Tutorial on Control and State Constrained Optimal Control Problems and Applications – Part 3: Pure State Constraints

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Outline

1. Theory of Optimal Control Problems with Pure State Constraints
2. Academic Example: order $q = 1$ of the state constraint
3. Van der Pol Oscillator: order $q = 1$ of the state constraint
4. Example: Immune Response
5. Example: Optimal Control of a Model of Climate Change
Optimal Control Problem with Pure State Constraints

State \( x(t) \in \mathbb{R}^n \), Control \( u(t) \in \mathbb{R} \) (for simplicity). All functions are assumed to be sufficiently smooth.

Dynamics and Boundary Conditions

\[
\dot{x}(t) = f(x(t), u(t)), \quad t \in [0, t_f],
\]

\[
x(0) = x_0 \in \mathbb{R}^n, \quad \psi(x(t_f)) = 0 \in \mathbb{R}^k,
\]

\[
(0 = \varphi(x(0), x(t_f)), \text{ mixed boundary conditions})
\]

Pure State Constraints and Control Bounds

\[
s(x(t)) \leq 0, \quad t \in [0, t_f], \quad (s : \mathbb{R}^n \to \mathbb{R})
\]

\[
\alpha \leq u(t) \leq \beta, \quad t \in [0, t_f].
\]

Minimize

\[
J(u, x) = g(x(t_f)) + \int_0^{t_f} f_0(x(t), u(t)) \, dt
\]
Hamiltonian

\[ H(x, \lambda, u) = \lambda_0 f(x, u) + \lambda f(x, u) \quad \lambda \in \mathbb{R}^n \]  
(row vector)

Let \((u, x) \in \mathcal{L}^\infty([0, t_f], \mathbb{R}) \times \mathcal{W}^{1,\infty}([0, t_f], \mathbb{R}^n)\) be a locally optimal pair of functions. Then there exist

- an adjoint (costate) function \(\lambda \in \mathcal{W}^{1,\infty}([0, t_f], \mathbb{R}^n)\) and a scalar \(\lambda_0 \geq 0\),

- a multiplier function of bounded variation \(\mu \in \mathcal{B}V^\infty([0, t_f], \mathbb{R})\),

- a multiplier \(\rho \in \mathbb{R}\) associated to the boundary condition \(\psi(x(t_f)) = 0\),

that satisfy the following conditions for a.a. \(t \in [0, t_f]\), where the argument \((t)\) denotes evaluations along the trajectory \((x(t), u(t), \lambda(t))\) :
Minimum Principle of Pontryagin et al. (Hestenes)

(i) Adjoint integral equation and transversality condition:

\[ \lambda(t) = \int_{t}^{t_f} H_x(s) \, ds + \int_{t}^{t_f} s_x(x(s)) \, d\mu(s) \]

\[ + (\lambda_0 g + \rho \psi)_x(x(t_f)) \quad \text{(if } s(x(t_f)) < 0 \text{)}, \]

(iiia) Minimum Condition for Hamiltonian:

\[ H(x(t), \lambda(t), u(t)) = \min \{ H(x(t), \lambda(t), u) \mid \alpha \leq u \leq \beta \} \]

(iii) positive measure \( d\mu \) and complementarity condition:

\[ d\mu(t) \leq 0 \text{ and } \int_{0}^{t_f} s_x(t)) \, d\mu(t) = 0 \]
Order of a state constraint $s(x(t)) \leq 0$

Define recursively functions $s^{(k)}(x, u)$ by

$$s^{(0)}(x, u) = s(x),$$

$$s^{(k+1)}(x, u) = \frac{\partial s^{(k)}}{\partial x}(x, u) f(x, u), \quad (k = 0, 1, \ldots)$$

Suppose there exist $q \in \mathbb{N}$ with

$$\frac{\partial s^{(k)}}{\partial u}(x, u) \equiv 0, \text{ i.e., } s^{(k)} = s^{(k)}(x), \quad (k = 0, 1, \ldots, q - 1),$$

$$\frac{\partial s^{(q)}}{\partial u}(x, u) \neq 0.$$

Then along a solution of $\dot{x}(t) = f(x(t), u(t))$ we have

$$s^{(k)}(x(t)) = \frac{d^k}{dt^k} s(x(t)) \quad (k = 0, 1, \ldots, q - 1),$$

$$s^{(q)}(x(t), u(t)) = \frac{d^q}{dt^q} s(x(t))$$
Regularity conditions to ensure \( d\mu(t) = \eta(t)\,dt \)

Regularity assumption on a boundary arc \( s(x(t)) = 0, \ t_1 \leq t \leq t_2 \)

\[
\frac{\partial s^{(q)}}{\partial u}(x(t), u(t)) \neq 0 \quad \forall \ t_1 \leq t \leq t_2.
\]

Assumption on boundary control

There exists a sufficiently smooth boundary control \( u = u_b(x) \) with \( s(x, u_b(x)) \equiv 0 \).

