Theory and Applications of Constrained Optimal Control Problems with Delays

PART 2 : Pure State Constraints

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ANOC Spring School and Workshop
ENSTA Paris, April 23–27, 2012
Overview of Part 2

- Delayed Optimal Control Problems with Pure State Constraints
- Minimum Principle for Delayed Optimal Control Problems with Pure State Constraints
- Optimal Investment and Dividend of a Firm
- Optimal Control of Growth and Climate Change
- Optimal Control of the Innate Immune Response
Delayed Optimal Control Problem with State Constraints

State $x(t) \in \mathbb{R}^n$, Control $u(t) \in \mathbb{R}^m$, Delays $d_x, d_u \geq 0$.

**Dynamics and Boundary Conditions**

$$
\dot{x}(t) = f(x(t), x(t - d_x), u(t), u(t - d_u)), \text{ a.e. } t \in [0, t_f],
$$

$$
x(t) = x_0(t), \quad t \in [-d_x, 0],
$$

$$
u(t) = u_0(t), \quad t \in [-d_u, 0),
$$

$$
\psi(x(t_f)) = 0
$$

**Control and State Constraints**

$$
u(t) \in U \subset \mathbb{R}^m, \quad S(x(t)) \leq 0, \quad t \in [0, t_f] \quad (S : \mathbb{R}^n \rightarrow \mathbb{R}^k).
$$

**Minimize**

$$
J(u, x) = g(x(t_f)) + \int_0^{t_f} f_0(x(t), x(t - d_x), u(t), u(t - d_u)) \, dt
$$

All functions are assumed to be sufficiently smooth.
Hamiltonian (Pontryagin) Function

\[
H(x, y, \lambda, u, v) := \lambda_0 f_0(t, x, y, u, v) + \lambda f(t, x, y, u, v)
\]

- \( y \) variable with \( y(t) = x(t - d_x) \)
- \( v \) variable with \( v(t) = u(t - d_u) \)
- \( \lambda \in \mathbb{R}^n, \lambda_0 \in \mathbb{R} \) adjoint (costate) variable

Let \( (u, x) \in L^\infty([0, t_f], \mathbb{R}^m) \times W^{1,\infty}([0, t_f], \mathbb{R}^n) \) be a locally optimal pair of functions. Then there exist
  - an adjoint function \( \lambda \in BV([0, t_f], \mathbb{R}^n) \) and \( \lambda_0 \geq 0 \),
  - a multiplier \( \rho \in \mathbb{R}^q \) (associated with terminal conditions),
  - a multiplier function (measure) \( \mu \in BV([0, t_f], \mathbb{R}^k) \),

such that the following conditions are satisfied for a.e. \( t \in [0, t_f] \):
Minimum Principle

(i) Advanced adjoint equation and transversality condition:

\[ \lambda(t) = \int_t^{t_f} \left( H_x(s) + \chi_{[0,t_f-d_x]}(t) H_y(s + d_x) \right) ds + \int_t^{t_f} S_x(x(s)) d\mu(s) \]

\[ + \left( \lambda_0 g + \rho \psi \right) x(x(t_f)) \quad (\text{if } S(x(t_f)) < 0), \]

where \( H_x(t) \) and \( H_y(t + d_x) \) denote evaluations along the optimal trajectory and \( \chi_{[0,t_f-d_x]} \) is the characteristic function.

(ii) Minimum Condition:

\[ H(t) + \chi_{[0,t_f-d_u]}(t) H(t + d_u) \]

\[ = \min_{w \in U} \left[ H(x(t), y(t), \lambda(t), w, \nu(t)) \right. \]

\[ + \left. \chi_{[0,t_f-d_u]}(t) H(t + d_u) H(x(t + d_u), y(t), \lambda(t + d_u), u(t + d_u), w) \right] \]

(iii) Multiplier condition and complementarity condition:

\[ d\mu(t) \geq 0, \quad \int_0^{t_f} S(x(t)) d\mu(t) = 0 \]
Minimum Principle

(i) Advanced adjoint equation and transversality condition:

\[
\lambda(t) = \int_t^{t_f} (H_x(s) + \chi[0, t_f - d_x](t) H_y(s + d_x)) \, ds + \int_t^{t_f} S_x(x(s)) \, d\mu(s) \\
+ (\lambda_0 g + \rho \psi) x(x(t_f)) \quad \text{(if } S(x(t_f)) < 0 \text{),}
\]

where \(H_x(t)\) and \(H_y(t + d_x)\) denote evaluations along the optimal trajectory and \(\chi[0, t_f - d_x]\) is the characteristic function.

