathcal{B}_b([0,T]\times E,\mathbb{R})$ such that $(s,x,t) \longmapsto \mathbb{E}^{s,x}[f(t,X_t)]$ is Borel. Then $\mathcal{I}$ is a linear space stable by uniform convergence and by monotone convergence and containing $\mathcal{C}_b([0,T] \times E,\mathbb{R})$ which is stable by multiplication and generates the Borel σ-field $\mathcal{B}([0,T]) \otimes \mathcal{B}(E)$. So by Theorem 21 in Chapter I of [11], $\mathcal{I} = \mathcal{B}_b([0,T] \times E,\mathbb{R})$. This theorem is sometimes called the functional

monotone class theorem.
Now for any positive Borel function f, we can set $f_n = f \wedge n$ which is bounded Borel. Since by monotonic convergence, $\mathbb{E}^{s,x}[f_n(t, X_t)]$ tends to $\mathbb{E}^{s,x}[f(t, X_t)]$, then $(s, x, t) \longmapsto \mathbb{E}^{s,x}[f(t, X_t)]$ is Borel as the pointwise limit of Borel functions. Finally for a general f it is enough to decompose it into $f = f^+ - f^-$ where f^+, f^- are positive functions. □

B Technicalities concerning homogeneous Markov classes and martingale problems

We start by introducing homogeneous Markov classes. In this section, we are given a Polish space E and some $T \in \mathbb{R}^*$.

Notation B.1. *A mapping $\tilde{p} : E \times [0, T] \times \mathcal{B}(E)$ will be called a* ***homogeneous transition function*** *if $p : (s, x, t, A) \longmapsto \tilde{p}(x, t - s, A)\mathbb{1}_{s<t} + \mathbb{1}_A(x)\mathbb{1}_{s\geq t}$ is a transition function in the sense of Definition A.3. This in particular implies $\tilde{p} = p(0, \cdot, \cdot, \cdot)$.*
A set of probability measures $(\mathbb{P}^x)_{x\in E}$ on the canonical space associated to T and E (see Notation A.1) will be called a ***homogeneous Markov class*** *associated to a homogeneous transition function $\tilde{p}$ if*

$$\begin{cases} \forall t \in [0, T] \quad \forall A \in \mathcal{B}(E) & , \mathbb{P}^x(X_t \in A) = \tilde{p}(x, t, A) \\ \forall 0 \leq t \leq u \leq T & , \mathbb{P}^x(X_u \in A|\mathcal{F}_t) = \tilde{p}(X_t, u - t, A) \quad \mathbb{P}^{s,x} a.s. \end{cases} \tag{B.1}$$

Given a homogeneous Markov class $(\mathbb{P}^x)_{x\in E}$ associated to a homogeneous transition function $\tilde{p}$, one can always consider the Markov class $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times E}$ associated to the transition function $p : (s, x, t, A) \longmapsto \tilde{p}(x, t - s, A)\mathbb{1}_{s<t} + \mathbb{1}_A(x)\mathbb{1}_{s\geq t}$. In particular, for any $x \in E$, we have $\mathbb{P}^{0,x} = \mathbb{P}^x$.

We show that a homogeneous transition function necessarily produces a measurable in time non homogeneous transition function.

Proposition B.2. *Let $\tilde{p}$ be a homogeneous transition function and let p be the associated non homogeneous transition function as described in Notation B.1. Then p is measurable in time in the sense of Definition A.4.*

Proof. Given that $p : (s, x, t, A) \longmapsto \tilde{p}(x, t-s, A)\mathbb{1}_{s<t}+\mathbb{1}_A(x)\mathbb{1}_{s\geq t}$, it is actually enough to show that $\tilde{p}(\cdot, \cdot, A)$ is Borel for any $A \in \mathcal{B}(E)$. We can also write $\tilde{p} = p(0, \cdot, \cdot, \cdot)$, so p is measurable in time if $p(0, \cdot, \cdot, A)$ is Borel for any $A \in \mathcal{B}(E)$, and this holds thanks to Proposition A.12 applied to $f := \mathbb{1}_A$. □

We then introduce below the notion of homogeneous martingale problems.

