Hamilton-Jacobi equations and scalar conservation laws

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Abstract

This note is concerned with the link between the viscosity solution of a Hamilton-Jacobi equation and the entropy solution of a scalar conservation law. The framework is a one dimensional space.

Keywords. Hamilton-Jacobi equations, scalar conservation laws, method of characteristics, optimal control theory, parabolic regularization

1 Introduction

The equations at stake: the following Hamilton-Jacobi equation

\[
\begin{aligned}
\partial_t U + H(\partial_x U) &= 0 & t > 0, x \in \mathbb{R}, \\
U(0, x) &= U_0(x) & x \in \mathbb{R}, t > 0.
\end{aligned}
\]

(1)

and the following scalar conservation law

\[
\begin{aligned}
\partial_t u + \partial_x (H(u)) &= 0 & t > 0, x \in \mathbb{R}, \\
u(0, x) &= u_0(x) & x \in \mathbb{R}, t > 0.
\end{aligned}
\]

(2)

Hamilton-Jacobi equations (under more general form) play a very important role in classical mechanics. They are useful in identifying conserved quantities for mechanical systems. More generally, they appear in the optimal control theory via a dynamic programming principle; we will say more about this later.

Scalar conservation laws appear in many areas of physics. As soon as an extensive quantity \( u \) is considered (such as mass, momentum, \( \text{etc.} \)) and its associated flux function is related to \( u \), Eq. (2) expresses the fact that \( u \) is conserved.

At least formally, it is clear that if \( U \) is a solution of (1), then \( u = \partial_x U \) is a solution of (2) with \( u_0 = \partial_x U_0 \). We would like to make this statement rigorous. This is not straightforward because, in general, solutions of (1) and (2) are not smooth; for instance, even if \( u_0 \) is indefinitely differentiable, there are no smooth solutions of (2).

Standing example. The standing example for (1) and (2) is the case where \( f(u) = u^2/2 \). In this case, (2) is known as Burgers equation.

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The Standing Example is very important in physics since (2) in this case is commonly used as a model for gas dynamics or traffic flow. Shock waves can be modelled with such an equation.

Standing assumptions.

\[ H, U_0 \] are Lipschitz continuous.

For further comments on the link between Hamilton-Jacobi equations and scalar conservation laws, see also [5].

This note is organized as follows. In Section 2, we present the method of characteristics for scalar conservation laws. In Section 3, we explain that when \( H \) is convex, the solution of (1) can be represented as an infimum over a finite space. In Section 4, definitions of entropy solution and viscosity solution are recalled and the link between the solutions of (1) and (2) is proved. In the last section, we give final comments.

2 Method of characteristics for scalar conservation laws

We explain here a classical method to solve scalar conservation laws. It permits, in some cases, to compute a classical solution for small times and smooth initial data and flux function \( H \).

2.0.1 The linear case

We start with the linear case. Assume first that \( H(p) = V_0 p \). We look for a trajectory that “carries” information concerning the equation. Hence, we consider an absolutely continuous function \( x(\cdot) \): \( x(t) = x + \int_0^t \dot{x}(s) ds \). We next compute the value of the solution \( u \) of (2) along this trajectory.

\[ \partial_t(u(t, x(t))) = \partial_t u(t, x(t)) + \partial_x u(t, x(t)) x'(t) . \]

Hence, we see that if we choose \( \dot{x}(t) = V_0 \), we obtain that \( u \) is constant along this curve, which is in fact a line. We conclude that \( u(t, x(t)) = u_0(x) \) that is to say \( u(t, x+V_0 t) = u_0(x) \) which implies that for all \( t > 0 \) and \( y \in \mathbb{R} \), we have

\[ u(t, y) = u_0(y - V_0 t) . \]

2.0.2 The general case

In the general case, we apply exactly the same method. We conclude that the trajectory have to satisfy

\[ \dot{x}(t) = H'(u(t, x(t))) . \]

Assume that we are able to find solution to this equation. We then conclude that \( u \) is constant along the trajectory and this implies that \( \dot{x}(t) = H'(u_0(x)) \). In other words, the...
trajectory is a straight line. Hence, the characteristics is the curve parametrized by \((t, x(t))\) with
\[x(t) = x + H'(u_0(x))t.\]

Here is a rigourous result (see for instance [18]).