Assumption: \( \alpha < u(t) = u_b(x(t)) < \beta \quad \forall \ t_1 < t < t_2 \).

Regularity of multiplier (measure) \( \mu \)

The regularity and assumption on boundary control imply that there exist a smooth multiplier \( \eta(t) \) with

\[
d\mu(t) = \eta(t)\,dt \quad t_1 < t < t_2
\]
Minimum Principle under regularity

### Augmented Hamiltonian

\[
H(x, \lambda, \eta, u) = H(x, \lambda, u) + \eta \cdot s(x)
\]

### Adjoint equation and jump conditions

\[
\frac{d\lambda}{dt}(t) = -H_x(t) = -H_x(t) - \eta(t)s_x(x(t))
\]

\[
\lambda(t_{k+}) = \lambda(t_{k-}) - \nu_k s_x(x(t_k)), \quad \nu_k \geq 0
\]

at each contact or junction time \(t_k\), \(\nu_k = \mu(t_{k+}) - \mu(t_{k-})\)

### Minimum condition

\[
H(x(t), \lambda(t), u(t)) = \min \{ H(x(t), \lambda(t), u) \mid \alpha \leq u \leq \beta \}
\]

\[
H_u(t) = 0 \quad \text{on boundary arcs} \quad t_1^+ < t < t_2^-
\]
Academic Example

Minimize \[ J(x, u) = \int_0^2 (u^2 + x^2) \, dt \]
subject to \[ \dot{x} = x^2 - u, \quad x(0) = 1, \; x(2) = 1, \]
and the state constraint \[ x(t) \geq a, \quad \forall \quad 0 \leq t \leq 2. \]

Boundary arc \( x(t) \equiv a = 0.7 \) for \( t_1 = 0.614 \leq t \leq t_2 = 1.386 \).
Academic Example: state and control trajectories

- **State Trajectories $x(t)$**
  - Data files: "x.dat", "x-a=0.6.dat", "x-a=0.7.dat", "x-a=0.85.dat"
  - Range: $0 \leq t \leq 2$

- **Control $u$**
  - Data files: "u.dat", "u-a=0.6.dat", "u-a=0.7.dat", "u-a=0.85.dat"
  - Range: $-0.5 \leq u \leq 2.5$
Academic Example: Computation of Multiplier $\eta$

State constraint: $s(x) = a - x \leq 0$.

Order of the state constraint is $q = 1$, since

$$s^{(1)}(x, u) = -\dot{x} = -x^2 + u, \quad (s^{(1)})_u = 1.$$  

Boundary control $u = u_b(x)$ with $s^{(1)}(x, u_b(x)) \equiv 0$ is given by

$$u_b(x) = x^2 = a^2.$$  

Augmented Hamiltonian and adjoint equation:

$$\mathcal{H}(x, \lambda, \eta, u) = u^2 + x^2 + \lambda(x^2 - u) + \eta(a - x),$$  

$$\dot{\lambda} = -\mathcal{H}_x = -2x - 2\lambda x + \eta.$$  

The minimum condition $0 = H_u = 2u - \lambda$ gives $u = \lambda/2$. Since $u = a^2$ holds on a boundary arc, we get $\dot{\lambda} = 0$ and hence the multiplier $\eta$ on the boundary

$$\eta(t) \equiv 2a(1 + 2a^2) > 0.$$
Theory of Optimal Control Problems with Pure State Constraints

**Academic Example:** order $q = 1$ of the state constraint

Van der Pol Oscillator: order $q = 1$ of the state constraint

Example: Immune Response

Example: Optimal Control of a Model of Climate Change

Academic Example: state and control trajectories

![State trajectories](image1)

State trajectories $x(t)$

![Multiplier eta](image2)

Multiplier $\eta$
Van der Pol Oscillator

Minimize $J(x, u) = \int_0^{t_f} (u^2 + x_1^2 + x_2^2) \, dt \quad (t_f = 4)$,

subject to

$\dot{x}_1 = x_2$, $\dot{x}_2 = -x_1 + x_2(1 - x_1^2) + u$,

$x_1(0) = x_2(0) = 1$, $x_1(t_f) = x_2(t_f) = 0$,

state constraint $x_2(t) \geq a \quad \forall \ 0 \leq t \leq t_f$.

