Fokker-Planck equations with terminal condition and related McKean probabilistic representation

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Abstract

Usually Fokker-Planck type partial differential equations (PDEs) are well-posed if the initial condition is specified. In this paper, alternatively, we consider the inverse problem which consists in prescribing final data: in particular we give sufficient conditions for uniqueness. In the second part of the paper we provide a probabilistic representation of those PDEs in the form of a solution of a McKean type equation corresponding to the time-reversal dynamics of a diffusion process.

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1 Introduction

The main objective of the paper consists in studying well-posedness and probabilistic representation of the Fokker-Planck PDE with terminal condition

$$\begin{cases} \partial_t \mathbf{u} &= \frac{1}{2}\sum_{i,j=1}^{d} \partial_{ij}^2 \left((\sigma\sigma^\top)_{i,j}(t,x)\mathbf{u}\right) - div\,(b(t,x)\mathbf{u}) \\ \mathbf{u}(T) &= \mu, \end{cases} \tag{1.1}$$

where $\sigma : [0,T] \times \mathbb{R}^d \to M_d(\mathbb{R})$, $b : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ and μ is a prescribed finite Borel (most often non-negative) measure on $\mathbb{R}^d$. When $\mathbf{u}(t)$ admits a density for some $t \in [0,T]$ we write $\mathbf{u}(t) = u(t,x)dx$. This equation is motivated by applications in various domains of physical sciences and engineering, as heat conduction [3], material science [23] or hydrology [2]. In particular, *hydraulic inversion* is interested in inverting a diffusion process representing the concentration of a pollutant to identify the pollution source location

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when the final concentration profile is observed. Those models give often rise to ill-posed problems because, either the solution is not unique or it is not stable. In this specific case, the existence is ensured by the fact that the observed contaminant is necessarily originated at a given time (as soon as the model is correct). Several authors have handled the lack of uniqueness by introducing regularization methods and approaching the problem using well-posed PDEs, see typically [28] and [17]. In particular for the PDE (1.1) there are very few results even concerning existence and uniqueness. The first objective of the paper is precisely to investigate uniqueness for (1.1).

The second objective is to propose a probabilistic representation of PDE (1.1). Our approach relies on the existence and uniqueness for that PDE. Although it is beyond the scope of this paper, it is important to emphasize the interests of probabilistic representation in possibly bringing new insights in stability analysis or numerical approximation of PDE (1.1). For instance, based on probabilistic representation of nonlinear PDEs [4, 15] have developed stochastic particle methods in the spirit of McKean to provide original Monte Carlo approximation schemes approaching several class of PDEs. For recent contributions in that direction, one can refer to [19, 18, 21, 20] and the survey paper [13]. In the same spirit, one may develop Monte Carlo approximation schemes for PDE (1.1) based on the probabilistic representation provided in the present paper, which will be the object of future works. Besides, the probabilistic representation of PDE (1.1) has already been exploited in [12], in the specific setting of Gaussian diffusions to propose an original approximation scheme for solving semi-linear PDEs with applications to stochastic control.

To realize the probabilistic representation of the PDE (1.1), when μ is non-negative, we consider the renormalized PDE

$$\left\{ \begin{array}{lcl} \partial_t \bar{\mathbf{u}} & = & \frac{1}{2} \sum_{i,j=1}^{d} \partial_{ij}^2 \left((\sigma\sigma^\top)_{i,j}(t,x) \bar{\mathbf{u}} \right) - div\left(b(t,x) \bar{\mathbf{u}} \right) \\ \bar{\mathbf{u}}(T) & = & \bar{\mu}, \end{array} \right. \tag{1.2}$$

where $\bar{\mu} = \frac{\mu}{\mu(\mathbb{R}^d)}$ is a probability measure. We remark that the PDEs (1.2) and (1.1) are equivalent in the sense that a solution to (1.2) (resp. (1.1)) provides a solution to the other one. The program consists in considering the McKean type stochastic differential equation (SDE)

$$\begin{cases} Y_t = Y_0 - \displaystyle\int_0^t b\left(T-r, Y_r\right) dr + \int_0^t \left\{ \frac{div_y \left(\Sigma_{i.} \left(T-r, Y_r\right) p_r\left(Y_r\right) \right)}{p_r\left(Y_r\right)} \right\}_{i \in [\![1,d]\!]} dr + \int_0^t \sigma\left(T-r, Y_r\right) d\beta_r, \\ p_t \text{ density of } \mathbf{p_t} = \text{law of } \mathrm{Y_t}, \mathrm{t} \in]0, \mathrm{T}[, \\ Y_0 \sim \bar{\mu}, \end{cases} \tag{1.3}$$

where β is a d-dimensional Brownian motion and $\Sigma = \sigma\sigma^\top$, whose solution is the couple $(Y, \mathbf{p})$. Indeed an application of Itô formula (see Proposition 4.3) shows that whenever $(Y, \mathbf{p})$ is a solution of the SDE (1.3) then $t \mapsto \mathbf{p}_{T-t}$ is a solution of (1.2).

The idea of considering (1.3) comes from the SDE verified by time-reversal of a diffusion. Time-reversal of Markov processes was explored by several authors: see for instance [11] for the diffusion case in finite dimension, [9] for the diffusion case in infinite dimension and [14] for the jump case. We also mention the two very interesting recent preprints [6, 7] in relation with entropy.

Consider a *forward* diffusion process X solution of

$$X_t = X_0 + \int_0^t b(s, X_s) ds + \int_0^t \sigma(s, X_s) dW_s, \ t \in [0, T], \tag{1.4}$$

where σ and b are Lipschitz coefficients with linear growth and W is a standard Brownian motion on $\mathbb{R}^d$. X

is a probabilistic representation of

$$\begin{cases} \partial_t \mathbf{u} & = & \frac{1}{2}\sum_{i,j=1}^{d} \partial_{ij}^2 \left((\sigma\sigma^\top)_{i,j}(t,x)\mathbf{u}\right) - div\left(b(t,x)\mathbf{u}\right) \\ \mathbf{u}(0) & = & \nu, \end{cases} \tag{1.5}$$

where $X_0 \sim \nu$. Indeed, whenever X is a solution of the SDE (1.4) then the function $t \mapsto \mathbf{u}(t)$, where $\mathbf{u}(t)$ is the law of X_t is a solution in the sense of distributions of the PDE (1.5). We remark also that $t \mapsto \mathbf{u}(t)$ solves the PDE (1.1), μ being the law of X_T. Let us now denote $\hat{X}_t := X_{T-t}, t \in [0,T]$ the time-reversal process of the solution X of (1.4). In [11] the authors gave sufficient general conditions on σ, b and on the marginal laws p_t of X_t so that $Y := \hat{X}$ is a solution (in law) of the SDE

$$Y_t = X_T - \int_0^t b\left(T-r, Y_r\right) dr + \int_0^t \left\{ \frac{div_y\left(\Sigma_{i.}\left(T-r, Y_r\right) p_{T-r}\left(Y_r\right)\right)}{p_{T-r}\left(Y_r\right)} \right\}_{i \in [\![1,d]\!]} dr + \int_0^t \sigma\left(T-r, Y_r\right) d\beta_r. \tag{1.6}$$

This constitutes an essential tool that we will exploit to prove existence of the McKean SDE (1.3).

As far as uniqueness for (1.3) is concerned, we repeat that the key idea relies on uniqueness for the PDE (1.2) (or (1.1)). First of all Proposition 4.3, states the following. If $(Y, \mathbf{p})$ is a solution of (1.3), then $\mathbf{p}\left(T - \cdot\right)$ is a solution of the PDE (1.1), with $\mu = \mathbf{p}(0)$. This fact justifies the terminology that (1.3) constitutes a probabilistic representation of (1.1). Now, if the PDE (1.1) admits at most one solution then $\mathbf{p}$ is completely identified, so (1.3) reduces to an ordinary SDE for which uniqueness in law (resp. pathwise) can be established whenever the coefficients are shown to be locally bounded (resp. locally Lipschitz).

As we have mentioned earlier, there are not many articles analyzing uniqueness for Fokker-Planck PDEs with terminal condition. For introductory purposes, we present two simple situations when this problem can be easily tackled: one by analytical means and one by probabilistic techniques.

a) The heat equation with terminal condition admits uniqueness. Suppose indeed that $u : [0,T] \mapsto \mathcal{S}'\left(\mathbb{R}^d\right)$ solves

$$\begin{cases} \partial_t \mathbf{u} = \Delta \mathbf{u} \\ \mathbf{u}\left(T\right) = \mu. \end{cases} \tag{1.7}$$

Then, the Fourier transform of u, $v\left(t,\cdot\right) := \mathcal{F}\mathbf{u}\left(t,\cdot\right), t \in [0,T]$ solves the ODE (for fixed $\xi \in \mathbb{R}^d$)

$$\begin{cases} \frac{d}{dt} v\left(t,\xi\right) = -\left|\xi\right|^2 v\left(t,\xi\right), \left(t,\xi\right) \in [0,T] \times \mathbb{R}^d \\ v\left(T,\cdot\right) = \mathcal{F}\mu. \end{cases} \tag{1.8}$$

This admits at most one solution, since setting $\mathcal{F}\mu = 0$ the unique solution of (1.8) is the null function.

b) Another relatively simple situation is described below to study uniqueness among the solutions of the PDE (1.1) whose initial value belongs to the class of Dirac measures. Consider the example when σ is continuous bounded non-degenerate and the drift b is affine i.e. $b(s,y) = b_0\left(s\right) + b_1\left(s\right) y, (s,y) \in [0,T] \times \mathbb{R}^d$, b_0 (resp. b_1) being mappings from $[0,T]$ to $\mathbb{R}^d$ (resp. to $M_d\left(\mathbb{R}\right)$). Suppose for a moment that the PDE in the first line of (1.1), but with initial condition (see (3.2)) is well-posed. Sufficient conditions for this will be provided in Remark 3.3.

Let $x \in \mathbb{R}^d$ and u be a solution of the PDE (1.1) such that $u(0,\cdot) = \delta_x$. If X^x is the solution of (1.4) with initial condition x, it is well-known that the family of laws of $X_t^x, t \in [0,T]$, is a solution of (1.1). So this coincides with $u(t,\cdot)$ and in particular μ is the law of X_T^x. To conclude we only need to determine

x. Taking the expectation in the SDE fulfilled by X^x, we show that the function $t \mapsto E^x(t) := \mathbb{E}(X^x_t)$ is solution of

$$E^x(t) = \int_{\mathbb{R}^d} y\mu(dy) - \int_t^T (b_0(s) + b_1(s)E^x(s))\, ds.$$

Previous linear ODE has clearly a unique solution. At this point $x = E(0)$ is uniquely determined.

Those examples give a flavor of how to tackle the uniqueness issue for the PDE (1.1). However, generalizing those approaches is far more complicated and constitutes the first part of the present work. The contributions of the paper are twofold.

1. We investigate uniqueness for the Fokker-Planck PDE with terminal condition (1.1). This is done in Section 3 in two different situations: the case when the coefficients are bounded and the situation of a PDE associated with an inhomogeneous Ornstein-Uhlenbeck (OU) semigroup. In Section 3.2 we show uniqueness for bounded continuous coefficients when solutions start in the class $\mathcal{C}$ of multiples of Dirac measures. In Proposition 3.9 we discuss dimension $d = 1$. Theorem 3.10 is devoted to the case $d \geq 2$. We distinguish the non-degenerate case from the possibly degenerate case but with smooth coefficients proving uniqueness for small time horizon T. In Section 3.3 we show uniqueness when the coefficients are stepwise time-homogeneous. In Theorem 3.13 the coefficients are time-homogeneous, bounded and Hölder, with non-degenerate diffusion. Corollary 3.16 extends previous results to the case of stepwise time-inhomogeneous coefficients. In Section 3.4, Theorem 3.19 treats the Ornstein-Uhlenbeck case.

2. We study existence and uniqueness in law for the McKean SDE (1.3), with some specific remarks concerning strong existence and pathwise uniqueness. After some preliminary considerations in Section 4.1, Proposition 4.10 and Theorem 4.12 discuss the case of bounded coefficients. Theorem 4.15 is devoted to the case of Ornstein-Uhlenbeck (with not necessarily Gaussian terminal condition), where strong existence and pathwise uniqueness are established.

2 Notations and preliminaries

Let us fix $d \in \mathbb{N}^*$, $T > 0$. $\mathcal{C}_c^\infty\left(\mathbb{R}^d\right)$ is the linear space of smooth functions with compact support. For a given $p \in \mathbb{N}^*$, $[\![1, p]\!]$ denotes the set of all integers between 1 and p included. $M_d(\mathbb{R})$ stands for the set of $d \times d$ matrices. $\langle , \rangle$ denotes the usual scalar product on $\mathbb{R}^d$, with associated norm $|.|$. For a given $A \in M_d(\mathbb{R})$, $Tr\,(A)$ (resp. $A^\top$) symbolizes the trace (resp. the transpose) of the matrix A. $||A||$ denotes the usual Frobenius norm. We also introduce the function Jf from $\mathbb{R}^p$ to $M_d(\mathbb{R})$ such that $Jf : z \mapsto \left(\partial_j f^i(z)\right)_{(i,j)\in[\![1,d]\!]\times[\![1,d]\!]}$.

Let $\alpha \in]0, 1[, n \in \mathbb{N}$. $\mathcal{C}_b(\mathbb{R}^d)$ (resp. $\mathcal{C}_b^n(\mathbb{R}^d)$) indicates the space of bounded continuous functions (resp. bounded functions of class $\mathcal{C}^n$ such that all the derivatives are bounded). $\mathcal{C}^\alpha(\mathbb{R}^d), 0 < \alpha < 1$, is the Banach space of bounded α-Hölder functions $\mathbb{R}^d \to \mathbb{R}$ equipped with the norm $|.|_\alpha := ||.||_\infty + [.]_\alpha$, where

$$[f]_\alpha := \sup_{x,y\in\mathbb{R}^d, x\neq y} \frac{|f(x) - f(y)|}{|x - y|^\alpha} < \infty$$

and $\|\cdot\|_\infty$ is the sup-norm. If n is some integer $\mathcal{C}^{\alpha+n}(\mathbb{R}^d)$ is the Banach space of bounded functions $f : \mathbb{R}^d \to \mathbb{R}$ such that all its derivatives up to order n are bounded and such that the derivatives of order n are α-Hölder continuous. This is equipped with the norm obtained as the sum of the $C_b^n(\mathbb{R}^d)$-norm plus the sum of the quantities $[g]_\alpha$ where g is an n-order derivative of f. For more details, see Section 0.2 of [22]. If E is a linear Banach space, we denote by $||.||_E$ the associated operator norm and by $\mathcal{L}(E)$ the space of linear bounded operators $E \to E$. Often in the sequel we will have $E = \mathcal{C}^{2\alpha}(\mathbb{R}^d)$.

$\mathcal{P}\left(\mathbb{R}^d\right)$ (resp. $\mathcal{M}_+\left(\mathbb{R}^d\right), \mathcal{M}_f\left(\mathbb{R}^d\right)$) denotes the set of probability (resp. non-negative finite valued, finite signed) measures on $\left(\mathbb{R}^d, \mathcal{B}\left(\mathbb{R}^d\right)\right)$. We also denote by $\mathcal{S}\left(\mathbb{R}^d\right)$ the space of Schwartz functions and by $\mathcal{S}'\left(\mathbb{R}^d\right)$ the space of tempered distributions. For all $\phi \in \mathcal{S}\left(\mathbb{R}^d\right)$ and $\mu \in \mathcal{M}_f\left(\mathbb{R}^d\right)$, we set the notations

$$\mathcal{F}\phi : \xi \mapsto \int_{\mathbb{R}^d} e^{-i\langle \xi, x\rangle} \phi(x)\, dx,\ \mathcal{F}\mu : \xi \mapsto \int_{\mathbb{R}^d} e^{-i\langle \xi, x\rangle} \mu(dx).$$

Given a mapping $\mathbf{u} : [0,T] \to \mathcal{M}_f\left(\mathbb{R}^d\right)$, we convene that when for $t \in [0,T]$, $\mathbf{u}(t)$ has a density, this is denoted by $u(t,\cdot)$. Recalling $\Sigma = \sigma\sigma^\top$, let us introduce, for a given t in $[0,T]$, the differential operator,

$$L_t f := \frac{1}{2}\sum_{i,j=1}^{d} \Sigma_{ij}(t,\cdot)\partial_{ij} f + \sum_{i=1}^{d} b_i(t,\cdot)\,\partial_i f, \tag{2.1}$$

$f \in C^2(\mathbb{R}^d)$ and denote by L_t^* its formal adjoint, which means that for a given signed measure η

$$L_t^* \eta := \frac{1}{2}\sum_{i,j=1}^{d} \partial_{ij}^2 \left(\Sigma_{i,j}(t,x)\eta\right) - div\left(b(t,x)\eta\right)\ . \tag{2.2}$$

With this notation, the PDE (1.1) rewrites

$$\begin{cases} \partial_t \mathbf{u} = L_t^* \mathbf{u} \\ \mathbf{u}(T) = \mu. \end{cases} \tag{2.3}$$

In the sequel we will often make use of the following assumptions.