**Lemma 1.** Assume that \(H, u_0 \in C^\infty\) and \(u_0\) and all its derivatives are bounded. Let us define
\[T^* = \begin{cases} +\infty & \text{if } \inf_{x \in \mathbb{R}} H''(u_0(x))u_0'(x) \geq 0 \\ -\inf_{x \in \mathbb{R}} H''(u_0(x))u_0'(x) & \text{if not.} \end{cases}\]

There exists a \(a < 0\) such that the application \(\Psi : (a, T^*) \times \mathbb{R} \to (a, T^*) \times \mathbb{R}\) that maps \((t, x)\) on \((t, x + H'(u_0(x))t)\) is a \(C^\infty\)-diffeomorphism.

We next can define on \((a, T)\) the regular function \(u(t, x) = u_0((\Psi^{-1}(t, x))_2)\) (where \(y_2\) denotes the second component of a vector of \(\mathbb{R}^2\)) and it is clear that it is a classical \((C^\infty)\) solution of (2).

**Remark 1.** The method of characteristics can be applied to (1). It leads to Hamilton’s equations satisfied by \(x(t)\) and \(p(t) = \partial_x u(t, x(t))\). See for instance [11].

### 3 Optimal control interpretation for (1) with convex Hamiltonian

In this paragraph, we explain that the solution of (1) can be represented as an infimum of an integral functional (called the cost function) over curves (called trajectories) of the function \(H\) is convex.

We first define the Legendre-Fenchel transform of the convex function \(H\)
\[H^*(q) = \sup_{p \in \mathbb{R}} \{pq - H(p)\}.\]

The function \(H^*\) is referred to as the Lagrangian (see classical mechanics).

This function can take the value \(+\infty\). It is finite if \(H\) is 1-coercive, i.e. if it satisfies
\[
\frac{H(p)}{|p|} \to +\infty \text{ as } |p| \to +\infty.
\]

Let us consider such \(H^*\)’s for a while.

We now introduce the function \(J : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}\) defined as follows
\[J(t, x, X) = U_0(X(0)) + \int_0^t H^*(\dot{X}(s))ds\]

where \(X : [0, t] \to \mathbb{R}\) is absolutely continuous, \(X(0) = x\) and \(\dot{x}\) denotes its derivative.
Example 1. Here are the Legendre-Fenchel transform of several functions.

\[
\begin{align*}
\left(\frac{1}{2}u^2\right)^* &= \frac{1}{2}u^2 \\
|u|^* &= \begin{cases} 0 & \text{if } |u| \leq 1 \\
+\infty & \text{if not.} \end{cases} \\
(V_0u)^* &= \begin{cases} 0 & \text{if } u = V_0 \\
+\infty & \text{if not.} \end{cases}
\end{align*}
\]

Remark 2. If the function \( H \) is not convex but still Lipschitz, an integral representation of the solution is still available by making use of the theory of differential games.

3.1 Oleinik-Lax formula

We are going to see that optimal trajectories are in fact straight lines. Hence, we will obtain an “explicit” formula both for (1) and (2).

In order to prove that optimal trajectories are straight lines, we simply prove that if we consider a trajectory \( X(\cdot) \) such that \( X(0) = x \) and \( X(t) = y \), then the trajectory \( \bar{X}(\cdot) \) such that \( \bar{X}(t) \equiv \frac{1}{t} \int_0^t \dot{X}(s)ds \) has a cost that is smaller than the one of \( X(\cdot) \). Indeed, Jensen’s inequality implies

\[
\int_0^t H^*(\dot{X}(s)) \frac{ds}{t} \geq H^*(\int_0^t \dot{X}(s) \frac{ds}{t})
\]

and this implies that

\[
J(t, x, X(\cdot)) \geq J(t, x, \bar{X}(\cdot)).
\]

We conclude that the value function of the optimal control problem is given by the following formula.

\[
U_{\text{Lax}}(t, x) = \inf\{y \in \mathbb{R} : U_0(x - y) + tH^*(\frac{y}{t})\}.
\]

Remark 3. When \( H(p) = p^2/2 \), the Lax function coincides with the Moreau-Yosida regularization of the function \( U_0 \) where \( t \) is the “small” parameter.

It is not very difficult to prove that \( U_{\text{Lax}} \) is Lipschitz continuous and satisfies a.e. (1). Hence, the function \( \partial_x U_{\text{Lax}} \) is a good candidate for a solution of (2). Lax introduced this function in order to solve (2).

