ABSTRACT

We analyze the convergence of the gradient descent (GD) method to solve large-scale inverse problems, where the corresponding forward and adjoint problems are solved iteratively by fixed-point iteration methods.

1. INTRODUCTION

We study the linear forward problem:

\[ u = Bu + Mσ + F \]

where \( u \in \mathbb{R}^n \) is the state variable; \( σ \in \mathbb{R}^m \) is the design variable; \( B, M, H \) are real matrices.

**Inverse problem:** Find \( σ \) from \( f = Hu(σ) \in \mathbb{R}^r \). Assumption: \( p(B) < 1 \) and \( H(I - B)^{-1}M \) is injective.

Method of least squares with the cost function \( J(σ) = \frac{1}{2} ∥Hu(σ) - f∥^2 \).

Lagrangian technique to define the adjoint state \( p = p(µ) \): \( p = B^∗p + H^∗(Hu(σ) - f) \).

Usual GD with fixed step \( τ > 0 \):

\[
\begin{align*}
σ^{n+1} &= σ^n - M^∗p^n, \\
u^{n+1} &= Bu^n + Mσ^n + F, \\
p^{n+1} &= B^∗p^n + H^∗(Hu^n - f).
\end{align*}
\]

Converge to the usual GD as \( k \to ∞ \).

Shifted GD with fixed step \( τ > 0 \):

\[
\begin{align*}
σ^{n+1} &= σ^n - M^∗p^n, \\
u^{n+1} &= Bu^n + Mσ^n + F, \\
p^{n+1} &= B^∗p^n + H^∗(Hu^n - f).
\end{align*}
\]

Converge to the shifted GD as \( k \to ∞ \).

Wait for \( σ \) before updating \( u, p \).

2. MULTI-STEP ONE-SHOT ALGORITHMS

<table>
<thead>
<tr>
<th>Method</th>
<th>Usual GD</th>
<th>Shifted GD</th>
<th>k-step one-shot</th>
<th>Shifted k-step one-shot</th>
</tr>
</thead>
<tbody>
<tr>
<td>( σ )</td>
<td>( σ^n - M^∗p^n )</td>
<td>( σ^n - M^∗p^n )</td>
<td>( σ^n - M^∗p^n )</td>
<td>( σ^n - M^∗p^n )</td>
</tr>
<tr>
<td>( u )</td>
<td>( u^n - Bu^n + Mσ^n + F )</td>
<td>( u^n - Bu^n + Mσ^n + F )</td>
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</tr>
</tbody>
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Converge to the usual GD as \( k \to ∞ \). Converge to the shifted GD as \( k \to ∞ \).

3. CONVERGENCE ANALYSIS IN 1D

**Necessary and sufficient condition for the convergence**

\[
\begin{align*}
τ &< \frac{2(1-k)}{\|h\|_M^2}, \quad τ < \frac{(1-δ)}{\|h\|_M^2}, \quad τ < \frac{1}{\nu (k, h)} \frac{1}{\|h\|_M^2}, \quad τ < 1 - (1 - δ) τ \frac{1}{\|h\|_M^2}.
\end{align*}
\]

where \( k \) and \( δ \) are plotted below (\( m = h = 1 \)):

4. MAIN RESULTS

**Theorem 1.** \( \exists r > 0 \) such that \( k \)-step one-shot converges. If \( 0 ≤ ∥B∥ < 1 \), take

\[
τ < \frac{ψ(k_r ∥B∥)}{∥H∥^2 ∥M∥^2}. \quad ψ \text{ is an explicit function.}
\]

**Theorem 2.** \( \exists r > 0 \) such that shifted \( k \)-step one-shot converges. If \( 0 ≤ ∥B∥ < 1 \), take

\[
τ < \frac{χ(k_r ∥B∥)}{∥H∥^2 ∥M∥^2}. \quad χ \text{ is an explicit function.}
\]

5. NUMERICAL RESULTS FOR A TOY PROBLEM

**Linearized conductivity inverse problem**

Medical application to EIT (Electrical Impedance Tomography).

**Forward problem** \( (δ > 0) \):

\[
\begin{align*}
-(1 + δ) \ \operatorname{div}(σ\nabla u) + u &= - \operatorname{div}(σ\nabla u_0) \text{ in } Ω, \\
\partial u / ∂ν &= 0, \quad σ = 0 \text{ on } ∂Ω
\end{align*}
\]

where \( u_0 \) satisfies

\[
\begin{align*}
-(1 + δ) \ \operatorname{div}(σ\nabla u_0) + u_0 &= 0 \text{ in } Ω, \\
\partial u_0 / ∂ν &= g \text{ on } ∂Ω.
\end{align*}
\]

**Inverse problem:** Recover \( σ \) from the measurement \( f = Hu(σ) \coloneqq u(σ) |_{∂Ω} \).

**Log-plot of the cost function** for each methods at different iterations:

6. REFERENCES