Assumption 1. *b, σ are Lipschitz in space uniformly in time, with linear growth.*

Assumption 2. *b and Σ are bounded.*

Assumption 3. *Σ is continuous.*

Assumption 4. *There exists $\epsilon > 0$ such that for all $t \in [0,T]$, $\xi \in \mathbb{R}^d$, $x \in \mathbb{R}^d$*

$$\langle \Sigma(t,x)\xi, \xi\rangle \geq \epsilon\, |\xi|^2. \tag{2.4}$$

For a given random variable X on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, $\mathcal{L}_{\mathbb{P}}(X)$ denotes its law under $\mathbb{P}$ and $\mathbb{E}_{\mathbb{P}}(X)$ its expectation under $\mathbb{P}$. When self-explanatory, the subscript will be omitted in the sequel.

3 A Fokker-Planck PDE with terminal condition

3.1 Preliminary results on uniqueness

In this section, we consider a Fokker-Planck type PDE with terminal condition for which the notion of solution is clarified in the following definition.

Definition 3.1. *Fix $\mu \in \mathcal{M}_f\left(\mathbb{R}^d\right)$. We say that a mapping $\mathbf{u}$ from $[0,T]$ to $\mathcal{M}_f\left(\mathbb{R}^d\right)$ solves the PDE* (1.1)*, if for all $\phi \in \mathcal{C}_c^\infty\left(\mathbb{R}^d\right)$ and all $t \in [0,T]$*

$$\int_{\mathbb{R}^d} \phi(y)\,\mathbf{u}(t)(dy) = \int_{\mathbb{R}^d} \phi(y)\,\mu(dy) - \int_t^T \int_{\mathbb{R}^d} L_s\phi(y)\,\mathbf{u}(s)(dy)\,ds. \tag{3.1}$$

We consider the following property related to a given class $\mathcal{C} \subseteq \mathcal{M}_+\left(\mathbb{R}^d\right)$. Later we will establish uniqueness results for (1.1) provided that the solution starts in $\mathcal{C}$.

Property 1. *For all $\nu \in \mathcal{C}$, the PDE*

$$\begin{cases} \partial_t \mathbf{u} = L_t^* \mathbf{u} \\ \mathbf{u}(0) = \nu \end{cases} \tag{3.2}$$

admits at most one solution $\mathbf{u} : [0,T] \to \mathcal{M}_+\left(\mathbb{R}^d\right)$.

We recall that, for a given $\nu \in \mathcal{M}_f\left(\mathbb{R}^d\right)$, $\mathbf{u} : [0,T] \to \mathcal{M}_f\left(\mathbb{R}^d\right)$ is a solution of the PDE (3.2) if for all $\phi \in \mathcal{C}_c^\infty\left(\mathbb{R}^d\right)$ and all $t \in [0,T]$,

$$\int_{\mathbb{R}^d} \phi(y)\, \mathbf{u}(t)(dy) = \int_{\mathbb{R}^d} \phi(y)\, \nu(dy) + \int_0^t \int_{\mathbb{R}^d} L_s \phi(y)\, \mathbf{u}(s)(dy)\, ds. \tag{3.3}$$

Suppose there is an $\mathcal{M}_+\left(\mathbb{R}^d\right)$-valued solution $\mathbf{u}$ of (3.2) such that $\mathbf{u}(0) \in \mathcal{C}$ for some class $\mathcal{C}$. We also suppose that Property 1 holds with respect to $\mathcal{C}$. Then this unique solution will be denoted by $\mathbf{u}^\nu$ in the sequel. We remark that, whenever Property 1 holds with respect to a given $\mathcal{C} \subseteq \mathcal{P}\left(\mathbb{R}^d\right)$, then the PDE (3.2) admits at most one $\mathcal{M}_+\left(\mathbb{R}^d\right)$-valued solution with any initial value belonging to $\mathbb{R}_+^* \mathcal{C} := (\alpha\nu)_{\alpha>0, \nu\in\mathcal{C}}$.

We start with a simple but fundamental observation.

Proposition 3.2. *Let us suppose σ, b to be locally bounded, ν be a Borel probability on $\mathbb{R}^d$, $\alpha > 0$, ξ be a r.v. distributed according to ν. Suppose that there is a solution X of SDE*

$$X_t = \xi + \int_0^t b(r, X_r)\, dr + \int_0^t \sigma(r, X_r)\, dW_r,\ t \in [0,T],\ \mathbb{P}\text{--a.s.}, \tag{3.4}$$

where W is a d-dimensional standard Brownian motion. Then the $\mathcal{M}_+\left(\mathbb{R}^d\right)$-valued function $t \mapsto \alpha\mathcal{L}(X_t)$ is a solution of the PDE (3.2) *with initial value $\alpha\nu$.*

Proof. One first applies Itô formula to $\varphi(X_t)$, where φ is a smooth function with compact support and then one takes the expectation. □

Remark 3.3.

1. *Suppose that the coefficients b, Σ are bounded. Property 1 holds with respect to $\mathcal{C} := \mathcal{M}_+\left(\mathbb{R}^d\right)$ as soon as the martingale problem associated with b, Σ admits uniqueness for all initial condition of the type $\delta_x, x \in \mathbb{R}^d$. Indeed, this is a consequence of Lemma 2.3 in [8].*

2. *Suppose b and σ with linear growth. Let $\nu \in \mathcal{M}_+\left(\mathbb{R}^d\right)$ not trivially null (resp. $\nu \in \mathcal{P}\left(\mathbb{R}^d\right)$). By Proposition 3.2, the existence of an $\mathcal{M}_+\left(\mathbb{R}^d\right)$-valued (resp. $\mathcal{P}\left(\mathbb{R}^d\right)$-valued) solution for the PDE* (3.2) *(even on $t \geq 0$) is ensured when the martingale problem associated to b and Σ admits existence (and consequently when the SDE* (3.4) *admits weak existence) with initial condition $\frac{\nu}{\|\nu\|}$. We remark that, for example, this happens when the coefficients b, σ are continuous with linear growth: see Theorem 12.2.3 in [27] for the case of bounded coefficients, the unbounded case can be easily obtained by truncation.*

3. *The martingale problem associated to b and Σ is well-posed for all deterministic initial condition, for instance in the following cases.*

    - *When Σ, b have linear growth and Σ is continuous and non-degenerate (i.e. Assumptions 2 and 4 hold), see [27] Corollary 7.1.7 and Theorem 10.2.2.*

- *Suppose $d = 1$ and σ is bounded. When σ is lower bounded by a positive constant on each compact set, see [27], Exercise 7.3.3.*
- *When $d = 2$, Σ is non-degenerate and σ and b are time-homogeneous and bounded, see [27], Exercise 7.3.4.*
- *When σ, b are Lipschitz with linear growth (with respect to the space variable); in this case one obtains even strong solutions of the corresponding stochastic differential equation.*

The lemma below provides in particular sufficient conditions for the validity of Property 1.

Lemma 3.4. 1. *Let $\nu \in \mathcal{P}(\mathbb{R}^d)$. We suppose Assumptions 2, 3 and 4. Then there is a unique $\mathcal{M}_+(\mathbb{R}^d)$-valued solution $\mathbf{u}$ to the PDE* (3.2) *with $\mathbf{u}(0) = \nu$. Moreover $\mathbf{u}^\nu$ takes values in $\mathcal{P}(\mathbb{R}^d)$. In particular Property 1 related to the class $\mathcal{C} = \mathcal{P}(\mathbb{R}^d)$ is verified.*

2. *Under Assumptions 1 and 2, Property 1 is fulfilled for $\mathcal{C} = \mathcal{M}_+(\mathbb{R}^d)$.*

Proof.

1. Existence follows by items 2. and 3. of Remark 3.3. Uniqueness is a consequence of items 1. and 3. of the same Remark.

2. Since b and σ are Lipschitz, Property 1 is fulfilled, see items 1. and 3. of Remark 3.3.

□

In Propositions 3.5 and 3.6 below we give two equivalent formulations for uniqueness of PDE (1.1).

Proposition 3.5. *Suppose Property 1 holds with respect to a given $\mathcal{C} \subseteq \mathcal{M}_+(\mathbb{R}^d)$. Suppose that for all $\nu \in \mathcal{C}$ there exists an $\mathcal{M}_+(\mathbb{R}^d)$-valued solution of* (3.2) *with initial value ν. Then, the following properties are equivalent.*

1. *The mapping from $\mathcal{C}$ to $\mathcal{M}_+(\mathbb{R}^d)$ $\nu \mapsto \mathbf{u}^\nu(T)$ is injective.*

2. *For all $\mu \in \mathcal{M}_+(\mathbb{R}^d)$, the PDE* (1.1) *with terminal value μ admits at most a solution in the sense of Definition 3.1, among all $\mathcal{M}_+(\mathbb{R}^d)$-valued solutions starting in the class $\mathcal{C}$.*

Proof. Concerning the converse implication, suppose that uniqueness holds for equation (1.1) in the sense of Definition 3.1, among non-negative measure-valued solutions starting in the class $\mathcal{C}$. Consider $\nu, \nu' \in \mathcal{C}$ such that $\mathbf{u}^\nu(T) = \mathbf{u}^{\nu'}(T)$. We remark that $\mathbf{u}^\nu, \mathbf{u}^{\nu'}$ are such solutions of PDE (1.1) with same terminal condition. Uniqueness gives $\mathbf{u}^\nu = \mathbf{u}^{\nu'}$ and in particular $\nu = \nu'$ and the injectivity stated in item 1. holds.

Concerning the direct implication, consider $\mathbf{u}^1, \mathbf{u}^2$ two non-negative measure-valued solutions of equation (1.1) in the sense of Definition 3.1, with the same terminal value in $\mathcal{M}_+(\mathbb{R}^d)$, such that $\mathbf{u}^i(0), i \in \{1,2\}$, belong to $\mathcal{C}$ and suppose that $\nu \mapsto \mathbf{u}^\nu(T)$ is injective from $\mathcal{C}$ to $\mathcal{M}_+(\mathbb{R}^d)$. Setting $\nu^i := \mathbf{u}^i(0)$, we remark that for a given $i \in \{1,2\}$ we have

$$\begin{cases} \partial_t \mathbf{u}^i = L_t^* \mathbf{u}^i \\ \mathbf{u}^i(0) = \nu_i, \end{cases} \tag{3.5}$$

in the sense of equation (3.3). Then, the fact $\mathbf{u}^1(T) = \mathbf{u}^2(T)$ gives $\mathbf{u}^{\nu_1}(T) = \mathbf{u}^{\nu_2}(T)$. By injectivity $\nu_1 = \nu_2$ and the statement 2. follows by Property 1. □

Proceeding in the same way as for the proof of Proposition 3.5, for the case of signed measures, we obtain the following.

Proposition 3.6. *Suppose that for all $\nu \in \mathcal{M}_f\left(\mathbb{R}^d\right)$, there exists a unique solution $\mathbf{u}^\nu : [0,T] \to \mathcal{M}_f\left(\mathbb{R}^d\right)$ of the PDE* (3.2) *with initial value ν. Then, the following properties are equivalent.*

1. *The function $\nu \mapsto \mathbf{u}^\nu(T)$ is injective.*

2. *For all $\mu \in \mathcal{M}_f(\mathbb{R}^d)$, the PDE* (1.1) *with terminal value μ admits at most a solution in the sense of Definition 3.1.*

Remark 3.7. 1. *Suppose that the coefficients Σ, b are bounded. Then, any measure-valued solution $\mathbf{u} : [0,T] \to \mathcal{M}_+(\mathbb{R}^d)$ of the PDE* (3.2) *such that $\mathbf{u}(0) \in \mathcal{P}(\mathbb{R}^d)$ takes values in $\mathcal{P}(\mathbb{R}^d)$. Indeed, this can be shown approaching the function $\varphi \equiv 1$ from below by smooth functions with compact support.*

2. *Replacing $\mathcal{M}_+(\mathbb{R}^d)$ with $\mathcal{P}(\mathbb{R}^d)$ in Property 1, item 2. in Proposition 3.5 can be stated also replacing $\mathcal{M}_+(\mathbb{R}^d)$ with $\mathcal{P}(\mathbb{R}^d)$.*

3.2 Uniqueness: the case of Dirac initial conditions

In this section we will make use of a probabilistic technique for discussing uniqueness of the PDE (1.1) among $\mathcal{M}_+(\mathbb{R}^d)$-valued solutions starting in $\mathcal{C} := (\alpha\delta_x)_{\alpha>0, x\in\mathbb{R}^d}$. We will make use of a probabilistic technique. Given a solution $\mathbf{u}$ of (1.1), we associate a process X being a solution of the SDE (1.4) whose (marginal) law is $\mathbf{u}(\mathbf{t})$. The idea consists in identifying uniquely the law of X_0. That approach only works with multiple Dirac initial conditions.

Remark 3.8. *Let $\alpha \geq 0$ and $x \in \mathbb{R}^d$. Suppose that there is a solution X^x of SDE* (3.4) *with $\xi = x$.*

1. *By Proposition 3.2, the $\mathcal{M}_+\left(\mathbb{R}^d\right)$-valued mapping $t \mapsto \alpha\mathcal{L}\left(X_t^x\right)$ is a solution of the PDE* (3.2) *with initial value $\alpha\delta_x$.*

2. *Under Property 1 (with respect to $\mathcal{C}$), $t \mapsto \alpha\mathcal{L}\left(X_t^x\right)$ can be identified with $\mathbf{u}^{\alpha\delta_x}$ and in particular*

$$\int_{\mathbb{R}^d} \mathbf{u}^{\alpha\delta_x}(t)(dy) = \alpha, \ \forall t \in [0,T].$$

In the sequel, whenever Assumption 1 holds, X^x denotes the unique solution of the SDE (3.4) with initial value $x \in \mathbb{R}^d$.

We start with the case of dimension $d = m = 1$.

Proposition 3.9. *(***Uniqueness: Dirac initial conditions, one-dimensional case***).*

We set $\mathcal{C} = (\alpha\delta_x)_{\alpha>0, x\in\mathbb{R}}$. Suppose the validity of Assumption 1 with $d = m = 1$. We moreover suppose the validity of one of the two hypotheses below.

1. *Assumption 2.*

2. *Property 1 holds with respect to $\mathcal{C}$.*

Then, for all $\mu \in \mathcal{M}_+(\mathbb{R})$, the PDE (1.1) *with terminal value μ admits at most one solution in the sense of Definition 3.1 among the $\mathcal{M}_+(\mathbb{R})$-valued solutions starting in $\mathcal{C}$.*

Proof. By Lemma 3.4 item 2. Property 1 is fulfilled with respect to $\mathcal{C}$.
Fix $(x,y) \in \mathbb{R}^2$ and $\alpha, \beta \geq 0$ such that

$$\mathbf{u}^{\alpha\delta_x}(T) = \mathbf{u}^{\beta\delta_y}(T). \tag{3.6}$$

Thanks to Proposition 3.5, to conclude, it suffices to show that $\alpha = \beta$ and $x = y$. By item 2. of Remark 3.8, we have $\alpha = \beta$ and consequently $\mathcal{L}(X_T^x) = \mathcal{L}(X_T^y)$. In particular $\mathbb{E}(X_T^x) = \mathbb{E}(X_T^y)$. Since b, σ are Lipschitz in space, they have bounded derivatives in the sense of distributions that we denote by $\partial_x b$ and $\partial_x \sigma$.

Set $Z^{x,y} := X^y - X^x$. We have

$$Z_t^{x,y} = (y-x) + \int_0^t b_s^{x,y} Z_s^{x,y} ds + \int_0^t \sigma_s^{x,y} Z_s^{x,y} dW_s, \forall t \in [0,T], \tag{3.7}$$

where for a given $s \in [0,T]$

$$b_s^{x,y} = \int_0^1 \partial_x b\left(s, aX_s^y + (1-a)X_s^x\right) da,\ \sigma_s^{x,y} = \int_0^1 \partial_x \sigma\left(s, aX_s^y + (1-a)X_s^x\right) da.$$

The unique solution of (3.7) is well-known to be

$$Z^{x,y} = \exp\left(\int_0^\cdot b_s^{x,y} ds\right) \mathcal{E}\left(\int_0^\cdot \sigma_s^{x,y} dW_s\right)(y-x),$$

where $\mathcal{E}(\cdot)$ denotes the Doléans exponential. Finally, we have

$$\mathbb{E}\left(\exp\left(\int_0^T b_s^{x,y} ds\right) \mathcal{E}\left(\int_0^\cdot \sigma_s^{x,y} dW_s\right)_T\right)(y-x) = 0.$$

Since the quantity appearing in the expectation is strictly positive, we conclude $x = y$. □

We continue now with a discussion concerning the multidimensional case $d \geq 2$. The uniqueness result below only holds when the time-horizon is small enough. Theorem 3.10 distinguishes two cases: the first one with regular, possibly degenerate, coefficients, the second one with non-degenerate, possibly irregular, coefficients. Later, in Section 3.3, we will present in a framework of piecewise time-homogeneous coefficients results which are valid for any time-horizon.

Theorem 3.10. *(***Uniqueness: Dirac initial conditions, multi-dimensional case***).*
We set $\mathcal{C} = (\alpha\delta_x)_{\alpha>0, x\in\mathbb{R}^d}$. We suppose the validity of either item (a) or (b) below.

(a) Assumptions 1 and Property 1 (for instance if Assumption 2 holds) with respect to $\mathcal{C}$.

(b) Assumptions 2, 3 and 4.