Example 2. If \( U_0(x) = |x| \) and \( H(p) = p^2/2 \), then

\[
U_{\text{Lax}}(t, x) = \begin{cases} 
|x| - \frac{t}{2} & \text{if } x \leq -t \\
\frac{x^2}{2t} & \text{if } -t \leq x \leq t \\
|x| - \frac{t}{2} & \text{if } x \geq t
\end{cases}
\]

Example 3. Assume here that \( U_0 \) is arbitrary but we look at a linear equation. In this case,

\[
U_{\text{Lax}}(t, x) = U_0(x - V_0t)
\]

and this is the solution we found above.
Example 4. Assume again that $U_0$ is arbitrary and consider the case $H(p) = |p|$. In this case,

$$U_{Lax}(t, x) = \inf_{|y| \leq 1} U_0(x - ty).$$

The interested reader is referred to [16, 15, 1] for further information.

4 Generalized solutions for (1) and (2)

In this section, we give examples illustrating the fact that classical solutions of (1) and (2) do not exist in general and that generalized solutions have to be define. If the notion of generalized solution is too weak, uniqueness is lost.

4.1 Entropy solutions for scalar conservation laws

We start with the study of scalar conservation laws. The interested reader is referred to [10, 18] (for instance) for further information.

4.1.1 Weak solutions

We would like to solve (2). Regular solutions cannot exist and this can be illustrated by the following simple example. Consider the Burgers equation with $u_0(x) = \text{Arctan}(x)$ as a smooth bounded initial condition. 

**Faire un dessin.** The left part of the initial condition moves faster than the right part. Hence, it is going to overlap it in finite time. Hence, shocks can occur even with very regular initial conditions $u_0$.

Next, we can look for solutions $u$ of (2) in the sense of distributions. The problem is that such a solution (called a weak solution) is not unique anymore. Consider the Burgers equation with the following bounded initial condition

$$u_0(x) = \begin{cases} 
-1 & \text{if } x < 0 \\
1 & \text{if } x > 0.
\end{cases}$$

If we apply the method of characteristics we described above, we obtain the picture in Figure 1. Hence, the solution is not defined everywhere. In order to define $u(t, x)$ when $|x| \leq t$, the simplest idea is to look for a piecewise linear function by setting

$$u(t, x) = \begin{cases} 
-1 & \text{if } x < -t \\
\frac{x}{t} & \text{if } |x| \leq t \\
+1 & \text{if } x > t.
\end{cases}$$

This solution is indeed a weak solution.

But there is another naive idea in order to get a weak solution: it is to choose (see Figure 2)
Figure 1: Characteristics for Burgers equation

Figure 2: A first weak solution

\[ u(t, x) = \begin{cases} 
-1 & \text{if } x < 0 \\
+1 & \text{if } x > 0 
\end{cases} \]

From these two examples, it is possible to construct an infinite family of weak solutions; see Figure 3.

4.1.2 Parabolic regularization

In some physical cases, (2) is an approximation for the following equation

\[ \partial_t u^\epsilon + \partial_x (H(u^\epsilon)) = \epsilon \Delta u^\epsilon \quad (3) \]

where \( \epsilon > 0 \) is a small parameter. For instance, \( \epsilon \) can represent the viscosity of a fluid in motion. In this case, under the Standing Assumptions we made, we have the following theorem.

**Theorem 1** (Solution of (3)). Assume that \( H \in C^\infty(\mathbb{R}) \) is Lipschitz continuous and \( u_0 \in L^\infty(\mathbb{R}) \). Then for all \( T > 0 \), there exists a unique \( u \in C^\infty((0, T) \times \mathbb{R}) \cap C([0, T] \times \mathbb{R}) \) which is a solution of (3).
Hence, we have a regular solution of (3) which is unique and when we pass to the limit $\epsilon \to 0$, we can construct many (weak) solutions. This means that the notion of solution we chose for (2) is too weak. We need to understand what is lost when passing to the limit.

The idea of Kružkov [14] is to consider a convex function $\eta$ (called an entropy) and look for the equation satisfied by $\eta(u)$. Multiplying (3) with $\eta'(u)$, we obtain

$$\partial_t(\eta(u^\epsilon)) + \eta'(u^\epsilon)\Phi'(u^\epsilon)\partial_x u^\epsilon = \epsilon\eta'(u^\epsilon)\Delta u^\epsilon. \quad (4)$$

Next, consider a function $\Phi$ such that $\Phi' = \eta'H'$ and write

$$\eta'(u^\epsilon)\Delta u^\epsilon = \Delta(\eta(u^\epsilon)) - \eta''(u)(\partial_x u^\epsilon)^2 \leq \Delta(\eta(u^\epsilon)).$$

The function $\Phi$ associated with $\eta$ is called the *entropy flux function* and $(\eta, \Phi)$ is called an *entropy flux pair*.

