McKean SDEs with singular coefficients

Elena Issoglio and Francesco Russo

Abstract. The paper investigates existence and uniqueness for a stochastic differential equation (SDE) with distributional drift depending on the law density of the solution. Those equations are known as McKean SDEs. The McKean SDE is interpreted in the sense of a suitable singular martingale problem. A key tool used in the investigation is the study of the corresponding Fokker-Planck equation.

Key words and phrases. Stochastic differential equations; distributional drift; McKean; Martingale problem.
2020 MSC. 60H10; 60H30; 35C99; 35D99; 35K10.

1. Introduction

In this paper we are concerned with the study of *singular McKean SDEs* of the form

$$\begin{cases} X_t = X_0 + \int_0^t F(v(s,X_s))b(s,X_s)\mathrm{d}s + W_t \\ v(t,\cdot) \text{ is the law density of } X_t, \end{cases} \tag{1}$$

for some given initial condition X_0 with density v_0. The terminology *McKean* refers to the fact that the coefficient of the SDE depends on the law of the solution process itself, while *singular* reflects the fact that one of the coefficients is a Schwartz distribution. The main aim of this paper is to solve the singular McKean problem (1), that is, to define rigorously the meaning of equation (1) and to find a (unique) solution to the equation. The key novelty is the *irregularity* of the drift, which is encoded in the term b.

The problem is d-dimensional, in particular the process X takes values in $X \in \mathbb{R}^d$, the function F is $F : \mathbb{R} \to \mathbb{R}^{d\times n}$, the term b is formally $b : [0,T] \times \mathbb{R}^d \to \mathbb{R}^n$ and W is a d-dimensional Brownian motion, where n, d are two integers. We assume that $b(t,\cdot) \in \mathcal{C}^{-\beta}(\mathbb{R}^n)$ for some $0 < \beta < 1/2$ (see below for the definition of Besov spaces $\mathcal{C}^{-\beta}(\mathbb{R}^n)$), which means that $b(t,\cdot)$ is a Schwartz distribution and thus the term $b(t,X_t)$, as well as its product with F, are only formal at this stage. The function F is nonlinear.

When b is a function, equation (1) has been recently studied by several authors. For example [19] study existence and uniqueness of the solution under some regularity assumptions on the drift, while [22] requires the drift to be of a special form, being Lipschitz-continuous with respect to the variable v, uniformly in time and space, and measurable with respect to space. We

Date: 16th July 2021.

also mention [2], where the authors obtain existence of the solution when assuming that the drift is a measurable function. For other past contributions see [18].

In the literature we also find some contributions on (1) with $F \equiv 1$, i.e. when there is no dependence on the law v but the drift b is a Schwartz distribution. In this case equation (1) becomes an SDE with singular drift. Ordinary SDEs with distributional drift were investigated by several authors, starting from [10, 9, 3, 23] in the one-dimensional case. In the multidimensional case it was studied by [8] with b being a Schwartz distribution living in a fractional Sobolev space of negative order (up to $-\frac{1}{2}$). Afterwards, [5] extended the study to a smaller negative order (up to $-\frac{2}{3}$) and formulated the problem as a martingale problem. We also mention [17], where the singular SDE is studied as a martingale problem, with the same setting as in the present paper (in particular the drift belongs to a negative Besov space rather than a fractional Sobolev space). Backwards SDEs with similar singular coefficients have also been studied, see [15, 16].

The main analytical tool in the works cited above is the study of an associated singular PDE (either Kolmogorov or Fokker-Planck). In the McKean case, the relevant PDE associated to equation (1) is the nonlinear Fokker-Planck equation

$$\begin{cases} \partial_t v = \frac{1}{2}\Delta v - \operatorname{div}(\tilde{F}(v) b) \\ v(0) = v_0, \end{cases} \tag{2}$$

where $\tilde{F}(v) := vF(v)$. PDEs with similar (ir)regular coefficients were studied in the past, see for example [8, 13] for the study of singular Kolmogorov equations. One can then use results on existence, uniqueness and continuity of the solution to the PDE (e.g. with respect to the initial condition and the coefficients) to infer results about the stochastic equation. For example in [8], the authors use the singular Kolmogorov PDE to define the meaning of the solution to the SDE and find a unique solution.

Let us remark that the PDEs mentioned above are a classical tool in the study of McKean equations when the dependence on the law density of the process is pointwise, which is the case in the present paper where we have $F(v(t,x))$. There is, however, a large body of literature that studies McKean equations where the drift depends on the law more regularly, typically it is assumed to be Lipschitz-continuous with respect to the Wasserstein metric. In this case the McKean equation is treated with different techniques than the ones explained above, in particular it is treated with probabilistic tools. This is nowadays a well-known approach, for more details see for example the recent books by Carmona and Delarue [6, 7], see also [21, 20].

Our contribution to the literature is twofold. The first and main novel result concerns the notion of solution to the singular McKean equation (1) (introduced in Definition 6.1) and its existence and uniqueness (proved in Theorem 6.4). The second contribution is the study of the singular Fokker-Plank equation (2), in particular we find a unique solution $v \in C([0,T]; C^{\beta+})$

in the sense of Schwartz distributions, see Theorem 3.7 for existence and Proposition 4.7 for uniqueness.

The paper is organised as follows. In Section 2 we introduce the notation and recall some useful results on semigroups and Besov spaces. We also recall briefly some results on the singular martingale problem. In Section 3 we study the singular Fokker-Planck PDE (2). Then we consider a mollified version of the PDE and the SDE in Sections 4 and 5, respectively. Finally in Section 6 we use the mollified PDEs and SDEs and their limits to study (1) and we prove our main theorem of existence and uniqueness of a solution to (1). In Appendix A we recall a useful fractional Gronwall's inequality. In Appendix B we show a characterization of continuity and compactness in inductive spaces.

2. Setting and useful results

2.1. **Notation and definitions.** Let us denote by $C^{1,2}_{buc} := C^{1,2}_{buc}([0,T]\times\mathbb{R}^d)$ the space of all $C^{1,2}$-functions such that the function and its gradient in x are bounded, and the Hessian matrix is uniformly continuous. We also use the notation $C^{0,1} := C^{0,1}([0,T]\times\mathbb{R}^d)$ to indicate the space of functions with gradient in x uniformly continuous in (t,x). By a slight abuse of notation we use the same notation $C^{1,2}_{buc}$ and $C^{0,1}$ for functions which are $\mathbb{R}^d$-valued. When $f:\mathbb{R}^d\to\mathbb{R}^d$ is differentiable, we denote by ∇f the matrix given by $(\nabla f)_{i,j} = \partial_i f_j$. When $f:\mathbb{R}^d\to\mathbb{R}$ we denote the Hessian matrix of f by $\mathrm{Hess}(f)$.

We denote by $\mathcal{S} = \mathcal{S}(\mathbb{R}^d)$ the space of Schwartz functions on $\mathbb{R}^d$ and by $\mathcal{S}' = \mathcal{S}'(\mathbb{R}^d)$ the space of Schwartz distributions. For $\gamma\in\mathbb{R}$ we denote by $\mathcal{C}^\gamma = \mathcal{C}^\gamma(\mathbb{R}^d)$ the Besov space or Hölder-Zygmund space and by $\|\cdot\|_\gamma$ its norm (see for example [1, Section 2.7]). We recall that for $\gamma'<\gamma$ one has $\mathcal{C}^\gamma\subset\mathcal{C}^{\gamma'}$. If $\gamma\in\mathbb{R}^+\setminus\mathbb{N}$ then the space coincides with the classical Hölder space of functions which are $\lfloor\gamma\rfloor$-times differentiable and such that the $\lfloor\gamma\rfloor$th derivative is $(\gamma-\lfloor\gamma\rfloor)$-Hölder continuous. For example if $\gamma\in(0,1)$ the classical γ-Hölder norm

$$\|f\|_\infty + \sup_{x\neq y, |x-y|<1} \frac{|f(x)-f(y)|}{|x-y|^\gamma}, \tag{3}$$

is an equivalent norm in $\mathcal{C}^\gamma$. With an abuse of notation we use $\|f\|_\gamma$ to denote (3). For this and for more details see, for example, [25, Chapter 1] or [1, Section 2.7]. Notice that we use the same notation $\mathcal{C}^\gamma$ to indicate $\mathbb{R}$-valued functions but also $\mathbb{R}^d$ or $\mathbb{R}^{d\times d}$-valued functions. It will be clear from the context which space is needed.

We denote by $C_T\mathcal{C}^\gamma$ the space of continuous functions on $[0,T]$ taking values in $\mathcal{C}^\gamma$, that is $C_T\mathcal{C}^\gamma := C([0,T];\mathcal{C}^\gamma)$. For any given $\gamma\in\mathbb{R}$ we denote by $\mathcal{C}^{\gamma+}$ and $\mathcal{C}^{\gamma-}$ the spaces given by

$$\mathcal{C}^{\gamma+} := \cup_{\alpha>\gamma}\mathcal{C}^\alpha, \qquad \mathcal{C}^{\gamma-} := \cap_{\alpha<\gamma}\mathcal{C}^\alpha.$$

Notice that $\mathcal{C}^{\gamma+}$ is an inductive space. We will also use the spaces $C_T C^{\gamma+} := C([0,T];\mathcal{C}^{\gamma+})$, with the meaning that $f \in C_T C^{\gamma+}$ if and only if there exists $\alpha > \gamma$ such that $f \in C_T C^\alpha$, see Lemma B.2 in Appendix B for a proof of the latter fact.

Similarly, we use the space $C_T C^{\gamma-} := C([0,T];\mathcal{C}^{\gamma-})$, meaning that $f \in C_T C^{\gamma-}$ if and only if for any $\alpha < \gamma$ we have $f \in C_T C^\alpha$. Notice that if f is continuous and such that $\nabla f \in C_T \mathcal{C}^{0+}$ then $f \in C^{0,1}$.

Let $(P_t)_t$ denote the semigroup generated by $\frac{1}{2}\Delta$ on $\mathcal{S}$, in particular for all $\phi \in \mathcal{S}$ we define $(P_t\phi)(x) := \int_{\mathbb{R}^d} p_t(x-y)\phi(y)\mathrm{d}y$, where the kernel p is the usual heat kernel

$$p(t,z) = \frac{1}{(2\pi t)^{d/2}} \exp\{-\frac{|z|^2}{t}\}. \tag{4}$$

It is easy to see that $P_t : \mathcal{S} \to \mathcal{S}$. Moreover we can extend it to $\mathcal{S}'$ by dual pairing (and we denote it with the same notation for simplicity). One has $\langle P_t\psi, \phi\rangle = \langle \psi, P_t\phi\rangle$ for each $\phi \in \mathcal{S}$ and $\psi \in \mathcal{S}'$, using the fact that the kernel is symmetric.

Lemma 2.1. *Let $g : [0,T] \to \mathcal{S}'(\mathbb{R}^d)$ be continuous and $w_0 \in \mathcal{S}'(\mathbb{R}^d)$. The unique (weak) solution of*

$$\begin{cases} \partial_t w = \frac{1}{2}\Delta w + g \\ w(0) = w_0 \end{cases}$$

is given by

$$P_t w_0 + \int_0^t P_{t-s} g(s)\mathrm{d}s, \quad t \in [0,T]. \tag{5}$$

By weak solution we mean, for every $\varphi \in \mathcal{S}(\mathbb{R}^d)$ and $t \in [0,T]$ we have $\langle w(t), \varphi\rangle = \langle w_0, \varphi\rangle + \int_0^t \langle w(s), \frac{1}{2}\Delta\varphi\rangle \mathrm{d}s + \int_0^t \langle g(s), \varphi\rangle \mathrm{d}s$.

Proof. The fact that (5) is a solution is done by inspection. The uniqueness is a consequence of Fourier transform. □

In the whole article the letter c or C will denote a generic constant which may change from line to line.

