Optimal control, Hamilton-Jacobi equations and singularities in euclidean and riemannian spaces

Piermarco CANNARSA & Carlo SINESTRARI

Università di Roma “Tor Vergata”

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NEW TRENDS IN OPTIMAL CONTROL

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Outline

1. Introduction to semiconcave functions, generalized differentials, and singularities
   - Semiconcave functions
   - Semiconcavity of value functions
   - Generalized differentials
   - Optimal synthesis
   - Singular sets, rectifiability
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Historical remarks

Oleinik 1957 One-sided Lipschitz estimate as a uniqueness criterion for scalar hyperbolic conservation laws

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Hrustalev 1978 Semiconcavity of the value function in optimal control
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Semiconcave functions have been widely applied in other fields, e.g. nonlinear second order PDEs (Krylov), geometry of Alexandrov spaces (Perelman, Petrunin), etc.
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Semiconcave functions

Definition

A function \( u \in C(A) \), with \( A \subset \mathbb{R}^n \) is called semiconcave in \( A \) (with a linear modulus) if there exists \( C \geq 0 \) such that

\[
    u(x + h) + u(x - h) - 2u(x) \leq C|h|^2,
\]

for all \( x, h \in \mathbb{R}^n \) such that \([x - h, x + h] \subset A\).

C is called a semiconcavity constant for \( u \) in \( A \).

\( u \) semiconvex if \(-u \) is semiconcave.
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Equivalent formulations

Proposition

The following properties are equivalent:

- $u$ is semiconcave with constant $C$;
- the function $x \rightarrow u(x) - \frac{C}{2}|x|^2$ is concave in $A$;
- $u = u_1 + u_2$, with $u_1$ concave and $u_2 \in C^2(A)$ such that $\|D^2 u_2\|_{\infty} \leq C$;
- for any $\nu \in \mathbb{R}^n$ such that $|\nu| = 1$ we have $\frac{\partial^2 u}{\partial \nu^2} \leq C$ in $A$ weakly;
- $u(x) = \inf_{i \in I} u_i(x)$, where $\{u_i\}_{i \in I} \subset C^2(A)$ such that $\|D^2 u_i\|_{\infty} \leq C$ for all $i \in I$.

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Optimal control, HJ eqns, singularities  
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Generalizations

Definition

A function $u : A \to \mathbb{R}$ is called semiconcave with modulus $\omega(\cdot)$, where $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ is nondecreasing and satisfies $\lim_{\rho \to 0^+} \omega(\rho) = 0$, if

$$
\lambda u(x) + (1 - \lambda) u(y) - u(\lambda x + (1 - \lambda)y) \leq \lambda(1 - \lambda)|x - y|\omega(|x - y|)
$$

for any pair $x, y \in A$, such that $[x, y] \subset S$ is contained in $S$ and for any $\lambda \in [0, 1]$.

Standard definition: linear modulus $\omega(h) = Ch$. 

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for any pair $x, y \in A$, such that $[x, y] \subset S$ is contained in $S$ and for any $\lambda \in [0, 1]$.

Standard definition: linear modulus $\omega(h) = Ch$. 
Generalizations (II)

- \( u \) semiconcave with modulus \( \omega \) iff \( u = \inf u_i \), with \( u_i \in C^1 \) and \( Du_i \) has a uniform modulus of continuity \( \omega(\cdot) \), for every \( i \).

- \( u \) semiconcave with modulus \( \omega \) does NOT imply that \( u = u_1 + u_2 \) with \( u_1 \) concave, \( u_2 \in C^1 \).
Generalizations (II)

- $u$ semiconcave with modulus $\omega$ iff $u = \inf u_i$, with $u_i \in C^1$ and $Du_i$ has a uniform modulus of continuity $\omega(\cdot)$, for every $i$.
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The distance function

Given any $S \subset \mathbb{R}^n$ closed, define

$$d_S(x) = \min_{y \in S} |y - x|, \quad x \in \mathbb{R}^n,$$

*distance function* from the set $S$.