(ii) Minimum Condition:

\[
H(t) + \chi[0, t_f - d_u](t) H(t + d_u) = \min_{w \in U} \left[ H(x(t), y(t), \lambda(t), w, v(t)) \\
+ \chi[0, t_f - d_u](t) H(t + d_u)H(x(t + d_u), y(t), \lambda(t + d_u), u(t + d_u), w) \right]
\]

(iii) Multiplier condition and complementarity condition:

\[
d\mu(t) \geq 0, \quad \int_0^{t_f} S(x(t)) \, d\mu(t) = 0
\]
(i) **Advanced adjoint equation and transversality condition:**

$$\lambda(t) = \int_h^t \left( H_x(s) + \chi[0,t_f-d_x](t) H_y(s + d_x) \right) ds + \int_h^t S_x(x(s)) d\mu(s)$$

$$+ (\lambda_0 g + \rho \psi) x(x(t_f))$$

( if $S(x(t_f)) < 0$ ),

where $H_x(t)$ and $H_y(t + d_x)$ denote evaluations along the optimal trajectory and $\chi[0,t_f-d_x]$ is the characteristic function.

(ii) **Minimum Condition:**

$$H(t) + \chi[0,t_f-d_u](t) H(t + d_u)$$

$$= \min_{w \in U} \left[ H(x(t), y(t), \lambda(t), w, v(t))ight.$$

$$+ \chi[0,t_f-d_u](t) H(t + d_u) H(x(t + d_u), y(t), \lambda(t + d_u), u(t + d_u), w) \left. \right]$$

(iii) **Multiplier condition and complementarity condition:**

$$d\mu(t) \geq 0, \quad \int_0^{t_f} S(x(t)) d\mu(t) = 0$$
Proof Ideas

Use the "delay–chain–argument" as in the proof of the minimum principle for retarded control problems with mixed control-state constraints in


Augment the state dimension and apply the minimum principle for state-constrained optimal control problems:


Regularity conditions for \( d\mu(t) = \eta(t)dt \) if \( d_u = 0 \)

Boundary arc: \( S(x(t)) = 0 \) for \( t_1 \leq t \leq t_2 \).

Assumption: \( u(t) \in \text{int}(U) \) for \( t_1 < t < t_2 \).

Under certain regularity conditions we have \( d\mu(t) = \eta(t)dt \) with a smooth multiplier \( \eta(t) \) for all \( t_1 < t < t_2 \).

Adjoint equation and jump conditions

\[
\dot{\lambda}(t) = -H_x(t) - \chi_{[0, t_f - d_x]} H_y(t + d_x) - \eta(t)S_x(x(t))
\]

\[
\lambda(t_k^+) = \lambda(t_k^-) - \nu_k S_x(x(t_k)) , \quad \nu_k \geq 0
\]

at each contact or junction time \( t_k \), \( \nu_k = \mu(t_k^+) - \mu(t_k^-) \)

Minimum condition on the boundary

\[
H_u(t) = 0 .
\]

This condition allows to compute the multiplier \( \eta = \eta(x, \lambda) \).
Regularity conditions for $d\mu(t) = \eta(t)dt$ if $d_u = 0$

**Boundary arc**: $S(x(t)) = 0$ for $t_1 \leq t \leq t_2$.

**Assumption**: $u(t) \in \text{int}(U)$ for $t_1 < t < t_2$.

Under certain **regularity conditions** we have $d\mu(t) = \eta(t)dt$ with a smooth multiplier $\eta(t)$ for all $t_1 < t < t_2$.

**Adjoint equation and jump conditions**

$$\dot{\lambda}(t) = -H_x(t) - \chi_{[0,t_f-d_x]} H_y(t + d_x) - \eta(t)S_x(x(t))$$

$$\lambda(t_k^+) = \lambda(t_k^-) - \nu_k S_x(x(t_k)),$$  \(\nu_k \geq 0\)

at each contact or junction time $t_k$, $\nu_k = \mu(t_k^+) - \mu(t_k^-)$

**Minimum condition on the boundary**

$$H_u(t) = 0.$$ 

This condition allows to compute the multiplier $\eta = \eta(x, \lambda)$.
A non-convex control problem with state delays