2.2. **Some useful results.** In the sections below, we are interested in the action of P_t on elements of Besov spaces $\mathcal{C}^\gamma$. These estimates are known as *Schauder's estimates*. For a proof we refer to [11], see also [12] for similar results.

Lemma 2.2 (Schauder's estimates)**.** *Let $f \in \mathcal{C}^\gamma$ for some $\gamma \in \mathbb{R}$. Then for any $\theta \geq 0$ there exists a constant c such that*

$$\|P_t f\|_{\gamma+2\theta} \leq c t^{-\theta} \|f\|_\gamma, \tag{6}$$

for all $t > 0$.

Moreover for $f \in \mathcal{C}^\gamma$ and for any $\theta \in (0,1)$ we have

$$\|P_t f - f\|_\gamma \leq c t^\theta \|f\|_{\gamma+2\theta}. \tag{7}$$

Notice that from (7) it readily follows that if $f \in \mathcal{C}^\gamma$ for some $0 < \theta < 1$, then for $t > s > 0$ we have

$$\|P_t f - P_s f\|_\gamma \le c(t-s)^\theta \|f\|_{\gamma+2\theta}. \tag{8}$$

In other words, this means that if $f \in \mathcal{C}^{\gamma+\theta}$ then $P_\cdot f \in C_T\mathcal{C}^\gamma$ (and in fact it is θ-Hölder continuous in time). We also recall that Bernstein's inequalities hold (see [1, Lemma 2.1] and [12, Appendix A.1]), that is for $\gamma \in \mathbb{R}$ there exists a constant $c > 0$ such that

$$\|\nabla g\|_\gamma \le c\|g\|_{\gamma+1}, \tag{9}$$

for all $g \in \mathcal{C}^{1+\gamma}$. Using Schauder's and Bernstein's inequalities we can easily obtain a useful estimate on the gradient of the semigroup, as we see below.

Lemma 2.3. *Let $\gamma \in \mathbb{R}$ and $\theta \in (0,1)$. If $g \in \mathcal{C}^\gamma$ then for all $t > 0$ we have $\nabla(P_t g) \in \mathcal{C}^{\gamma+2\theta-1}$ and*

$$\|\nabla(P_t g)\|_{\gamma+2\theta-1} \le c t^{-\theta}\|g\|_\gamma. \tag{10}$$

The following is an important estimate which allows to define the so called *pointwise product* between certain distribution and functions, which is based on Bony's estimates. For details see [4] or [12, Section 2.1]. Let $f \in \mathcal{C}^\alpha$ and $g \in \mathcal{C}^{-\beta}$ with $\alpha - \beta > 0$ and $\alpha, \beta > 0$. Then the pointwise product $f\,g$ is well-defined as an element of $\mathcal{C}^{-\beta}$ and there exists a constant $c > 0$ such that

$$\|f\,g\|_{-\beta} \le c\|f\|_\alpha \|g\|_{-\beta}. \tag{11}$$

Moreover if f and g are continuous functions defined on $[0,T]$ with values in the above Besov spaces, then the product is also continuous with values in $\mathcal{C}^{-\beta}$, and

$$\|f\,g\|_{C_T\mathcal{C}^{-\beta}} \le c\|f\|_{C_T\mathcal{C}^\alpha}\|g\|_{C_T\mathcal{C}^{-\beta}}. \tag{12}$$

2.3. **Assumptions.** We now collect the assumptions on the distributional term b, the nonlinearity F and $\tilde F$ and on the initial condition v_0 that will be used later on in order for PDE (2) to be well-defined and for the McKean-Vlasov problem (1) to be solved.

Assumption 1. *Let $0 < \beta < 1/2$ and $b \in C_T\mathcal{C}^{-\beta}$.*

Assumption 2. *Let (b^n) be a sequence of bounded functions in $C_T\mathcal{C}^{-\beta}$ that converges to b in $C_T\mathcal{C}^{-\beta}$. Moreover for each n, let $b^n(t,\cdot) \in C_b^\infty$ for all $t \in [0,T]$.*

Example 2.4. *Assumption 2 is satisfied for instance in the following case. Let $\beta' < \beta$ and let $b \in C_T\mathcal{C}^{-\beta'}$. We define the sequence (b^n) for any fixed $t \in [0,T]$ and for all $n \ge 1$ by*

$$b^n(t,\cdot) := \phi_n * b(t,\cdot),$$

where $\phi_n(x) := p_{1/n^2}(x)$ and p is the Gaussian kernel defined in (4). If $\psi \in \mathcal{S}'$ then $\phi_n * \psi = P_{1/n^2}\psi$, thus we have $b_n(t,\cdot) = P_{1/n^2} b(t,\cdot)$.

(i) *We now show that* $t \mapsto b^n(t,\cdot)$ *is continuous in* $\mathcal{C}^{-\beta}$. *For any* $t,s \in [0,T]$ *we have*

$$\begin{aligned}\|b^n(t,\cdot)-b^n(s,\cdot)\|_{-\beta} =&\|P_{1/n^2}b(t,\cdot)-P_{1/n^2}b(s,\cdot)\|_{-\beta}\\ =&\|P_{1/n^2}(b(t,\cdot)-b(s,\cdot))\|_{-\beta}\\ \leq& c\|b(t,\cdot)-b(s,\cdot)\|_{-\beta}\\ \leq& c\|b(t,\cdot)-b(s,\cdot)\|_{-\beta'},\end{aligned}$$

having used estimate (6) *in Lemma 2.2 (with* $\theta=0$*). The conclusion now follows.*

(ii) *We show* $b^n \to b$ *in* $C_T\mathcal{C}^{-\beta}$. *For* $t \in [0,T]$ *we have*

$$\begin{aligned}\|b^n(t,\cdot)-b(t,\cdot)\|_{-\beta} =&\|P_{1/n^2}b(t,\cdot)-b(t,\cdot)\|_{-\beta}\\ \leq& c\left(\frac{1}{n^2}\right)^{\frac{\beta-\beta'}{2}}\|b(t,\cdot)\|_{-\beta'},\end{aligned}$$

having used (7) *in Lemma 2.2. Now we take the sup over* $t\in[0,T]$ *and we have* $\|b^n-b\|_{C_T\mathcal{C}^{-\beta}}\to 0$ *as* $n\to\infty$*, since* $\beta-\beta'>0$.

Assumption 3. *Let* F *be Lipschitz and bounded.*

Assumption 4. *Let* $\tilde F(z) := zF(z)$ *be globally Lipschitz.*

We believe that Assumption 4 is unnecessary. Indeed by Assumption 3 one gets that $\tilde F$ is locally Lipschitz with linear growth. This condition could be sufficient to show that a solution PDE (2) exists, for example using techniques similar to the ones appearing in [14, Proposition 3.1] and [22, Theorem 22]. However we assume here $\tilde F$ to be Lipschitz to improve the readability of the paper.

Assumption 5. *Let* $v_0 \in \mathcal{C}^{\beta+}$.

Assumption 6. *Let* v_0 *be a bounded probability density.*

2.4. **The singular Martingale Problem.** We conclude this section with a short recap of useful results from [17], where the authors consider the Martingale Problem for SDEs of the form

$$X_t = X_0 + \int_0^t B(s,X_s)\mathrm{d}s + W_t, \tag{13}$$

where B satisfies Assumption 1 (with $b=B$). Notice that this SDE can be considered as the linear counterpart of the McKean-Vlasov problem (1), which can be obtained for example by 'fixing' a suitable function v and considering $B=F(v)b$ in the SDE in (1).

First of all, let us recall the definition of the operator $\mathcal{L}$ associated to SDE (13) given in [17]. The operator $\mathcal{L}$ is defined as

$$\begin{array}{rccl}\mathcal{L}: & \mathcal{D}^0_{\mathcal{L}} & \to & \{\mathcal{S}'\text{-valued integrable functions}\}\\ & f & \mapsto & \mathcal{L}f := \dot f + \frac12\Delta f + \nabla f\, B,\end{array} \tag{14}$$

where

$$\mathcal{D}^0_{\mathcal{L}} := C_T\mathcal{C}^{(1+\beta)+} \cap AC([0,T];\mathcal{S}')$$

and AC stands for "absolutely continuous". Here $f : [0,T]\times\mathbb{R}^d \to \mathbb{R}$ and the function $\dot f : [0,T] \to \mathcal{S}'$ is defined for any $f \in \mathcal{D}^0_{\mathcal{L}}$ as the unique $\dot f$ such that $f(t) - f(0) = \int_0^t \dot f(s) ds$, which always exists because f is absolutely continuous. Note also that $\nabla f\, B$ is well-defined using (11) and Assumption 1. The Laplacian Δ is intended in the sense of distributions.

Next we give the definition of solution to the martingale problem in [17]: a couple $(X,\mathbb{P})$ is a *solution to the martingale problem with distributional drift B* (for shortness, solution of MP) if and only if for every $f \in \mathcal{D}_{\mathcal{L}}$

$$f(t,X_t) - f(0,X_0) - \int_0^t (\mathcal{L}f)(s,X_s)\mathrm{d}s \tag{15}$$

is a local martingale under $\mathbb{P}$, where the domain $\mathcal{D}_{\mathcal{L}}$ is given by

$$\begin{aligned}\mathcal{D}_{\mathcal{L}} := \quad & \{f \in \mathcal{D}^0_{\mathcal{L}} : \text{ such that } \exists g \in C_T\mathcal{C}^{0+} \\ & \text{such that } f \text{ is a weak solution of } \mathcal{L}f = g\},\end{aligned} \tag{16}$$

and $\mathcal{L}$ has been defined in (14). We say that *the martingale problem with distributional drift B admits uniqueness* if, whenever we have two solutions $(X^1,\mathbb{P}^1)$ and $(X^2,\mathbb{P}^2)$, then the law of X^1 under $\mathbb{P}^1$ equals the law of X^2 under $\mathbb{P}^2$. With this definition at hand, we show in [17] that MP admits existence and uniqueness.

3. Fokker-Planck singular PDE

This section is devoted to the study of the singular Fokker-Planck equation (2), recalled here for ease of reading

$$\begin{cases} \partial_t v = \frac{1}{2}\Delta v - \mathrm{div}(\tilde F(v) b) \\ v(0) = v_0. \end{cases}$$

After introducing the notions of solution for this PDE (weak and mild, which turns out to be equivalent), we will show that there exists a solution in Theorem 3.7 with Schaefer's fixed point theorem. We will show with different techniques (see Section 4, in Proposition 4.7), that such solution is unique.

Below we will need mapping properties of the function $\tilde F$ when viewed as operator acting on $\mathcal{C}^\alpha$, for some $\alpha \in (0,1)$. To this aim, we make a slight abuse of notation and denote by $\tilde F$ the function when viewed as an operator, that is for $f \in \mathcal{C}^\alpha$ we have $\tilde F(f) := \tilde F(f(\cdot))$. We sometimes omit the brackets and write $\tilde F f$ in place of $\tilde F(f)$. The result below on $\tilde F$ is taken from [14], Proposition 3.1 and equation (32).

Lemma 3.1 (Issoglio [14]). *Under Assumption 4 and if $\alpha \in (0,1)$ then*

- $\tilde F : \mathcal{C}^\alpha \to \mathcal{C}^\alpha$ *and for all $f,g \in \mathcal{C}^\alpha$*

$$\|\tilde F f - \tilde F g\|_\alpha \le c(1 + \|f\|^2_\alpha + \|g\|^2_\alpha)^{1/2}\|f-g\|_\alpha;$$

- *for all* $f \in \mathcal{C}^\alpha$ *then* $\|\tilde F f\|_\alpha \le c(1+\|f\|_\alpha)$.

This mapping property allows us to define weak and mild solutions for the singular Fokker-Planck equation.

Definition 3.2. *Let Assumptions 1, 4 and 5 hold and let* $v \in C_T\mathcal{C}^{\beta+}$.