There is $T > 0$ small enough such that the following holds. For all $\mu \in \mathcal{M}_+(\mathbb{R}^d)$, the PDE (1.1) *admits at most one solution in the sense of Definition 3.1 among the $\mathcal{M}_+(\mathbb{R}^d)$-valued solutions starting in $\mathcal{C}$.*

The proof of Theorem 3.10 in case (a) relies on a basic lemma of moments estimates.

Lemma 3.11. *We suppose Assumption 1. Let $(x,y) \in \mathbb{R}^d \times \mathbb{R}^d$. Then, $\sup_{t\in[0,T]} \mathbb{E}\left(|X_t^x - X_t^y|^2\right) \leq |y-x|^2 e^{KT}$, with $K := 2K^b + \sum_{j=1}^d \left(K^{\sigma,j}\right)^2$, where*

$$K^b := \sup_{s\in[0,T]} ||\, ||Jb(s,\cdot)||\, ||_\infty$$

and for all $j \in [\![1,d]\!]$

$$K^{\sigma,j} := \sup_{s\in[0,T]} ||\, ||J\sigma_{.j}(s,\cdot)||\, ||_\infty\,,$$

where $\|\cdot\|$ stands for the sup-norm.

Proof (of Theorem 3.10).

Taking into account Property 1 we fix $(x_1, x_2) \in \mathbb{R}^d \times \mathbb{R}^d, \alpha, \beta \geq 0$ such that

$$\mathbf{u}^{\alpha\delta_{x_1}}(T) = \mathbf{u}^{\beta\delta_{x_2}}(T). \tag{3.8}$$

To conclude, by Proposition 3.5, it suffices again to show $\alpha = \beta$ and $x_1 = x_2$.

1. We write the proof in the case (a), in particular under Assumption 1. Once again, item 2. of Remark 3.8 gives $\alpha = \beta$ and

$$\mathbb{E}\left(X_T^{x_1}\right) = \mathbb{E}\left(X_T^{x_2}\right). \tag{3.9}$$

   Adopting the same notations as in the proof of Lemma 3.11, a similar argument as in (5.12), together with (5.10) (in the Appendix) allows to show that the local martingale part of $Z^{x_1,x_2} = X^{x_2} - X^{x_1}$ defined in (5.8) is a true martingale. So, taking the expectation in (5.12) with $x = x_1, y = x_2$, by Lemma 3.11 we obtain

$$\begin{aligned} |\mathbb{E}\left(X_T^{x_2} - X_T^{x_1}\right) - (x_2 - x_1)| &\leq K_b \int_0^T \mathbb{E}|X_r^{x_2} - X_r^{x_1}|dr \\ &\leq K_b \int_0^T \sqrt{\mathbb{E}\left(|X_r^{x_2} - X_r^{x_1}|\right)^2} dr \\ &\leq \frac{K}{2} T e^{\frac{K}{2}T} |x_2 - x_1|. \end{aligned}$$

   Remembering (3.9), this implies

$$\left(1 - \frac{K}{2} T e^{\frac{K}{2}T}\right) |x_2 - x_1| \leq 0.$$

   Taking T such that $\frac{K}{2}T < M$ with $Me^M < 1$, we have $1 - \frac{K}{2}Te^{\frac{K}{2}T} > 0$, which implies $|x_2 - x_1| = 0$.

2. We discuss the case (b), i.e. we suppose Assumptions 2, 3, and 4. Firstly, point 1. of Theorem 1. in [29] ensures the existence of probability spaces $\left(\Omega^i, \mathcal{F}^i, \mathbb{P}^i\right),\ i \in \{1,2\}$ on which are defined respectively two m-dimensional Brownian motions W^1, W^2 and two processes X^1, X^2 such that

$$X_t^i = x_i + \int_0^t b\left(s, X_s^i\right) ds + \int_0^t \sigma\left(s, X_s^i\right) dW_s^i,\ \mathbb{P}^i\text{–a.s., t} \in [0, \text{T}].$$

   Again item 2. of Remark 3.8 implies $\alpha_1 = \alpha_2$ and

$$\mathcal{L}_{\mathbb{P}^1}\left(X_T^1\right) = \mathcal{L}_{\mathbb{P}^2}\left(X_T^2\right). \tag{3.10}$$

   Secondly, point b. of Theorem 3 in [29] shows that for every given bounded $D \subset \mathbb{R}^d$, for all $\phi : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ belonging to $W_p^{1,2}([0,T] \times D)$ (see Definition of that space in [29]) for a given $p > d + 2$, for all $t \in [0,T], i \in \{1,2\}$, we have

$$\phi\left(t, X_t^i\right) = \phi\left(0, x_i\right) + \int_0^t \left(\partial_t + L_s\right) \phi\left(s, X_s^i\right) ds + \int_0^t J\phi\left(s, X_s^i\right) \sigma\left(s, X_s^i\right) dW_s^i,\ \mathbb{P}^i\text{–a.s.} \tag{3.11}$$

   where the application of $\partial_t + L_t, t \in [0,T]$ has to be understood componentwise.

Thirdly, Theorem 2. in [29] shows that if T is sufficiently small, then the system of d PDEs

$$\forall (t,x) \in [0,T] \times \mathbb{R}^d, \begin{cases} \partial_t \phi(t,x) + L_t \phi(t,x) = 0, \\ \phi(T,x) = x, \end{cases} \tag{3.12}$$

admits a solution ϕ in $W_p^{1,2}([0,T] \times D)$ for all $p > 1$ and all bounded $D \subset \mathbb{R}^d$. Moreover the partial derivatives of ϕ in space are bounded (in particular $J\phi$ is bounded) and $\phi(t,\cdot)$ is injective for all $t \in [0,T]$.

Combining now (3.12) with identity (3.11), we observe that $\phi\left(., X^i\right), i \in \{1,2\}$, are local martingales. Using additionally the fact that $J\phi$ and σ are bounded, it is easy to show that they are true martingales. Taking the expectation in (3.11) with respect to $\mathbb{P}^i, i = 1,2$, gives

$$\phi(0,x_i) = \mathbb{E}_{\mathbb{P}^i}\left(\phi\left(T, X_T^i\right)\right), i \in \{1,2\}.$$

In parallel, identity (3.10) gives

$$\mathbb{E}_{\mathbb{P}^1}\left(\phi\left(T, X_T^1\right)\right) = \mathbb{E}_{\mathbb{P}^2}\left(\phi\left(T, X_T^2\right)\right).$$

So, $\phi(0,x_1) = \phi(0,x_2)$. We conclude that $x_1 = x_2$ since $\phi(0,\cdot)$ is injective.

□

3.3 Uniqueness: the case of bounded non-degenerate coefficients

In this section we consider the case of (possibly piecewise) time-homogeneous coefficients in dimension $d \geq 1$. We make use of an analytic technique based on semigroups which requires bounded coefficients (Assumption 2), non-degeneracy (Assumption 4) and an additional Hölder regularity assumption of the coefficients.

We start with the time-homogeneouse case stating the following.

Assumption 5. *1. b, Σ are time-homogeneous.*

2. For all $(i,j) \in [\![1,d]\!]^2$, $b_i, \Sigma_{ij} \in \mathcal{C}^{2\alpha}\left(\mathbb{R}^d\right)$, for a given $\alpha \in]0, \frac{1}{2}[$.

We refer to the differential operator L_t introduced in (2.1) and we simply set here $L \equiv L_t$.

Remark 3.12. *Suppose the validity of Assumptions 2, 4, 5.*

1. *Let $T > 0$. Proposition 4.2 in [8] implies that, for every $\nu \in \mathcal{M}_f\left(\mathbb{R}^d\right)$, there exists a unique $\mathcal{M}_f\left(\mathbb{R}^d\right)$-valued solution of the PDE* (3.2) *with initial value ν, which will be again denoted by $\mathbf{u}^\nu$. We notice in particular that Property 1 holds.*

   In the sequel T will be omitted.

2. *We remark that the uniqueness result mentioned in item 1. is unknown in the case of general bounded coefficients. In the general framework, only a uniqueness result for non-negative solutions is available, see Remark 3.3 point 1.*

3. *Since L is time-homogeneous, taking into account Assumptions 4, 5, operating a shift, uniqueness for the PDE* (3.2) *also holds replacing the initial time 0 by any other initial time, for every initial value in $\mathcal{M}_f\left(\mathbb{R}^d\right)$, with any other maturity T.*

It is significant to remark that the uniqueness theorem below holds in the class finite signed measures valued functions.

Theorem 3.13. *(***Uniqueness: the case of non-degenerate time-homogeneous coefficients***).*

Suppose the validity of Assumptions 2, 4 and 5. Then, for all $\mu \in \mathcal{M}_f\left(\mathbb{R}^d\right)$, the PDE (1.1) *with terminal value μ admits at most one $\mathcal{M}_f\left(\mathbb{R}^d\right)$-valued solution in the sense of Definition 3.1.*

By Theorems 3.1.12, 3.1.14 and Corollary 3.1.16 in [22] the differential operator L suitably extends as a map $\mathcal{D}(L) = \mathcal{C}^{2\alpha+2}(\mathbb{R}^d) \subset \mathcal{C}^{2\alpha}(\mathbb{R}^d) \mapsto \mathcal{C}^{2\alpha}\left(\mathbb{R}^d\right)$ and that extension is sectorial, see Definition 2.0.1 in [22]. We set $E := \mathcal{C}^{2\alpha}\left(\mathbb{R}^d\right)$. By the considerations below that Definition, in (2.0.2) and (2.0.3) therein, one defines $P_t := e^{tL}, P_t : E \to E, t \geq 0$. By Proposition 2.1.1 in [22], $(P_t)_{t\geq 0}$ is a semigroup and $t \mapsto P_t$ is analytical on $]0, +\infty[$ with values in $\mathcal{L}(E)$, with respect to $||.||_E$.

Before proving the theorem, we provide two lemmata.

Lemma 3.14. *Suppose the validity of Assumptions 2, 4 and 5. Then, for all $\phi \in E$ and all $\nu \in \mathcal{M}_f\left(\mathbb{R}^d\right)$, the function from $\mathbb{R}_+^*$ to $\mathbb{R}$*

$$t \mapsto \int_{\mathbb{R}^d} P_t\phi(x)\,\nu(dx)$$

is analytic.

Proof. The result can be easily established using the fact that $\phi \mapsto P_t\phi$ with values in $\mathcal{L}(E)$ is analytic and the fact that the map $\psi \mapsto \int_{\mathbb{R}^d} \psi(x)\nu(dx)$ is linear and bounded.

□

Lemma 3.15. *Suppose the validity of Assumptions 2, 4 and 5. Let $T > 0$. Then for all $\nu \in \mathcal{M}_f\left(\mathbb{R}^d\right)$, $t \in [0, T]$ and $\phi \in E$ we have the identity*

$$\int_{\mathbb{R}^d} P_t\phi(x)\,\nu(dx) = \int_{\mathbb{R}^d} \phi(x)\,\mathbf{u}^\nu(t)(dx), \tag{3.13}$$

where $\mathbf{u}^\nu$ was defined in point 1. of Remark 3.12.

Proof. Let $\nu \in \mathcal{M}_f\left(\mathbb{R}^d\right)$. We denote by $\mathbf{v}^\nu$ the mapping from $[0, T]$ to $\mathcal{M}_f\left(\mathbb{R}^d\right)$ such that $\forall t \in [0, T]$, $\forall \phi \in E$

$$\int_{\mathbb{R}^d} \phi(x)\mathbf{v}^\nu(t)(dx) = \int_{\mathbb{R}^d} P_t\phi(x)\nu(dx). \tag{3.14}$$

Previous expression defines the measure $\mathbf{v}^\nu(t, \cdot)$ since $\phi \mapsto \int_{\mathbb{R}^d} P_t\phi(x)\nu(dx)$ is continuous with respect to the sup-norm, using $\|P_t\phi\|_\infty \leq \|\phi\|_\infty$, and Lebesgue's dominated convergence theorem. By approximating the elements of E with elements of $\mathcal{C}_c^\infty\left(\mathbb{R}^d\right)$, it will be enough to prove (3.13) for $\phi \in \mathcal{C}_c^\infty\left(\mathbb{R}^d\right)$.

Our idea is to show that $\mathbf{v}^\nu$ is an $\mathcal{M}_f\left(\mathbb{R}^d\right)$-valued solution of (3.2) with initial value ν, so that $\mathbf{v}^\nu = \mathbf{u}^\nu$ via point 1. of Remark 3.12. This will prove (3.13) for $\phi \in \mathcal{C}_c^\infty\left(\mathbb{R}^d\right)$. Let $t \in [0, T]$ and $\phi \in \mathcal{C}_c^\infty\left(\mathbb{R}^d\right)$. On the one hand, point (i) of Proposition 2.1.1 in [22] gives

$$LP_t\phi = P_tL\phi, \tag{3.15}$$

since $\mathcal{C}_c^\infty\left(\mathbb{R}^d\right) \subset \mathcal{D}(L) = \mathcal{C}^{2\alpha+2}\left(\mathbb{R}^d, \mathbb{R}\right)$. On the other hand, for all $s \in [0, t]$, we have

$$\begin{aligned} |LP_s\phi|_E &= |P_sL\phi|_{2\alpha} \\ &\leq ||P_s||_E\,|L\phi|_E \\ &\leq M_0e^{\omega s}\,|L\phi|_E\,, \end{aligned}$$

with M_0, ω the real parameters appearing in Definition 2.0.1 in [22] and using point (iii) of Proposition 2.1.1 in the same reference. Then the mapping $s \mapsto LP_s\phi$ belongs obviously to $L^1([0,t];E)$ and point (ii) of Proposition 2.1.4 in [22] combined with identity (3.15) gives

$$P_t\phi = \phi + \int_0^t P_s L\phi ds.$$

Back to our main goal, using in particular Fubini's theorem, we have

$$\begin{aligned}\int_{\mathbb{R}^d} P_t\phi(x)\,\nu(dx) &= \int_{\mathbb{R}^d} \phi(x)\,\nu(dx) + \int_{\mathbb{R}^d}\int_0^t P_s L\phi(x)\,ds\nu(dx) \\ &= \int_{\mathbb{R}^d} \phi(x)\,\nu(dx) + \int_0^t\int_{\mathbb{R}^d} P_s L\phi(x)\,\nu(dx)\,ds \\ &= \int_{\mathbb{R}^d} \phi(x)\,\nu(dx) + \int_0^t\int_{\mathbb{R}^d} L\phi(x)\,\mathbf{v}^\nu(s)(dx)\,ds.\end{aligned}$$

This shows that $\mathbf{v}^\nu$ is a solution of the PDE (3.2). □

Proof (of Theorem 3.13).
Let $\nu, \nu' \in \mathcal{M}_f(\mathbb{R}^d)$ such that

$$\mu_T := \mathbf{u}^\nu(T) = \mathbf{u}^{\nu'}(T).$$

Thanks to Proposition 3.6, it suffices to show that $\nu = \nu'$ i.e.

$$\forall \phi \in \mathcal{C}_c^\infty(\mathbb{R}^d),\ \int_{\mathbb{R}^d} \phi(x)\,\nu(dx) = \int_{\mathbb{R}^d} \phi(x)\,\nu'(dx).$$

Since $T > 0$ is arbitrary, by Remark 3.12 we can consider $\mathbf{u}^{\nu,2T}$ and $\mathbf{u}^{\nu',2T}$, defined as the corresponding $\mathbf{u}^\nu$ and $\mathbf{u}^{\nu'}$ functions obtained replacing the horizon T with $2T$. They are defined on $[0,2T]$ and by Remark 3.12 1. (uniqueness on $[0,T]$), they constitute extensions of the initial $\mathbf{u}^\nu$ and $\mathbf{u}^{\nu'}$.
By Remark 3.12 3., the uniqueness of an $\mathcal{M}_f(\mathbb{R}^d)$-valued solution of the PDE (3.2) (for $t \in [T,2T]$, with T as initial time) holds for

$$\begin{cases}\partial_t\mathbf{u}(\tau) = L^*\mathbf{u}(\tau),\ T \le \tau \le 2T \\ \mathbf{u}(T) = \mu_T.\end{cases} \tag{3.16}$$

Now, the functions $\mathbf{u}^{\nu,2T}$ and $\mathbf{u}^{\nu',2T}$ solve (3.16) on $[T,2T]$. This gives in particular

$$\forall \tau \ge T,\ \forall \phi \in \mathcal{C}_c^\infty(\mathbb{R}^d),\ \int_{\mathbb{R}^d} \phi(x)\,\mathbf{u}^{\nu,2T}(\tau)(dx) = \int_{\mathbb{R}^d} \phi(x)\,\mathbf{u}^{\nu',2T}(\tau)(dx). \tag{3.17}$$

Fix $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^d)$. Combining now the results of Lemmata 3.14 and 3.15, we obtain that the function

$$\tau \mapsto \int_{\mathbb{R}^d} \phi(x)\,\mathbf{u}^{\nu,2T}(\tau)(dx) - \int_{\mathbb{R}^d} \phi(x)\,\mathbf{u}^{\nu',2T}(\tau)(dx), \tag{3.18}$$

defined on $[0,2T]$, is zero on $[T,2T]$ and analytic on $]0,2T]$. Hence it is zero on $]0,2T]$. By (3.13) we obtain

$$\int_{\mathbb{R}^d} P_\tau\phi(x)\,(\nu - \nu')\,(dx) = 0,\ \forall \tau \in]0,2T]. \tag{3.19}$$

Separating ν and ν' in positive and negative components, we can finally apply dominated convergence theorem in (3.18) to send τ to $0+$. This is possible thanks to points (i) of Proposition 2.1.4 and (iii) of Proposition 2.1.1 in [22] together with the representation (3.13). Indeed $P_\tau\phi(x) \to \phi(x)$ for every $\phi \in E, x \in \mathbb{R}^d$ when $\tau \to 0+$. This shows $\nu = \nu'$ and ends the proof.