It is a special case of the *minimum time function*, corresponding to

$$y' = a(t) \in A = B_1 \text{ (unit ball)}.$$
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Proposition

- The squared distance function $d_S^2$ is semiconcave in $\mathbb{R}^n$ with semiconcavity constant $2$.

- $d_S$ is locally semiconcave in $\mathbb{R}^n \setminus S$. More precisely, given $\Omega$ such that $\text{dist}(S, \Omega) > 0$, $d_S$ is semiconcave in $\Omega$ with semiconcavity constant equal to $\text{dist}(S, \Omega)^{-1}$. 
Semiconcavity of the distance function

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Semiconcavity of the distance function (II)

Proof of the semiconcavity of $d_S^2$

For any $x \in \mathbb{R}^n$ we have

$$d_S^2(x) - |x|^2 = \min_{y \in S} |x - y|^2 - |x|^2 = \min_{y \in S} \left( |y|^2 - 2\langle x, y \rangle \right).$$

$\implies d_S^2(x) - |x|^2$ is concave (infimum of linear functions)

$\implies d_S^2(\cdot)$ semiconcave with constant 2.  \hfill \square
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Proof of the semiconcavity of $d^2_S$

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Semiconcavity of the distance function (III)

Proof of the local semiconcavity of \( d_S \)

Take \( z, h \in \mathbb{R}^n, z \neq 0 \). By a direct computation

\[
|z + h| + |z - h| - 2|z| \leq \frac{|h|^2}{|z|}.
\]

Let now \( \Omega \) be a set with positive distance from \( S \). For any \( x, h \) such that \([x - h, x + h] \subset \Omega\), let \( \bar{x} \in S \) be a projection of \( x \) onto \( S \). Then

\[
d_S(x + h) + d_S(x - h) - 2d_S(x) \\
\leq |x + h - \bar{x}| + |x - h - \bar{x}| - 2|x - \bar{x}| \\
\leq \frac{|h|^2}{|x - \bar{x}|} \leq \frac{|h|^2}{\text{dist}(S, \Omega)}. \quad \square
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$$\leq \frac{|h|^2}{|x - \bar{x}|} \leq \frac{|h|^2}{\text{dist}(S, \Omega)}. \quad \square$$
Interior sphere property

We say that \( S \subset \mathbb{R}^n \) satisfies the \textit{interior sphere property} for some \( r > 0 \) if, for any \( x \in S \) there exists \( y \) such that \( x \in B_r(y) \subset S \).

Proposition

If \( S \) satisfies the interior sphere property for some \( r > 0 \), then \( d_S \) is semiconcave in \( \mathbb{R}^n \setminus S \) with constant equal to \( r^{-1} \).
Interior sphere property

We say that $S \subset \mathbb{R}^n$ satisfies the *interior sphere property* for some $r > 0$ if, for any $x \in S$ there exists $y$ such that $x \in \overline{B_r(y)} \subset S$.

Proposition

*If $S$ satisfies the interior sphere property for some $r > 0$, then $d_S$ is semiconcave in $\mathbb{R}^n \setminus S$ with constant equal to $r^{-1}$.***
The Mayer problem with fixed horizon

- \((f, A)\) control process in \(\mathbb{R}^n\), \(T > 0\)
- given \((t, x)\) and a control \(\alpha : [t, T] \rightarrow A\)
  
  \[ y(\cdot; t, x, \alpha) \text{ solution of } \begin{cases} y(s) = f(y(s), \alpha(s)) & s \in [t, T] \\ y(t) = x \end{cases} \]

- \(\psi : \mathbb{R}^n \rightarrow \mathbb{R}\) final cost

Problem (Mayer with fixed horizon)

\[
\text{minimize } \psi(y(T; t, x, \alpha)) \text{ over all } \alpha \in L^1(t, T; A)
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Problem (Mayer with fixed horizon)

minimize $\psi(y(T; t, x, \alpha))$ over all $\alpha \in L^1(t, T; A)$
Semiconcavity of the value function

Value function

\[ V(t, x) = \min_{\alpha} \psi(y(T; t, x, \alpha)), \quad (t, x) \in [0, T] \times \mathbb{R}^n. \]

Theorem

(Cannarsa-Frankowska, 1991) Suppose that
- The control set \( A \) is compact.
- \( f(x, a) \) is differentiable w.r.t. \( x \).
- \( f(x, a) \) and \( f_x(x, a) \) are Lipschitz continuous w.r.t. \( x \), uniformly in \( a \).
- \( \psi \) is semiconcave in \( \mathbb{R}^n \).