(i) *We say that* v *is a* mild solution *for the singular Fokker-Planck equation* (2) *if for all* $t \in [0,T]$ *the following integral equation is satisfied*

$$v(t) = P_t v_0 - \int_0^t P_{t-s}[\mathrm{div}(\tilde F(v(s))b(s))]\mathrm{d}s. \tag{17}$$

(ii) *We say that* v *is a* weak solution *for the singular Fokker-Planck equation* (2) *if for all* $\varphi \in \mathcal{S}(\mathbb{R}^d)$ *and all* $t \in [0,T]$ *we have*

$$\langle \varphi, v(t)\rangle = \langle \varphi, v_0\rangle + \int_0^t \langle \frac{1}{2}\Delta\varphi, v(s)\rangle \mathrm{d}s + \int_0^t \langle \nabla\varphi, \tilde F(v)(s)b(s)\rangle \mathrm{d}s. \tag{18}$$

Note that the term $\tilde F(v)(s)b(s)$ *appearing in both items is well-defined as an element of* $\mathcal{C}^{-\beta}$ *thanks to* (11) *and Assumption 1 together with Lemma 3.1.*

Proposition 3.3. *Let* $v \in C_T\mathcal{C}^{\beta+}$. *The function* v *is a weak solution of PDE* (2) *if and only if it is a mild solution.*

Proof. This is a consequence of Lemma 2.1 with $g(s) := -\mathrm{div}(\tilde F(v(s))b(s))$. □

We are interested in finding a mild solution of (2) in the space $C_T\mathcal{C}^{\beta+}$. To do so we will apply Schafer's fixed point theorem, following similar ideas as done in [14, Section 4]. To this end, we state and prove a few preparatory results, including a priori estimates and mapping properties of the solution map. Let us denote by J the solution map for the mild solution of PDE (2), that is for $v \in C_T\mathcal{C}^\alpha$ for some $\alpha \in (0,1)$ we have

$$J_t(v) := P_t v_0 - \int_0^t P_{t-s}[\mathrm{div}(\tilde F(v(s))b(s))]\mathrm{d}s.$$

Then a mild solution of (2) is a solution of $v = J(v)$, in other words it is a fixed point of J. In the proofs below we will also use the notation

$$G_s(v) := \tilde F(v(s))b(s) \tag{19}$$

for brevity.

We present now an a priori bound for mild solutions, if they exist.

Proposition 3.4. *Let Assumptions 1, 4 and 5 hold. Let* $\alpha \in (\beta, 1-\beta)$. *If* $v \in C_T\mathcal{C}^\alpha$ *is such that* $v = \lambda J(v)$ *for some* $\lambda \in [0,1]$, *then we have*

$$\|v\|_{C_T\mathcal{C}^\alpha} \le K,$$

where K *is a constant depending on* $\|v_0\|_\alpha, \|b\|_{C_T\mathcal{C}^{-\beta}}, T$ *but independent of* λ. *Moreover* K *is an increasing function of* $\|b\|_{C_T\mathcal{C}^{-\beta}}$.

It is obvious that choosing $\lambda = 1$ we get a mild solution of (2). Here we state the result for more general λ because it will be needed later on.

Proof. Using Bernstein's inequality (9) we get

$$\|\text{div} G_s(v)\|_{-\beta-1} \leq \sum_{i=1}^{d} \|\frac{\partial}{\partial x_i} G_s(v)\|_{-\beta-1} \leq c \sum_{i=1}^{d} \|G_s(v)\|_{-\beta}.$$

Then using the definition of G from (19), pointwise product property (11) (since $\alpha - \beta > 0$) and Lemma 3.1 we have

$$\|\text{div} G_s(v)\|_{-\beta-1} \leq c\|\tilde{F}(v(s))\|_\alpha \|b(s)\|_{-\beta} \leq c(1 + \|v(s)\|_\alpha)\|b(s)\|_{-\beta}. \quad (20)$$

Now using this, together with Schauder's estimates (Lemma 2.2 with $\theta := \frac{\alpha+\beta+1}{2}$) and the fact that $\theta < 1$, for fixed $t \in [0, T]$, one obtains

$$\begin{aligned}
\|v(t)\|_\alpha &\leq \|\lambda P_t v_0\|_\alpha + \int_0^t \|\lambda P_{t-s}[\text{div} G_s(v)]\|_\alpha \text{d}s \\
&\leq c\|v_0\|_\alpha + \int_0^t c(t-s)^{-\frac{\alpha+\beta+1}{2}} \|\text{div} G_s(v)\|_{-\beta-1} \text{d}s \\
&\leq c\|v_0\|_\alpha + c\int_0^t c(t-s)^{-\frac{\alpha+\beta+1}{2}} (1 + \|v(s)\|_\alpha)\|b(s)\|_{-\beta} \text{d}s \\
&\leq c\|v_0\|_\alpha + c\|b\|_{C_T\mathcal{C}^{-\beta}} \int_0^t (t-s)^{-\frac{\alpha+\beta+1}{2}} (1 + \|v(s)\|_\alpha) \text{d}s \\
&\leq c\|v_0\|_\alpha + c\|b\|_{C_T\mathcal{C}^{-\beta}} T^{\frac{1-\alpha-\beta}{2}} + c\|b\|_{C_T\mathcal{C}^{-\beta}} \int_0^t (t-s)^{-\frac{\alpha+\beta+1}{2}} \|v(s)\|_\alpha \text{d}s.
\end{aligned}$$

Now by a generalised Gronwall's inequality (see Lemma A.1) we have

$$\|v(t)\|_\alpha \leq [c\|v_0\|_\alpha + c\|b\|_{C_T\mathcal{C}^{-\beta}} T^{\frac{1-\alpha-\beta}{2}}] E_\eta(c\|b\|_{C_T\mathcal{C}^{-\beta}} \Gamma(\eta) t^\eta),$$

with $\eta = -\frac{\alpha+\beta+1}{2} + 1 = \frac{1-\alpha-\beta}{2} > 0$ and where E_η is the Mittag-Leffler function, see Lemma A.1. Now taking the sup over $t \in [0, T]$ and using the fact that E_η is increasing we get

$$\begin{aligned}
&\|v\|_{C_T\mathcal{C}^\alpha} \\
&\quad \leq \left[c\|v_0\|_\alpha + c\|b\|_{C_T\mathcal{C}^{-\beta}} T^{\frac{1-\alpha-\beta}{2}}\right] E_\eta \left(c\|b\|_{C_T\mathcal{C}^{-\beta}} \Gamma\left(\frac{1-\alpha-\beta}{2}\right) T^{\frac{1-\alpha-\beta}{2}}\right) \\
&\quad \leq \left[c\|v_0\|_\alpha + c\|b\|_{C_T\mathcal{C}^{-\beta}} T\right] E_\eta \left(c\|b\|_{C_T\mathcal{C}^{-\beta}} \Gamma(1) T\right) \\
&\quad =: K.
\end{aligned}$$

This concludes the proof. □

The next result is about mapping properties of the solution map J.

Proposition 3.5. *Let Assumptions 1, 4 and 5 hold. Let us fix $\alpha, \alpha', \varepsilon'$ such that $\beta < \alpha \leq \alpha' < 1 - \beta$, $1 - \alpha' - \beta - 2\varepsilon' > 0$, $\varepsilon' > 0$ and such that $v_0 \in \mathcal{C}^{\alpha'+2\varepsilon'}$, which is always possible thanks to Assumptions 1 and 5.*

(i) *For any* $v \in C_T\mathcal{C}^\alpha$ *we have*

$$\|J(v)\|_{C_T^{\varepsilon'}\mathcal{C}^{\alpha'}} \leq c\|v_0\|_{\alpha'+2\varepsilon'} + c\|b\|_{C_T\mathcal{C}^{-\beta}} T^{\frac{1-\alpha'-\beta-2\varepsilon'}{2}}(1+\|v\|_{C_T\mathcal{C}^\alpha}) \quad (21)$$

and for any $v, u \in C_T\mathcal{C}^\alpha$ *we have*

$$\sup_{t\in[0,T]} \|J_t(v) - J_t(u)\|_{\alpha'} \leq cT^{\frac{1-\alpha'-\beta}{2}}(1+\|u\|^2_{C_T\mathcal{C}^\alpha} + \|v\|^2_{C_T\mathcal{C}^\alpha})^{1/2} \times \|v-u\|_{C_T\mathcal{C}^\alpha}\|b\|_{C_T\mathcal{C}^{-\beta}}, \quad (22)$$

$$\|J_t(v) - J_t(u) - (J_s(v) - J_s(u))\|_{\alpha'} \leq c(t-s)^{\varepsilon'} T^{\frac{1-\alpha'-\beta-2\varepsilon'}{2}}\|b\|_{C_T\mathcal{C}^{-\beta}} \times (1+\|u\|^2_{C_T\mathcal{C}^\alpha} + \|v\|^2_{C_T\mathcal{C}^\alpha})^{1/2}\|v-u\|_{C_T\mathcal{C}^\alpha}, \qquad \forall s < t \in [0,T]. \quad (23)$$

(ii) $J : C_T\mathcal{C}^\alpha \to C_T^{\varepsilon'}\mathcal{C}^{\alpha'}$ *and it is continuous.*
(iii) $J : C_T\mathcal{C}^{\beta+} \to C_T\mathcal{C}^{\beta+}$ *and it is continuous.*

Proof. Item (iii) is a direct consequence of Item (ii). Item (ii) is a consequence of Item (i). In fact, the mapping property follows from (21). As far as continuity is concerned, equations (22) and (23) allow us to bound the norm

$$\|J(v) - J(u)\|_{C_T^{\varepsilon'}\mathcal{C}^{\alpha'}} = \sup_{t\in[0,T]} \|J_t(v) - J_t(u)\|_{\alpha'} + \sup_{0\leq s<t\leq T} \frac{\|J_t(v) - J_t(u) - (J_s(v) - J_s(u))\|_{\alpha'}}{(t-s)^{\varepsilon'}}. \quad (24)$$

We now show Item (i) in 5 steps. Notice that $v_0 \in \mathcal{C}^{\alpha'}$ because $\alpha' < \alpha' + 2\varepsilon'$. Let $v \in C_T\mathcal{C}^\alpha$.

Step 1. Let $0 \leq t \leq T$. *We show that* $J_t(v) \in \mathcal{C}^{\alpha'}$.