Hence, we deduce from (4) the following inequality

$$\partial_t(\eta(u^\epsilon)) + \partial_x(\Phi(u^\epsilon)) \leq \epsilon\Delta(\eta(u^\epsilon)).$$

We would like to pass to the limit in this inequality. In order to do so, we use a weak formulation: consider a test function $\phi \in C_0^\infty((0, +\infty) \times \mathbb{R})$, $\phi \geq 0$ and integrate by parts in order to get

$$\int_0^{+\infty} \int_\mathbb{R} (\eta(u^\epsilon)\partial_t\phi + \Phi(u^\epsilon)\partial_x\phi) + \int_\mathbb{R} \eta(0, x)\phi(0, x)dx \geq -\epsilon \int_0^{+\infty} \int_\mathbb{R} \eta(u^\epsilon)\Delta\phi.$$ 

It can be proved that there exists a constant $C_0 > 0$ which does not depend on $\epsilon$ and such that

$$\|u^\epsilon\|_{L^\infty((0, +\infty) \times \mathbb{R})} \leq C_0$$
$$\|\partial_x u^\epsilon\|_{L^\infty((0, +\infty), L^1(\mathbb{R}))} \leq C_0$$
$$\|\partial_t u^\epsilon\|_{L^\infty((0, +\infty), L^1(\mathbb{R}))} \leq C_0$$
In particular, the sequence $u^\epsilon$ is bounded in $W^{1,1}((0, T) \times (-M, M))$ for all $T, M > 0$. Hence, by Rellich Theorem, we can assume that $u^\epsilon$ converges towards $u \in L^\infty((0, +\infty) \times \mathbb{R})$ a.e. as $\epsilon \to 0$. It is now easy to get the following inequality
\[
\int_0^{+\infty} \int_\mathbb{R} \eta(u) \partial_t \phi + \Phi(u) \partial_x \phi + \int_\mathbb{R} \eta(u_0(x)) \phi(0, x) \geq 0.
\] (5)

4.1.3 Existence and uniqueness of entropy solution

**Definition 1** (Entropy solution for (2)). Assume that $u_0 \in L^\infty(\mathbb{R})$ and $H$ is Lipschitz continuous. A function $u \in L^\infty(\mathbb{R}^+ \times \mathbb{R})$ is an entropy solution of (2) if, for all convex function $\phi \in C^1(\mathbb{R})$ and all test function $\phi \in C_\infty_c(\mathbb{R}^+ \times \mathbb{R})$, we have (5) where $\Phi$ denotes the entropy flux function associated with $\eta$.

**Theorem 2** (Existence and uniqueness of an entropy solution of (2)). Assume that $u_0 \in L^\infty(\mathbb{R})$ and $H$ is Lipschitz continuous. There then exists a unique entropy solution of (2). Moreover, if $u$ is an entropy solution associated with the initial condition $u_0$ and $v$ is an entropy solution associated with $v_0$, we have for all $R > 0$ and a.e. $t > 0$
\[
\int_{|x| \leq R} |u(t, x) - v(t, x)|^+ \leq L_H \int_{|x| \leq R + L_H t} (u_0(x) - v_0(x))^+ dx.
\] (6)

where $L_H$ denotes the Lipschitz constant of $H$.

Such a result can be found in [4]. There is a lot of information in (6). Here are important comments.

- Maximum principle. Remark that constant functions are trivial solutions of (2). Then if $u_0 \leq M$, we can deduce from (6) that $u \leq M$.

- $L^1$-contraction property. Assume that $u_0, v_0 \in L^\infty$ satisfies $u_0 - v_0 \in L^1(\mathbb{R})$. Then by exchanging the role of $u$ and $v$ and by letting $R \to +\infty$ we obtain
\[
\int_\mathbb{R} |u(t, x) - v(t, x)| \leq L_H \int_\mathbb{R} |u_0(x) - v_0(x)| dx.
\]

This property is known as the $L^1$ contraction property of scalar conservation laws.

- Finite speed of propagation. If $u_0$ and $v_0$ differs only outside the ball of radius $R + L_H t$ centered at the origin, the corresponding solutions are the same on the ball of radius $R$ centered at the origin.