□

For the sake of applications it is useful to formulate a piecewise time-homogeneous version of Theorem 3.13.

Corollary 3.16. *(***Uniqueness: the case of non-degenerate piecewise time-homogeneous coefficients***).*
Let $n \in \mathbb{N}^*$. Let $0 = t_0 < \ldots < t_n = T$ be a partition. For $k \in [\![2, n]\!]$ (resp. $k = 1$) we denote $I_k =]t_{k-1}, t_k]$ (resp. $[t_0, t_1]$). Suppose that the following holds.

1. *For all $k \in [\![1, n]\!]$, the restriction of σ (resp. b) to $I_k \times \mathbb{R}^d$ is a time-homogeneous function $\sigma^k : \mathbb{R}^d \to M_d(\mathbb{R})$ (resp. $b^k : \mathbb{R}^d \to \mathbb{R}^d$).*

2. *Assumption 4.*

3. *Assumptions 2 and 5 are verified for each σ^k, b^k and Σ^k, where we have set $\Sigma^k := \sigma^k {\sigma^k}^\top$.*

Then, for all $\mu \in \mathcal{M}_f\left(\mathbb{R}^d\right)$, the PDE (1.1) *with terminal value μ admits at most one $\mathcal{M}_f\left(\mathbb{R}^d\right)$-valued solution in the sense of Definition 3.1.*

Proof. For each given $k \in [\![1, n]\!]$, we introduce the PDE operator L^k defined by

$$L^k := \frac{1}{2}\sum_{i,j=1}^{d} \Sigma^k_{ij}\partial_{ij} + \sum_{i=1}^{d} b^k_i \partial_i. \tag{3.20}$$

Let now $\mathbf{u}^1, \mathbf{u}^2$ be two solutions of (1.1) with same terminal value μ.
The measure-valued functions $\mathbf{v}^i := \mathbf{u}^i\left(\cdot + t_{n-1}\right), i \in \{1,2\}$ defined on $[0, T - t_{n-1}]$ are solutions of

$$\begin{cases} \partial_t \mathbf{v} = \left(L^n\right)^* \mathbf{v} \\ \mathbf{v}\left(T - t_{n-1}, \cdot\right) = \mu, \end{cases} \tag{3.21}$$

in the sense of Definition 3.1 replacing T by $T - t_{n-1}$ and L by L^n. Then, Theorem 3.13 gives $\mathbf{v}^1 = \mathbf{v}^2$ and consequently $\mathbf{u}^1 = \mathbf{u}^2$ on $[t_{n-1}, T]$. To conclude, we proceed by backward induction.

□

3.4 Uniqueness: the case of Ornstein-Uhlenbeck semigroup

In this section, we consider the case $b := (s,x) \mapsto C(s)x$ with C continuous from $[0,T]$ to $M_d\left(\mathbb{R}\right)$ and σ continuous from $[0,T]$ to $M_d\left(\mathbb{R}\right)$. Here we perform an analytic approach based on Fourier analysis.

We recall that $\Sigma := \sigma\sigma^\top$. In that setting, the classical Fokker-Planck PDE (1.5) for finite measures reads

$$\begin{cases} \partial_t \mathbf{u}\left(t\right) = \frac{1}{2}\sum_{i,j=1}^{d} \Sigma(t)_{ij}\partial_{ij}\mathbf{u}\left(t\right) - \sum_{i=1}^{d}\partial_i\left(\left(C(t)x\right)_i \mathbf{u}\left(t\right)\right) \\ \mathbf{u}(0) = \nu \in \mathcal{M}_f\left(\mathbb{R}^d\right). \end{cases} \tag{3.22}$$

In the sequel we will denote by $\mathcal{D}\left(t\right),\ t \in [0,T]$, the unique solution of

$$\mathcal{D}(t) = I - \int_0^t C(s)^\top \mathcal{D}(s) ds,\ t \in [0,T]. \tag{3.23}$$

We recall that for every $t \in [0,T]$, $\mathcal{D}(t)$ is invertible and

$$\mathcal{D}^{-1}(t) = I + \int_0^t \mathcal{D}^{-1}(s) C(s)^\top ds,\ t \in [0,T]. \tag{3.24}$$

For previous and similar properties, see Chapter 8 of [5].

Proposition 3.17. *For all $\nu \in \mathcal{M}_f\left(\mathbb{R}^d\right)$, the PDE (3.22) with initial value ν admits at most one $\mathcal{M}_f\left(\mathbb{R}^d\right)$-valued solution. In particular Property 1 holds for $\mathcal{C} = \mathcal{M}_+\left(\mathbb{R}^d\right)$.*

Proof.

1. Let $\nu \in \mathcal{M}_f\left(\mathbb{R}^d\right)$ and $\mathbf{u}$ be a solution of the PDE (3.2) with initial value ν. Identity (3.3) can be extended to $\mathcal{S}\left(\mathbb{R}^d\right)$ since for all $t \in [0,T]$, $\mathbf{u}(t)$ belongs to $\mathcal{M}_f\left(\mathbb{R}^d\right)$. Then, $t \mapsto \mathcal{F}\mathbf{u}(t)$ verifies

$$\mathcal{F}\mathbf{u}(t)(\xi) = \mathcal{F}\nu(\xi) + \int_0^t \left\langle C(s)^\top \xi, \nabla \mathcal{F}\mathbf{u}(s) \right\rangle ds - \frac{1}{2}\int_0^t \left\langle \Sigma(s)\xi, \xi \right\rangle \mathcal{F}\mathbf{u}(s)\, ds,\ (t,\xi) \in [0,T] \times \mathbb{R}^d. \quad (3.25)$$

In fact, the integrand inside the first integral has to be understood as a Schwartz distribution: in particular the symbol ∇ is understood in the sense of distributions and for each given $s \in [0,T]$, $\left\langle C(s)^\top \xi, \nabla \mathcal{F}\mathbf{u}(s) \right\rangle$ denotes the tempered distribution

$$\varphi \mapsto \sum_{i=1}^d \partial_i \mathcal{F}\mathbf{u}(s) \left(\xi \mapsto \left(C(s)^\top \xi\right)_i \varphi(\xi)\right).$$

Indeed, even though for any t, $\mathcal{F}\mathbf{u}(t)$ is a function, the equation (3.25) has to be understood in $\mathcal{S}'\left(\mathbb{R}^d\right)$. Hence, for all $\phi \in \mathcal{S}\left(\mathbb{R}^d\right)$, this gives

$$\begin{aligned}
&\int_{\mathbb{R}^d} \phi(\xi)\, \mathcal{F}\mathbf{u}(t)(\xi)\, d\xi - \int_{\mathbb{R}^d} \phi(\xi)\, \mathcal{F}\nu(\xi)\, \phi(\xi) d\xi && (3.26)\\
&= -i \sum_{k,l=1}^d \int_0^t C(s)_{kl} \int_{\mathbb{R}^d} \xi_l \mathcal{F}\phi_k(\xi)\, \mathbf{u}(s)(d\xi)\, ds - \frac{1}{2}\int_0^t \int_{\mathbb{R}^d} \left\langle \Sigma(s)\xi, \xi \right\rangle \mathcal{F}\mathbf{u}(s)(\xi)\, \phi(\xi) d\xi ds\\
&= -\sum_{k,l=1}^d \int_0^t C(s)_{kl} \int_{\mathbb{R}^d} \mathcal{F}\left(\partial_l \phi_k\right)(\xi)\, \mathbf{u}(s)(d\xi)\, ds - \frac{1}{2}\int_0^t \int_{\mathbb{R}^d} \left\langle \Sigma(s)\xi, \xi \right\rangle \mathcal{F}\mathbf{u}(s)(\xi)\, d\xi ds\\
&= -\int_0^t \int_{\mathbb{R}^d} \left(div_\xi \left(C(s)^\top \xi \phi(\xi)\right) + \frac{1}{2}\left\langle \Sigma(s)\xi, \xi \right\rangle \phi(\xi) \right) \mathcal{F}\mathbf{u}(s)(\xi) d\xi ds,
\end{aligned}$$

where $\phi_k : \xi \mapsto \xi_k \phi(\xi)$ for a given $k \in [\![1, d]\!]$.

2. Let now $\mathbf{v} : [0,T] \to \mathcal{M}_f\left(\mathbb{R}^d\right)$ be defined by

$$\int_{\mathbb{R}^d} \phi(x)\, \mathbf{v}(t)(dx) = \int_{\mathbb{R}^d} \phi\left(\mathcal{D}(t)^\top x\right) \mathbf{u}(t)(dx), \quad (3.27)$$

$t \in [0,T], \phi \in \mathcal{C}_b(\mathbb{R}^d)$. For every $\xi \in \mathbb{R}^d$, we set $\phi(x) = \exp(-i\langle \xi, x\rangle)$ in (3.27) to obtain

$$\mathcal{F}\mathbf{v}(t)(\xi) = \mathcal{F}\mathbf{u}(t)\left(\mathcal{D}(t)\xi\right), \quad (3.28)$$

for all $\xi \in \mathbb{R}^d$, for all $t \in [0,T]$.

3. We want now to show that, for each ξ, $t \mapsto \mathcal{F}\mathbf{v}(t)$ fulfills an ODE. To achieve this, suppose for a moment that $(t,\xi) \mapsto \mathcal{F}\mathbf{u}(t)(\xi)$ is differentiable with respect to the variable ξ. Then, on the one hand, for all $(t,\xi) \in [0,T] \times \mathbb{R}^d$, we have

$$\mathcal{F}\mathbf{u}(t)(\xi) = \mathcal{F}\nu(\xi) + \int_0^t \left\langle C(s)^\top \xi, \nabla_\xi \mathcal{F}\mathbf{u}(s)(\xi) \right\rangle ds - \frac{1}{2}\int_0^t \left\langle \Sigma(s)\xi, \xi \right\rangle \mathcal{F}\mathbf{u}(s)(\xi)\, ds, \quad (3.29)$$

thanks to identity (3.25). This means in particular that, for each given $\xi \in \mathbb{R}^d$, $t \mapsto \mathcal{F}\mathbf{u}(t)(\xi)$ is differentiable almost everywhere on $[0,T]$.

On the other hand, for almost every $t \in [0,T]$ and all $\xi \in \mathbb{R}^d$, we have

$$\begin{aligned}\partial_t \mathcal{F}\mathbf{v}(t)(\xi) &= \partial_t \mathcal{F}\mathbf{u}(t)(\mathcal{D}(t)\xi) + \sum_{i=1}^{d} \left(\frac{d}{dt}(\mathcal{D}(t)\xi)\right)_i \partial_i \mathcal{F}\mathbf{u}(t)(\mathcal{D}(t)\xi), \\ &= \partial_t \mathcal{F}\mathbf{u}(t)(\mathcal{D}(t)\xi) - \sum_{i=1}^{d} \left(C(t)^\top \mathcal{D}(t)\xi\right)_i \partial_i \mathcal{F}\mathbf{u}(t)(\mathcal{D}(t)\xi), \\ &= -\frac{1}{2}\langle \Sigma(t)\mathcal{D}(t)\xi, \mathcal{D}(t)\xi\rangle \mathcal{F}\mathbf{v}(t)(\xi), \end{aligned} \tag{3.30}$$

where from line 1 to line 2, we have used the fact $\frac{d}{dt}(\mathcal{D}(t)\xi) = -C(t)^\top \mathcal{D}(t)\xi$ for all $(t,\xi) \in [0,T] \times \mathbb{R}^d$ and from line 2 to line 3, the identity (3.29). Since $t \mapsto \mathcal{F}\mathbf{v}(t)(\xi)$ is absolutely continuous by (3.28), (3.30) implies

$$\mathcal{F}\mathbf{v}(t)(\xi) = \mathcal{F}\nu(\xi) - \frac{1}{2}\int_0^t \langle \Sigma(s)\mathcal{D}(s)\xi, \mathcal{D}(s)\xi\rangle \mathcal{F}\mathbf{v}(s)(\xi)\, ds, \xi \in \mathbb{R}^d, \tag{3.31}$$

for all $t \in [0,T]$.

4. Now, if $(t,\xi) \mapsto \mathcal{F}\mathbf{u}(t)(\xi)$ is not necessarily differentiable in the variable ξ, we will be able to prove (3.31) still holds by making use of calculus in the sense of distributions.

5. Suppose that (3.31) holds. This gives

$$\mathcal{F}\mathbf{u}(t)(\xi) = e^{-\int_0^t \frac{\left|\sigma(s)^\top \xi\right|^2}{2} ds} \mathcal{F}\nu\left(\mathcal{D}^{-1}(t)\xi\right) \tag{3.32}$$

and so $\mathbf{u}$ is completely determined.

6. The proof is now concluded after we have established (3.31). Since both sides of it are continuous in (t,ξ), it will be enough to show the equality as $\mathcal{S}'(\mathbb{R}^d)$-valued. This can be done differentiating (3.25), considered as an equality in $\mathcal{S}'(\mathbb{R}^d)$. For this we will apply Lemma 3.18 below setting $\Phi := \mathcal{F}\mathbf{u}(t)$ for every fixed $t \in [0,T]$ and differentiating in time. We set $\Phi_t(\xi) = \mathcal{F}\mathbf{v}(t)(\xi),\ \xi \in \mathbb{R}^d$ and $\Phi_t(\varphi) = \int_{\mathbb{R}^d} \varphi(\xi)\Phi_t(\xi)\mathrm{d}\xi, \varphi \in \mathcal{S}(\mathbb{R}^d)$. (3.31) follows from Lemma 3.18 below remarking that Φ_t is compatible with the one defined in (3.33).

□

Lemma 3.18. *Let $\Phi \in \mathcal{S}'\left(\mathbb{R}^d\right), t \in [0,T]$. We denote by Φ_t the element of $\mathcal{S}'\left(\mathbb{R}^d\right)$ such that for all $\varphi \in \mathcal{S}\left(\mathbb{R}^d\right)$*

$$\Phi_t(\varphi) := \det\left(\mathcal{D}^{-1}(t)\right)\Phi\left(\varphi\left(\mathcal{D}^{-1}(t)\cdot\right)\right). \tag{3.33}$$

Then, for all $t \in [0,T]$

$$\Phi_t(\varphi) = \Phi(\varphi) - \sum_{i=1}^{d}\int_0^t (\partial_i \Phi)_s \left(x \mapsto \left(C(s)^\top \mathcal{D}(s)x\right)_i \varphi(x)\right) ds. \tag{3.34}$$

Proof. We begin with the case $\Phi \in \mathcal{S}\left(\mathbb{R}^d\right)$ (or only $\mathcal{C}^\infty\left(\mathbb{R}^d\right)$). In this case,

$$\Phi_t(x) = \Phi(\mathcal{D}(t)x),\ x \in \mathbb{R}^d, t \in [0,T].$$

Hence, for every $t \in [0,T]$

$$\frac{d}{dt}\Phi_t(x) = \left\langle \frac{d}{dt}(\mathcal{D}(t)x), \nabla\Phi(\mathcal{D}(t)x)\right\rangle$$
$$= -\left\langle C(t)^\top \mathcal{D}(t)x, \nabla\Phi(\mathcal{D}(t)x)\right\rangle$$
$$= -\sum_{i=1}^d \left(C(t)^\top \mathcal{D}(t)x\right)_i (\partial_i\Phi)_t(x),$$

Now, coming back to the general case, let $\Phi \in \mathcal{S}'(\mathbb{R}^d)$ and $(\phi_\epsilon)_{\epsilon>0}$ be a sequence of mollifiers in $\mathcal{S}(\mathbb{R}^d)$, converging to the Dirac measure. Then for all $\epsilon > 0$, the function $\Phi * \phi_\epsilon : x \mapsto \Phi(\phi_\epsilon(x - \cdot))$ belongs to $\mathcal{S}'(\mathbb{R}^d) \cap \mathcal{C}^\infty(\mathbb{R}^d)$. By the first part of the proof, (3.34) holds replacing Φ with $\Phi * \varphi_\varepsilon$. Now, this converges to Φ in $\mathcal{S}'(\mathbb{R}^d)$ when ϵ tends to $0+$. (3.34) follows sending ϵ to $0+$. Indeed, for all $\varphi \in \mathcal{S}(\mathbb{R}^d)$, $t \in [0,T]$, setting $\check{\phi}_\epsilon : y \mapsto \phi_\epsilon(-y)$, we have

$$\Phi_t(\varphi) = \lim_{\epsilon\to 0+}\int_{\mathbb{R}^d}\varphi(x)\,(\Phi*\phi_\epsilon)_t(x)\,dx$$
$$= \lim_{\epsilon\to 0+}\int_{\mathbb{R}^d}\varphi(x)\Phi*\phi_\epsilon(x)\,dx - \lim_{\epsilon\to 0+}\sum_{i=1}^d\int_0^t \det\left(\mathcal{D}^{-1}(s)\right)\int_{\mathbb{R}^d}\left(C(s)^\top x\right)_i\varphi\left(\mathcal{D}^{-1}(s)x\right)\partial_i\Phi*\phi_\epsilon(x)dxds$$
$$= \Phi(\varphi) - \lim_{\epsilon\to 0+}\sum_{i=1}^d\int_0^t \det\left(\mathcal{D}^{-1}(s)\right)\partial_i\Phi\left(\left(\left(C(s)^\top\cdot\right)_i\varphi\left(\mathcal{D}^{-1}(s)\cdot\right)\right)*\check{\phi}_\epsilon\right)ds$$
$$= \Phi(\varphi) - \sum_{i=1}^d\int_0^t \det\left(\mathcal{D}^{-1}(s)\right)\partial_i\Phi\left(\left(C(s)^\top\cdot\right)_i\varphi\left(\mathcal{D}^{-1}(s)\cdot\right)\right)ds$$
$$= \Phi(\varphi) - \sum_{i=1}^d\int_0^t (\partial_i\Phi)_s\left(x\mapsto\left(C(s)^\top\mathcal{D}(s)x\right)_i\varphi(x)\right)ds.$$

To conclude, it remains to justify the commutation between the limit in ϵ and the integral in time from line 3 to line 4 using Lebesgue's dominated convergence theorem. On the one hand, for a given $i \in [\![1,d]\!]$, the fact $\partial_i\Phi$ belongs to $\mathcal{S}'(\mathbb{R}^d)$ implies that there exists $C>0$, $N\in\mathbb{N}$ such that for all $\varphi\in\mathcal{S}(\mathbb{R}^d)$

$$|\partial_i\Phi(\varphi)| \le C\sup_{|\alpha|\le N}\sup_{x\in\mathbb{R}^d}\left(1+|x|^2\right)^N|\partial_x^\alpha\varphi(x)|,$$

see Chapter 1, Exercise 8 in [26]. On the other hand, the quantities

$$\sup_{x\in\mathbb{R}^d}\left(1+|x|^2\right)^N\left|\partial_x^\alpha\left(x_j\varphi(\mathcal{D}^{-1}(s)\cdot)\right)*\check{\phi}_\epsilon\right|$$

are bounded uniformly in the couple (s,ϵ), for all $j\in[\![1,d]\!]$, $\alpha\in\mathbb{N}^d$, taking also into account that the function $s\mapsto\mathcal{D}^{-1}(s)$ is continuous and therefore bounded. Since C is also continuous on $[0,T]$, we are justified to use Lebesgue's dominated convergence theorem.