Definition B.3. *Given A an operator mapping a linear algebra $\mathcal{D}(A) \subset \mathcal{B}_b(E, \mathbb{R})$ into $\mathcal{B}_b(E, \mathbb{R})$, we say that a set of probability measures $(\mathbb{P}^x)_{x\in E}$ on the measurable space $(\Omega, \mathcal{F})$ (see Notation A.1) solves the* ***homogeneous Martingale Problem*** *associated to $(\mathcal{D}(A), A)$ if for any $x \in E$, $\mathbb{P}^x$ satisfies*

- *for every* $\phi \in \mathcal{D}(A)$, $\phi(X_\cdot) - \int_0^\cdot A\phi(X_r)dr$ *is a* $(\mathbb{P}^x, (\mathcal{F}_t)_{t\in[0,T]})$*-local martingale;*
- $\mathbb{P}^x(X_0 = x) = 1$.

We say that this **homogeneous Martingale Problem is well-posed** *if for any* $x \in E$, $\mathbb{P}^x$ *is the only probability measure on* $(\Omega, \mathcal{F})$ *verifying those two items.*

Remark B.4. *If* $(\mathbb{P}^x)_{x\in E}$ *is a homogeneous Markov class solving the homogeneous Martingale Problem associated to some* $(\mathcal{D}(A), A)$*, then the corresponding* $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times E}$ *(see Notation B.1) solves the Martingale Problem associated to* $(\mathcal{D}(A), A, V_t \equiv t)$ *in the sense of Definition 2.3. Moreover if the homogeneous Martingale Problem is well-posed, so is the latter one.*

So a homogeneous Markov process solving a homogeneous martingale problem falls into our setup. We will now see how we can pass from an operator A which only acts on time-independent functions to an evolution operator $\partial_t + A$, and see how our Markov class still solves the corresponding martingale problem.

Notation B.5. *Let* E *be a Polish space and let* A *be an operator mapping a linear algebra* $\mathcal{D}(A) \subset \mathcal{B}_b(E, \mathbb{R})$ *into* $\mathcal{B}_b(E, \mathbb{R})$.
If $\phi \in \mathcal{B}([0,T] \times E, \mathbb{R})$ *is such that for every* $t \in [0,T]$, $\phi(t,\cdot) \in \mathcal{D}(A)$*, then* $A\phi$ *will denote the mapping* $(t,x) \longmapsto A(\phi(t,\cdot))(x)$.

We now introduce the time-inhomogeneous domain associated to A *which we denote* $\mathcal{D}^{max}(\partial_t + A)$ *and which consists in functions* $\phi \in \mathcal{B}_b([0,T] \times E, \mathbb{R})$ *verifying the following conditions:*

- $\forall x \in E$, $\phi(\cdot, x) \in \mathcal{C}^1([0,T], \mathbb{R})$ *and* $\forall t \in [0,T]$, $\phi(t,\cdot) \in \mathcal{D}(A)$;
- $\forall t \in [0,T]$, $\partial_t\phi(t,\cdot) \in \mathcal{D}(A)$ *and* $\forall x \in E$, $A\phi(\cdot, x) \in \mathcal{C}^1([0,T], \mathbb{R})$;
- $\partial_t \circ A\phi = A \circ \partial_t\phi$;
- $\partial_t\phi$, $A\phi$ *and* $\partial_t \circ A\phi$ *belong to* $\mathcal{B}_b([0,T] \times E, \mathbb{R})$.

On $\mathcal{D}^{max}(\partial_t + A)$ *we will consider the operator* $\partial_t + A$.

Remark B.6. *With these notations, it is clear that* $\mathcal{D}^{max}(\partial_t + A)$ *is a sub-linear space of* $\mathcal{B}_b([0,T] \times E, \mathbb{R})$*. It is in general not a linear algebra, but always contains* $\mathcal{D}(A)$*, and even* $\mathcal{C}^1([0,T], \mathbb{R}) \otimes \mathcal{D}(A)$*, the linear algebra of functions which can be written* $\sum\limits_{k\leq N} \lambda_k\psi_k\phi_k$ *where* $N \in \mathbb{N}$*, and for any* k, $\lambda_k \in \mathbb{R}$, $\psi_k \in \mathcal{C}^1([0,T], \mathbb{R})$, $\phi_k \in \mathcal{D}(A)$*. We also notice that* $\partial_t + A$ *maps* $\mathcal{D}^{max}(\partial_t + A)$ *into* $\mathcal{B}_b([0,T] \times E, \mathbb{R})$.

Lemma B.7. *Let us consider the same notations and under the same assumptions as in Notation B.5. Let* $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times E}$ *be a Markov class solving the well-posed Martingale Problem associated to* $(A, \mathcal{D}(A), V_t \equiv t)$ *in the sense of Definition 2.3. Then it also solves the well-posed martingale problem associated to* $(\partial_t + A, \mathcal{A}, V_t \equiv t)$ *for any linear algebra* $\mathcal{A}$ *included in* $\mathcal{D}^{max}(\partial_t + A)$.