Using the definition of J, Schauder's estimate for the semigroup, estimate (9) and the bound (20) we have

$$\begin{aligned}
\|J_t(v)\|_{\alpha'} &\leq \|P_t v_0\|_{\alpha'} + \int_0^t \|P_{t-s}[\mathrm{div} G_s(v)]\|_{\alpha'} \mathrm{d}s \\
&\leq c\|v_0\|_{\alpha'} + c\int_0^t (t-s)^{-\frac{\alpha'+\beta+1}{2}} \|\mathrm{div} G_s(v)\|_{-\beta-1} \mathrm{d}s \\
&\leq c\|v_0\|_{\alpha'} + c\int_0^t (t-s)^{-\frac{\alpha'+\beta+1}{2}} (1+\|v(s)\|_\alpha)\|b(s)\|_{-\beta} \mathrm{d}s \\
&\leq c\|v_0\|_{\alpha'} + cT^{\frac{1-\alpha'-\beta}{2}}(1+\|v\|_{C_T\mathcal{C}^\alpha})\|b\|_{C_T\mathcal{C}^{-\beta}}. \quad (25)
\end{aligned}$$

Step 2. Let $0 \leq s < t \leq T$. *We show*

$$\|J_t(v) - J_s(v)\|_{\alpha'} \leq c(t-s)^{\varepsilon'}\|v_0\|_{\alpha'+2\varepsilon'} + c(t-s)^{\varepsilon'} T^{\frac{1-\alpha'-\beta-2\varepsilon'}{2}}(1+\|v\|_{C_T\mathcal{C}^\alpha})\|b\|_{C_T\mathcal{C}^{-\beta}}. \quad (26)$$

We have

$$\begin{aligned}\|J_t(v)-J_s(v)\|_{\alpha'} \leq& \|(P_{t-s}-I)(P_s v_0)\|_{\alpha'}\\ &+\|\int_0^s (P_{t-s}-I)(P_{s-r}[\mathrm{div} G_r(v))]\mathrm{d}r\|_{\alpha'}\\ &+\|\int_s^t P_{t-r}[\mathrm{div} G_r(v)]\mathrm{d}r\|_{\alpha'}\\ =:& M_1+M_2+M_3.\end{aligned}$$

For M_1 we recall that $v_0 \in \mathcal{C}^{\alpha'+2\varepsilon'}$ and we use Schauder's estimate (7) (with $\gamma-2\theta=\alpha', \theta=\varepsilon'$) and continuity of the semigroup to get

$$M_1 \leq c(t-s)^{\varepsilon'}\|P_s v_0\|_{\alpha'+2\varepsilon'} \leq c(t-s)^{\varepsilon'}\|v_0\|_{\alpha'+2\varepsilon'}.$$

For M_2 we use (7) as well as (6) and (20) to get

$$\begin{aligned}M_2 &\leq c(t-s)^{\varepsilon'}\int_0^s \|(P_{s-r}[\mathrm{div} G_r(v))]\|_{\alpha'+2\varepsilon'}\mathrm{d}r\\ &\leq c(t-s)^{\varepsilon'}\int_0^s (s-r)^{-\frac{\alpha'+\beta+1+2\varepsilon'}{2}}\|\mathrm{div} G_r(v)\|_{-\beta-1}\mathrm{d}r\\ &\leq c(t-s)^{\varepsilon'}T^{\frac{1-\alpha'-\beta-2\varepsilon'}{2}}(1+\|v\|_{C_T\mathcal{C}^\alpha})\|b\|_{C_T\mathcal{C}^{-\beta}}.\end{aligned}$$

For M_3 we use only (6) and (20) to get

$$\begin{aligned}M_3 &\leq c\int_s^t (t-r)^{-\frac{\alpha'+\beta+1}{2}}\|\mathrm{div} G_r(v)\|_{-\beta-1}\mathrm{d}r\\ &\leq c\int_s^t (t-r)^{-\frac{\alpha'+\beta+1}{2}}\mathrm{d}r(1+\|v\|_{C_T\mathcal{C}^\alpha})\|b\|_{C_T\mathcal{C}^{-\beta}}\\ &\leq c(t-s)^{\frac{1-\alpha'-\beta}{2}}(1+\|v\|_{C_T\mathcal{C}^\alpha})\|b\|_{C_T\mathcal{C}^{-\beta}}\\ &\leq c(t-s)^{\varepsilon'}T^{\frac{1-\alpha'-\beta-2\varepsilon'}{2}}(1+\|v\|_{C_T\mathcal{C}^\alpha})\|b\|_{C_T\mathcal{C}^{-\beta}}.\end{aligned}$$

Putting the three bounds for M_1, M_2, M_3 together we can conclude that (26) holds.

Step 3. We show bound (21).

Using Step 1 and Step 2 we have

$$\begin{aligned}\|J(v)\|_{C_T^{\varepsilon'}\mathcal{C}^{\alpha'}} =& \sup_{t\in[0,T]}\|J_t(v)\|_{\alpha'}+\sup_{0\leq s<t\leq T}\frac{\|J_t(v)-J_s(v)\|_{\alpha'}}{(t-s)^{\varepsilon'}}\\ \leq& c\|v_0\|_{\alpha'}+c\|v_0\|_{\alpha'+2\varepsilon'}\\ &+cT^{\frac{1-\alpha'-\beta}{2}}(1+\|v\|_{C_T\mathcal{C}^\alpha})\|b\|_{C_T\mathcal{C}^{-\beta}}\\ &+2T^{\frac{1-\alpha'-\beta-2\varepsilon'}{2}}(1+\|v\|_{C_T\mathcal{C}^\alpha})\|b\|_{C_T\mathcal{C}^{-\beta}}\\ \leq& c\|v_0\|_{\alpha'+2\varepsilon'}+c\|b\|_{C_T\mathcal{C}^{-\beta}}T^{\frac{1-\alpha'-\beta-2\varepsilon'}{2}}(1+\|v\|_{C_T\mathcal{C}^\alpha}),\end{aligned}$$

which concludes the proof of (21).

Step 4. We show bound (22).

By Schauder's estimates and Lemma 3.1 we get

$$\begin{aligned}
&\sup_{t\in[0,T]} \|J_t(v)-J_t(u)\|_{\alpha'} \\
&= \sup_{t\in[0,T]} \| \int_0^t P_{t-s}[\mathrm{div}G_s(v)-\mathrm{div}G_s(u)]\mathrm{d}s\|_{\alpha'} \\
&= \sup_{t\in[0,T]} \| \int_0^t P_{t-s}[\mathrm{div}(\tilde F(v(s))-\tilde F(u(s))b(s))]\mathrm{d}s\|_{\alpha'} \\
&\le \sup_{t\in[0,T]} \| \int_0^t (t-s)^{-\frac{\alpha'+\beta+1}{2}} \|\tilde F(v(s))-\tilde F(u(s))\|_\alpha \|b(s)\|_{-\beta}\mathrm{d}s \\
&\le \sup_{t\in[0,T]} \| \int_0^t (t-s)^{-\frac{\alpha'+\beta+1}{2}} \mathrm{d}s\,(1+\|u\|^2_{C_T\mathcal{C}^\alpha}+\|v\|^2_{C_T\mathcal{C}^\alpha})^{1/2}\|v-u\|_{C_T\mathcal{C}^\alpha}\|b\|_{C_T\mathcal{C}^{-\beta}} \\
&\le cT^{\frac{1-\alpha'-\beta}{2}}(1+\|u\|^2_{C_T\mathcal{C}^\alpha}+\|v\|^2_{C_T\mathcal{C}^\alpha})^{1/2}\|v-u\|_{C_T\mathcal{C}^\alpha}\|b\|_{C_T\mathcal{C}^{-\beta}}.
\end{aligned}$$

Step 5. We show bound (23).

We have

$$\begin{aligned}
&\|J_t(v)-J_t(u)-(J_s(v)-J_s(u))\|_\alpha \\
&\quad\le \|P_tv_0-P_tv_0-(P_sv_0-P_sv_0)\|_\alpha \\
&\qquad + \left\| \int_0^s (P_{t-s}-I)P_{s-r}[\mathrm{div}([\tilde F(v(r))-\tilde F(u(r))]b(r))]\mathrm{d}r \right\|_\alpha \\
&\qquad + \left\| \int_s^t P_{t-r}[\mathrm{div}([\tilde F(v(r))-\tilde F(u(r))]b(r))]\mathrm{d}r \right\|_\alpha \\
&\quad =: N_1+N_2+N_3.
\end{aligned}$$

Notice that $N_1=0$. For N_2 and N_3 we do similar computations as for M_2 and M_3 in Step 2, respectively, and using also Lemma 3.1 we get

$$\begin{aligned}
N_2 &\le c(t-s)^{\varepsilon'} \int_0^s (s-r)^{-\frac{\alpha'+\beta+1+2\varepsilon'}{2}}\mathrm{d}r\|b\|_{C_T\mathcal{C}^{-\beta}}\times \\
&\qquad \times (1+\|u\|^2_{C_T\mathcal{C}^\alpha}+\|v\|^2_{C_T\mathcal{C}^\alpha})^{1/2}\|v-u\|_{C_T\mathcal{C}^\alpha} \\
&\le c(t-s)^{\varepsilon'}T^{\frac{1-\alpha'-\beta-2\varepsilon'}{2}}\|b\|_{C_T\mathcal{C}^{-\beta}}(1+\|u\|^2_{C_T\mathcal{C}^\alpha}+\|v\|^2_{C_T\mathcal{C}^\alpha})^{1/2}\|v-u\|_{C_T\mathcal{C}^\alpha}
\end{aligned}$$

and

$$\begin{aligned}
N_3 &\le c(t-s)^{\frac{1-\alpha'-\beta}{2}}(1+\|u\|^2_{C_T\mathcal{C}^\alpha}+\|v\|^2_{C_T\mathcal{C}^\alpha})^{1/2}\|v-u\|_{C_T\mathcal{C}^\alpha} \\
&\le c(t-s)^{\varepsilon'}T^{\frac{1-\alpha'-\beta-2\varepsilon'}{2}}\|b\|_{C_T\mathcal{C}^{-\beta}}(1+\|u\|^2_{C_T\mathcal{C}^\alpha}+\|v\|^2_{C_T\mathcal{C}^\alpha})^{1/2}\|v-u\|_{C_T\mathcal{C}^\alpha},
\end{aligned}$$

where $\frac{1-\alpha'-\beta-2\varepsilon'}{2}>0$ by choice of the parameters. □

A consequence of Proposition 3.5 is the corollary below.

Corollary 3.6. *Let α, α' and ε' be chosen according to Proposition 3.5. Then for all ε such that $\varepsilon' \geq \varepsilon > 0$ we have $J : C_T^\varepsilon \mathcal{C}^\alpha \to C_T^{\varepsilon'} \mathcal{C}^{\alpha'}$ is continuous.*

Proof. We observe that $C_T\mathcal{C}^\alpha$ is continuously embedded in $C_T^\varepsilon\mathcal{C}^\alpha$ for all $\varepsilon > 0$. By part (ii) of Proposition 3.5 we then get $J : C_T^\varepsilon\mathcal{C}^\alpha \to C_T^{\varepsilon'}\mathcal{C}^{\alpha'}$ and the mapping is continuous. □

We are now ready to show that a solution to (2) exists.

Theorem 3.7. *Let Assumptions 1, 4 and 5 hold. Then there exists a mild solution v of (2). Moreover there exists $\varepsilon > 0$ and $\alpha > \beta$, only depending on v_0, such that $v \in C_T^\varepsilon\mathcal{C}^\alpha \subset C_T\mathcal{C}^{\beta+}$.*

Proof. Let us fix here $\varepsilon' > 0$ and $\alpha' > \beta$ such that $v_0 \in \mathcal{C}^{\alpha'+2\varepsilon'}$ and such that $\alpha' < 1-\beta$ and $1-\alpha'-\beta-2\varepsilon' > 0$, which is always possible thanks to Assumptions 1 and 5. We choose α such that $\beta < \alpha \leq \alpha'$ and we choose ε such that $0 < \varepsilon \leq \varepsilon'$. Therefore we can apply Corollary 3.6 which tells that $J : C_T^\varepsilon\mathcal{C}^\alpha \to C_T^{\varepsilon'}\mathcal{C}^{\alpha'}$ is continuous and thus also $J : C_T^\varepsilon\mathcal{C}^\alpha \to C_T^\varepsilon\mathcal{C}^\alpha$ is continuous. We next show that the same operator J is compact. Indeed, by bound (21) and $\|v\|_{C_T\mathcal{C}^\alpha} \leq \|v\|_{C_T^\varepsilon\mathcal{C}^\alpha}$ we have that the image of a ball in $C_T^\varepsilon\mathcal{C}^\alpha$ is a ball in $C_T^{\varepsilon'}\mathcal{C}^{\alpha'}$, and balls in the latter are precompact sets in $C_T^\varepsilon\mathcal{C}^\alpha$. The idea is to apply Schaefer's fixed point theorem, see [24, Theorem 4.3.2]. For this we further need to show that the set

$$\Lambda := \{v \in C_T^\varepsilon\mathcal{C}^\alpha : v = \lambda J(v) \text{ for some } \lambda \in [0,1]\}$$

is bounded in $C_T^\varepsilon\mathcal{C}^\alpha$. Notice that $(\varepsilon, \alpha, \alpha)$ satisfy the assumptions on $(\varepsilon', \alpha, \alpha')$ from Proposition 3.5, in particular $1-\alpha-\beta-2\varepsilon > 0$ and $v_0 \in \mathcal{C}^{\alpha+2\varepsilon}$. If $v \in \Lambda$, then by (21) in Proposition 3.5 and by Proposition 3.4 we have

$$\begin{aligned}
\|v\|_{C_T^\varepsilon\mathcal{C}^\alpha} &= \lambda\|J(v)\|_{C_T^\varepsilon\mathcal{C}^\alpha} \\
&\leq \|J(v)\|_{C_T^\varepsilon\mathcal{C}^\alpha} \\
&\leq c\|v_0\|_{\alpha+2\varepsilon} + c\|b\|_{C_T\mathcal{C}^{-\beta}} T^{\frac{1-\alpha-\beta-2\varepsilon}{2}}(1+\|v\|_{C_T\mathcal{C}^\alpha}) \\
&\leq c\|v_0\|_{\alpha+2\varepsilon} + c\|b\|_{C_T\mathcal{C}^{-\beta}} T^{\frac{1-\alpha-\beta-2\varepsilon}{2}}(1+K) \\
&=: K' < \infty,
\end{aligned}$$

where K' is independent of v. Thus by Schaefer's fixed point theorem we can conclude that there exists a fixed point v^* of J in $C_T^\varepsilon\mathcal{C}^\alpha$, $v^* = J(v^*)$, and such v^* is a mild solution of (2) in $C_T\mathcal{C}^{\beta+}$ since $C_T^\varepsilon\mathcal{C}^\alpha \subset C_T^\varepsilon\mathcal{C}^{\beta+} \subset C_T\mathcal{C}^{\beta+}$. □