- Comparison principle. Assume that $u_0 \leq v_0$. Then (6) implies that $u \leq v$. In other words, if initial conditions are ordered, so are the corresponding solutions.
4.2 Viscosity solutions for Hamilton-Jacobi equations

4.2.1 Weak solutions

From the weak solutions we constructed for (2), it is easy to deduce an infinite family of functions that satisfy (1) in a weak sense. Here, a weak solution is a Lipschitz (in time and space) function which satisfies (1) almost everywhere.

4.2.2 Parabolic regularization

A parabolic regularization can be used in order to construct solutions to (1). Hence, we consider

$$\partial_t u^\varepsilon + H(\partial_x u^\varepsilon) = \varepsilon \Delta u^\varepsilon.$$  

(7)

Hence, we have the following result.

**Theorem 3** (Solution of (7)). Assume that $H \in C^\infty(\mathbb{R})$, $U_0 \in L^\infty$ and $H$ and $U_0$ are Lipschitz continuous. Then for all $T > 0$, there exists a unique $U^\varepsilon \in C^\infty((0,T) \times \mathbb{R}) \cap C([0,T) \times \mathbb{R})$ which is a solution of (3). Moreover, there exists a constant $C_0 > 0$ which does not depend on $\varepsilon$ such that

$$\|U^\varepsilon\|_{L^\infty((0,\infty) \times \mathbb{R})} \leq C_0.$$

**Remark 4.** Here, $C_0 = \|U_0\|_{L^\infty(\mathbb{R})}$.

We would like to pass to the limit in (7) and get enough information at the limit in order to ensure the uniqueness of the solution.

Assume that we are able to prove that $U^\varepsilon$ converges locally uniformly towards a continuous function $U : (0, +\infty) \times \mathbb{R} \to \mathbb{R}$ and consider a smooth test function $\phi \in C^\infty((0, +\infty) \times \mathbb{R})$. Assume moreover that $U - \phi$ attains a strict local maximum at a point $(t_0, x_0)$, that is to say there exists $r > 0$ such that for all $x \in B(x_0, r)$,

$$U(t, x) - \phi(t, x) \leq U(t_0, x_0) - \phi(t_0, x_0).$$

It is therefore possible to prove that the maximum of $U^\varepsilon - \phi$ on $B(x_0, r)$ is not attained at the boundary and this implies that there exists $(t_\varepsilon, x_\varepsilon)$ such that

$$\forall (t, x) \in B((t_0, x_0), r), \quad U^\varepsilon(t, x) - \phi(t, x) \leq U^\varepsilon(t_0, x_0) - \phi(t_0, x_0)$$

$$(t_\varepsilon, x_\varepsilon) \to (t_0, x_0) \text{ as } \varepsilon \to 0$$

In particular, we have

$$\partial_t U^\varepsilon(t_\varepsilon, x_\varepsilon) = \partial_t \phi(t, x)$$

$$\partial_x U^\varepsilon(t_\varepsilon, x_\varepsilon) = \partial_x \phi(t, x)$$

$$\partial_{x,x} U^\varepsilon(t_\varepsilon, x_\varepsilon) \leq \partial_{x,x} \phi(t_\varepsilon, x_\varepsilon).$$
Since $U^\epsilon$ satisfies (7), we conclude that $\phi$ satisfies
$$\partial_t \phi(t_\epsilon, x_\epsilon) + H(\partial_x \phi(t_\epsilon, x_\epsilon) \leq \epsilon \Delta \phi(t_\epsilon, x_\epsilon).$$
Letting $\epsilon \to 0$, we finally obtain
$$\partial_t \phi(t_0, x_0) + H(\partial_x \phi(t_0, x_0) \leq 0.$$ (8)
If now $U^\epsilon - \phi$ attains a strict local minimum at $(t_0, x_0)$, the reader can check that the previous
argument can be adapted and the final inequality is reversed:
$$\partial_t \phi(t_0, x_0) + H(\partial_x \phi(t_0, x_0) \geq 0.$$ (9)
This was first remarked by Evans [11]. Next, Crandall and Lions [8] proved that this permits
to prove that the limit is unique.

4.2.3 Definition of a viscosity solution

**Definition 2** (Viscosity solution for (1)). Consider a continuous function $U : [0, T] \times \mathbb{R} \to \mathbb{R}$.
- The function $U$ is a viscosity subsolution of (1) if, for all $\phi \in C^\infty([0, T] \times \mathbb{R})$ such that $U - \phi$ has a strict local maximum at $(t, x)$, we have (8).
- The function $U$ is a viscosity supersolution of (1) if, for all $\phi \in C^\infty([0, T] \times \mathbb{R})$ such that $U - \phi$ has a strict local minimum at $(t, x)$, we have (9).
- The function $U$ is a viscosity solution of (1) if it is both a subsolution and a supersolution.