□

We state now the main result of this section.

Theorem 3.19. *(***Uniqueness: the case of OU semigroup***).*

For all $\mu\in\mathcal{M}_f(\mathbb{R}^d)$, the PDE (1.1) with terminal value μ admits at most one $\mathcal{M}_f(\mathbb{R}^d)$-valued solution in the sense of Definition 3.1.

Proof. Let $\mu \in \mathcal{M}_f\left(\mathbb{R}^d\right)$ and $\mathbf{u}$ be a solution of (1.1) with terminal value μ. Then, $\mathbf{u}$ solves the PDE (3.2) with initial value $\mathbf{u}(0)$. As a consequence, by (3.32) appearing at the end of the proof of Proposition 3.17, for all $\xi \in \mathbb{R}^d$,

$$\mathcal{F}\mu(\xi) = e^{-\int_0^T \frac{\left|\sigma(s)^\top \xi\right|^2}{2} ds} \mathcal{F}\mathbf{u}(0)\left(\mathcal{D}^{-1}(T)\xi\right),$$

so that

$$\mathcal{F}\mathbf{u}(0)(\xi) = e^{\int_0^T \frac{\left|\sigma(s)^\top \xi\right|^2}{2} ds} \mathcal{F}\mu(\mathcal{D}(T)\xi).$$

Hence, $\mathbf{u}(0)$ is entirely determined by μ and Proposition 3.17 gives the result. □

4 McKean SDE related to the PDE with terminal condition

In this section, we concentrate on the analysis of the well-posedness of the McKean SDE (1.3) that we relate to the PDE (1.1). The existence results for the SDE (1.3) will be based on two pillars: the reachability condition constituted by the existence of a solution of the Fokker-Planck PDE with terminal condition and the time-reversal techniques of [11]. They follow from general statements of Section 5.1, in the Appendix. The uniqueness results for the SDE (1.3) will be a consequence of results stated in Section 5.2.

4.1 Preliminary considerations

Regarding $b : [0,T] \times \mathbb{R}^d \mapsto \mathbb{R}^d$, $\sigma : [0,T] \times \mathbb{R}^d \mapsto M_d(\mathbb{R})$, we set $\widehat{b} := b(T - .,\cdot)$, $\widehat{\sigma} := \sigma(T - .,\cdot)$, $\widehat{\Sigma} := \widehat{\sigma}^\top \widehat{\sigma}$. Given a probability-valued function $\mathbf{p} : [0,T] \to \mathcal{P}(\mathbb{R}^d)$, we denote by p_t the density of $\mathbf{p}(t)$, for $t \in [0,T]$, whenever it exists. In this section μ will denote the terminal condition of the PDE (1.1) supposed to be a probability. For the McKean type SDE (1.3), remarking that $\mu = \bar{\mu}$, we consider the following notion of solution.

Definition 4.1. *On a given filtered probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P}\right)$ equipped with an d-dimensional $(\mathcal{F}_t)_{t\in[0,T]}$-Brownian motion β, a* **solution** *of the SDE* (1.3) *is a couple $(Y, \mathbf{p})$ fulfilling* (1.3)*, such that Y is $(\mathcal{F}_t)_{t\in[0,T]}$-adapted and such that for all $i \in [\![1,d]\!]$, all compact $K \subset \mathbb{R}^d$, all $\tau < T$*

$$\int_0^\tau \int_K \left| div_y\left(\widehat{\Sigma}_{i.}(r,y)\, p_r(y)\right)\right| dy dr < \infty. \tag{4.1}$$

Remark 4.2. *For a given solution $(Y, \mathbf{p})$ of equation* (1.3)*, identity* (4.1) *appearing in Definition 4.1 implies in particular that, for all $i \in [\![1,d]\!]$, all $\tau < T$*

$$\mathbb{E}\left(\int_0^\tau \left| \frac{div_y\left(\widehat{\Sigma}_{i.}(r,Y_r)\, p_r(Y_r)\right)}{p_r(Y_r)} \right| dr\right) < \infty.$$

The terminology stating that the SDE (1.3) constitutes a probabilistic representation of the PDE (1.1) is justified by the result below.

Proposition 4.3. *Suppose b, σ locally bounded. If $(Y, \mathbf{p})$ is a solution of* (1.3) *in the sense of Definition 4.1, then $\mathbf{p}(T - \cdot)$ is a solution of* (1.1)*, with $\mu = \mathbf{p}(0)$ in the sense of Definition 3.1.*

Proof. Let $(Y, \mathbf{p})$ be a solution of (1.3) in the sense of Definition 4.1 with a Brownian motion symbolized by β. Let $\phi \in \mathcal{C}_c^\infty\left(\mathbb{R}^d\right)$ and $t \in]0, T]$. Itô's formula gives

$$\phi\left(Y_{T-t}\right) = \phi\left(Y_0\right) + \int_0^{T-t} \left\langle \tilde{b}(s, Y_s; p_s), \nabla\phi\left(Y_s\right) \right\rangle + \frac{1}{2} Tr\left(\widehat{\Sigma}\left(s, Y_s\right) \nabla^2 \phi\left(Y_s\right)\right) ds + \int_0^{T-t} \nabla\phi\left(Y_s\right)^\top \sigma\left(s, Y_s\right) d\beta_s, \tag{4.2}$$

with

$$\tilde{b}\left(s, y; p_s\right) := \left\{ \frac{div_y\left(\widehat{\Sigma}_{j\cdot}\left(s, y\right) p_s\left(y\right)\right)}{p_s\left(y\right)} \right\}_{j \in [\![1,d]\!]} - \widehat{b}\left(s, y\right), \quad (s, y) \in]0, T[\times \mathbb{R}^d.$$

We now want to take the expectation in identity (4.2). On the one hand, Remark 4.2, implies that, for all $i \in [\![1, d]\!]$ and $s \in]0, T[$ a.e.

$$\mathbb{E}\left(\left|\frac{div_y\left(\widehat{\Sigma}_{i\cdot}\left(s, Y_s\right) p_s\left(Y_s\right)\right)}{p_s\left(Y_s\right)} \partial_i \phi\left(Y_s\right)\right|\right) < \infty.$$

On the other hand

$$\int_0^T \mathbb{E}\left\{ Tr\left(\widehat{\Sigma}\left(s, Y_s\right) \nabla^2 \phi\left(Y_s\right)\right)\right\} ds = \sum_{i,j=1}^d \int_0^T \int_{\mathbb{R}^d} \widehat{\Sigma}_{ij}\left(s, y\right) \partial_{ij} \phi\left(y\right) p_s\left(y\right) dy ds.$$

Previous expression is finite since Σ is bounded on compact sets and the partial derivatives of ϕ have compact supports. With similar arguments we prove that $\int_0^T ds \mathbb{E}\left|\left\langle \widehat{b}\left(s, Y_s\right), \nabla\phi\left(Y_s\right)\right\rangle\right| < \infty, \ s \in]0, T[$. Moreover, fixing $s \in]0, T[$ a.e., integrating by parts we have

$$\begin{aligned} \mathbb{E}\left\{\left\langle \tilde{b}\left(s, Y_s; p_s\right), \nabla\phi\left(Y_s\right)\right\rangle\right\} &= \sum_{k,j=1}^d \int_{\mathbb{R}^d} \partial_k\left(\widehat{\Sigma}_{jk}\left(s, y\right) p_s\left(y\right)\right) \partial_j \phi\left(y\right) dy - \int_{\mathbb{R}^d} \left\langle \widehat{b}\left(s, y\right), \nabla\phi\left(y\right)\right\rangle p_s\left(y\right) dy \\ &= -\int_{\mathbb{R}^d} Tr\left(\widehat{\Sigma}\left(s, y\right) \nabla^2 \phi\left(y\right)\right) p_s\left(y\right) dy - \int_{\mathbb{R}^d} \left\langle \widehat{b}\left(s, y\right), \nabla\phi\left(y\right)\right\rangle p_s\left(y\right) dy. \end{aligned} \tag{4.3}$$

Now, the quadratic variation of the local martingale $M^Y := \int_0^\cdot \nabla\phi\left(Y_s\right)^\top \sigma\left(s, Y_s\right) d\beta_s$ yields

$$\left[M^Y\right] = \int_0^\cdot \nabla\phi\left(Y_s\right)^\top \Sigma\left(s, Y_s\right) \nabla\phi\left(Y_s\right) ds.$$

We remark in particular that $\mathbb{E}\left(\left[M^Y\right]_T\right) < \infty$ since Σ is bounded on compact sets and ϕ has compact support. This shows M^Y is a true (even square integrable) martingale and all terms involved in (4.2) are integrable.

At this point we evaluate the expectation in (4.2) taking the considerations above together with (4.1) and (4.3) into account. We obtain

$$\mathbb{E}\left(\phi\left(Y_{T-t}\right)\right) = \int_{\mathbb{R}^d} \phi\left(y\right) \mu\left(dy\right) - \int_0^{T-t} \int_{\mathbb{R}^d} L_{T-s} \phi\left(y\right) p_s\left(y\right) dy ds.$$

Applying the change of variable $t \mapsto T - t$, we finally obtain the identity

$$\int_{\mathbb{R}^d} \phi\left(y\right) p_{T-t}\left(y\right) dy = \int_{\mathbb{R}^d} \phi\left(y\right) \mu\left(dy\right) - \int_t^T \int_{\mathbb{R}^d} L_s \phi\left(y\right) p_{T-s}\left(y\right) dy ds,$$

which means that $\mathbf{p}\left(T - \cdot\right)$ solves the PDE (1.1) in the sense of Definition 3.1 with terminal value μ. $\square$

4.2 Notion of existence and uniqueness for the McKean SDE in a given class

We provide the different notions of existence and uniqueness for (1.3) we will use in the sequel.

Definition 4.4. *Let $\mathcal{A}$ be a class of measure-valued functions from $[0,T]$ to $\mathcal{P}\left(\mathbb{R}^d\right)$.*

1. *We say that the SDE* (1.3) *admits* **existence in law** *in $\mathcal{A}$, if there exists a complete filtered probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P}\right)$ equipped with an m-dimensional $(\mathcal{F}_t)_{t\in[0,T]}$-Brownian motion β and a couple $(Y, \mathbf{p})$ solution of* (1.3) *in the sense of Definition 4.1 such that $\mathbf{p}$ belongs to $\mathcal{A}$.*

2. *Let $\left(Y^1, \mathbf{p^1}\right)$, $\left(Y^2, \mathbf{p^2}\right)$ be two solutions of* (1.3) *in the sense of Definition 4.1 associated to some complete filtered probability spaces $\left(\Omega^1, \mathcal{F}^1, \left(\mathcal{F}^1_t\right)_{t\in[0,T]}, \mathbb{P}^1\right)$, $\left(\Omega^2, \mathcal{F}^2, \left(\mathcal{F}^2_t\right)_{t\in[0,T]}, \mathbb{P}^2\right)$ respectively, equipped with Brownian motions β^1, β^2 respectively and such that $\mathbf{p^1}, \mathbf{p^2}$ belong to $\mathcal{A}$. We say that* (1.3) *admits* **uniqueness in law** *in $\mathcal{A}$, if Y^1_0, Y^2_0 have the same law implies that Y^1, Y^2 have the same law.*

3. *We say that* (1.3) *admits* **strong existence** *in $\mathcal{A}$ if for any complete filtered probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P}\right)$ equipped with an m-dimensional $(\mathcal{F}_t)_{t\in[0,T]}$-Brownian motion β, there exists a solution $(Y, \mathbf{p})$ of equation* (1.3) *in the sense of Definition 4.1 such that $\mathbf{p}$ belongs to $\mathcal{A}$.*

4. *We say that* (1.3) *admits* **pathwise uniqueness** *in $\mathcal{A}$ of if for any complete filtered probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P}\right)$ equipped with an m-dimensional $(\mathcal{F}_t)_{t\in[0,T]}$-Brownian motion β, for any solutions $\left(Y^1, \mathbf{p^1}\right)$, $\left(Y^2, \mathbf{p^2}\right)$ of* (1.3) *in the sense of Definition 4.1 such that $Y^1_0 = Y^2_0$, $\mathbb{P}-\text{a.s.}$ and $\mathbf{p^1}, \mathbf{p^2}$ belong to $\mathcal{A}$, we have $Y^1 = Y^2$, $\mathbb{P}-\text{a.s.}$.*

5. *If the mention to a specific class $\mathcal{A}$ is omitted as far as uniqueness (in law or pathwise), the class $\mathcal{A}$ is the one of all possible probability valued functions verifying* (4.1).

We finally define the sets in which we will formulate existence and uniqueness results in the sequel.

Notation 1.

1. *For a given $\mathcal{C} \subseteq \mathcal{P}\left(\mathbb{R}^d\right)$, $\mathcal{A}_\mathcal{C}$ denotes the set of measure-valued functions $\mathbf{p}$ from $[0,T]$ to $\mathcal{P}\left(\mathbb{R}^d\right)$ such that $\mathbf{p}(T)$ belongs to $\mathcal{C}$. Furthermore, for a given measure-valued function $\mathbf{p} : [0,T] \mapsto \mathcal{P}\left(\mathbb{R}^d\right)$, we will write*
$$b(t, \cdot; \mathbf{p}_t) := \left\{ \frac{div_y\left(\widehat{\Sigma}_{i.} p_t\right)}{p_t} \right\}_{i\in[\![1,d]\!]}, \tag{4.4}$$
for almost all $t \in [0,T]$ whenever p_t exists and the right-hand side quantity of (4.4) *is well-defined. The function $(t,x) \mapsto b(t,x;\mathbf{p}_t)$ is defined on $[0,T]\times\mathbb{R}^d$ with values in $\mathbb{R}^d$.*

2. *Let $\mathcal{A}_1$ (resp. $\mathcal{A}_2$) denote the set of measure-valued functions from $[0,T]$ to $\mathcal{P}\left(\mathbb{R}^d\right)$ $\mathbf{p}$ such that, for all $t \in [0,T[$, $\mathbf{p}(t)$ admits a density p_t with respect to the Lebesgue measure on $\mathbb{R}^d$ and such that $(t,x) \mapsto b(t,x;\mathbf{p}_t)$ is locally bounded (resp. is locally Lipschitz in space with linear growth) on $[0,T[\times\mathbb{R}^d$.*

4.3 Well-posedness for the McKean SDE: the bounded coefficients case

In this section, we state a significant result about existence and uniqueness in law together with pathwise uniqueness for the SDE (1.3). We exploit here in particular the uniqueness results related to the PDE (1.1) obtained in Section 3.2 and Section 3.3. As far as uniqueness is concerned, given a solution $(Y, \mathbf{p})$ of (1.3), we insist that the basic idea consists in showing that $\mathbf{p}$ solves (1.1), see Proposition 4.3. At this point Y solves an ordinary SDE and we only need to show that the coefficients fulfill the assumptions which guarantee uniqueness, see e.g. Lemma 4.5. On the other hand, the existence results for (1.3) are based on the techniques of [11] of determining the dynamics of the time-reversal of a diffusion.