Remark 3.8. *We can show that the solution v is more regular if we suppose that $v_0 \in \mathcal{C}^{(1-\beta)-}$ in place of Assumption 5. In this case with similar arguments we could get that a solution v exists in $C_T\mathcal{C}^{(1-\beta)-}$.*

Remark 3.9. *We will prove below (with other techniques) that the solution v found in Theorem 3.7 is actually unique in $C_T\mathcal{C}^{\beta+}$, see Proposition 4.7.*

4. The regularised PDE and its limit

Let Assumptions 1, 2, 3, 4 and 5 hold throughout in this section.

When the term b is replaced by b^n satisfying Assumption 2 we get a smoothed PDE, that is, we get the Fokker-Planck equation

$$\begin{cases} \partial_t v^n = \frac{1}{2}\Delta v^n - \mathrm{div}(\tilde{F}(v^n)b^n) \\ v^n(0) = v_0, \end{cases} \tag{27}$$

where we recall that $\tilde{F}(v^n) = v^n F(v^n)$. For ease of reading, we recall that the mild solution of (27) is given by an element $v^n \in C_T\mathcal{C}^{\beta+}$ such that

$$v^n(t) = P_t v_0 - \int_0^t P_{t-s}[\mathrm{div}(\tilde{F}(v^n(s))b^n(s))]\mathrm{d}s. \tag{28}$$

Remark 4.1. *We observe that, since $b^n \in C_T\mathcal{C}^{-\beta}$, then all results from Section 3 are still valid, in particular the bound from Proposition 3.5 and the existence result from Theorem 3.7 still apply to* (27)*.*

At this point we introduce the notation and some useful results on a very similar semilinear PDE studied in [22]. We consider the PDE

$$\begin{cases} \partial_t u(t,x) = \frac{1}{2}\Delta u(t,x) - \mathrm{div}(u(t,x)\mathfrak{b}(t,x,u(t,x))) \\ u(0,\mathrm{d}x) = \mathfrak{u}_0(\mathrm{d}x), \end{cases} \tag{29}$$

where $\mathfrak{u}_0$ is a Borel probability measure which admits v_0 as bounded density with respect to the Lebesgue measure. We set

$$\mathfrak{b}(t,x,z) := F(z)b^n(t,x). \tag{30}$$

Thanks to Assumptions 3 and 2 we have that the term $\mathfrak{b}(t,x,z)$ is uniformly bounded.

Definition 4.2. *We will call a* semigroup solution *of the PDE* (29) *a function $u \in L^\infty([0,T]\times\mathbb{R}^d)$ that satisfies the integral equation*

$$\begin{aligned} u(t,x) = &\int_{\mathbb{R}^d} p_t(x-y)v_0(y)\mathrm{d}y \\ &+ \sum_{j=1}^d \int_0^t \int_{\mathbb{R}^d} u(s,y)\mathfrak{b}_j(s,y,u(s,y))\partial_{y_j}p_{t-s}(x-y)\mathrm{d}y\,\mathrm{d}s, \end{aligned} \tag{31}$$

where p is the Gaussian heat kernel introduced in (4)*.*

Notice that this definition is inspired by [22, Definition 6], but we modified it here to include the condition $u \in L^\infty([0,T]\times\mathbb{R}^d)$, rather than $u \in L^1([0,T]\times\mathbb{R}^d)$ (the latter as in [22], where moreover the solution is called 'mild solution'). Indeed integrability of u is sufficient for the integrals in the semigroup solution to make sense, because $\mathfrak{b}$ is also bounded and the heat kernel and its derivative are integrable.

The first result we have on (29) is about uniqueness of the semigroup solution in $L^\infty([0,T]\times\mathbb{R}^d)$. This result is not included in [22], but we were inspired by proofs therein, in particular by the proof of [22, Lemma 20].

Lemma 4.3. *There exists at most one semigroup solution of* (29)*.*

Proof. First of all we remark that since $p_t(y)$ is the heat kernel then we have two positive constants c_p, C_p such that

$$|\partial_{y_j} p_t(y)| \le \frac{C_p}{\sqrt{t}} q_t(y), \tag{32}$$

for all $j = 1, \dots, d$, where $q_t(y) = \left(\frac{c_p}{t\pi}\right)^{d/2} e^{-c_p \frac{|y|^2}{t}}$ is a Gaussian probability density. This can be easily calculated from the explicit form of $p_t(y)$, since one gets

$$\partial_{y_j} p_t(y) = -\frac{y_j}{t} p_t(y)$$

so that

$$|\partial_{y_j} p_t(y)| = \frac{|y_j|}{t} p_t(y) = \frac{1}{\sqrt{t}} \frac{|y_j|}{\sqrt{t}} p_t(y).$$

Now we observe that when $\frac{|y_j|}{\sqrt{t}} \le 1$ then $\frac{|y_j|}{\sqrt{t}} p_t(y) \le p_t(y)$ so it can be bounded by $q_t(y)$ for suitable constant c_p and (32) follows. On the other hand, when $\frac{|y_j|}{\sqrt{t}} > 1$, we have that $p_t(y)$ goes to zero exponentially with respect to $\frac{|y_j|}{\sqrt{t}}$, in particular there exists two positive constants C_p, c_p such that

$$\frac{|y_j|}{\sqrt{t}} p_t(y) = \frac{|y_j|}{\sqrt{t}} c \frac{1}{t^{d/2}} e^{-c\frac{|y_j|^2}{t}} \le \ C_p \left(\frac{c_p}{t\pi}\right)^{d/2} e^{-c_p \frac{|y|^2}{t}},$$

and (32) follows.

Let us consider two semigroup solutions u_1, u_2 of (29). We denote by $\Pi(u)$ the semigroup solution map, which is the right-hand side of (31). Notice that $v_0 \in L^\infty(\mathbb{R}^d)$ by Assumption 5, and the function $z \mapsto z\mathfrak{b}(t,x,z)$ is Lipschitz, uniformly in t, x because $\tilde{F}$ is assumed to be Lipschitz in Assumption 4. Using this together with the bound (32) for fixed $t \in (0,T]$ we get

$$\begin{aligned}
&\|\Pi(u_1)(t,\cdot) - \Pi(u_2)(t,\cdot)\|_\infty \\
&= \Big\| \sum_{j=1}^d \int_0^t \int_{\mathbb{R}^d} \Big(u_1(s,y)\mathfrak{b}_j(s,y,u_1(s,y)) - u_2(s,y)\mathfrak{b}_j(s,y,u_2(s,y)) \Big) \\
&\qquad\qquad \cdot \partial_{y_j} p_{t-s}(x-y) \mathrm{d}y \, \mathrm{d}s \Big\|_\infty \\
&\le C \int_0^t \int_{\mathbb{R}^d} |u_1(s,y) - u_2(s,y)| \frac{1}{\sqrt{t-s}} C_u q_{t-s}(x-y) \mathrm{d}y \, \mathrm{d}s \\
&\le C \int_0^t \|u_1(s,\cdot) - u_2(s,\cdot)\|_\infty \frac{1}{\sqrt{t-s}} \mathrm{d}s \cdot \int_{\mathbb{R}^d} q_{t-s}(x-y) \mathrm{d}y \\
&\le C \int_0^t \|u_1(s,\cdot) - u_2(s,\cdot)\|_\infty \frac{1}{\sqrt{t-s}} \mathrm{d}s.
\end{aligned}$$

Now, by an application of a fractional Gronwall's inequality (see Lemma A.1) we conclude that $\|u_1(t,\cdot) - u_2(t,\cdot)\|_\infty \le 0$ for all $t \in [0,T]$, so in particular we have

$$\|u_1 - u_2\|_{L^\infty([0,T]\times\mathbb{R}^d)} = 0,$$

hence the semigroup solution is unique in $L^\infty([0,T]\times\mathbb{R}^d)$. □

At this point we want to compare the concept of mild solution and that of semigroup solution. Recall that $\mathfrak{b}(t,x,z) = F(z)b^n(t,x)$ so in fact PDE (29) is exactly (27). First we state and prove a preparatory lemma, where $\mathfrak{f}$ is vector-valued and will be taken to be $u(t,x)\mathfrak{b}(t,x,u(t,x))$ for fixed t in the following result.

Lemma 4.4. *Let $\mathfrak{f} \in L^\infty(\mathbb{R}^d;\mathbb{R}^d)$, $t\in[0,T]$. Then*

$$P_t(div\,\mathfrak{f}) = -\sum_{j=1}^d \int_{\mathbb{R}^d} \mathfrak{f}(y)\partial_{y_j} p_t(\cdot - y)\mathrm{d}y, \tag{33}$$

almost everywhere.

Proof. We will show that the left-hand side (LHS) and the right-hand side (RHS) are the same object in $\mathcal{S}'$. Moreover we know that both sides are functions in L^∞ so we conclude. Notice that the heat kernel $p_t(x)$ is the same kernel associated to the semigroup P_t, namely if $\phi\in\mathcal{S}$, then $P_t\phi\in\mathcal{S}$ with $P_t\phi(x) = \int_{\mathbb{R}^d} p_t(x-y)\phi(y)dy$. Let $\phi\in\mathcal{S}$. The LHS gives

$$\begin{aligned}
\langle P_t(\mathrm{div}\,\mathfrak{f}),\phi\rangle &= \langle \mathrm{div}\,\mathfrak{f}, P_t\phi\rangle \\
&= -\sum_{j=1}^d \langle \mathfrak{f}, \partial_{y_j}(P_t\phi)\rangle \\
&= -\sum_{j=1}^d \left\langle \mathfrak{f}, \partial_{y_j}\left(\int_{\mathbb{R}^d} p_t(\cdot - x)\phi(x)\mathrm{d}x\right)\right\rangle \\
&= -\sum_{j=1}^d \left\langle \mathfrak{f}, \int_{\mathbb{R}^d} \partial_{y_j} p_t(\cdot - x)\phi(x)\mathrm{d}x\right\rangle.
\end{aligned}$$

The RHS of (33), on the other hand, gives

$$
\begin{aligned}
\langle -\sum_{j=1}^{d}\int_{\mathbb{R}^d}\mathfrak{f}(y)\partial_{y_j}p_t(\cdot-y)\mathrm{d}y,\phi\rangle &= -\sum_{j=1}^{d}\langle\int_{\mathbb{R}^d}\mathfrak{f}(y)\partial_{y_j}p_t(\cdot-y)\mathrm{d}y,\phi\rangle\\
&= -\sum_{j=1}^{d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\mathfrak{f}(y)\partial_{y_j}p_t(x-y)\mathrm{d}y\,\phi(x)\mathrm{d}x\\
&= -\sum_{j=1}^{d}\int_{\mathbb{R}^d}\mathfrak{f}(y)\int_{\mathbb{R}^d}\partial_{y_j}p_t(x-y)\phi(x)\mathrm{d}x\,\mathrm{d}y\\
&= -\sum_{j=1}^{d}\int_{\mathbb{R}^d}\mathfrak{f}(y)\int_{\mathbb{R}^d}\partial_{y_j}p_t(y-x)\phi(x)\mathrm{d}x\,\mathrm{d}y\\
&= -\sum_{j=1}^{d}\left\langle\mathfrak{f},\int_{\mathbb{R}^d}\partial_{y_j}p_t(\cdot-x)\phi(x)\mathrm{d}x\right\rangle,
\end{aligned}
$$

having used the symmetry of $p_t(\cdot)$. □

We are now ready to prove that any mild solution is also a semigroup solution.