Then the value function \( V \) is semiconcave in \([0, T] \times \mathbb{R}^n\) (jointly in \((t, x)\)).
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Semiconcavity of the value function

Value function

\[ V(t, x) = \min_{\alpha} \psi(y(T; t, x, \alpha)), \quad (t, x) \in [0, T] \times \mathbb{R}^n. \]

Theorem

*(Cannarsa-Frankowska, 1991)* Suppose that

- The control set \( A \) is compact.
- \( f(x, a) \) is differentiable w.r.t. \( x \).
- \( f(x, a) \) and \( f_x(x, a) \) are Lipschitz continuous w.r.t. \( x \), uniformly in \( a \).
- \( \psi \) is semiconcave in \( \mathbb{R}^n \).

Then the value function \( V \) is semiconcave in \([0, T] \times \mathbb{R}^n\) (jointly in \((t, x)\)).
Proof of the semiconcavity (I)

Estimates on trajectories starting at different points but following the same control.

Lemma

There exists $c > 0$ such that

$$|y(T; t, x_0, \alpha) - y(T; t, x_1, \alpha)| \leq c|x_0 - x_1|,$$

and

$$
|y(T; t, x_0, \alpha) + y(T; t, x_1, \alpha) - 2y(T; t, \frac{x_0 + x_1}{2}, \alpha)| \leq c|x_0 - x_1|^2
$$

for all $\alpha : [t, T] \to U$ and $x_0, x_1 \in \mathbb{R}^n$.

Regularity of $f$, Gronwall Lemma.
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Regularity of $f$, Gronwall Lemma.
Proof of the semiconcavity (II)

For simplicity, we only prove semiconcavity w.r.t. $x$.

Consider $x - h, x, x + h \in \mathbb{R}^n$ and $t \in [0, T)$. Let $\alpha : [t, T] \rightarrow A$ be an optimal control for the middle point $(t, x)$.

Let us set for simplicity

\[ y(\cdot) = y(\cdot ; t, x, \alpha), \quad y_-(\cdot) = y(\cdot ; t, x - h, \alpha), \quad y_+(\cdot) = y(\cdot ; t, x + h, \alpha). \]

By the previous lemma

\[ |y_+(T) - y_-(T)| \leq c|h|, \quad |y_+(T) + y_-(T) - 2y(T)| \leq c|h|^2. \]
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Proof of the semiconcavity (III)

It follows,

\[
V(t, x + h) + V(t, x - h) - 2V(t, x) \\
\leq \psi(y_+(T)) + \psi(y_-(T)) - 2\psi(y(T)) \\
= \psi(y_+(T)) + \psi(y_-(T)) - 2\psi\left(\frac{y_+(T) + y_-(T)}{2}\right) \\
\quad + 2\psi\left(\frac{y_+(T) + y_-(T)}{2}\right) - 2\psi(y(T)) \\
\leq C_\psi |y_+(T) - y_-(T)|^2 + L_\psi |y_+(T) + y_-(T) - 2y(T)| \\
\leq (C_\psi c^2 + L_\psi c)|h|^2,
\]

which proves the semiconcavity w.r.t. \( x \). \( \square \)
Minimum time function

- \((f, A)\) control process in \(\mathbb{R}^n\),
- \(f(x, a)\) Lipschitz w.r.t. \(x, A\) compact;
- given \(\alpha : [0, \infty) \rightarrow A\) control,

\[ y(\cdot; x, \alpha) \text{ solution of } \begin{cases} \dot{y}(t) = f(y(t), \alpha(t)) & (t \geq 0) \\ y(0) = x \end{cases} \]