Proof. We start by noticing that since $\mathcal{D}(A) \subset \mathcal{B}_b(E, \mathbb{R})$ and is mapped into $\mathcal{B}_b(E, \mathbb{R})$, then for any $(s,x) \in [0,T] \times E$ and $\phi \in \mathcal{D}(A)$, $M^{s,x}[\phi]$ is bounded and is therefore a martingale.

We fix $(s,x) \in [0,T] \times E$, $\phi \in \mathcal{D}^{max}(\partial_t + A)$ and $s \leq t \leq u \leq T$ and we will show that

$$\mathbb{E}^{s,x}\left[\phi(u, X_u) - \phi(t, X_t) - \int_t^u (\partial_t + A)\phi(r, X_r)dr \middle| \mathcal{F}_t\right] = 0, \tag{B.2}$$

which implies that $\phi(\cdot, X_\cdot) - \int_s^\cdot (\partial_t + A)\phi(r, X_r)dr$, $t \in [s,T]$ is a $\mathbb{P}^{s,x}$-martingale. We have

$$\begin{array}{rl} & \mathbb{E}^{s,x}[\phi(u, X_u) - \phi(t, X_t)|\mathcal{F}_t] \\ = & \mathbb{E}^{s,x}[(\phi(u, X_t) - \phi(t, X_t)) + (\phi(u, X_u) - \phi(u, X_t))|\mathcal{F}_t] \\ = & \mathbb{E}^{s,x}\left[\int_t^u \partial_t\phi(r, X_t)dr + \left(\int_t^u A\phi(u, X_r)dr + (M^{s,x}[\phi(u,\cdot)]_u - M^{s,x}[\phi(u,\cdot)]_t)\right)|\mathcal{F}_t\right] \\ = & \mathbb{E}^{s,x}\left[\int_t^u \partial_t\phi(r, X_t)dr + \int_t^u A\phi(u, X_r)dr|\mathcal{F}_t\right] \\ = & I_0 - I_1 + I_2, \end{array}$$

where $I_0 = \mathbb{E}^{s,x}\left[\int_t^u \partial_t\phi(r, X_r)dr + \int_t^u A\phi(r, X_r)dr|\mathcal{F}_t\right]$; $I_1 = \mathbb{E}^{s,x}\left[\int_t^u (\partial_t\phi(r, X_r) - \partial_t\phi(r, X_t))dr|\mathcal{F}_t\right]$ $I_2 = \mathbb{E}^{s,x}\left[\int_t^u (A\phi(u, X_r) - A\phi(r, X_r))dr|\mathcal{F}_t\right]$. (B.2) will be established if one proves that $I_1 = I_2$. We do this below.
At fixed r and ω, $v \longmapsto A\phi(v, X_r(\omega))$ is $\mathcal{C}^1$, therefore $A\phi(u, X_r(\omega)) - A\phi(r, X_r(\omega)) = \int_r^u \partial_t A\phi(v, X_r(\omega))dv$ and $I_2 = \mathbb{E}^{s,x}\left[\int_t^u \int_r^u \partial_t A\phi(v, X_r)dvdr|\mathcal{F}_t\right]$. Then $I_1 = \mathbb{E}^{s,x}\left[\int_t^u \int_t^r A\partial_t\phi(r, X_v)dvdr|\mathcal{F}_t\right] + \mathbb{E}^{s,x}\left[\int_t^u (M^{s,x}[\partial_t\phi(r,\cdot)]_r - M^{s,x}[\partial_t\phi(r,\cdot)]_t)dr|\mathcal{F}_t\right]$.
Since $\partial_t\phi$ and $A\partial_t\phi$ are bounded, $M^{s,x}[\partial_t\phi(r,\cdot)]_r(\omega)$ is uniformly bounded in (r,ω), so by Fubini's theorem for conditional expectations we have

$$\begin{array}{rl} & \mathbb{E}^{s,x}[\int_t^u (M^{s,x}[\partial_t\phi(r,\cdot)]_r - M^{s,x}[\partial_t\phi(r,\cdot)]_t)dr|\mathcal{F}_t] \\ = & \int_t^u \mathbb{E}^{s,x}[M^{s,x}[\partial_t\phi(r,\cdot)]_r - M^{s,x}[\partial_t\phi(r,\cdot)]_t|\mathcal{F}_t]dr \\ = & 0. \end{array} \tag{B.3}$$

Finally since $\partial_t A\phi = A\partial_t\phi$ and again by Fubini's theorem for conditional expectations, we have $\mathbb{E}^{s,x}\left[\int_t^u \int_r^u \partial_t A\phi(v, X_r)dvdr|\mathcal{F}_t\right] = \mathbb{E}^{s,x}\left[\int_t^u \int_t^r A\partial_t\phi(r, X_v)dvdr|\mathcal{F}_t\right]$ so $I_1 = I_2$ which concludes the proof. ☐

In conclusion we can state the following.