Proposition 4.5. *Any mild solution v^n of (27) is also a semigroup solution.*

Proof. Recall that $F(z)b^n(t,x)=\mathfrak{b}(t,x,z)$ by (30). For v^n to be a semigroup solution it must be an a.e. bounded function that satisfies (31). First we notice that, since v^n is a mild solution, there exists $\alpha>\beta$ such that $v^n\in C_T\mathcal{C}^\alpha\subset L^\infty([0,T]\times\mathbb{R}^d)$ so the second term on the RHS of expression (31) is well-defined. We recall that by Assumption 3, F is bounded and by Assumption 2 hence $\mathfrak{b}$ is bounded. Moreover by Assumption 5 the initial condition $v_0\in\mathcal{C}^{\beta+}\subset L^\infty([0,T]\times\mathbb{R}^d)$ so also the first term on the RHS of expression (31) is well-defined.

Now we show that the two terms on the RHS of (28) are equal to the terms on the RHS of (31). We start with the initial condition term, which can be written as

$$(P_tv_0)(x)=\int_{\mathbb{R}^d}p_t(x-y)v_0(y)\mathrm{d}y,$$

since p_t is the kernel of the semigroup P_t. For the second term we use Lemma 4.4 with $\mathfrak{f}=u\mathfrak{b}$ to get

$$
\begin{aligned}
P_t(\mathrm{div}[u(t)F(u(t))b^n(t,\cdot)]) &= P_t(\mathrm{div}[u(t)\mathfrak{b}(t,u(t))])\\
&= -\sum_{j=1}^{d}\int_0^t\int_{\mathbb{R}^d}u(s,y)\mathfrak{b}_j(s,y,u(s,y))\partial_{y_j}p_t(\cdot-y)\mathrm{d}y\,\mathrm{d}s
\end{aligned}
$$

and so (31) becomes (28), i.e. the mild solution v^n is also a semigroup solution. □

Remark 4.6. *Let n be fixed. By Theorem 3.7 there is a mild solution v^n of (27); now v^n is unique in $C_T\mathcal{C}^{\beta+}$. Indeed, by Proposition 4.5, v^n is a semigroup solution and we know that the latter is unique in L^∞ by Lemma 4.3.*

The next result establishes, in particular, the uniqueness of the solution v in $C_T\mathcal{C}^{\beta+}$ and a continuity result with respect to $b \in C_T\mathcal{C}^{-\beta}$.

Proposition 4.7. (i) *Let b^1, b^2 satisfy Assumption 1. Let v^1 (resp. v^2) be a mild solution of (2) with $b = b^1$ (resp. $b = b^2$). For any $\alpha \in (\beta, 1-\beta)$ such that $v^1, v^2 \in C_T\mathcal{C}^\alpha$, there exists a function $\ell_\alpha : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$, increasing in the second variable, such that*

$$\|v^1(t) - v^2(t)\|_\alpha \leq \ell_\alpha(\|v_0\|_\alpha, \|b^1\| \vee \|b^2\|)\|b^1 - b^2\|_{C_T\mathcal{C}^{-\beta}},$$

for all $t \in [0,T]$.

(ii) *There is a unique mild solution v of (2) in $C_T\mathcal{C}^{\beta+}$.*

(iii) *Let $(b^m)_m$ be a sequence in $C_T\mathcal{C}^{-\beta}$. Let v^m be a mild solution of (2) with $b^m = b$ and v be a mild solution of (2). If $b^m \to b$ in $C_T\mathcal{C}^{-\beta}$ then $v^m \to v$ in $C_T\mathcal{C}^{\beta+}$.*

Proof. Item (i). Let v^1 (resp. v^2) be a solution in $C_T\mathcal{C}^{\beta+}$ to (27) with $b = b^1$ (resp. $b = b^2$); so there exists $\alpha \in (\beta, 1-\beta)$ such that $v^1, v^2 \in C_T\mathcal{C}^\alpha$. We fix $t \in [0,T]$. Using Schauder's estimates and Bernstein's inequalities, for the difference below we get the bound

$$\begin{aligned}
\|v^1(t) - v^2(t)\|_\alpha &= \left\|\int_0^t P_{t-s}\left(\mathrm{div}[\tilde F(v^1(s))b^1(s) - \tilde F(v^2(s))b^2(s)]\right)\mathrm{d}s\right\|_\alpha \\
&\leq c\int_0^t (t-s)^{-\frac{\alpha+\beta+1}{2}}\left\|\mathrm{div}[\tilde F(v^1(s))b^1(s) - \tilde F(v^2(s))b^2(s)]\right\|_{-\beta-1}\mathrm{d}s \\
&\leq c\int_0^t (t-s)^{-\frac{\alpha+\beta+1}{2}}\left\|\tilde F(v^1(s))b^1(s) - \tilde F(v^2(s))b^2(s)\right\|_{-\beta}\mathrm{d}s. \qquad (34)
\end{aligned}$$

Now, in order to bound the term inside the integral we use the mapping properties of $\tilde F$ from Lemma 3.1, the property (11) of the pointwise product, and the fact that v^1 and v^2 are mild solutions. We get

$$
\begin{aligned}
&\left\|\tilde F(v^1(s))b^1(s) - \tilde F(v^2(s))b^2(s)\right\|_{-\beta}\\
=&\left\|\tilde F(v^1(s))b^1(s) - \tilde F(v^2(s))b^1(s) + \tilde F(v^2(s))b^1(s) - \tilde F(v^2(s))b^2(s)\right\|_{-\beta}\\
\le&\left\|[\tilde F(v^1(s)) - \tilde F(v^2(s))]b^1(s)\right\|_{-\beta} + \left\|\tilde F(v^2(s))[b^1(s) - b^2(s)]\right\|_{-\beta}\\
\le& c\left\|\tilde F(v^1(s)) - \tilde F(v^2(s))\right\|_{\alpha}\|b^1(s)\|_{-\beta} + c\left\|\tilde F(v^2(s))\right\|_{\alpha}\|b^1(s) - b^2(s)\|_{-\beta}\\
\le& c\left(1+\|v^1(s)\|_\alpha^2 + \|v^2(s)\|_\alpha^2\right)^{1/2}\left\|v^1(s)-v^2(s)\right\|_\alpha\|b^1(s)\|_{-\beta}\\
&+ c(1+\|v^1(s)\|_\alpha)\|b^1(s)-b^2(s)\|_{-\beta}\\
\le& c\left(1+\|v^1\|_{C_T\mathcal{C}^\alpha}^2 + \|v^2\|_{C_T\mathcal{C}^\alpha}^2\right)^{1/2}\|v^1(s)-v^2(s)\|_\alpha\|b^1\|_{C_T\mathcal{C}^{-\beta}}\\
&+ c\left(1+\|v^2\|_{C_T\mathcal{C}^\alpha}\right)\|b^1-b^2\|_{C_T\mathcal{C}^{-\beta}}.
\end{aligned}
$$

At this point we use the a priori bound K_1 for v^1 (resp. K_2 for v^2) found in Proposition 3.4, which depends on $\|v_0\|_\alpha$ and $\|b^1\|_{C_T\mathcal{C}^{-\beta}}$ (resp. $\|b^2\|_{C_T\mathcal{C}^{-\beta}}$) and is increasing with respect to the latter. Thus we get

$$
\begin{aligned}
&\left\|\tilde F(v^1(s))b^1(s) - \tilde F(v^2(s))b^2(s)\right\|_{-\beta}\\
&\le c\left(1+K_1^2+K_2^2\right)^{1/2}\|v^1(s)-v^2(s)\|_\alpha\|b^1\|_{C_T\mathcal{C}^{-\beta}}\\
&+ c\left(1+K_2\right)\|b^1-b^2\|_{C_T\mathcal{C}^{-\beta}}\\
&\le \tilde\ell_\alpha\left(\|v_0\|_\alpha, \|b^1\|_{C_T\mathcal{C}^{-\beta}}\vee\|b^2\|_{C_T\mathcal{C}^{-\beta}}\right)\|v^1(s)-v^2(s)\|_\alpha\\
&+ \tilde\ell_\alpha\left(\|v_0\|_\alpha, \|b^1\|_{C_T\mathcal{C}^{-\beta}}\vee\|b^2\|_{C_T\mathcal{C}^{-\beta}}\right)\|b^1-b^2\|_{C_T\mathcal{C}^{-\beta}},
\end{aligned}
$$

where $\tilde\ell_\alpha(\cdot,\cdot)$ is a function increasing in the second variable. Putting this into (34) we get

$$
\begin{aligned}
&\|v^1(t)-v^2(t)\|_\alpha\\
&= c\,\tilde\ell_\alpha\left(\|v_0\|_\alpha, \|b^1\|_{C_T\mathcal{C}^{-\beta}}\vee\|b^2\|_{C_T\mathcal{C}^{-\beta}}\right)\|b^1-b^2\|_{C_T\mathcal{C}^{-\beta}}T^{\frac{1-\alpha-\beta}{2}}\\
&+ \tilde\ell_\alpha\left(\|v_0\|_\alpha, \|b^1\|_{C_T\mathcal{C}^{-\beta}}\vee\|b^2\|_{C_T\mathcal{C}^{-\beta}}\right)\int_0^t (t-s)^{-\frac{\alpha+\beta+1}{2}}\|v^1(s)-v^2(s)\|_\alpha \mathrm{d}s,
\end{aligned}
$$

and by a generalised Gronwall's inequality (see Lemma A.1) we get

$$
\begin{aligned}
&\|v^1(t)-v^2(t)\|_\alpha\\
&\le c\,\tilde\ell_\alpha\left(\|v_0\|_\alpha, \|b^1\|_{C_T\mathcal{C}^{-\beta}}\vee\|b^2\|_{C_T\mathcal{C}^{-\beta}}\right)\|b^1-b^2\|_{C_T\mathcal{C}^{-\beta}}T^{\frac{1-\alpha-\beta}{2}}\\
&\times E_{\frac{1-\alpha-\beta}{2}}\left(\tilde\ell_\alpha\left(\|v_0\|_\alpha, \|b^1\|_{C_T\mathcal{C}^{-\beta}}\vee\|b^2\|_{C_T\mathcal{C}^{-\beta}}\right)\Gamma\left(\tfrac{1-\alpha-\beta}{2}\right)T^{\frac{1-\alpha-\beta}{2}}\right)\\
&=: \ell_\alpha\left(\|v_0\|_\alpha, \|b^1\|_{C_T\mathcal{C}^{-\beta}}\vee\|b^2\|_{C_T\mathcal{C}^{-\beta}}\right)\|b^1-b^2\|_{C_T\mathcal{C}^{-\beta}},
\end{aligned}
$$

where $\ell_\alpha(\cdot,\cdot)$ is again a function increasing in the second variable.