- target \(S \subset \mathbb{R}^n\) nonempty closed set
- transition time \(\tau(x, \alpha) = \inf \{ t \geq 0 \mid y(t; x, \alpha) \in S \}\)
- controllable set \(C = \{ x \in \mathbb{R}^n \mid \exists \alpha : \tau(x, \alpha) < \infty \}\)
- minimum time function \(T(x) = \inf_\alpha \tau(x, \alpha) \quad x \in C\)
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Petrov condition

Definition

*Given* $y \in \partial S$, a vector $\nu \in \mathbb{R}^n$ is called a **proximal normal** to $S$ at $y$ if

$$\text{proj}_S(y + \varepsilon\nu) = \{y\}$$

for $\varepsilon > 0$ small enough.

Definition

*We say that* $(f, A)$ *satisfies the Petrov condition on* $S$ *if there exists* $\mu > 0$ *such that*

$$\min_{a \in A} f(x, a) \cdot \nu \leq -\mu |\nu|$$

*for any* $x \in \partial S$, $\nu$ **proximal normal** to $S$ *at* $x$. 
Local controllability

Theorem

(Petrov 1970, Bardi-Falcone 1990, . . .) Let the Petrov condition hold. Then

- $C$ is an open neighbourhood of $S$;
- there exist $k, \delta > 0$ such that
  \[ T(x) \leq kd_S(x), \quad \forall x \text{ s.t. } d_S(x) \leq \delta \]
- $T$ is locally Lipschitz continuous on $C$. 

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Local controllability

Theorem

\textit{(Petrov 1970, Bardi-Falcone 1990, \ldots ) Let the Petrov condition hold. Then}

- $\mathcal{C}$ is an open neighbourhood of $S$;
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$$T(x) \leq kd_S(x), \quad \forall x \text{ s.t. } d_S(x) \leq \delta$$

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- $T$ is locally Lipschitz continuous on $C$. 
Semiconcavity of $T$

**Theorem**

Let the Petrov condition hold and let $f(x, a)$ be $C^{1,1}$ w.r.t. $x$.

- If $S$ satisfies an interior sphere property, then $T$ is locally semiconcave in $\overline{C \setminus S}$. (Cannarsa-S., 1995)
- If $f(x, A)$ is convex and satisfies an interior sphere property for $x$ near $S$, then $T$ is locally semiconcave in $C \setminus S$. (Cannarsa-Frankowska, S., 2004)
- If $f(x, a) = Ax + a$ for some matrix $A$ and $S$ is convex, then $T$ is locally semiconvex in $\overline{C \setminus S}$. (Cannarsa-S., 1995)
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Outline

1. Introduction to semiconcave functions, generalized differentials, and singularities
   - Semiconcave functions
   - Semiconcavity of value functions
   - Generalized differentials
   - Optimal synthesis
   - Singular sets, rectifiability
Lipschitz continuity

**Proposition**

If $u : A \to \mathbb{R}$ is semiconcave (with a general modulus), it is locally Lipschitz continuous in the interior of $A$.

**Corollary**

Semiconcave functions are differentiable almost everywhere (Rademacher’s theorem).

**Theorem**

(Alexandroff) Semiconcave functions with linear modulus is twice differentiable almost everywhere.
**Proposition**

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Theorem

(Alexandroff) Semiconcave functions with linear modulus is twice differentiable almost everywhere.
Fréchet differentials

Let \( u : A \to \mathbb{R} \), with \( A \subset \mathbb{R}^n \) open.

**Definition**

*Given \( x \in A \), the sets*

\[
D^- u(x) = \left\{ p \in \mathbb{R}^n : \liminf_{y \to x} \frac{u(y) - u(x) - \langle p, y - x \rangle}{|y - x|} \geq 0 \right\},
\]

\[
D^+ u(x) = \left\{ p \in \mathbb{R}^n : \limsup_{y \to x} \frac{u(y) - u(x) - \langle p, y - x \rangle}{|y - x|} \leq 0 \right\}
\]

*are called, respectively, the (Fréchet) subdifferential and superdifferential of \( u \) at \( x \).*
Reachable gradients

Definition

Given \( u : A \rightarrow \mathbb{R} \) and \( x \in A \), we say that \( p \) is a reachable gradient of \( u \) at \( x \) if there exists \( \{x_n\} \subset A \) such that \( u \) is differentiable at \( x_n \) and

\[
x = \lim_{n \to \infty} x_n \quad \text{and} \quad p = \lim_{n \to \infty} Du(x_n).
\]

We denote by \( D^*u(x) \) the set of reachable gradients.