State $x(t) \in \mathbb{R}$, control $u(t) \in \mathbb{R}$, delay $d \geq 0$

Dynamics and Boundary Conditions

$\dot{x}(t) = x(t-d)^2 - u(t), \quad t \in [0, 2],$

$x(t) = x_0(t) = 1, \quad t \in [-d, 0],$

$x(2) = 1$

Control and State Constraints

$x(t) \geq \alpha$, i.e., $S(x(t)) = -x(t) + \alpha \leq 0, \quad t \in [0, 2]$

Minimize

$J(u, x) = \int_0^2 (x(t)^2 + u(t)^2) \, dt$
Optimal solutions without state constraints

\[
\min \int_0^2 (x(t)^2 + u(t)^2) \, dt \quad \text{s.t.} \quad \dot{x}(t) = x(t - d)^2 - u(t), \; x_0(t) \equiv 1, \; x(2) = 1
\]

optimal state and control for delays \( d = 0.0, \; d = 0.1, \; d = 0.2, \; d = 0.5, \)
Optimal solutions with state constraint $x(t) \geq \alpha = 0.7$

Optimal state and control for delays $d = 0.0$, $d = 0.1$, $d = 0.2$, $d = 0.5$
Minimum Principle

Augmented Hamiltonian: \( y(t) = x(t - d) \)

\[
\mathcal{H}(x, y, \lambda, \eta, u) = u^2 + x^2 + \lambda(y^2 - u) + \eta(-x + \alpha)
\]

Adjoint equation

\[
\dot{\lambda}(t) = -\mathcal{H}_x(t) - \chi_{[0,2-d]} \mathcal{H}_y(t + d)
\]

\[
= \begin{cases} 
-2x(t) - 2\lambda(t + d)x(t) + \eta(t), & 0 \leq t \leq 2 - d \\
-2x(t) + \eta(t), & 2 - d \leq t \leq 2
\end{cases}
\]

Minimum condition

\[
\mathcal{H}_u(t) = 0 \Rightarrow u(t) = \frac{\lambda(t)}{2}
\]
Boundary arc \( x(t) = \alpha = 0.7 \) for \( t_1 \leq t \leq t_2 \)

\[
x(t) \equiv \alpha \implies \dot{x}(t) = 0 \implies x(t - d)^2 = u(t) = \lambda(t)/2
\]

Computation of multiplier \( \eta(t) \) by differentiation

\[
\eta(t) = 2(2x(t-d)(x(t-2d)^2 - \lambda(t-d)/2) + x(t) + \lambda(t+d)x(t))
\]

delays \( d = 0.0, d = 0.1, d = 0.2, d = 0.5, d = 1.0 \)

state \( x \)

multiplier for state constraint \( x(t) \geq \alpha \)
The optimal control model was proposed by Peter Kort (Tilburg). Stefan Winnemöller: diploma thesis, Münster, 2011.

State Variables:

\[ K(t) \quad : \quad \text{Capital of a firm at time } t \in [0, t_f] \]
\[ X(t) \quad : \quad \text{Equity capital} \]
\[ B(t) = K(t) - X(t) \quad : \quad \text{borrowed capital} \]

Control Variables:

\[ I(t) \quad : \quad \text{Investment} \]
\[ D(t) \quad : \quad \text{Dividend} \]
Optimal Control Model

Interest paid on borrowed capital:
\[ r(K, B) = 0.05 \cdot (1 + 2 (B/X)^{0.01}) , \quad B = K - X . \]

Dynamics

\[
\begin{align*}
\dot{K}(t) &= I(t - d) - \delta \cdot K(t) , \\
\dot{X}(t) &= I(t - d) - \delta \cdot K(t) - \eta \cdot K(t)^\gamma \\
&\quad - (I(t) + \mu \cdot I(t)^2) - r(B(t), X(t)) \cdot B(t) - D(t) , \\
K(0) &= K_0 , \quad X(0) = X_0 , \\
I(t) &= 0 , \quad -d \leq t < 0 .
\end{align*}
\]

Control and State Constraints

\[
0 \leq I(t) \leq I_{\text{max}} , \quad 0 \leq D(t) \leq D_{\text{max}} \quad \text{for } 0 \leq t \leq T , \\
0 \leq B(t) = K(t) - X(t) .
\]
Utility function \[ U(D, K) = D^\alpha (a + b \cdot K^\beta) \quad (0 < \alpha \leq 1) \]

Maximize the Dividend

\[
J(K, X, I, D) = e^{-\rho \cdot t_f} X(t_f) + \int_0^{t_f} e^{-\rho \cdot t} D(t)^\alpha (a + b \cdot K(t)^\beta)
\]