Item (ii). We choose $b^1 = b^2 = b$ and assume that v^1 and v^2 are any two solutions of (2) in $C_T\mathcal{C}^{\beta+}$. There exists $\alpha \in (\beta, 1-\beta)$ such that $v^1, v^2 \in C_T\mathcal{C}^\alpha$. By Item (i) we have $\|v^1(t) - v^2(t)\|_\alpha \le 0$ for all $t \in [0,T]$, hence $v^1 = v^2$ in $C_T\mathcal{C}^\alpha$, so the solution is unique.

Item (iii). Let $(b^m)_m$ be a sequence in $C_T\mathcal{C}^{-\beta}$. Let us assume that v^m is the unique solution of (2) with $b = b^m$. Notice that since the solution is unique by Item (ii), it corresponds to the one constructed in Theorem 3.7 and thus it lives in $C_T\mathcal{C}^\alpha$, where α depends only on v_0, hence not on m. Let v be the unique solution of (2). We apply again Item (i) with $b^1 = b^m$ and $b^2 = b$ to get

$$\|v^m(t) - v(t)\|_\alpha \le \ell_\alpha\left(\|v_0\|_\alpha, \|b^m\|_{C_T\mathcal{C}^{-\beta}} \vee \|b\|_{C_T\mathcal{C}^{-\beta}}\right)\|b^m - b\|_{C_T\mathcal{C}^{-\beta}}. \quad (35)$$

We have $\sup_m \|b^m\|_{C_T\mathcal{C}^{-\beta}} < \infty$ because $b^m \to b$ in $C_T\mathcal{C}^{-\beta}$, and

$$\ell_\alpha\left(\|v_0\|_\alpha, \|b^m\|_{C_T\mathcal{C}^{-\beta}} \vee \|b\|_{C_T\mathcal{C}^{-\beta}}\right) \le \ell_\alpha\left(\|v_0\|_\alpha, \sup_m \|b^m\|_{C_T\mathcal{C}^{-\beta}} \vee \|b\|_{C_T\mathcal{C}^{-\beta}}\right)$$

because $\ell_\alpha(\|v_0\|_\alpha, \cdot)$ is increasing. Therefore plugging this into (35) we have

$$\|v^m(t) - v(t)\|_\alpha \le c\|b^m - b\|_{C_T\mathcal{C}^{-\beta}},$$

where $c := \ell_\alpha\left(\|v_0\|_\alpha, \sup_m \|b^m\|_{C_T\mathcal{C}^{-\beta}} \vee \|b\|_{C_T\mathcal{C}^{-\beta}}\right)$. Thus taking the sup over t we get that $v^m \to v$ in $C_T\mathcal{C}^\alpha$ if $b^m \to b$ in $C_T\mathcal{C}^{-\beta}$, which implies the convergence of $v^m \to v$ in $C_T\mathcal{C}^{\beta+}$ because $\alpha > \beta$. □

5. The regularised SDEs

In this section we consider the regularised version of the McKean SDE introduced in (1), when b is replaced by a bounded b^n with $b^n(t,\cdot) \in C_b^\infty$: the sequence will be chosen according to Assumption 2 in the next section. Further, let us suppose that Assumptions 3, 4, 5 and 6 hold throughout the section. We get

$$\begin{cases} X_t^n = X_0 + \int_0^t F(v^n(s, X_s^n)) b^n(s, X_s^n) \mathrm{d}s + W_t \\ v^n(t,\cdot) \text{ is the law density of } X_t^n, \end{cases} \quad (36)$$

for some given $X_0 \sim v_0$. In order to show existence and uniqueness of a solution of (36) and its link to the mild (and semigroup) solution v^n of (27), we make use of Theorem 12 and 13 from [22], as we see below.

Let $\mathfrak{b} := Fb^n$. Since F is Lipschitz and bounded by Assumption 3 and b^n is bounded by Assumption 2, then $\mathfrak{b}$ is uniformly bounded ad uniformly Lipschitz in the last variable.

Proposition 5.1. *Let $(W_t)_{t\in[0,T]}$ be a Brownian motion on some probability space and let $X_0 \sim v_0$. Let $b^n : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ be a bounded Borel function with $b^n(t,\cdot) \in C_b^\infty$. Let moreover Assumptions 3, 4, 5 and 6 hold.*

(i) *There exists a couple (X^n, v^n) with v^n bounded, verifying* (36).
(ii) *Given two solutions (X^n, v^n) and $(\hat{X}^n, \hat{v}^n)$ of* (36) *with v^n and $\hat{v}^n$ bounded, then $(X^n, v^n) = (\hat{X}^n, \hat{v}^n)$.*

(iii) *If (X^n, v^n) is a solution to* (36) *with v^n bounded, then v^n is a semigroup solution of* (29).

Proof. We will apply the results [22, Theorem 12 point 1 and Theorem 13 point 3] to our setting. In particular, (36) is the special case of equation (1) in [22] when $\Lambda = 0, b_0 = 0, (a_{i,j}) = I, \Phi = I$ and $\mathfrak{u}_0$ has a density v_0 with respect to the Lebesgue measure. Notice that all assumptions in Theorems 12 and 13 are satisfied. Indeed, the drift $\mathfrak{b} = Fb^n$ is bounded and Lipschitz because F is Lipschitz and bounded by Assumption 3 and b^n is bounded by Assumption 2.

(i) and (ii). We apply the result [22, Theorem 13 point 3].

(iii). We apply the result [22, Theorem 12 point 1] to get that v^n is a weak solution of (29). Under [22, Assumption C], [1] weak and semigroup solutions are equivalent, see [22, Proposition 16]. □

6. Solving the McKean problem

Let Assumptions 1, 2, 3, 4, 5 and 6 be standing assumptions in this section. For ease of reading, we recall the problem at hand, which was illustrated in (1). We want to solve the McKean equation

$$\begin{cases} X_t = X_0 + \int_0^t F(v(s, X_s))b(s, X_s)\mathrm{d}s + W_t \\ v(t, \cdot) \text{ is the law density of } X_t, \end{cases} \tag{37}$$

for some given initial condition $X_0 \sim v_0$. The corresponding Fokker-Planck singular equation (already introduced in (2) and recalled here for ease of reading) is

$$\begin{cases} \partial_t v = \frac{1}{2}\Delta v - \operatorname{div}(\tilde{F}(v)b) \\ v(0) = v_0, \end{cases} \tag{38}$$

where $\tilde{F}(v) := vF(v)$, to which we gave a proper meaning and which we solved in Section 3.

Definition 6.1. *A* solution (in law) of the McKean problem (37) *is a triple $(X, \mathbb{P}, v)$ such that $\mathbb{P}$ is a probability measure on some measurable space $(\Omega, \mathcal{F})$, the function v is defined on $[0, T] \times \mathbb{R}^d$ and belongs to $C_T\mathcal{C}^{\beta+}$, the couple $(X, \mathbb{P})$ is a solution to the martingale problem with distributional drift $B(t, \cdot) := F(v(s, \cdot))b(s, \cdot)$, and $v(t, \cdot)$ is the law density of X_t.*

We say that the McKean problem (37) admits uniqueness *if, whenever we have two solutions $(X, \mathbb{P}, v)$ and $(\hat{X}, \hat{\mathbb{P}}, \hat{v})$, then $v = \hat{v}$ in $C_T\mathcal{C}^{\beta+}$ and the law of X under $\mathbb{P}$ equals the law of $\hat{X}$ under $\hat{\mathbb{P}}$.*

Using the tools developed in the previous sections, in Theorem 6.4 we will construct a solution $(X, \mathbb{P}, v)$ to the McKean problem (37) and show that this solution is unique. We first recall two useful results from [17]. Let us

[1] which postulates uniqueness of weak solutions for $\partial_t u = L^* u, u_0 = 0$ in the class of measure valued functions, which is true if $L^* = \Delta$, see [22, Remark 7].

consider a distributional drift $B \in C_T\mathcal{C}^{-\beta}$ that satisfies Assumption 1 with $b = B$.

The first result concerns convergence in law when the distributional drift B is approximated by a sequence of smooth functions B^n. This result is crucial to show existence of the McKean equation.

Proposition 6.2 (Issoglio Russo, [17])**.** *Let (B^n) be a sequence in $C_T\mathcal{C}^{-\beta}$ converging to B in $C_T\mathcal{C}^{-\beta}$. Let $(X,\mathbb{P})$ (respectively $(X^n,\mathbb{P}^n)$) be a solution to the (linear) MP with distributional drift B (respectively B^n). Then the sequence $(X^n,\mathbb{P}^n)$ converges in law to $(X,\mathbb{P})$. In particular, if B^n is a bounded function and X^n is a strong solution of*

$$X^n_t = X_0 + \int_0^t B^n(s, X^n_s)\mathrm{d}s + W_t,$$

then X^n converges to $(X,\mathbb{P})$ in law.

The second result is the fact that the law of the solution X to a (linear) martingale problem with distributional drift B solves the Fokker-Planck equation in the weak sense. This result is crucial to show uniqueness of the McKean equation.

Proposition 6.3 (Issoglio Russo, [17])**.** *Let $(X,\mathbb{P})$ be a solution to the martingale problem with distributional drift B. Let $v(t,\cdot)$ be the law density of X_t and let us assume that $v \in C_T\mathcal{C}^{\beta+}$. Then v is a weak solution (in the sense of Definition 3.2 part (ii)) of the Fokker-Plank equation*

$$\begin{cases} \partial_t v = \frac{1}{2}\Delta v - \mathrm{div}(vB) \\ v(0) = v_0. \end{cases}$$

We can now state and prove the main result of this paper.

Theorem 6.4. *Let Assumptions 1, 2, 3, 4, 5 and 6 hold. Then there exists a solution $(X,\mathbb{P},v)$ to the McKean problem* (37)*. Furthermore, the McKean problem admits uniqueness according to Definition 6.1.*

Proof. Existence. Let us consider a sequence $(b^n) \to b$ satisfying Assumption 2. The corresponding smoothed McKean problem is

$$\begin{cases} X^n_t = X_0 + \int_0^t F(v^n(s, X^n_s))b^n(s, X^n_s)\mathrm{d}s + W_t, \\ v^n(t,\cdot) \text{ is the law density of } X^n_t. \end{cases} \tag{39}$$

By Proposition 5.1 part (i) we have a solution (X^n, v^n) of (39) where v^n is bounded and X^n is a strong solution. By Proposition 5.1 part (iii) we have that v^n is a semigroup solution of (27). On the other hand, we know by Remark 4.1 that a mild solution u^n of the same equation exists. By Proposition 4.5 we know that u^n is a semigroup solution and moreover it is bounded (because it is a mild solution). By uniqueness of semigroup solutions (see Lemma 4.3) we have $v^n = u^n$.

Now we notice that $B^n := F(v^n)b^n$ converges to $B := F(v)b$ in $C_T\mathcal{C}^{-\beta}$ because of (12), the linearity of the pointwise product, the Lipschitz property of F, Lemma 3.1, the convergence $b^n \to b$ by assumption and the convergence $v^n \to v$ by Proposition 4.7. From Proposition 6.2 we have that $X^n \to X$ in law (as $B^n \to B$), and since v^n is the law density of X^n we have that v must be the law density of X.