If \( u \in \text{Lip}_{loc}(A) \), then \( D^*u(x) \neq \emptyset \) for any \( x \in A \).

If \( u \in \text{Lip}_{loc}(A) \), the convex hull of \( D^*u(x) \) coincides with Clarke's generalized gradient.
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Differential properties

Proposition

Let $u : A \to \mathbb{R}$ be semiconcave (with general modulus). Then

- $D^+ u(x) = \text{co}(D^* u(x))$.
- $D^+ u(x) \neq \emptyset$.
- $D^* u(x) \subset \partial D^+ u(x)$.
- If $x_k \to x$ and if $p_k \in D^+ u(x_k)$ satisfy $p_k \to p$, then $p \in D^+ u(x)$ (upper semicontinuity of $D^+ u$).
- If $D^+ u(x)$ is a singleton, then $u$ is differentiable at $x$. 
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For simplicity, linear modulus of semiconcavity, \( A \) open convex.

**Proposition**

Let \( u : A \to \mathbb{R} \) be semiconcave with constant \( C \). Then

\[
\text{if and only if}
\]

\[
\forall p \in D^+ u(x), \quad u(y) \leq u(x) + \langle p, y - x \rangle + \frac{C}{2} |x - y|^2
\]

for all \( y \in A; \)

\[
\text{given } x, y \text{ and } p \in D^+ u(x), q \in D^+ u(y), \text{ we have}
\]

\[
\langle q - p, y - x \rangle \leq C |x - y|^2 \quad \text{(monotonicity of } D^+ u).\]
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- $p \in D^+ u(x)$ if and only if

$$u(y) \leq u(x) + \langle p, y - x \rangle + \frac{C}{2}|x - y|^2$$

for all $y \in A$;

- given $x, y$ and $p \in D^+ u(x), q \in D^+ u(y)$, we have

$$\langle q - p, y - x \rangle \leq C|x - y|^2 \quad \text{(monotonicity of } D^+ u).$$
An approximation lemma

Proposition

Let $u : A \to \mathbb{R}$ be semiconcave, $x_0 \in A$ and $V$ an open set such that $x_0 \in V \subset \overline{V} \subset A$. Then, for any $p \in D^+ u(x_0)$ there is a sequence $u_k \in C^\infty(V)$ such that

- $u_k \to u$ uniformly in $V$
- $D u_k(x_0) \to p$
- $\|u_k\|_{\infty} \leq M$, $\|D u_k\|_{\infty} \leq L$, $\|D^2 u_k\|_{\infty} \leq C$, for all $k$,

where $M$, $L$ and $C$ are respectively the supremum, the Lipschitz constant and the semiconcavity constant of $u$ on $A$. 
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- $u_k \to u$ uniformly in $V$
- $Du_k(x_0) \to p$
- $\|u_k\|_\infty \leq M$, $\|Du_k\|_\infty \leq L$, $\|D^2 u_k\|_\infty \leq C$, for all $k$, where $M$, $L$ and $C$ are respectively the supremum, the Lipschitz constant and the semiconcavity constant of $u$ on $A$. 