Parameters

\[
I_{\text{max}} = 1.5, \quad D_{\text{max}} = 3, \quad K(0) = 0.2, \quad X(0) = 0.1,
\eta = 1.0, \quad a = 1.0, \quad b = 0.5, \quad \beta = 0.5,
\mu = 0.5, \quad \gamma = 0.8, \quad \rho = 0.1, \quad \delta = 0.2,
t_f = 10.
\]

Case \( \alpha = 1 \): control \( D \) appears linearly (bang-bang or singular).
Recall functions

\[ U(D, X) = D^\alpha(a + b \cdot X^\beta) \]
\[ r(K, B) = 0.05 \cdot (1 + 2 (B/X)^{0.01}), \quad B = K - X. \]

Hamiltonian with delayed variable \( l_d \), \( l_d(t) = l(t - d) \).

\[
H(K, X, \lambda_K, \lambda_X, D, l, l_d) = e^{-\rho \cdot t} U(D, X) + \lambda_K (l_d - \delta \cdot K) \\
+ \lambda_X (l_d - \delta K^\gamma + \eta \cdot K^\gamma - (I + \mu \cdot l^2) - r(B, X) \cdot B - D)
\]

The minimum condition with respect to investment \( l \) gives

\[
0 = H_l(t) + \chi [0, t_f - d] H_{l_d}(t + d)
\]

\[
I(t) = \begin{cases} 
\frac{\lambda_K(t+d)+\lambda_X(t+d)-\lambda_X(t)}{2\mu\lambda_K(t)} & , \quad 0 \leq t \leq t_f - d \\
-\lambda_K(t)/(2\mu\lambda_X(t)) & , \quad t_f - d \leq t \leq t_f
\end{cases}
\]
Dividend control from Maximum Principle

Minimum condition with respect to dividend $D$:

Case $0 < \alpha < 1$: The condition $H_D(t) = 0$ gives

$$D(t) = -\frac{\lambda_K(t)}{\exp(-\rho \cdot t) \alpha (a + b \cdot X^\beta)} , \quad 0 \leq t \leq t_f$$

Case $\alpha = 1$: control $D$ appears linearly.

The switching function

$$\sigma(t) = \exp(-\rho \cdot t)(a + b X(t)^\beta)\lambda_X(t)$$

defines the dividend control according to

$$D(t) = \begin{cases} 
D_{\max}, & \sigma(t) > 0 \\
D_{\min}, & \sigma(t) < 0 \\
\text{singular}, & \sigma(t) = 0, \ t \in l_s \subset [0, t_f] 
\end{cases}$$
Delay $d = 0$ : $U(K, D) = D^\alpha(a + b \cdot K^\beta)$, $\alpha = 0.5$

Top row: Capital $K$ and Equity $X$, Dividend $D$ and Investment $I$.

Bottom row: debt rate $V = B/X$, adjoint variables $\lambda_K$, $\lambda_X$. 
Delay $d = 0 : U(K, D) = D^\alpha(a + b \cdot K^\beta)$, $\alpha = 1.0$

(a): Capital $K$, Equity $X$ and Borrowed Capital $B$

(b): Dividend $D$ is bang-singular-bang and Investment $I$
$d = 0$ and $d = 0.5$ : $U(K, D) = D^\alpha(a + b \cdot K^\beta)$, $\alpha = 0.5$

Delay $d = 0$

Delay $d = 0.5$

Capital $K$, Equity $X$ and Borrowed Capital $B$
\[ d = 0 \text{ and } d = 0.5 : \ U(K, D) = D^\alpha (a + b \cdot K^\beta), \ \alpha = 0.5 \]
\( d = 0 \) and \( d = 0.5 \): \( U(K, D) = D^\alpha(a + b \cdot K^\beta) \), \( \alpha = 1 \)

**Delay \( d = 0 \)**

**Delay \( d = 0.5 \)**

**Capital** \( K \), **Equity** \( X \) and **Borrowed Capital** \( B \)
$d = 0$ and $d = 0.5 : U(K, D) = D^\alpha(a + b \cdot K^\beta), \alpha = 1$

Delay $d = 0$

Delay $d = 0.5$

Dividend $D$ (bang-singular-bang) and Investment $I$


Dynamic Model of Growth and Climate Change

State Variables:

\[ K(t) : \text{Capital (per capita)} \]
\[ M(t) : \text{CO}_2 \text{ concentration in the atmosphere} \]
\[ T(t) : \text{Temperature (Kelvin)} \]