**Theorem 4** (Comparison principle for (1)). Assume that $U_0$ is Lipschitz continuous. If $U$ is a subsolution of (1), $V$ is a supersolution of (1) and $U(0, x) \leq U_0(x) = V(0, x)$, then $U \leq V$ on $(0, +\infty) \times \mathbb{R}$.

In particular, there exists a unique viscosity solution of (1) and we deduce that $U^\epsilon \to U$ locally uniformly as $\epsilon \to 0$.

4.3 The link between the viscosity solution $U$ and the entropy
solution $u$

Consider $u^\epsilon$ given by Theorem 1 and $U^\epsilon$ given by Theorem 7. The uniqueness result for (3)
implies that $u_\epsilon = \partial_x U^\epsilon$. We now that $U^\epsilon \to U$ locally uniformly; hence $U^\epsilon \to U$ in the sense of
distributions and this implies that $\partial_x U^\epsilon \to \partial U$ in the sense of distributions. We also have
$u^\epsilon \to u$ almost everywhere. Hence $u^\epsilon \to u$ in the sense of distributions, hence we deduce
that $u = \partial_x U$.

**Theorem 5**. Assume that $U_0 \in W^{1,\infty}$ and $H$ is Lipschitz continuous. Then the unique
viscosity solution $U$ of (1) supplemented with the initial condition $U(0, x)U_0(x)$ is Lipschitz
continuous and the unique entropy solution $u$ of (2) supplemented with the initial condition
$u(0, x) = \partial_x U_0(x)$ coincides with the space derivative of $U$.

**Remark 5**. There exists a direct proof of Theorem 5 in the stationary case [6]
4.4 Application to numerical schemes

In this paragraph, we would like to say a few words about numerical schemes for Hamilton-Jacobi equations and scalar conservation laws which are based on (or related to) Theorem 5.

First, Theorem 5 is proved in [13] by using the front tracking method [9, 12] for scalar conservation laws. Precisely, it is proved that a front tracking method can be developed for Hamilton-Jacobi equation by using the one for conservation laws. When using the front tracking method for (2), the initial condition $u_0$ is supposed to be piecewise constant with compact support and the flux function $H$ is piecewise linear. Hence, explicit formulae are obtained for the viscosity solution of (1) under this assumption on $H$ and by assuming that $U_0$ is piecewise linear and Lipschitz continuous. These formulae also appear in [2].

We also point out that numerical schemes for scalar conservation laws with convex flux function $H$ are developed in [7] by using Theorem 5 and the optimal control interpretation of the Hamilton-Jacobi equation. Eventually, Prof. Shu [19] will explain in his last lecture how he developed numerical schemes for Hamilton-Jacobi equation by using Theorem 5.

5 Further comments

5.1 The case of $x$-dependent Hamiltonians

We decided to assume that $H$ does not depend on $x$ in order to simplify the presentation. However, results can be obtained if (1) and (2) are replaced respectively with

$$\partial_t U + H(x, \partial_x U) = 0$$

(10)

and

$$\partial_t u + \partial_x (H(x, u)) = 0$$

(11)

and $H$ is assumed to be Lipschitz continuous and satisfy (for instance)

$$|\partial_x H(x, p)| \leq C_0(1 + |p|) \quad \text{and} \quad |\partial_p H(x, p)| \leq C_0.$$

In particular, eikonal equations can be treated

$$\partial_t U + c(x)|DU| = 0.$$

An optimal control interpretation of the Hamilton-Jacobi equation is still available but optimal trajectories are not straight lines anymore.

5.2 Boundary conditions

The problem of boundary conditions for scalar conservation laws is a difficult one. Boundary Dirichlet condition can be imposed as soon as the boundary condition is $L^\infty$ for instance
Hence, (11) can be studied on $\Omega$ and supplemented with the following boundary condition

$$u = u_b \text{ on } (0, T) \times \partial \Omega.$$ 

Similarly, (10) can be studied on $\Omega$ and supplemented with

$$\partial_x U = u_b \text{ on } (0, T) \times \partial \Omega.$$ 

It is not clear that this Hamilton-Jacobi equation is well-posed in this case and that a comparison principle holds true.

References