Corollary B.8. *Given a homogeneous Markov class* $(\mathbb{P}^x)_{x \in E}$ *solving a well-posed homogeneous Martingale Problem associated to some* $(\mathcal{D}(A), A)$*, there exists a Markov class* $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times E}$ *which transition function is measurable in time and such that for any algebra* $\mathcal{A}$ *included in* $\mathcal{D}^{max}(\partial_t + A)$*,* $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times E}$ *solves the well-posed Martingale Problem associated to* $(\partial_t + A, \mathcal{A}, V_t \equiv t)$ *in the sense of Definition 2.3.*

References

[1] C. D. Aliprantis and K. C. Border. *Infinite-dimensional analysis*. Springer-Verlag, Berlin, second edition, 1999. A hitchhiker's guide.

[2] D. G. Aronson. Bounds for the fundamental solution of a parabolic equation. *Bull. Amer. Math. Soc.*, 73:890–896, 1967.

[3] D. Bakry, I. Gentil, and M. Ledoux. *Analysis and geometry of Markov diffusion operators*, volume 348. Springer Science & Business Media, 2013.

[4] V. Bally, E. Pardoux, and L. Stoica. Backward stochastic differential equations associated to a symmetric Markov process. *Potential Anal.*, 22(1):17–60, 2005.

[5] G. Barles, R. Buckdahn, and E. Pardoux. Backward stochastic differential equations and integral-partial differential equations. *Stochastics: An International Journal of Probability and Stochastic Processes*, 60(1-2):57–83, 1997.

[6] G. Barles and E. Lesigne. SDE, BSDE and PDE. In *Backward stochastic differential equations (Paris, 1995–1996)*, volume 364 of *Pitman Res. Notes Math. Ser.*, pages 47–80. Longman, Harlow, 1997.

[7] A. Barrasso and F. Russo. Backward Stochastic Differential Equations with no driving martingale, Markov processes and associated Pseudo Partial Differential Equations. 2017. Preprint, hal-01431559, v2.

[8] J.M. Bismut. Conjugate convex functions in optimal stochastic control. *J. Math. Anal. Appl.*, 44:384–404, 1973.

[9] G. Da Prato and J. Zabczyk. *Stochastic equations in infinite dimensions*, volume 152 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, second edition, 2014.

[10] F. Delarue and R. Diel. Rough paths and 1d SDE with a time dependent distributional drift: application to polymers. *Probab. Theory Related Fields*, 165(1-2):1–63, 2016.

[11] C. Dellacherie and P.-A. Meyer. *Probabilités et potentiel*, volume A. Hermann, Paris, 1975. Chapitres I à IV.

[12] C. Dellacherie and P.-A. Meyer. *Probabilités et potentiel. Chapitres XII–XVI*. Publications de l'Institut de Mathématiques de l'Université de Strasbourg [Publications of the Mathematical Institute of the University of Strasbourg], XIX. Hermann, Paris, second edition, 1987. Théorie des processus de Markov. [Theory of Markov processes].

[13] E. Di Nezza, G. Palatucci, and E. Valdinoci. Hitchhiker's guide to the fractional Sobolev spaces. *Bull. Sci. Math.*, 136(5):521–573, 2012.

[14] E. B. Dynkin. *Markov processes and related problems of analysis*, volume 54 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge-New York, 1982.

[15] N. El Karoui, S. Peng, and M. C. Quenez. Backward stochastic differential equations in finance. *Mathematical finance*, 7(1):1–71, 1997.

[16] S. N. Ethier and T. G. Kurtz. *Markov processes*. Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics. John Wiley & Sons, Inc., New York, 1986. Characterization and convergence.

[17] F. Flandoli, E. Issoglio, and F. Russo. Multidimensional stochastic differential equations with distributional drift. *Trans. Amer. Math. Soc.*, 369(3):1665–1688, 2017.

[18] F. Flandoli, F. Russo, and J. Wolf. Some SDEs with distributional drift. I. General calculus. *Osaka J. Math.*, 40(2):493–542, 2003.