We formulate the following hypothesis for the couple (b, Σ), where we recall that $\Sigma = \sigma\sigma^\top$.

Assumption 6. *$\Sigma : [0,T] \times \mathbb{R}^d \to M_d(\mathbb{R})$, $b : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$) are Borel functions such that the following holds.*

- *For each $t \in [0,T]$, $(\nabla_x b_i(t,\cdot))_{i\in[\![1,d]\!]}$, $(\nabla_x \Sigma_{ij}(t,\cdot))_{i,j\in[\![1,d]\!]}$ exist and they are continuous;*
- *For each $t \in [0,T]$, $\nabla_x^2 \Sigma(t,\cdot)$ exists and $\nabla_x^2 \Sigma$ is Hölder continuous in space with some exponent $\alpha \in]0,1[$ uniformly in time.*
- *$\nabla \Sigma$ and ∇b are uniformly bounded.*

Assumption 7. *Σ is supposed to be Hölder continuous in time.*

The first step consists in proving existence and uniqueness in law for the SDE (1.3) in the class $\mathcal{A}_1$. For this we will state a fundamental lemma whose proof will appear in the Appendix.

Lemma 4.5. *Suppose the validity of Assumptions 2, 4, 6 and 7. Then, for all $\nu \in \mathcal{P}\left(\mathbb{R}^d\right)$, $\mathbf{u}^\nu(t)$ admits a density $u^\nu(t,\cdot) \in \mathcal{C}^1\left(\mathbb{R}^d\right)$ for all $t \in]0,T]$. Furthermore, for each compact K of $]0,T] \times \mathbb{R}^d$, there are strictly positive constants C_1^K, C_2^K, C_3^K, also depending on ν such that*

$$C_1^K \le u^\nu(t,x) \quad \le \quad C_2^K \tag{4.5}$$

$$|\partial_i u^\nu(t,x)| \quad \le \quad C_3^K,\ i \in [\![1,d]\!], \tag{4.6}$$

for all $(t,x) \in K$.

Remark 4.6. *Under Assumptions 2, 3 (which is a consequence of Assumptions 6 and 7) together with 4, for every $\nu \in \mathcal{P}(\mathbb{R}^d)$, by Lemma 3.4, there exists a unique $\mathcal{P}\left(\mathbb{R}^d\right)$-valued solution $\mathbf{u}^\nu$ of the PDE* (3.2).

Lemma 4.7. *Let μ be the probability measure introduced at the beginning of Section 4.1. Suppose that $\mu = \mathbf{u}^\nu(T)$ for some $\nu \in \mathcal{P}\left(\mathbb{R}^d\right)$. We assume the following.*

1. *Assumptions 1, 2, 4 and 6.*
2. *$\mathbf{u}^\nu(t)$ admits a density $u^\nu(t,\cdot) \in W^{1,1}_{\rm loc}(\mathbb{R}^d)$, for all $t \in]0,T]$.*
3. *For each compact K of $]0,T] \times \mathbb{R}^d$, there are strictly positive constants C_1^K, C_2^K, C_3^K, also depending on ν such that* (4.5) *and* (4.6) *hold $\forall (t,x) \in K$.*

Then the SDE (1.3) *admits existence in law in $\mathcal{A}_1$.*

A consequence of the two lemmata above is the proposition below, which states in particular existence in law in $\mathcal{A}_1$.

Proposition 4.8. *We suppose the validity of Assumptions 1, 2, 4, 6 and 7.*

1. *Suppose the existence of $\nu \in \mathcal{P}(\mathbb{R}^d)$ such that $\mathbf{u}^\nu(T) = \mu$. Then, the SDE* (1.3) *admits existence in law in $\mathcal{A}_1$. Moreover, if ν is a Dirac mass, existence in law occurs in $\mathcal{A}_{(\delta_x)_{x\in\mathbb{R}^d}} \cap \mathcal{A}_1$.*
2. *Otherwise* (1.3) *does not admit existence in law.*

Remark 4.9. *For a class of coefficients b, Σ, an interesting problem would be to determine the reachability set of possible μ, i.e. of the set of μ for which there exists ν such that $\mu = \mathbf{u}^\nu$. This however goes beyond the scope of our paper.*

Proof (of Proposition 4.8).

1. The first part is a direct consequence of Lemma 4.5, Lemma 4.7 and expression (4.4). If in addition, ν is a Dirac mass, then $\mathbf{u}^\nu(0)$ belongs to $\mathcal{C} := (\delta_x)_{x\in\mathbb{R}^d}$, hence existence in law occurs in $\mathcal{A}_\mathcal{C} \cap \mathcal{A}_1$ by Proposition 5.2 in the Appendix.

2. Otherwise suppose ab absurdo that $(Y, \mathbf{p})$ is a solution of the SDE(1.3). By Proposition 4.3 $\mathbf{p}(T-\cdot)$ is a solution of the PDE (1.1). We set $\nu_0 = \mathbf{p}(T)$ so that $\mathbf{p}(T-\cdot)$ verifies also the PDE (3.2) with initial value ν_0. Since, by Lemma 3.4 uniqueness holds for (3.2), it follows that $\mathbf{p}(T-\cdot) = \mathbf{u}^{\nu_0}$ which concludes the proof of item 2.

□

Proof (of Lemma 4.7). Suppose $\mu = \mathbf{u}^\nu(T)$ for some $\nu \in \mathcal{P}(\mathbb{R}^d)$. We recall that Property 1 holds with respect to $\mathcal{C} := \mathcal{P}(\mathbb{R}^d)$ by Lemma 3.4. In view of applying again Proposition 5.2 stated in the Appendix, we need to check the validity of Property 2 with respect to $\mathcal{C}$ and Property 3. Property 2 is verified by $\mathbf{u} = \mathbf{u}^\nu$. Indeed the function $\mathbf{u}^\nu$ is a $\mathcal{P}(\mathbb{R}^d)$-valued solution of the PDE (1.1) with terminal value μ and such that $\mathbf{u}^\nu(0)$ belongs to $\mathcal{C}$. Condition (5.1) appearing in Property 2 is satisfied with $\mathbf{u} = \mathbf{u}^\nu$ thanks to the right-hand side of inequalities (4.5) and (4.6) and the fact that Σ is bounded. Hence Property 2 holds with respect to $\mathcal{C}$. It remains to show Property 3 holds i.e. that

$$(t,x) \mapsto \frac{div_x\left(\widehat{\Sigma}_{i.}(t,x)u^\nu(T-t,x)\right)}{u^\nu(T-t,x)}$$

is locally bounded on $[0,T[\times\mathbb{R}^d$. To achieve this, we fix $i \in [\![1,d]\!]$ and a bounded open subset $\mathcal{O}$ of $[0,T[\times\mathbb{R}^d$. For $(t,x) \in \mathcal{O}$ we have

$$\left|\frac{div_x\left(\widehat{\Sigma}_{i.}(t,x)\,u^\nu(T-t,x)\right)}{u^\nu(T-t,x)}\right| \le \left|div_x\left(\widehat{\Sigma}_{i.}(t,x)\right)\right| + \left|\widehat{\Sigma}_{i.}(t,x)\right|\frac{|\nabla_x u^\nu(T-t,x)|}{u^\nu(T-t,x)}.$$

The latter quantity is locally bounded in t,x thanks to the boundedness of $\Sigma, div_x\left(\widehat{\Sigma}_{i.}\right)$ and inequalities (4.5) and (4.6). Hence, Property 3 holds. The application of Proposition 5.2 ends the proof. □

Proposition 4.10. *(***The McKean SDE: well-posedness in the case of Dirac initial conditions.***)* *Suppose the validity of Assumptions 1, 2, 4, 6 and 7. The following results hold.*

1. *Let us suppose $d = 1$. Suppose μ equals $\mathbf{u}^{\delta_{x_0}}(T)$ for some $x_0 \in \mathbb{R}$. Then* (1.3) *admits existence and uniqueness in law in $\mathcal{A}_{(\delta_x)_{x\in\mathbb{R}^d}} \cap \mathcal{A}_1$, pathwise uniqueness in $\mathcal{A}_{(\delta_x)_{x\in\mathbb{R}^d}} \cap \mathcal{A}_2$.*

2. *Let $d \ge 2$. There is a maturity T sufficiently small (only depending on the Lipschitz constant of b, σ) such that the following result holds. Suppose μ equals $\mathbf{u}^{\delta_{x_0}}(T)$ for some $x_0 \in \mathbb{R}^d$. Then* (1.3) *admits existence and uniqueness in law in $\mathcal{A}_{(\delta_x)_{x\in\mathbb{R}^d}} \cap \mathcal{A}_1$, pathwise uniqueness in $\mathcal{A}_{(\delta_x)_{x\in\mathbb{R}^d}} \cap \mathcal{A}_2$.*

Proof. By Assumptions 1, 2, 4, 6 and 7, Proposition 4.8 implies that the SDE (1.3) admits existence in law in the two cases in the specific classes. To check the uniqueness in law and pathwise uniqueness results, we wish to apply Corollary 5.5 stated in the Appendix. It suffices now to check Property 5 with respect to $(\delta_x)_{x\in\mathbb{R}}$, for the separate two cases.

1. Fix $x_0 \in \mathbb{R}^d$. This will follow from Proposition 3.9 that holds under Assumption 1.

2. We proceed as for previous case but applying Theorem 3.10 instead of Proposition 3.9.

□

Previous Proposition 4.10 provides uniqueness in law only among the solutions $(Y, \mathbf{p})$ such $\mathbf{p}$ belongs to a subclass of $\mathcal{A}_1$. We state now the two most important results of the section which in particular provide uniqueness in law for the SDE (1.3) among all possible solutions.

Theorem 4.11. *(***The McKean SDE: well-posedness in the case of non-degenerate time-homogeneous coefficients.***)*

Suppose b, σ are time-homogeneous, Assumptions 1, 2, 4, 6 and suppose there is $\nu \in \mathcal{P}\left(\mathbb{R}^d\right)$ (a priori not known) such that $\mu = \mathbf{u}^\nu (T)$.

1. *The SDE* (1.3) *admits existence and uniqueness in law. Moreover existence in law holds in $\mathcal{A}_1$.*

2. (1.3) *admits pathwise uniqueness in $\mathcal{A}_2$.*

Proof.
1. (a) First, Assumption 7 trivially holds since b, σ are time-homogeneous. Then, point 1. of Proposition 4.8 implies that the SDE (1.3) admits existence in law (in $\mathcal{A}_1$).

   (b) Let $(Y, \mathbf{p})$ be a solution of (1.3). Proceeding as in the proof of item 2. of Proposition 4.8, we obtain that $\mathbf{p}(T - \cdot) = \mathbf{u}^{\nu_0}$ with $\nu_0 = \mathbf{p}(T)$. Then, Lemma 4.5 shows that $\mathbf{p}$ belongs to $\mathcal{A}_1$, see (4.4) in Notation 1.

   (c) To conclude it remains to show uniqueness in law in $\mathcal{A}_1$. For this we wish to apply point 1. of Corollary 5.5 in the Appendix. To achieve this, we check Property 5 with respect to $\mathcal{P}(\mathbb{R}^d)$. This is a consequence of Assumptions 2, 3, 4 and 5 and Theorem 3.13. This concludes the proof of item 1.

2. Concerning pathwise uniqueness in $\mathcal{A}_2$, we proceed as for uniqueness in law but applying point 2. of Corollary 5.5 in the Appendix. This is valid since σ are bounded and Lipschitz by Assumptions 1, 2 and 3.

□

In the result below we extend Theorem 4.11 to the case when the coefficients b, σ are piecewise time-homogeneous.

Theorem 4.12. *(***The McKean SDE: well-posedness with non-degenerate piecewise time-homogeneous coefficients.***)*

Let $n \in \mathbb{N}^*$. Let $0 = t_0 < \ldots < t_n = T$ be a partition. For $k \in [\![2, n]\!]$ (resp. $k = 1$) we denote $I_k =]t_{k-1}, t_k]$ (resp. $[t_0, t_1]$). Suppose that the following holds.

1. *For all $k \in [\![1, n]\!]$ the restriction of σ (resp. b) to $I_k \times \mathbb{R}^d$ is a time-homogeneous function $\sigma^k : \mathbb{R}^d \to M_d(\mathbb{R})$ (resp. $b^k : \mathbb{R}^d \to \mathbb{R}^d$).*

2. *Assumptions 2 and 4.*

3. *σ is Lipschitz (in space uniformly in time).*

4. *Assumption 6 holds for the couples (b^k, Σ^k).*

Suppose μ equals $\mathbf{u}^{\nu}(T)$ for some $\nu \in \mathcal{P}\left(\mathbb{R}^d\right)$. Then SDE (1.3) *admits existence and uniqueness in law. Moreover, existence in law holds in $\mathcal{A}_1$.*

Remark 4.13. *A similar remark as in Proposition 4.8 holds for the Theorems 4.11 and 4.12. If there is no $\nu \in \mathcal{P}(\mathbb{R}^d)$ such that $\mathbf{u}^{\nu}(T) = \mu$, then* (1.3) *does not admit existence in law.*

Proof (of Theorem 4.12). We recall that by Lemma 3.4, $\mathbf{u}^{\nu_0}$ is well-defined for all $\nu_0 \in \mathcal{P}\left(\mathbb{R}^d\right)$.

1. We first show that $\mathbf{u}^{\nu_0}$ verifies (4.5) and (4.6). Indeed, fix $k \in [\![1, n]\!]$. The restriction $\mathbf{u_k}$ of $\mathbf{u}^{\nu_0}$ to $\bar{I}_k$ is a solution $\mathbf{v}$ of the first line of the PDE (3.2) replacing $[0, T]$ with $\bar{I}_k$, L by L^k defined in (3.20), with initial condition $\mathbf{v}(t_{k-1}) = \mathbf{u}^{\nu_0}(t_{k-1})$. That restriction is even the unique solution, using Lemma 3.4 replacing $[0, T]$ with $\bar{I}_k$. We apply Lemma 4.5 replacing $[0, T]$ with $\bar{I}_k$, taking into account Assumption 7, which holds trivially with respect to Σ^k. This implies that $\mathbf{u}^{\nu_0}$ verifies (4.5) and (4.6) replacing $[0, T]$ with $\bar{I}_k$, and therefore on the whole $[0, T]$.

2. Existence in law in $\mathcal{A}_1$, follows now by Lemma 4.7.

3. It remains to show uniqueness in law. Let $(Y, \mathbf{p})$ be a solution of the SDE (1.3). We set $\nu_0 := \mathbf{p}(T)$. Since $\mathbf{u}^{\nu_0}$ and $\mathbf{p}(T - \cdot)$ solve the PDE (3.2), Lemma 3.4 implies that $\mathbf{p}$ is uniquely determined. Similarly as in item 1.(b) of the proof of Theorem 4.11, item 1. of the present proof and Lemma 4.5 allow to show that $\mathbf{p}$ belongs to $\mathcal{A}_1$.

4. It remains to show uniqueness in law in $\mathcal{A}_1$. For this, Corollary 3.16 implies Property 5 in the Appendix with $\mathcal{C} = \mathcal{P}(\mathbb{R}^d)$. Uniqueness of (1.3) in the class $\mathcal{A}_1$ follows now by Corollary 5.5 in the Appendix, which ends the proof.

□

4.4 Well-posedness for the McKean SDE: the OU semigroup

In this section we investigate existence and uniqueness for the SDE (1.3) in the context of an OU semigroup. As for Section 4.3, the uniqueness statement for the related PDE (1.1) (see Section 3.4), appears to be crucial. The only limitation here is that the matrix function Σ has to be invertible, otherwise the additive drift in (1.3) would not be defined.

Suppose that b is of the form $(s, x) \mapsto C(s)\, x$ with C continuous from $[0, T]$ to $\mathbb{R}^d$ and σ continuous from $[0, T]$ to $M_d(\mathbb{R})$. We also suppose that for all $t \in [0, T]$, $\Sigma(t)$ is invertible. We denote by $\mathcal{C}(t) := (\mathcal{D}(t)^{-1})^{\top}, t \in [0, T]$ where $\mathcal{D}$ is the unique solution of (3.23). Evaluating the transposed matrix on both sides of (3.24), we remark that $\mathcal{C}$ is solution of the matrix-valued ODE,

$$\mathcal{C}(t) = I + \int_0^t C(s)\mathcal{C}(s) ds, \ t \in [0, T].$$

For a given $x_0 \in \mathbb{R}^d$ and a given t $\in]0, T]$, we denote by $p_t^{x_0}$ the density of a Gaussian random vector with mean $m_t^{x_0} = \mathcal{C}(t) x_0$ and covariance matrix $Q_t = \mathcal{C}(t) \int_0^t \mathcal{C}^{-1}(s)\Sigma(s)\mathcal{C}^{-1}(s)^{\top} ds\ \mathcal{C}(t)^{\top}$. Note that for all $t \in]0, T]$, Q_t is strictly positive definite, in particular it is invertible. Indeed, for every $t \in [0, T]$, $\Sigma(t)$ is strictly positive definite. By continuity in t, $\int_0^t \mathcal{C}^{-1}(s)\Sigma(s)\mathcal{C}^{-1}(s)^{\top} ds$ is also strictly positive definite and finally the same holds for Q_t. For a given $\nu \in \mathcal{P}\left(\mathbb{R}^d\right)$, $t \in]0, T]$, we set the notation

$$p_t^{\nu} : x \mapsto \int_{\mathbb{R}^d} p_t^{x_0}(x)\, \nu(dx_0). \tag{4.7}$$

At this level, we need a lemma.