P. Cannarsa & C. Sinestrari (Rome 2)
An approximation lemma

Proposition

Let \( u : A \to \mathbb{R} \) be semiconcave, \( x_0 \in A \) and \( V \) an open set such that \( x_0 \in V \subset \overline{V} \subset A \). Then, for any \( p \in D^+ u(x_0) \) there is a sequence \( u_k \in C^\infty(V) \) such that

- \( u_k \to u \) uniformly in \( V \)
- \( Du_k(x_0) \to p \)
- \( ||u_k||_\infty \leq M, ||Du_k||_\infty \leq L, ||D^2u_k||_\infty \leq C \), for all \( k \),

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\begin{itemize}
  \item \( u_k \rightarrow u \) uniformly in \( V \)
  \item \( D u_k(x_0) \rightarrow p \)
  \item \( \|u_k\|_\infty \leq M, \|D u_k\|_\infty \leq L, \|D^2 u_k\|_\infty \leq C \), for all \( k \),
\end{itemize}

where \( M, L \) and \( C \) are respectively the supremum, the Lipschitz constant and the semiconcavity constant of \( u \) on \( A \).
Consider the Hamilton-Jacobi equation

\[(HJ) \quad H(x, u, Du) = 0, \quad x \in \Omega \subset \mathbb{R}^n. \]

with \(H\) a continuous function.

\(u \in C(\Omega)\) is a viscosity solution of \((HJ)\) if it satisfies, for any \(x \in \Omega\),

\[H(x, u(x), p) \leq 0 \quad \forall p \in D^+ u(x),\]

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Proposition

Suppose that $H(x, u, p)$ is convex w.r.t. $p$. Let $u : \Omega \to \mathbb{R}$ be a semiconcave function which satisfies $(HJ)$ at all points of differentiability. Then $u$ is a viscosity solution of $(HJ)$.

Proof — At the points $x$ where $u$ is differentiable — trivial.

If $u$ is not differentiable at $x$, then $D^- u(x) = \emptyset$, while $D^+ u(x) = \text{co}(D^* u(x))$.

By continuity, $H(x, u(x), p) = 0$ for all $p \in D^* u(x)$.

By convexity, $H(x, u(x), p) \leq 0$ for all $p \in D^+ u(x)$. □
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Semiconcavity and viscosity (II)

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Marginal functions

Marginal functions: infimum of smooth functions

\[ u(x) = \min_{s \in S} F(s, x) \]

Then \( u \) is semiconcave.
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\( A \subset \mathbb{R}^n \) open, \( S \subset \mathbb{R}^m \) compact.
\( F = F(s, x) \) continuous in \( S \times A \) together with \( D_x F \).

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Marginal functions (II)

Theorem

Let \( u(x) = \min_{s \in S} F(s, x) \) as above. Given \( x \in A \), define

\[
M(x) = \{ s \in S : u(x) = F(s, x) \},
\]

\[
Y(x) = \{ D_x F(s, x) : s \in M(x) \}.
\]

Then, for any \( x \in A \)

\[
D^+ u(x) = \text{co} Y(x).
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In particular, \( u \) is differentiable at \( x \) if and only if \( Y(x) \) is a singleton.
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Corollary

Let $S$ be a nonempty closed subset of $\mathbb{R}^n$. Then

- $d_S$ is differentiable at $x \notin S$ if and only if $\text{proj}_S(x)$ is a singleton and in this case

$$Dd_S(x) = \frac{x - y}{|x - y|}$$

where $y$ is the unique element of $\text{proj}_S(x)$.

- If $\text{proj}_S(x)$ is not a singleton then we have

$$D^+ d_S(x) = \text{co} \left\{ \frac{x - y}{|x - y|} : y \in \text{proj}_S(x) \right\},$$

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Outline

1. Introduction to semiconcave functions, generalized differentials, and singularities
   - Semiconcave functions
   - Semiconcavity of value functions
   - Generalized differentials
   - Optimal synthesis
   - Singular sets, rectifiability
Back to the Mayer problem

- \((f, A)\) control process in \(\mathbb{R}^n\), \(T > 0\)
- given \((t, x)\) and a control \(\alpha : [t, T] \rightarrow A\)

\[ y(\cdot; t, x, \alpha) \text{ solution of } \begin{cases} \dot{y}(s) = f(y(s), \alpha(s)) & s \in [t, T] \\ y(t) = x \end{cases} \]