Control Variables

\[ C(t) : \text{Consumption} \]
\[ A(t) : \text{Abatement per capita} \]
Dynamical Model of Growth and Climate Change

Production: \( Y = K^{0.18} \cdot D(T - T_o), \quad T_o = 288 (K) \)

Damage: \( D(T - T_o) = (0.025 (T - T_o)^2 + 1)^{-0.025} \)

Dynamics of per-capita capital \( K \)

\[
\dot{K} = Y - C - A - (\delta + n)K, \quad K(0) = K_0.
\]

( \( \delta = 0.075, \quad n = 0.03 \) )

Emission: \( E = 3.5 \cdot 10^{-4} \cdot K/A \)

Dynamics of \( CO_2 \) concentration \( M \)

\[
\dot{M} = 0.49 E - 0.1 M, \quad M(0) = M_0.
\]
Dynamical Model of Growth and Climate Change

**Albedo** (non-reflected energy) at temperature $T$ (Kelvin):

$$1 - \alpha_1(T) = k_1 \frac{2}{\pi} \arctan \left( \frac{\pi(T-293)}{2} \right) + k_2,$$

$k_1 = 5.6 \cdot 10^{-3}$, $k_2 = 0.1795$.

Radiative forcing: $5.35 \ln(M)$.

Outgoing radiative flux (Stefan-Boltzman-law): $\epsilon \sigma_T T^4$.

Parameters: $\epsilon = 0.95$, $\sigma_T = 5.67 \cdot 10^{-8}$,

$$c.th = 0.149707$$, $Q = 1367$.

**Dynamics of temperature $T$ with delay $d \geq 0$**

$$\dot{T}(t) = c.th \cdot \left[ \left( 1 - \alpha_1(T(t)) \right) \frac{Q}{4} - \frac{19}{116} \cdot \epsilon \cdot \sigma_T \cdot T(t)^4 \
+ 5.35 \cdot \ln(M(t - d)) \right],$$

$T(0) = T_0$. 
Control constraints for finite horizon $t_f < \infty$:

\[ C_{\text{min}} \leq C(t) \leq C_{\text{max}}, \quad A(t) \leq A_{\text{max}} = 3 \cdot 10^{-3}. \]

State constraints of order 1, 2 and 3:

\[ K(t) \geq K_{\text{max}} \text{ for } 0 \leq t_s \leq t \leq t_f, \]
\[ M(t) \leq M_{\text{max}} \text{ for } 0 \leq t_s \leq t \leq t_f, \]
\[ T(t) \leq T_{\text{max}} \text{ for } 0 \leq t_s \leq t \leq t_f. \]

Maximize consumption

\[ J(K, M, T, C, A) = \int_{0}^{t_f} e^{-(n-\rho)t} \ln(C(t)) \, dt \]

Infinite-horizon control problem for $t_f = \infty$. 

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Theory and Applications of Constrained Optimal Control Problems
Maximum Principle: Necessary Conditions

State vector $X = (K, M, T)$ and control vector $u = (C, A)$.

Current-Value Hamiltonian:

$$H(X, \lambda, C, A) = \ln(C) + \lambda \dot{X}, \quad \lambda = (\lambda_K, \lambda_M, \lambda_T).$$

Adjoint equations (delay $d = 0$):

$$\dot{\lambda} = (\rho - n) \lambda - H_X(X, \lambda, C, A).$$

Without control constraints the maximum condition gives $H_C = H_A = 0$ which yields

$$C = 1/\lambda_K, \quad A = \left(-\gamma \frac{\lambda_M}{\lambda_K} \beta_1 a^\gamma K^\gamma\right)^{1/(1+\gamma)}.$$

For active state constraints adjoint variables may have jumps.
Steady states (stationary points) of the canonical system

Steady states are determined by the 6 equations

\[
\dot{X} = f(X, u(X, \lambda)) = 0, \quad \dot{\lambda} = (\rho - n)\lambda - H_X(X, \lambda, u(X, \lambda)) = 0.
\]

Constant Abatement \( A = 1.21 \cdot 10^{-3} \): 3 steady states

\[
T_s := 291.607, \quad M_s = 2.05196, \quad K_s = 1.44720,
\]

\[
T_s := 294.258, \quad M_s = 1.90954, \quad K_s = 1.34726,
\]

\[
T_s := 294.969, \quad M_s = 2.07792, \quad K_s = 1.46606.
\]