Uniqueness. Suppose that we have two solutions of the McKean problem (37), $(X^1, \mathbb{P}^1, v^1)$ and $(X^2, \mathbb{P}^2, v^2)$. By definition we know that $(X^i, \mathbb{P}^i)$ is a solution to the (linear) martingale problem with distributional drift $B^i := F(v^i)b$. Thus by Proposition 6.3 we have that v^i is a weak solution to the Fokker-Planck equation

$$\begin{cases} \partial_t v^i = \frac{1}{2}\Delta v^i - \mathrm{div}(v^i F(v^i) b) \\ v^i(0) = v_0, \end{cases}$$

which is exactly PDE (38). Item (ii) in Proposition 4.7 guarantees uniqueness of the mild solution of (38) and Proposition 3.3 ensures that weak and mild solutions of the Fokker-Planck equation are equivalent, hence $v^1 = v^2 =: v$. Note that it is crucial the fact that $v^i \in C_T\mathcal{C}^{\beta+}$. This implies that $(X^i, \mathbb{P}^i)$ are both solutions of the same (linear) martingale problem with distributional drift $B := F(v)b$, so by uniqueness of the solution of MP (see Section 2.4) we conclude that the law of X^1 under $\mathbb{P}^1$ equals the law of X^2 under $\mathbb{P}^2$. □

Appendix A. A generalised Gronwall's inequality

Here we recall a useful generalised Gronwall's inequality (or fractional Gronwall's inequality). For a proof see [26, Corollary 2].

Lemma A.1. *Suppose $\eta > 0$, $a(t)$ is a nonnegative function locally integrable on $0 \le t < T$ (some $T \le \infty$) and nondecreasing on $[0, T)$. Let $g(t)$ be a nonnegative, nondecreasing continuous function defined on $0 \le t < T$, $g(t) \le M$ (constant), and suppose $f(t)$ is nonnegative and locally integrable on $0 \le t < T$ with*

$$f(t) \le a(t) + g(t) \int_0^t (t-s)^{\eta-1} f(s)\mathrm{d}s$$

on this interval. Then

$$f(t) \le a(t) E_\eta(g(t)\Gamma(\eta)t^\eta),$$

where E_η is the Mittag-Leffler function defined by $E_\eta(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(k\eta+1)}$.

Remark A.2. *In* [14], *the end of the proof of Proposition 4.1 incorrectly uses Gronwall's lemma. The proper argument should instead cite a generalised Gronwall's inequality, like the one stated above.*

Appendix B. Compactness and continuity in inductive spaces

This appendix is devoted to the proof of a continuity result in inductive spaces. We show in two steps that a function belongs to $C_T\mathcal{C}^{\gamma+}$ if and only if it belongs to $C_T\mathcal{C}^\alpha$ for some $\alpha > \gamma$.

The first step is about compactness of sets in inductive spaces $\mathcal{C}^{\gamma+}$.

Lemma B.1. *Let $\gamma > 0$. A set $K \subset \mathcal{C}^{\gamma+}$ is a compact in $\mathcal{C}^{\gamma+}$ if and only if there exists $\alpha > \gamma$ such that $K \subset \mathcal{C}^{\alpha}$ and K is a compact in $\mathcal{C}^{\alpha}$.*

Proof. "$\Rightarrow$". Let $K \subset \mathcal{C}^{\gamma+}$ be a compact. For any $x \in K$, we know that $x \in C^{\alpha(x)}$ for some $\alpha(x) > \gamma$ and we pick an arbitrary open neighbourhood $V(x)$ in $\mathcal{C}^{\alpha(x)}$. Thus $V(x)$ is an open set of $\mathcal{C}^{\gamma+}$. We have $K \subset \cup_{x\in K} V(x)$, and since K is compact in $\mathcal{C}^{\gamma+}$ there exists a finite subcovering $K \subset \cup_{i=1}^{N} V(x_i)$. Let $\alpha := \min_{i=1,\dots,N} \alpha(x_i)$. Thus $K \subset \mathcal{C}^{\alpha}$. Next we show that K is also a compact in $\mathcal{C}^{\alpha}$ for the chosen α. Let $(O_\nu)_\nu$ be any open covering of K in $\mathcal{C}^{\alpha}$, that is $K \subset \cup_\nu O_\nu$. Each O_ν is an open set of $\mathcal{C}^{\alpha}$ thus also of $\mathcal{C}^{\gamma+}$, therefore $(O_\nu)_\nu$ is also an open covering of $\mathcal{C}^{\gamma+}$, thus there exists a finite covering.

"$\Leftarrow$". Let K be a compact in $\mathcal{C}^{\alpha}$, for some $\alpha > \gamma$. The inclusion $K \subset \mathcal{C}^{\gamma+}$ is obvious. Now let us take an open covering of K in $\mathcal{C}^{\gamma+}$, that is $K \subset \cup_\nu O_\nu$, where each O_ν is an open set in $\mathcal{C}^{\gamma+}$. Since $K \subset \mathcal{C}^{\alpha}$, then $K \subset \cup_\nu (O_\nu \cap \mathcal{C}^{\alpha})$. Finally we notice that since O_ν is an open set in $\mathcal{C}^{\gamma+}$, by trace topology we have that $O_\nu \cap \mathcal{C}^{\alpha}$ is an open set of $\mathcal{C}^{\alpha}$ (because $\mathcal{C}^{\alpha}$ is a closed set of $\mathcal{C}^{\gamma+}$). Thus we can extract a finite subcovering in $\mathcal{C}^{\alpha}$, which will be also a finite subcovering of K in $\mathcal{C}^{\gamma+}$. □

Next we show the continuity result.

Lemma B.2. *Let $\gamma > 0$. Then $C([0,T]; \mathcal{C}^{\gamma+}) = \cup_{\alpha>\gamma} C([0,T]; \mathcal{C}^{\alpha})$.*

Proof. The inclusion $\supseteq$ is obvious.
Next we show the inclusion $\subseteq$. Let $f : [0,T] \to \mathcal{C}^{\gamma+}$ be continuous. We have to find $\alpha > \gamma$ such that $f \in C([0,T]; \mathcal{C}^{\alpha})$. Let $E_f := \{f(t), t \in [0,T]\}$, which is a compact in $\mathcal{C}^{\gamma+} = \cup_{\alpha>\gamma} \mathcal{C}^{\alpha}$ since it is the image of the compact $[0,T]$ via f which is continuous. By Lemma B.1 there exists $\alpha > \gamma$ such that E_f is a compact in $\mathcal{C}^{\alpha}$, in particular, $f : [0,T] \to \mathcal{C}^{\alpha}$. It remains to show that $f(t_n) \to f(t_0)$ in $\mathcal{C}^{\alpha}$ when $t_n \to t_0$. Since E_f is compact in $\mathcal{C}^{\alpha}$, there exists a subsequence $t_{n_k} \to t_0$ such that $f(t_{n_k}) \to l$ for some $l \in \mathcal{C}^{\alpha}$, thus $l \in \mathcal{C}^{\gamma+}$. On the other hand, $f \in C([0,T]; \mathcal{C}^{\gamma+})$ means that $f(t_n) \to f(t_0)$ in $\mathcal{C}^{\gamma+}$. Thus by uniqueness of the limit we have $l = f(t_0)$. □

References

[1] H. Bahouri, J-Y. Chemin, and R. Danchin. *Fourier Analysis and Nonlinear Partial Differential Equations*. Springer, 2011.

[2] V. Barbu and M. Röckner. Probabilistic representation for solutions to nonlinear Fokker-Planck equations. *SIAM J. Math. Anal.*, 50(4):4246–4260, 2018.

[3] R. F. Bass and Z.-Q. Chen. Brownian motion with singular drift. *Ann. Probab.*, 31(2):791–817, 2003.

[4] J.-M. Bony. Calcul symbolique et propagation des singularites pour les équations aux dérivées partielles non linéaires. *Ann. Sci. Ec. Norm. Super.*, 14:209–246, 1981.

[5] G. Cannizzaro and K. Chouk. Multidimensional SDEs with singular drift and universal construction of the polymer measure with white noise potential. *Ann. Probab.*, 46(3):1710–1763, 2018.

[6] R. Carmona and F. Delarue. *Probabilistic theory of mean field games with applications I*, volume 84 of *Probability Theory and Stochastic Modelling*. Springer International Publishing, 2018.

[7] R. Carmona and F. Delarue. *Probabilistic theory of mean field games with applications II*, volume 84 of *Probability Theory and Stochastic Modelling*. Springer International Publishing, 2018.

[8] F. Flandoli, E. Issoglio, and F. Russo. Multidimensional SDEs with distributional coefficients. *T. Am. Math. Soc.*, 369:1665–1688, 2017.

[9] F. Flandoli, F. Russo, and J. Wolf. Some SDEs with distributional drift. I. General calculus. *Osaka J. Math.*, 40(2):493–542, 2003.

[10] F. Flandoli, F. Russo, and J. Wolf. Some SDEs with distributional drift. II. Lyons-Zheng structure, Itô's formula and semimartingale characterization. *Random Oper. Stochastic Equations*, 12(2):145–184, 2004.

[11] M. Gubinelli. Private communication. 2020.

[12] M. Gubinelli, P. Imkeller, and N. Perkowski. Paracontrolled distributions and singular PDEs. *Forum of Mathematics, Pi*, 3:75 pages, 2015.

[13] E. Issoglio. Transport equations with fractal noise - existence, uniqueness and regularity of the solution. *J. Analysis and its App.*, 32(1):37–53, 2013.

[14] E. Issoglio. A non-linear parabolic PDE with a distributional coefficient and its applications to stochastic analysis. *J. Differential Equations*, 267(10):5976–6003, 2019.

[15] E. Issoglio and S. Jing. Forward-Backward SDEs with distributional coefficients. *Stochastic Processes and their Applications*, 130(1):47–78, 2020.

[16] E. Issoglio and F. Russo. A Feynman–Kac result via Markov BSDEs with generalised drivers. *Bernoulli*, 26(1):728–766, 2020.

[17] E. Issoglio and F. Russo. On the martingale problem for SDEs with drifts in Besov spaces. *preprint (in preparation)*, 2021.

[18] L. Izydorczyk, N. Oudjane, and F. Russo. McKean Feynman-Kac probabilistic representations of nonlinear partial differential equations. *Geometry and Invariance in Stochastic Dynamics. Eds. S. Ugolini et al. Preprint hal-02397045*, to appear, 2021.

[19] B. Jourdain and S. Méléard. Propagation of chaos and fluctuations for a moderate model with smooth initial data. *Ann. Inst. H. Poincaré Probab. Statist.*, 34(6):727–766, 1998.

[20] A. Le Cavil, N. Oudjane, and F. Russo. Particle system algorithm and chaos propagation related to a non-conservative McKean type stochastic differential equations. *Stochastics and Partial Differential Equations: Analysis and Computation.*, pages 1–37, 2016.

[21] A. Le Cavil, N. Oudjane, and F. Russo. Probabilistic representation of a class of non-conservative nonlinear partial differential equations. *ALEA Lat. Am. J. Probab. Math. Stat*, 13(2):1189–1233, 2016.

[22] J. Lieber, N. Oudjane, and F. Russo. On the well-posedness of a class of McKean Feynman-Kac equations. *Markov Process. Relat. Fields*, 25(5):821–862, 2019.

[23] F. Russo and G. Trutnau. Some parabolic PDEs whose drift is an irregular random noise in space. *Ann. Probab.*, 35(6):2213–2262, 2007.

[24] D. R. Smart. *Fixed point theorems*, volume 66. Cambridge University Press, Cambridge, 1980.

[25] H. Triebel. *Bases in Function Spaces, Sampling, Discrepancy, Numerical Integration*, volume 11 of *EMS Tracts in Mathematics*. European Mathematical Society, 2010.

[26] H. Ye, J. Gao, and Y. Ding. A generalized Gronwall inequality and its application to a fractional differential equation. *J. Math. Anal. Appl.*, 328(2):1075–1081, 2007.

(Elena Issoglio) Dipartimento di Matematica 'G. Peano', Universitá di Torino
Email address, Corresponding author: elena.issoglio@unito.it

(Francesco Russo) Unité de Mathématiques appliquées, ENSTA Paris, Institut Polytechnique de Paris
Email address: francesco.russo@ensta-paris.fr