- \(\psi : \mathbb{R}^n \rightarrow \mathbb{R}\) final cost

Mayer problem: minimize \(\psi(y(T; t, x, \alpha))\) over all \(\alpha \in L^1(t, T; A)\)

Value function \(V(t, x) = \min_{\alpha} \psi(y(T; t, x, \alpha))\).
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Value function \(V(t, x) = \min_{\alpha} \psi(y(T; t, x, \alpha))\).
We assume in the following

- $A$ compact
- $f(x, a)$ of class $C^{1,1}$ w.r.t. $x$
- $\psi : \mathbb{R}^n \to \mathbb{R}$ of class $C^1$ and semiconcave
Pontryagin’s maximum principle

**Theorem**

- \( \alpha^* \in L^1(0, T; A) \) and \( y^*(\cdot) := y(\cdot; x, \alpha^*) \) optimal pair

\[
\psi(y^*(T)) = \min_{\alpha \in L^1(0,T;A)} \psi(y(T; x, \alpha))
\]

- let \( p^* \) be the solution of the adjoint problem

\[
\begin{aligned}
\dot{p}(s) &= -f_x(y^*(s), \alpha^*(s))^\text{tr} p(s) \quad (s \in [0, T]) \\
p(T) &= D\psi(y^*(T))
\end{aligned}
\]

then

\[
p^*(s) \cdot f(y^*(s), \alpha^*(s)) = \min_{a \in A} p^*(s) \cdot f(y^*(s), a) \quad (s \in [0, T] \text{ a.e.})
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\]
Denote by $\nabla^+ V(t, x)$, $\nabla^- V(t, x)$ the super- and subdifferential of $V$ at $(t, x)$ with respect to the $x$ variable alone.

**Theorem**

*(Clarke-Vinter 1987, Cannarsa-Frankowska, 1991)* Under the previous assumptions, we have that

$$p(s) \in \nabla^+ V(s, y(s)), \quad \forall s \in [t, T].$$

If in addition $p(t) \in \nabla^- V(t, y(t))$, then we also have

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Dual arc inclusion for the Mayer problem

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Hamiltonian form of PMP

Assume that $f(x, A)$ is a ($n$-dimensional) uniformly convex set for all $x$.

This implies that $H(x, p) = \max_{a \in A} -p \cdot f(x, a)$ is smooth for $p \neq 0$.

**Theorem**

Let $(\alpha, y)$ be an optimal pair for the point $(t, x) \in [0, T] \times \mathbb{R}^n$ and let $p : [t, T] \to \mathbb{R}^n$ be a dual arc associated with $(\alpha, y)$ such that $p(\bar{s}) \neq 0$ for some $\bar{s} \in [t, T]$. Then $p(s) \neq 0$ for all $s \in [t, T]$ and $(y, p)$ solves the system

\[
\begin{align*}
    y'(s) &= -H_p(y(s), p(s)) \\
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optimal synthesis

Theorem

Given a point \((t, x) \in [0, T] \times \mathbb{R}^n\) and a vector \(\bar{p} = (\bar{p}_t, \bar{p}_x) \in D^* V(t, x)\) such that \(\bar{p} \neq 0\), let us associate with \(\bar{p}\) the pair \((y(\cdot), p(\cdot))\) which solves the Hamiltonian system with initial conditions \(y(t) = x, p(t) = \bar{p}_x\).

Then \(y(\cdot)\) is an optimal trajectory for \((t, x)\) and \(p(\cdot)\) is a dual arc associated with \(y(\cdot)\).

The map from \(D^* V(t, x)\) to the set of optimal trajectories from \((t, x)\) defined in this way is injective, and it is one-to-one if \(0 \notin D^* V(t, x)\).

Corollary

If \(0 \notin D^* V(t, x)\), then the optimal trajectory at \((t, x)\) is unique if and only if \(V\) is differentiable at \((t, x)\).
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1. Introduction to semiconcave functions, generalized differentials, and singularities
   - Semiconcave functions
   - Semiconcavity of value functions
   - Generalized differentials
   - Optimal synthesis
   - Singular sets, rectifiability
The singular set

Given $u : A \to \mathbb{R}$ semiconcave, the *singular set* of $u$ is

$$\Sigma(u) = \{ x \in A : u \text{ is not differentiable at } x \}$$
$$= \{ x \in A : D^+u(x) \text{ is not a singleton} \}.$$

We know: $\Sigma$ has measure zero.