Control variable abatement \( A(t) \): Social Optimum

\[
T_s := 288.286, \quad M_s = 1.28500, \quad K_s = 1.79647.
\]

Finite-horizon control problems with control and state constraints: discretize the control problem and apply NLP methods. use the Applied Modeling Language AMPL (Fourer et al.) and Interior-Point optimization code IPOPT (Wächter et al.) or the code NUOCCCS developed by C. Bückens (Bremen).
Infinite horizon: $T(0) = 290$, $M(0) = 2.0$, $K(0) = 1.4$
Infinite horizon: $T(0) = 290$, $M(0) = 2.0$, $K(0) = 1.4$
Infinite horizon: $T(0) = 293, \ M(0) = 2.0, \ K(0) = 1.4$
Finite horizon: $T(0) = 290, M(0) = 2.0, K(0) = 1.4$

![Graphs of Consumption, Capital, Temperature, CO₂ concentration over time](image-url)
Finite horizon: $T(0) = 292$, $X(t_f) = X_{s1}$
Finite horizon: $d = 10$, $T(0) = 292$, $X(t_f) = X_{s1}$
Finite horizon: $d = 10$, $T(0) = 292$, $X(t_f) = X_{s1}$
Finite horizon: $T(0) = 292$, $T(t_f) = 290$

Constraints:

$0.895 \leq C(t) \leq 0.95$, $K(t) \geq 1$, $M(t) \leq 1.8$, $t_s = 10 \leq t \leq t_f$. 

---

**Consumption C**

---

**Capital K**

---

**Temperature T (Kelvin)**

---

**CO₂ concentration M**

---
Finite horizon: $T(0) = 292$, $T(t) \leq 289$ for $t \geq t_s = 10$
Infinite horizon: Social Optimum $u = (C, A), \ T(0) = 292$
Infinite horizon: Social Optimum \( u = (C, A) \), \( T(0) = 292 \)

- Capital \( K \)
- Adjoint variable (shadow price) \( \lambda_K \)
- \( \lambda_K \) decreases over time.

- \( \lambda_K \) is the adjoint variable (shadow price) for capital.

- \( \lambda_K \) starts at a higher value and decreases as time progresses, approaching a steady state.

- \( \lambda_K \) is crucial for understanding the economic implications of capital accumulation.

- \( \lambda_K \) represents the marginal value of capital, indicating its economic worth in the context of the model.

- The graphs illustrate the dynamics of capital and its shadow price over time, providing insights into economic optimization.

- The model considers both capital and its shadow price, which is essential for understanding the economic implications of capital accumulation.
Finite horizon: Social Optimum $u = (C, A), \ T(0) = 292$

Consumption $C$

Abatement $A$

Temperature $T$ (Kelvin)

$\text{CO}_2$ concentration $M$
Social Optimum for infinite horizon and finite horizon

Temperature $T$ (Kelvin)

Consumption $C$

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Theory and Applications of Constrained Optimal Control Problems
Dynamic model of the immune response:


Optimal control:

Innate Immune Response: state and control variables

State variables:

\( x_1(t) \): concentration of \textit{pathogen} \\
(= concentration of associated \textit{antigen})

\( x_2(t) \): concentration of \textit{plasma cells}, \\
which are carriers and producers of antibodies

\( x_3(t) \): concentration of \textit{antibodies}, which kill the pathogen \\
(= concentration of \textit{immunoglobulins})

\( x_4(t) \): relative characteristic of a \textit{damaged organ} \\
( 0 = healthy, 1 = dead )

Control variables:

\( u_1(t) \): pathogen killer

\( u_2(t) \): plasma cell enhancer

\( u_3(t) \): antibody enhancer

\( u_4(t) \): organ healing factor
Innate Immune Response: state and control variables

State variables:

\( x_1(t) \) : concentration of **pathogen**
  (=concentration of associated **antigen**)
\( x_2(t) \) : concentration of **plasma cells**, which are carriers and producers of antibodies
\( x_3(t) \) : concentration of **antibodies**, which kill the pathogen
  (=concentration of **immunoglobulins**)
\( x_4(t) \) : relative characteristic of a **damaged organ**
  (0 = healthy, 1 = dead)

Control variables:

\( u_1(t) \) : pathogen killer
\( u_2(t) \) : plasma cell enhancer
\( u_3(t) \) : antibody enhancer
\( u_4(t) \) : organ healing factor
Generic dynamical model of the immune response

\[ \dot{x}_1(t) = (1 - x_3(t))x_1(t) - u_1(t), \]
\[ \dot{x}_2(t) = 3A(x_4(t))x_1(t - d)x_3(t - d) - (x_2(t) - 2) + u_2(t), \]
\[ \dot{x}_3(t) = x_2(t) - (1.5 + 0.5x_1(t))x_3(t) + u_3(t), \]
\[ \dot{x}_4(t) = x_1(t) - x_4(t) - u_4(t). \]

Immune deficiency function triggered by target organ damage

\[ A(x_4) = \begin{cases} 
\cos(\pi x_4), & 0 \leq x_4 \leq 0.5 \\
0, & 0.5 \leq x_4
\end{cases}. \]

For \( 0.5 \leq x_4(t) \) the production of plasma cells stops.

State delay \( d \geq 0 \) in variables \( x_1 \) and \( x_3 \)

Initial conditions \( (d = 0) \) : \( x_2(0) = 2, \ x_3(0) = 4/3, \ x_4(0) = 0 \)

Case 1 : \( x_1(0) = 1.5, \) decay, requires no therapy (control)
Case 2 : \( x_1(0) = 2.0, \) slower decay, requires no therapy
Case 3 : \( x_1(0) = 3.0, \) diverges without control (lethal case)
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Optimal control model: cost functional

State \( x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \),  
Control \( u = (u_1, u_2, u_3, u_4) \in \mathbb{R}^4 \)

\( L^2 \)-functional quadratic in control: Stengel et al.

Minimize \( J_2(x, u) = x_1(t_f)^2 + x_4(t_f)^2 \)

\[ + \int_0^{t_f} (x_1^2 + x_4^2 + u_1^2 + u_2^2 + u_3^2 + u_4^2) \, dt \]

\( L^1 \)-functional linear in control

Minimize \( J_1(x, u) = x_1(t_f)^2 + x_4(t_f)^2 \)

\[ + \int_0^{t_f} (x_1^2 + x_4^2 + u_1 + u_2 + u_3 + u_4) \, dt \]

Control constraints: \( 0 \leq u_i(t) \leq u_{\text{max}}, \ i = 1, \ldots, 4 \)

Final time: \( t_f = 10 \)
Optimal control model: cost functional

State \( x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \), Control \( u = (u_1, u_2, u_3, u_4) \in \mathbb{R}^4 \)

\( L^2 \)-functional quadratic in control: Stengel et al.

Minimize \( J_2(x, u) = x_1(t_f)^2 + x_4(t_f)^2 \)
\[
+ \int_0^{t_f} \left( x_1^2 + x_4^2 + u_1^2 + u_2^2 + u_3^2 + u_4^2 \right) dt
\]

\( L^1 \)-functional linear in control

Minimize \( J_1(x, u) = x_1(t_f)^2 + x_4(t_f)^2 \)
\[
+ \int_0^{t_f} \left( x_1^2 + x_4^2 + u_1 + u_2 + u_3 + u_4 \right) dt
\]

Control constraints: \( 0 \leq u_i(t) \leq u_{\text{max}}, \ i = 1, \ldots, 4 \)

Final time: \( t_f = 10 \)
Optimal control model: cost functional

**State** \( x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4, \)  **Control** \( u = (u_1, u_2, u_3, u_4) \in \mathbb{R}^4 \)

**\( L^2 \)-functional quadratic in control: Stengel et al.**

Minimize \( J_2(x, u) = x_1(t_f)^2 + x_4(t_f)^2 \)
\[ + \int_0^{t_f} \left( x_1^2 + x_4^2 + u_1^2 + u_2^2 + u_3^2 + u_4^2 \right) dt \]

**\( L^1 \)-functional linear in control**

Minimize \( J_1(x, u) = x_1(t_f)^2 + x_4(t_f)^2 \)
\[ + \int_0^{t_f} \left( x_1^2 + x_4^2 + u_1 + u_2 + u_3 + u_4 \right) dt \]

Control constraints: \( 0 \leq u_i(t) \leq u_{\text{max}}, \ i = 1, \ldots, 4 \)

Final time: \( t_f = 10 \)
$L^2$–functional, $d = 0$ : optimal state and control variables

State variables $x_1, x_2, x_3, x_4$ and optimal controls $u_1, u_2, u_3, u_4$:
second-order sufficient conditions via matrix Riccati equation
$L^2$–functional, $d = 0$ : state constraint $x_4(t) \leq 0.2$

State and control variables for state constraint $x_4(t) \leq 0.2$.