[19] F. Flandoli, F. Russo, and J. Wolf. Some SDEs with distributional drift. II. Lyons-Zheng structure, Itô's formula and semimartingale characterization. *Random Oper. Stochastic Equations*, 12(2):145–184, 2004.

[20] M. Fuhrman and G. Tessitore. Nonlinear Kolmogorov equations in infinite dimensional spaces: the backward stochastic differential equations approach and applications to optimal control. *Ann. Probab.*, 30(3):1397–1465, 2002.

[21] M. Fukushima, Y. Oshima, and M. Takeda. *Dirichlet forms and symmetric Markov processes*, volume 19 of *de Gruyter Studies in Mathematics*. Walter de Gruyter & Co., Berlin, 1994.

[22] W. Hoh. Pseudo differential operators generating markov processes. *Habilitations-schrift, Universität Bielefeld*, 1998.

[23] E. P. Hsu. *Stochastic analysis on manifolds*, volume 38. American Mathematical Soc., 2002.

[24] E. Issoglio and S. Jing. Forward-Backward SDEs with distributional coefficients. preprint - ArXiv 2016 (arXiv:1605.01558).

[25] N. Jacob. *Pseudo differential operators and Markov processes. Vol. I*. Imperial College Press, London, 2001. Fourier analysis and semigroups.

[26] N. Jacob. *Pseudo differential operators & Markov processes. Vol. II*. Imperial College Press, London, 2002. Generators and their potential theory.

[27] N. Jacob. *Pseudo Differential Operators & Markov Processes: Markov Processes And Applications*, volume 3. Imperial College Press, 2005.

[28] J. Jacod. *Calcul stochastique et problèmes de martingales*, volume 714 of *Lecture Notes in Mathematics*. Springer, Berlin, 1979.

[29] J. Jacod and A. N. Shiryaev. *Limit theorems for stochastic processes*, volume 288 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, second edition, 2003.

[30] J. Jost. *Riemannian Geometry and Geometric Analysis.* Berlin, Heidelberg: Springer Berlin Heidelberg, 2011.

[31] T. Klimsiak. Semi-Dirichlet forms, Feynman-Kac functionals and the Cauchy problem for semilinear parabolic equations. *J. Funct. Anal.*, 268(5):1205–1240, 2015.

[32] G. Liang, T. Lyons, and Z. Qian. Backward stochastic dynamics on a filtered probability space. *Ann. Probab.*, 39(4):1422–1448, 2011.

[33] P.A. Meyer. *Séminaire de Probabilités, X.* Lecture Notes in Mathematics, Vol. 511. Springer-Verlag, Berlin-New York, 1976. Tenu à l'Université de Strasbourg, Strasbourg. Première partie (année universitaire 1974/1975). Seconde partie: Théorie des intégrales stochastiques (année universitaire 1974/1975). Exposés supplémentaires, Edité par P. A. Meyer.

[34] É. Pardoux and S. Peng. Adapted solution of a backward stochastic differential equation. *Systems Control Lett.*, 14(1):55–61, 1990.

[35] É. Pardoux and S. Peng. Backward stochastic differential equations and quasilinear parabolic partial differential equations. In *Stochastic partial differential equations and their applications (Charlotte, NC, 1991)*, volume 176 of *Lecture Notes in Control and Inform. Sci.*, pages 200–217. Springer, Berlin, 1992.

[36] E. Pardoux and A. Răşcanu. *Stochastic differential equations, backward SDEs, partial differential equations*, volume 69 of *Stochastic Modelling and Applied Probability.* Springer, Cham, 2014.

[37] S. Peng. Probabilistic interpretation for systems of quasilinear parabolic partial differential equations. *Stochastics Stochastics Rep.*, 37(1-2):61–74, 1991.

[38] J.P. Roth. Opérateurs dissipatifs et semi-groupes dans les espaces de fonctions continues. *Ann. Inst. Fourier (Grenoble)*, 26(4):ix, 1–97, 1976.

[39] F. Russo and G. Trutnau. Some parabolic PDEs whose drift is an irregular random noise in space. *Ann. Probab.*, 35(6):2213–2262, 2007.

[40] F. Russo and L. Wurzer. Elliptic PDEs with distributional drift and backward SDEs driven by a càdlàg martingale with random terminal time. *To appear in Stochastics and Dynamics.*, 2015. arXiv:1407.3218v2.

[41] D. W. Stroock. Diffusion processes associated with Lévy generators. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, 32(3):209–244, 1975.

[42] D. W. Stroock and S. R. S. Varadhan. *Multidimensional diffusion processes.* Classics in Mathematics. Springer-Verlag, Berlin, 2006. Reprint of the 1997 edition.