Lemma 4.14. *Let $\nu \in \mathcal{P}\left(\mathbb{R}^d\right)$. The measure-valued function $t \mapsto p_t^\nu(x)dx$ is the unique solution of the PDE* (3.2) *with initial value ν. Consequently it coincides with $\mathbf{u}^\nu$. Furthermore, $\mathbf{u}^\nu\left(T-\cdot\right)$ belongs to $\mathcal{A}_2$.*

Proof.

1. By Chapter 5, Section 5.6 in [16], for every $t \in]0,T]$, $p_t^{x_0}$ is the density of the random variable $X_t^{x_0}$, where X^{x_0} is the unique strong solution of (3.4) with initial value x_0. The mapping $t \mapsto p_t^{x_0}(x)dx$ is a solution of (3.2) by Proposition 3.2, with initial condition δ_{x_0}. Consequently, by superposition, $t \mapsto p_t^\nu(x)dx$ is a solution of the PDE (3.2) with initial value ν.

2. By Proposition 3.17, Property 1 with respect to $\mathcal{C} = \mathcal{P}(\mathbb{R}^d)$ is verified so that $t \mapsto p_t^\nu(x)dx$ is the unique solution of (3.2) so that it coincides with $\mathbf{u}^\nu$.

3. It remains to show that $\mathbf{u}^\nu\left(T-\cdot\right)$ belongs to $\mathcal{A}_2$, namely that for all $i \in [\![1,d]\!]$

$$(t,x) \mapsto \frac{div_x\left(\Sigma\left(T-t\right)_{i\cdot}\, p_{T-t}^\nu\left(x\right)\right)}{p_{T-t}^\nu\left(x\right)},$$

is locally Lipschitz with linear growth in space on $[0,T[\times\mathbb{R}^d$.

Fix $i \in [\![1,d]\!]$, $t \in [0,T[$ and $x \in \mathbb{R}^d$. Remembering the fact that $p_{T-t}^{x_0}$ is a Gaussian law with mean $m_{T-t}^{x_0}$ and covariance matrix Q_{T-t} for a given $x_0 \in \mathbb{R}^d$, we have

$$\frac{div_x\left(\Sigma\left(T-t\right)_{i\cdot}\, p_{T-t}^\nu\left(x\right)\right)}{p_{T-t}^\nu\left(x\right)} = -\frac{1}{p_{T-t}^\nu\left(x\right)}\int_{\mathbb{R}^d}\left\langle \Sigma\left(T-t\right)_{i\cdot}\,, Q_{T-t}^{-1}\left(x-m_{T-t}^{x_0}\right)\right\rangle p_{T-t}^{x_0}\left(x\right)\nu\left(dx_0\right). \quad (4.8)$$

Let K be a compact subset of $]0,T] \times \mathbb{R}^d$; then there is $M_K > 0$ such that for all $(t,x) \in K$, $x_0 \in \mathbb{R}^d$,

$$\left|\left\langle \Sigma\left(T-t\right)_{i\cdot}\,, Q_{T-t}^{-1}\left(x-m_{T-t}^{x_0}\right)\right\rangle p_{T-t}^{x_0}\left(x\right)\right| \leq \left|\Sigma\left(T-t\right)_{i\cdot}\right|\left|\left|Q_{T-t}^{-1}\right|\right|\left|x-m_{T-t}^{x_0}\right| p_{T-t}^{x_0}\left(x\right) \leq M_K.$$

This follows because $t \mapsto \Sigma(T-t)$ and $t \mapsto Q_{T-t}^{-1}$ are continuous on $[0,T[$ and, setting

$$c_K := \inf\{t|(t,x) \in K\}, \quad m_K := \sup_{a\in\mathbb{R}}|a|\exp\left(-c_K\frac{a^2}{2}\right),$$

we have

$$|x-m_{T-t}^{x_0}|p_{T-t}^{x_0}(x) \leq m_K,\ \forall (t,x) \in K.$$

To show that left-hand side of (4.8) is locally bounded on $[0,T[\times\mathbb{R}^d$ it remains to show that $(t,x) \mapsto \int_{\mathbb{R}^d} p_{T-t}^{x_0}(x)\nu(dx_0)$ is lower bounded on K. Indeed, let I be a compact of $\mathbb{R}^d$. Since $(t,x,x_0) \mapsto p_{T-t}^{x_0}(x)$ is strictly positive and continuous is lower bounded by a constant $c(K,I)$. The result follows choosing I such that $\nu(I) > 0$.

To conclude, it remains to show that the functions $(t,x) \mapsto \frac{div_x\left(\Sigma(T-t)_{i\cdot}\, p_{T-t}^\nu(x)\right)}{p_{T-t}^\nu(x)},\ i \in [\![1,d]\!]$ defined on $[0,T[\times\mathbb{R}^d$ has locally bounded spatial derivatives, which implies that they are Lipschitz with linear growth on each compact of $[0,T[\times\mathbb{R}^d$. By technical but easy computations, the result follows using the fact the real functions $a \mapsto |a|^m \exp\left(-\frac{a^2}{2}\right)$, $m = 1,2$, are bounded.

□

We give now a global well-posedness result for the SDE (1.3).

Theorem 4.15. *(***The McKean SDE: well-posedness in the Ornstein-Uhlenbeck case.***)*

1. *Suppose the initial condition μ equals $\mathbf{u}^\nu\left(T\right)$ for some $\nu \in \mathcal{P}\left(\mathbb{R}^d\right)$. Then, equation* (1.3) *admits existence in law, strong existence, uniqueness in law and pathwise uniqueness.*

2. *Otherwise* (1.3) *does not admit any solution.*

Proof. Item 2. can be proved using similar arguments as for the proof of Proposition 4.8. Let $(Y, \mathbf{p})$ be a solution of (1.3) and set $\nu_0 = \mathbf{p}(T)$. By Proposition 4.3, $\mathbf{p}(T - \cdot)$ is a solution of the PDE (1.1), so that $\mathbf{p}(T - \cdot)$ verifies also the PDE (3.2) with initial value ν_0. Since, by Proposition 3.17, uniqueness holds for (3.2), it follows that $\mathbf{p}(T - \cdot) = \mathbf{u}^{\nu_0}$ which concludes the proof of item 2.

We prove now item 1. For this, taking into account Proposition 5.4, Yamada-Watanabe theorem and related results for classical SDEs, it suffices to show strong existence and pathwise uniqueness. We set $\mathcal{C} := \mathcal{P}\left(\mathbb{R}^d\right)$.

a) Concerning the strong existence statement, we want to apply Proposition 5.2 stated in the Appendix. Since b, σ are affine, Assumption 1 trivially holds; Property 1 with respect to $\mathcal{C}$ thanks to Proposition 3.17. It remains to verify Property 2 with respect to $\mathcal{C}$ and Property 4 (in the Appendix).

   By Lemma 4.14, for all $t \in]0, T]$, $\mathbf{u}^\nu(t)$ admits p_t^ν (see (4.7)) for density. Then, relation (5.1) below holds since, by (4.7) and the considerations above, $(t, x) \mapsto p_t^\nu(x)$ is locally bounded with locally bounded spatial derivatives. Hence, Property 2 holds with respect to $\mathcal{C}$. Finally, Lemma 4.14 implies that $\mathbf{u}^\nu\left(T - \cdot\right)$ belongs to $\mathcal{A}_2$. Hence, Property 4 holds with respect to $\mathcal{C}$ and so Proposition 5.2 implies existence in law.

b) Let $(Y, \mathbf{p})$ be a solution of equation (1.3). Proposition 4.3 implies that $\mathbf{p}\left(T - \cdot\right)$ solves (1.1). Then, Proposition 3.17 gives $\mathbf{p}\left(T - \cdot\right) = \mathbf{u}^{\nu_0}$ with $\nu_0 = \mathbf{p}\left(T\right)$. Lemma 4.14 implies $\mathbf{p}$ belongs to $\mathcal{A}_2$.

c) It remains to show pathwise uniqueness in $\mathcal{A}_2$ for which we will make use of Corollary 5.5, lying on Property 5, both stated in the Appendix. Indeed we check that Property 5 holds with respect to $\mathcal{C}$ thanks to Theorem 3.19. Now, point 2 of Corollary 5.5 implies pathwise uniqueness in $\mathcal{A}_2$ since b, σ are locally Lipschitz with linear growth in space.

□

5 Appendix

For ease of reading the paper, we have postponed some technical results in this appendix. Sections 5.1 and 5.2 link the well-posedness of the PDE (1.1) to the well-posedness of the McKean SDE (1.3). In particular Proposition 5.2 (resp. Corollary 5.5) links the existence (resp. uniqueness) of the PDE (1.2) with the SDE (1.3). Sections 5.3 and 5.4 give the proofs of two technical Lemma (Lemma 3.11 and 4.5).

5.1 PDE with terminal condition and existence for the McKean SDE

We suppose that Property 1 is in force for a fixed $\mathcal{C} \subseteq \mathcal{P}\left(\mathbb{R}^d\right)$ and consider the Property 2 with respect to $\mathcal{C}$ and Properties 3 and 4 related to a given function $\mathbf{u} : [0, T] \to \mathcal{M}_+(\mathbb{R}^d)$.

Property 2.

1. $\mathbf{u}\left(0\right)$ *belongs to* $\mathcal{C}$.

2. $\forall t \in]0,T[$, $\mathbf{u}(t)$ *admits a density with respect to the Lebesgue measure on* $\mathbb{R}^d$ *(denoted by* $u(t,\cdot)$*) and for all* $t_0 > 0$ *and all compact* $K \subset \mathbb{R}^d$

$$\int_{t_0}^{T} \int_K |u(t,x)|^2 + \sum_{i=1}^{d}\sum_{j=1}^{d} |\sigma_{ij}(t,x)\,\partial_i u(t,x)|^2\, dxdt < \infty. \tag{5.1}$$

Remark 5.1. *Suppose Assumption 1 holds and let* $\mathbf{u}$ *be a measure-valued function verifying Property 2. Then* (5.1) *implies that the family of densities* $u(T-t,\cdot)\,, t \in]0,T[$ *verifies condition* (4.1) *appearing in Definition 4.1. To show this, it suffices to check that for all* $t_0 > 0$, *all compact* $K \subset \mathbb{R}^d$ *and all* $(i,j,k) \in [\![1,d]\!]^2 \times [\![1,d]\!]$

$$\int_{t_0}^{T} \int_K |\partial_j (\sigma_{ik}(s,y)\,\sigma_{jk}(s,y)\,u(s,y))|\, dyds < \infty. \tag{5.2}$$

The integrand appearing in (5.2) *is well-defined. Indeed, in the sense of distributions we have*

$$\partial_j (\sigma_{ik}\sigma_{jk}u) = \sigma_{ik}\sigma_{jk}\partial_j u + u\,(\sigma_{ik}\partial_j\sigma_{jk} + \sigma_{jk}\partial_j\sigma_{ik})\,; \tag{5.3}$$

moreover the components of σ *are Lipschitz, so they are (together with their space derivatives) locally bounded. Also* u *and* $\sigma_{jk}\partial_j u$ *are square integrable by* (5.1), *which implies* (5.2).

We introduce two other properties possibly fulfilled by a function $\mathbf{u} : [0,T] \to \mathcal{M}_+(\mathbb{R}^d)$.

Property 3. $\mathbf{u}(T)$ *admits a density and* $\mathbf{u}(T-\cdot)\,|_{[0,T[\times\mathbb{R}^d}$ *belongs to* $\mathcal{A}_1$.

Property 4. $\mathbf{u}(T)$ *admits a density and* $\mathbf{u}(T-\cdot)\,|_{[0,T[\times\mathbb{R}^d}$ *belongs to* $\mathcal{A}_2$.

We remark that Property 4 implies 3.

Proposition 5.2. *Suppose the validity of Assumptions 1. We also suppose that the backward PDE* (1.1) *with terminal condition* μ *admits at least an* $\mathcal{M}_+(\mathbb{R}^d)$*-valued solution* $\mathbf{u}$ *in the sense of Definition 3.1, fulfilling Property 1 and Property 2 with respect to* $\mathcal{C}$. *Then* (1.3) *admits existence in law in* $\mathcal{A}_{\mathcal{C}}$.

Moreover, if $\mathbf{u}$ *fulfills Property 3 (resp. 4) then* (1.3) *admits existence in law in* $\mathcal{A}_{\mathcal{C}} \cap \mathcal{A}_1$ *(resp. strong existence in* $\mathcal{A}_{\mathcal{C}} \cap \mathcal{A}_2$*).*

Proof. Let $\mathbf{u}$ the function of the statement such that fulfilling Property 2, i.e. $\mathbf{u}(0)$ belongs to $\mathcal{C}$ We consider now a filtered probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P}\right)$ equipped with an $(\mathcal{F}_t)_{t\in[0,T]}$-Brownian motion W. Let X_0 be a r.v. distributed according to $\mathbf{u}(0)$. Under Assumption 1, it is well-known that there is a solution X to

$$X_t = X_0 + \int_0^t b(s,X_s)\,ds + \int_0^t \sigma(s,X_s)\,dW_s,\ t \in [0,T]. \tag{5.4}$$

Now, by Proposition 3.2, $t \mapsto \mathcal{L}(X_t)$ is a $\mathcal{P}(\mathbb{R}^d)$-valued solution of the PDE (3.2) in the sense of (3.3) with initial value $\mathbf{u}(0) \in \mathcal{C}$. Then Property 1 for $\mathbf{u}$ implies

$$\mathcal{L}(X_t) = \mathbf{u}(t)\,,\ t \in [0,T], \tag{5.5}$$

since $\mathbf{u}$ solves also the PDE (3.2) with initial value $\mathbf{u}(0) \in \mathcal{C}$. This implies in particular that $\mathbf{u}$ is probability valued and that for all $t \in]0,T[$, X_t has $u(t,\cdot)$ as a density fulfilling condition (5.1).

Combining this observation with Assumption 1, Theorem 2.1 in [11] states that there exists a filtered probability space $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t\in[0,T]}, \mathbb{Q})$ equipped with some Brownian motion β and a copy of $\hat{X}$ (still denoted by the same letter) such that $\widehat{X}$ fulfills the first line of the SDE (1.3) with β and

$$\mathbf{p}(t) := \mathbf{u}(T-t)\,,\ t \in]0,T[. \tag{5.6}$$

Finally, existence in law for the SDE (1.3) in the sense of Definition 4.1 holds since $(\widehat{X}, \mathbf{u}(T-\cdot))$ is a solution of (1.3) on the same filtered probability space and the same Brownian motion above. This occurs in $\mathcal{A}_{\mathcal{C}}$ since $\mathcal{L}\left(\widehat{X}_T\right) \in \mathcal{C}$ thanks to equality (5.5) for $t = T$.

We discuss rapidly the *moreover* point.

- Suppose that $\mathbf{u}$ fulfills Property 3. Then $\mathbf{u}(T-\cdot)$ belongs to $\mathcal{A}_{\mathcal{C}} \cap \mathcal{A}_1$ and we also have existence in law in $\mathcal{A}_{\mathcal{C}} \cap \mathcal{A}_1$.
- Suppose that $\mathbf{u}$ fulfills Property 4. Then, taking into account (5.6), strong existence and pathwise uniqueness for the first line of (1.3) holds by classical arguments since the coefficients are locally Lipschitz with linear growth, see [24] Exercise (2.10), and Chapter IX.2 and [24], Th. 12. section V.12. of [25]. By Yamada-Watanabe theorem this implies uniqueness in law, which shows that $\mathbf{u}(T-\cdot)$ constitutes the marginal laws of the considered strong solutions. This concludes the proof of strong existence in $\mathcal{A}_{\mathcal{C}} \cap \mathcal{A}_2$ since $\mathbf{u}(T-\cdot)$ belongs to $\mathcal{A}_{\mathcal{C}} \cap \mathcal{A}_2$, by Property 4.

□

Remark 5.3. *By* (5.6)*, the second component* $\mathbf{p}$ *of the solution of* (1.3) *is given by* $\mathbf{u}(T-\cdot)$ *.*

5.2 PDE with terminal condition and uniqueness for the McKean SDE

In this subsection we discuss some questions related to uniqueness for the PDE (1.3). We consider the following Property related to a given subset $\mathcal{C}$ of $\mathcal{P}\left(\mathbb{R}^d\right)$.

Property 5. *The PDE* (1.1) *with terminal condition* μ *admits at most a* $\mathcal{P}\left(\mathbb{R}^d\right)$*-valued solution* $\mathbf{u}$ *in the sense of Definition 3.1 such that* $\mathbf{u}(0)$ *belongs to* $\mathcal{C}$*.*

We recall that Section 3.2 provides various classes of examples where Property 5 holds.