Boundary arc $x_4(t) \equiv 0.2$ for $t_1 = 0.398 \leq t \leq t_2 = 1.35$. 
Compute multiplier $\eta$ as function of $(x, \lambda)$:

$$\eta(x, \lambda) = \lambda_2 3\pi \sin(\pi x_4) x_1 x_3 - \lambda_1 + 2\lambda_4 - 2x_3 x_1 + 2x_4$$

Scaled multiplier $0.1 \eta(t)$ and boundary arc $x_4(t) = 0.2$
\( L^2 \)-functional, delay \( d > 0 \), constraint \( x_4(t) \leq \alpha \)

**Dynamics with state delay \( d > 0 \)**

\[
\begin{align*}
\dot{x}_1(t) &= (1 - x_3(t))x_1(t) - u_1(t), \\
\dot{x}_2(t) &= 3 \cos(\pi x_4) x_1(t - d)x_3(t - d) - (x_2(t) - 2) + u_2(t), \\
\dot{x}_3(t) &= x_2(t) - (1.5 + 0.5x_1(t))x_3(t) + u_3(t), \\
\dot{x}_4(t) &= x_1(t) - x_4(t) - u_4(t) \\
x_4(t) &\leq \alpha \leq 0.5
\end{align*}
\]

**Initial conditions**

\[
\begin{align*}
x_1(t) &= 0, \quad -d \leq t < 0, \quad x_1(0) = 3, \\
x_3(t) &= 4/3, \quad -d \leq t \leq 0, \quad x_4(0) = 0.
\end{align*}
\]
$L^2$–functional: delay $d = 1$ and $x_4(t) \leq 0.2$

State variables for $d = 0$ and $d = 1$
$L^2$–functional: delay $d = 1$ and $x_4(t) \leq 0.2$

Optimal controls for $d = 0$ and $d = 1$
$L^2$–functional, $d = 1$ : multiplier $\eta(t)$ for $x_4(t) \leq 0.2$

Compute multiplier $\eta$ as function of $(x, \lambda)$:

$$
\eta(x, y, \lambda) = \lambda_2 3\pi \sin(\pi x_4) y_1 y_3 - \lambda_1 + 2\lambda_4 - 2x_3 x_1 + 2x_4
$$

Scaled multiplier $0.1\eta(t)$ and boundary arc $x_4(t) = 0.2$; $\eta(t)$ is discontinuous at $t = d = 1$
\[ L^1 \text{–functional and state delay } d \geq 0 \]

**Minimize**

\[
J_1(x, u) = p_{11} x_1(t_f)^2 + p_{44} x_4(t_f)^2 \\
+ \int_0^{t_f} \left( p_{11} x_1^2 + p_{44} x_4^2 + q_1 u_1 + q_2 u_2 + q_3 u_3 + q_4 u_4 \right) dt
\]

**Dynamics with delay \( d \) and control constraints**

\[
\begin{align*}
\dot{x}_1(t) &= (1 - x_3(t))x_1(t) - u_1(t), \\
\dot{x}_2(t) &= 3A(x_4(t))x_1(t - r)x_3(t - r) - (x_2(t) - 2) + u_2(t), \\
\dot{x}_3(t) &= x_2(t) - (1.5 + 0.5x_1(t))x_3(t) + u_3(t), \\
\dot{x}_4(t) &= x_1(t) - x_4(t) - u_4(t), \\
0 \leq u_i(t) &\leq u_{\text{max}}, \quad 0 \leq t \leq t_f \quad (i = 1, \ldots, 4)
\end{align*}
\]
$L^1$–functional: non-delayed problem $d = 0$: $u_{\text{max}} = 2$
$L^1$–functional : non-delayed problem $d = 1 : u_{\text{max}} = 2$
$L^1$–functional: non-delayed, time–optimal control for

\[ x_1(t_f) = x_4(t_f) = 0, \quad x_3(t_f) = \frac{4}{3} \]

$u_{\text{max}} = 1$: minimal time $t_f = 2.2151$, singular arc for $u_4(t)$
Future work and further applications

1. Minimum Principle for control problems with multiple delays.
2. Optimal oil extraction and exploration: state delay (Bruns, Maurer, Semmler)
4. Numerical methods for state-dependent delays (John Betts, Steve Campbell)