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Much sharper results can be given in terms of *rectifiability* properties.
Let $k \in \{0, 1, \ldots, n\}$ and let $C \subset \mathbb{R}^n$.

- $C$ is called a \textit{k–rectifiable} set if there exists a Lipschitz continuous function $f : \mathbb{R}^k \to \mathbb{R}^n$ such that $C \subset f(\mathbb{R}^k)$.

- $C$ is called a \textit{countably k–rectifiable} set if it is the union of a countable family of $k$–rectifiable sets.

- $C$ is called a \textit{countably $\mathcal{H}^k$–rectifiable set} if there exists a countably $k$–rectifiable set $E \subset \mathbb{R}^n$ such that $\mathcal{H}^k(C \setminus E) = 0$. Here $\mathcal{H}^k$ denotes the $k$-dimensional Hausdorff measure.
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rectifiability results

It is easy to see that, if \( u \) is semiconcave with a linear modulus, then \( Du \) is a function of bounded variation.

The singular set \( \Sigma(u) \) coincides with the *jump set* of \( Du \) in the theory of \( BV \) functions.

Standard results about \( BV \) functions then imply that \( \Sigma(u) \) is *countably \( H^{n-1} \)-rectifiable*.

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rectifiability results (II)

\[ D^+ u(x) \text{ is a convex set} \implies \text{it has an integer dimension.} \]

For \( k = 1, \ldots, n \), we define

\[ \Sigma^k(u) = \{ x \in \Sigma : \dim(D^+ u(x)) = k \}. \]

**Theorem**

If \( u : \Omega \to \mathbb{R} \) is semiconcave (with a general modulus) then the set \( \Sigma^k(u) \) is countably \((n - k)\)-rectifiable for any \( k = 1, \ldots, n \).

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example

Let \( u(x, y) = -|x| - |y| \), concave on \( \mathbb{R}^2 \).

Then \( \Sigma(u) = \{(x, y) : x = 0 \text{ or } y = 0\} \).

If \( x = 0 \) and \( y > 0 \), then \( D^+u(x, y) = [-1, 1] \times \{-1\} \). Similarly, any point with \( x = 0, y \neq 0 \), or with \( x \neq 0, y = 0 \) belongs to \( \Sigma^1(u) \).

Finally, \( D^+u(0, 0) = [-1, 1] \times [-1, 1] \), and \( \Sigma^2(u) = \{(0, 0)\} \).
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**Definition**

Let $S \subset \mathbb{R}^n$ and $x \in \overline{S}$ be given. The contingent cone (or Bouligand’s tangent cone) to $S$ at $x$ is the set

$$T(x, S) = \left\{ \lim_{i \to \infty} \frac{x_i - x}{t_i} : x_i \in S, x_i \to x, t_i \in \mathbb{R}_+, t_i \downarrow 0 \right\}.$$

The vector space generated by $T(x, S)$ is called tangent space to $S$ at $x$ and is denoted by $\text{Tan}(x, S)$. 

**sketch of the proof**
Theorem

Let $S \subset \mathbb{R}^n$ be a set such that $\dim \text{Tan}(x, S) \leq k$, for all $x \in S$, for a given integer $k \in [0, n]$. Then $S$ is countably $k$–rectifiable.

Given $\rho > 0$, we denote by $\Sigma^k_\rho(u)$ the set of all $x \in \Sigma^k(u)$ such that $D^+u(x)$ contains a $k$–dimensional sphere of radius $\rho$.

Theorem

If $u$ is semiconcave in $\Omega$, then $\text{Tan}(x, \Sigma^k_\rho(u)) \subset [D^+u(x)]^\perp$, $\forall x \in \Sigma^k_\rho(u)$.

The rectifiability theorem follows. □
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Thank you for your attention!