Proposition 5.4. *Suppose the validity of Property 5 with respect to* $\mathcal{C}$ *and suppose* b, σ *to be locally bounded.*

Let $\left(Y^i, \mathbf{p}^i\right),\ i \in \{1, 2\}$ *be two solutions of the SDE* (1.3) *in the sense of Definition 4.1 such that* $\mathbf{p}^1(T), \mathbf{p}^2(T)$ *belong to* $\mathcal{C}$*. Then,*

$$\mathbf{p}^1 = \mathbf{p}^2.$$

Proof. Proposition 4.3 shows that $\mathbf{p}^1(T-\cdot), \mathbf{p}^2(T-\cdot)$ are $\mathcal{P}\left(\mathbb{R}^d\right)$-valued solutions of the PDE (1.1) in the sense of Definition 3.1 with terminal value μ. Property 5 gives the result since $\mathbf{p}^1(T), \mathbf{p}^2(T)$ belong to $\mathcal{C}$. □

As a corollary, we establish some consequences about uniqueness in law and pathwise uniqueness results for the SDE (1.3) in the classes $\mathcal{A}_1$ and $\mathcal{A}_2$.

Corollary 5.5. *Suppose the validity of Property 5 with respect to* $\mathcal{C}$*. Then, the following results hold.*

1. *If* b *is locally bounded,* σ *is continuous and if the non-degeneracy Assumption 4 holds then the SDE* (1.3) *admits uniqueness in law in* $\mathcal{A}_{\mathcal{C}} \cap \mathcal{A}_1$*.*
2. *If* b, σ *are locally Lipschitz with linear growth in space, then* (1.3) *admits pathwise uniqueness in* $\mathcal{A}_{\mathcal{C}} \cap \mathcal{A}_2$*.*

Proof. If $(Y, \mathbf{p})$ is a solution of the SDE (1.3) and is such that $\mathbf{p}(T)$ belongs to $\mathcal{C}$, then by Proposition 5.4 $\mathbf{p}$ is determined by $\mu = \mathcal{L}(Y_0)$.

To show that item 1. (resp. 2.) holds, it suffices to show that the classical SDE

$$dX_t = \Big(b(t, X_t; \mathbf{p}_t) - \widehat{b}(t, X_t)\Big)\, dt + \widehat{\sigma}(t, X_t)\, dW_t,\ t \in [0, T[, \tag{5.7}$$

where b was defined in (4.4) and W an m-dimensional Brownian motion, admits uniqueness in law (resp. pathwise uniqueness). The mentioned uniqueness in law is a consequence of Theorem 10.1.3 in [27] and pathwise uniqueness holds by [24] Exercise (2.10), and Chapter IX.2 and [25] Th. 12. Section V.12. □

5.3 Proof of Lemma 3.11

Proof. For a given $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d$ we set

$$Z_t^{x,y} := X_t^y - X_t^x, t \in [0, T].$$

We have

$$Z_t^{x,y} = y - x + \int_0^t B_r^{x,y} Z_r^{x,y} dr + \sum_{j=1}^m \int_0^t C_r^{x,y,j} Z_r^{x,y} dW_r^j,\ t \in [0, T], \tag{5.8}$$

with, for all $r \in [0, T]$

$$B_r^{x,y} := \int_0^1 Jb\,(r, aX_r^y + (1-a)X_r^x)\, da, \quad C_r^{x,y,j} := \int_0^1 J\sigma_{.j}\,(r, aX_r^y + (1-a)X_r^x)\, da, \forall\, j \in [\![1, m]\!].$$

By the classical existence and uniqueness theorem for SDEs with Lipschitz coefficients we know that

$$\mathbb{E}(\sup_{s \le T} |X_s^z|^2) < \infty, \tag{5.9}$$

for all $z \in \mathbb{R}^d$. This implies

$$\mathbb{E}(\sup_{t \in [0,T]} |Z_t^{x,y}|^2) < \infty. \tag{5.10}$$

Now, Itô's formula gives, for all $t \in [0, T]$

$$|Z_t^{x,y}|^2 = |y - x|^2 + 2\int_0^t \langle B_r^{x,y} Z_r^{x,y}, Z_r^{x,y}\rangle\, dr + \sum_{j=1}^d \int_0^t \left|C_r^{x,y,j} Z_r^{x,y}\right|^2 dr + 2\sum_{i=1}^d M_t^{x,y,i}, \tag{5.11}$$

where, for a given $i \in [\![1, d]\!]$, $M^{x,y,i}$ denotes the local martingale $\int_0^\cdot Z_s^{x,y,i} \sum_{j=1}^d \left(C_s^{x,y,j} Z_s^{x,y}\right)_i dW_s^j$.
Consequently, for all $i \in [\![1, d]\!]$, we have

$$\begin{aligned}
\sqrt{[M^{x,y,i}]_T} &= \sqrt{\sum_{j=1}^d \int_0^T \left(Z_r^{x,y,i}\right)^2 \left(C_r^{x,y,j} Z_r^{x,y}\right)_i^2 dr}, \\
&\le \sqrt{\sum_{j=1}^d \int_0^T \left|C_r^{x,y,j} Z_r^{x,y}\right|^2 |Z_r^{x,y}|^2\, dr}, \\
&\le \sqrt{T \sum_{j=1}^d (K^{\sigma,j})^2 \sup_{r \in [0,T]} |Z_r^{x,y}|^2}\,.
\end{aligned} \tag{5.12}$$

By the latter inequality and (5.10), we know that $\mathbb{E}\left([M^{x,y,i}]_T^{\frac{1}{2}}\right) < \infty$, so for all $i \in [\![1, d]\!]$, $M^{x,y,i}$ is a true martingale. Taking expectation in identity (5.11), we obtain

$$\mathbb{E}\left(\left|Z_t^{x,y}\right|^2\right) = |y-x|^2 + \int_0^t \mathbb{E}\left(2\left\langle B_r^{x,y} Z_r^{x,y}, Z_r^{x,y}\right\rangle + \sum_{k=1}^d \left|C_r^{x,y,k} Z_r^{x,y}\right|^2\right) dr.$$

Hence, thanks to Cauchy-Schwarz inequality and to the definition of K^b and $K^{\sigma,j}$ for all $j \in [\![1,d]\!]$

$$\mathbb{E}\left(\left|Z_t^{x,y}\right|^2\right) \leq |y-x|^2 + K\int_0^t \mathbb{E}\left(\left|Z_r^{x,y}\right|^2\right) dr$$

and we conclude via Gronwall's Lemma. □

5.4 Proof of Lemma 4.5

Let $\nu \in \mathcal{P}\left(\mathbb{R}^d\right)$. For each given $t \in [0,T]$, we denote by G_t the differential operator such that for all $f \in \mathcal{C}^2\left(\mathbb{R}^d\right)$

$$G_t f = \frac{1}{2}\sum_{i,j=1}^d \partial_{ij}\left(\Sigma_{ij}(t,\cdot) f\right) - \sum_{i=1}^d \partial_i\left(b_i(t,\cdot) f\right).$$

Assumption 6 implies that for a given $f \in \mathcal{C}^2\left(\mathbb{R}^d\right)$, $G_t f$ can be rewritten in the two following ways:

$$G_t f = \frac{1}{2}\sum_{i,j=1}^d \Sigma_{ij}(t,\cdot)\partial_{ij} f + \sum_{i=1}^d (\sum_{j=1}^d \partial_i\Sigma_{ij}(t,\cdot) - b_i(t,\cdot))\partial_i f + c^1(t,\cdot) f, \tag{5.13}$$

with

$$c^1 : (t,x) \mapsto \frac{1}{2}\sum_{i,j=1}^d \partial_{ij}\Sigma_{ij}(t,x) - \sum_{i=1}^d \partial_i b_i(t,x).$$

$$G_t f = \frac{1}{2}\sum_{i,j=1}^d \partial_j(\partial_i\Sigma_{ij}(t,\cdot) f + \Sigma_{ij}(t,\cdot)\partial_i f - \sum_{i=1}^d b_i(t,\cdot)\partial_i f) - \sum_{i=1}^d \partial_i b_i(t,\cdot) f. \tag{5.14}$$

On the one hand, combining identity (5.13) with Assumptions 2, 3, 4 and 6, there exists a fundamental solution Γ (in the sense of Definition stated in Section 1. p.3 of [10]) of $\partial_t u = G_t u$, thanks to Theorem 10. Section 6 Chap. 1. in the same reference. Furthermore, there exists $C_1, C_2 > 0$ such that for all $i \in [\![1,d]\!]$, $x, \xi \in \mathbb{R}^d$, $\tau \in [0,T]$, $t > \tau$,

$$\left|\Gamma(x,t,\xi,\tau)\right| \leq C_1 (t-\tau)^{-\frac{d}{2}} \exp\left(-\frac{C_2 |x-\xi|^2}{4(t-\tau)}\right), \tag{5.15}$$

$$\left|\partial_{x_i}\Gamma(x,t,\xi,\tau)\right| \leq C_1 (t-\tau)^{-\frac{d+1}{2}} \exp\left(-\frac{C_2 |x-\xi|^2}{4(t-\tau)}\right), \tag{5.16}$$

thanks to identities (6.12), (6.13) in Section 6 Chap. 1 in [10].

On the other hand, combining Identity (5.14) with Assumption 6, there exists a so called weak fundamental solution Θ of $\partial_t u = G_t u$ thanks to Theorem 5 in [1]. In addition, there exists $K_1, K_2, K_3 > 0$ such that for almost every $x, \xi \in \mathbb{R}^d$, $\tau \in [0,T]$, $t \geq \tau$

$$\frac{1}{K_1}(t-\tau)^{-\frac{d}{2}} \exp\left(-\frac{K_2 |x-\xi|^2}{4(t-\tau)}\right) \leq \Theta(x,t,\xi,\tau) \leq K_1 (t-\tau)^{-\frac{d}{2}} \exp\left(-\frac{K_3 |x-\xi|^2}{4(t-\tau)}\right), \tag{5.17}$$

thanks to point (ii) of Theorem 10 in [1].

Our goal is now to show that Γ and Θ coincide. To this end, we adapt the argument developed at the beginning of Section 7 in [1]. Fix a function H from $[0,T]\times\mathbb{R}^d$ belonging to $\mathcal{C}_c^\infty\left([0,T]\times\mathbb{R}^d\right)$. Identity (7.6) in Theorem 12 Chap 1. Section 1. of [10] implies in particular that the function

$$u:(t,x)\mapsto\int_0^t\int_{\mathbb{R}^d}\Gamma(x,t,\xi,\tau)\,H(\tau,\xi)\,d\xi d\tau,$$

is continuously differentiable in time, two times continuously differentiable in space and is a solution of the Cauchy problem

$$\begin{cases}\partial_t u(t,x)=G_t u(t,x)+H(t,x)\,,\ (t,x)\in]0,T]\times\mathbb{R}^d,\\ u(0,\cdot)=0.\end{cases}\tag{5.18}$$

It is consequently also a weak (i.e. distributional) solution of (5.18), which belongs to $\mathcal{E}^2(]0,T]\times\mathbb{R}^d)$ (see definition of that space in [1]) since u is bounded thanks to inequality (5.15) and the fact that H is bounded. Then, point (ii) of Theorem 5 in [1] says that

$$(t,x)\mapsto\int_0^t\int_{\mathbb{R}^d}\Theta(x,t,\xi,\tau)\,H(\tau,\xi)\,d\xi d\tau$$

is the unique weak solution in $\mathcal{E}^2(]0,T]\times\mathbb{R}^d)$ of (5.18). This implies that for every $(t,x)\in]0,T]\times\mathbb{R}^d$ we have

$$\int_0^t\int_{\mathbb{R}^d}(\Gamma-\Theta)(x,t,\xi,\tau)\,H(\tau,\xi)\,d\xi d\tau=0.$$

Point (i) of Theorem 5 in [1] (resp inequality (5.15)) implies that Θ (resp. Γ) belongs to $L^p\left(]0,T]\times\mathbb{R}^d\right)$ as a function of (ξ,τ), for an arbitrary $p\geq d+2$. Then, we conclude that for all $(t,x)\in]0,T]\times\mathbb{R}^d$,

$$\Theta(x,t,\xi,\tau)=\Gamma(x,t,\xi,\tau)\,,\ d\xi d\tau a.e.\tag{5.19}$$

for all $(\tau,\xi)\in[0,t[\times\mathbb{R}^d$. This happens by density of $\mathcal{C}_c^\infty\left([0,T]\times\mathbb{R}^d\right)$ in $L^q\left(]0,T]\times\mathbb{R}^d\right)$, q being the conjugate of p.

This, together with (5.17) and the fact that Γ is continuous in (τ,ξ) implies that (5.17) holds for all $(\tau,\xi)\in[0,t[\times\mathbb{R}^d$ and therefore

$$\frac{1}{K_1}(t-\tau)^{-\frac{d}{2}}\exp\left(-\frac{K_2|x-\xi|^2}{4(t-\tau)}\right)\leq\Gamma(x,t,\xi,\tau)\leq K_1(t-\tau)^{-\frac{d}{2}}\exp\left(-\frac{K_3|x-\xi|^2}{4(t-\tau)}\right).\tag{5.20}$$

We introduce

$$q_t:=x\mapsto\int_{\mathbb{R}^d}\Gamma(x,t,\xi,0)\,\nu(d\xi)\,.$$

By (5.20), with $\tau=0$ we get

$$q_t(x)\geq\frac{1}{K_1}t^{-\frac{d}{2}}\int_{\mathbb{R}^d}\exp\left(-\frac{K_2|x-\xi|^2}{4t}\right)\nu(d\xi)\,.\tag{5.21}$$

We denote now by $\mathbf{v}^\nu$ the measure-valued mapping such that $\mathbf{v}^\nu(0,\cdot)=\nu$ and for all $t\in]0,T]$, $\mathbf{v}^\nu(t)$ has density q_t with respect to the Lebesgue measure on $\mathbb{R}^d$. We want to show that $\mathbf{v}^\nu$ is a solution of the PDE (3.2) with initial value ν to conclude $\mathbf{u}^\nu=\mathbf{v}^\nu$ thanks to the validity of Property 1 because of Lemma 3.4 and Assumptions 2, 3 and 4. To this end, we remark that the definition of a fundamental solution for $\partial_t u=G_t u$ says that u is a $C^{1,2}$ solution and consequently also a solution in the sense of distributions. In particular for all $\phi\in\mathcal{C}_c^\infty\left(\mathbb{R}^d\right)$, for all $t\geq\epsilon>0$

$$\int_{\mathbb{R}^d}\phi(x)\,\mathbf{v}^\nu(t)(dx)=\int_{\mathbb{R}^d}\phi(x)\,\mathbf{v}^\nu(\epsilon)(dx)+\int_\epsilon^t\int_{\mathbb{R}^d}L_s\phi(x)\,\mathbf{v}^\nu(s)(dx)\,ds.\tag{5.22}$$

To conclude, it remains to send ϵ to $0+$. Theorem 15 section 8. Chap 1. and point (ii) of the definition stated p. 27 in [10] imply in particular that for all $\phi \in \mathcal{C}_c^\infty\left(\mathbb{R}^d\right)$, $\xi \in \mathbb{R}^d$,

$$\int_{\mathbb{R}^d} \Gamma\left(x, \epsilon, \xi, 0\right) \phi\left(x\right) dx \underset{\epsilon \to 0+}{\longrightarrow} \phi\left(\xi\right).$$

Fix now $\phi \in \mathcal{C}_c^\infty\left(\mathbb{R}^d\right)$. In particular thanks to Fubini's theorem, (5.17) and Lebesgue's dominated convergence theorem we have

$$\begin{aligned}\int_{\mathbb{R}^d} \phi\left(x\right) \mathbf{v}^\nu\left(\epsilon\right)\left(dx\right) &= \int_{\mathbb{R}^d} \phi\left(x\right) \int_{\mathbb{R}^d} \Gamma\left(x, \epsilon, \xi, 0\right) \nu\left(d\xi\right) dx \\ &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \Gamma\left(x, \epsilon, \xi, 0\right) \phi\left(x\right) dx \nu\left(d\xi\right) \\ &\underset{\epsilon \to 0+}{\longrightarrow} \int_{\mathbb{R}^d} \phi\left(\xi\right) \nu\left(d\xi\right).\end{aligned}$$

By (5.22) $\mathbf{v}^\nu$ is a solution of the PDE (3.2) and consequently $\mathbf{u}^\nu = \mathbf{v}^\nu$, so that, for every $t \in]0, T]$, $\mathbf{u}^\nu\left(t\right)$ admits $u^\nu(t, \cdot) = q_t$ for density with respect to the Lebesgue measure on $\mathbb{R}^d$. Now, integrating the inequalities (5.15), (5.16) with respect to ν and combining this with inequality (5.21), we obtain the existence of $K_1, K_2, C_1, C_2 > 0$ such that for all $t \in]0, T]$, for all $x \in \mathbb{R}^d$, for all $i \in [\![1, d]\!]$

$$\frac{1}{K_1} t^{-\frac{d}{2}} \int_{\mathbb{R}^d} \exp\left(-\frac{K_2\left|x - \xi\right|^2}{4t}\right) \nu\left(d\xi\right) \leq u^\nu\left(t, x\right) \leq K_1 t^{-\frac{d}{2}},$$

$$\left|\partial_i u^\nu\left(t, x\right)\right| \leq C_1 t^{-\frac{d+1}{2}}.$$

Consequently, the upper bounds in (4.5) and (4.6) hold. Concerning the lower bound in (4.5), let I be a compact subset of $\mathbb{R}^d$ such that $\nu(I) > 0$, the result follows since $(t, x, \xi) \mapsto \exp\left(-\frac{K_2|x-\xi|^2}{4t}\right)$ is strictly positive, continuous and therefore lower bounded by a strictly positive constant on $K \times I$ for each compact K of $]0, T] \times \mathbb{R}^d$.

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