Boundary arc $x_2(t) \equiv a = -0.4$ for $t_1 = 0.887 \leq t \leq t_2 = 2.62$.
Van der Pol : Computation of Multiplier $\eta$

Order of the state constraint is $q = 1$, since $s(x) = a - x$ and

$$s^{(1)}(x, u) = -\dot{x}_2 = x_1 + x_2(x_1^2 - 1) - u, \quad (s^{(1)})_u = -1 \neq 0.$$  

Boundary control $u = u_b(x)$ with $s^{(1)}(x, u_b(x)) \equiv 0$ is given by

$$u_b(x) = x_1 + x_2(x_1^2 - 1) = x_1 + a(x_1^2 - 1).$$

Augmented Hamiltonian and adjoint equation:

$$\mathcal{H}(x, \lambda, \eta, u) = u^2 + x_1^2 + x_2^2 + \lambda_1 x_2 + \lambda_2 (-x_1 + x_2(1 - x_1^2) + u) + \eta(a - x),$$

$$\dot{\lambda}_1 = -\mathcal{H}_{x_1} = -2x_1 + \lambda_2(1 + 2x_1x_2),$$

$$\dot{\lambda}_2 = -\mathcal{H}_{x_2} = -2x_2 - \lambda_1 - \lambda_2(1 - x_1^2) + \eta.$$  

The minimum condition $0 = H_u = 2u + \lambda_2$ gives $u = -\lambda_2/2$.  
Hence, $x_1 - a + ax_1^2 = -\lambda_2/2$ holds on the boundary.

Differentiation yields

$$\eta = \eta(x, \lambda) = -4a^2x_1 + \lambda_1 + \lambda_2(1 - x_1^2).$$
Van der Pol Oscillator: optimal control

Boundary arc $x_2(t) \equiv a = -0.4$ for $t_1 = 0.887 \leq t \leq t_2 = 2.62$. 
Van der Pol Oscillator: multiplier $\eta$

Boundary arc $x_2(t) \equiv a = -0.4$ for $t_1 = 0.887 \leq t \leq t_2 = 2.62$. 
**CASE I : Hamiltonian is regular**

CASE I: control $u$ appears "nonlinearly" and $U = \mathbb{R}$

Assume that

- the Hamiltonian is regular, i.e., $H(x, \lambda, u)$ admits a unique minimum with respect to $u$,
- the strict Legendre condition $H_{uu}(t) > 0$ holds.

Let $q \geq 1$ be the order of the state constraint $s(x) \leq 0$ and consider a boundary arc with $s(x(t)) = 0 \ \forall \ t_1 \leq t \leq t_2$.

**Junction conditions**

- $q = 1$: The control $u(t)$ and the adjoint variable $\lambda(t)$ are continuous at $t_k$, $k = 1, 2$.
- $q = 2$: The control $u(t)$ is continuous but the adjoint variable may have jumps according to $\lambda(t_k^+) = \lambda(t_k^-) - \nu_k s_x(t_k)$.
- $q \geq 3$ and $q$ odd: If the control $u(t)$ is piecewise analytic then there are no boundary arcs but only contact points.
Case II: control $u$ appears linearly

Dynamics and Boundary Conditions

$$
\dot{x}(t) = f_1(x(t)) + f_2(x(t)) \cdot u(t), \text{ a.e. } t \in [0, t_f],
$$

$$
x(0) = x_0 \in \mathbb{R}^n, \quad \psi(x(t_f)) = 0 \in \mathbb{R}^k,
$$

Control and State Constraints

$$
\alpha \leq u(t) \leq \beta \quad s(x(t)) \leq 0 \quad \forall \ t \in [0, t_f].
$$

Minimize

$$
J(u, x) = g(x(t_f)) + \int_0^{t_f} (f_{01}(x(t)) + f_{02}(x(t)) \cdot u(t)) \, dt
$$
Case II: Hamiltonian and switching function

Normal Hamiltonian

\[ H(x, \lambda, u) = f_{01}(x) + \lambda f_1(x) + \left[ f_{02}(x) + \lambda f_2(x) \right] \cdot u. \]

Augmented Hamiltonian

\[ \mathcal{H}(x, \lambda, \mu, u) = H(x, \lambda, u) + \mu s(x). \]

Switching function

\[ \sigma(x, \lambda) = H_u(x, \lambda, u) = f_{02}(x) + \lambda f_2(x), \quad \sigma(t) = \sigma(x(t), \lambda(t)). \]

On a boundary arc we have

\[ s(x(t)) = 0, \quad t_1 \leq t \leq t_2, \]

\[ \alpha < u(t) < \beta, \quad t_1 < t < t_2. \]

The minimum condition for the control implies

\[ 0 = H_u(x(t), \lambda(t), u(t)) = \sigma(t), \quad t_1 + \leq t \leq t_2 -. \]

Formally, the boundary control behaves like a singular control.
Case II: Boundary Control and Junction Theorem

Let $q$ be the order of the state constraint:

$$s^{(q)}(x, u) = \frac{d^q}{dt^q} s(x) = s_1(x) + s_2(x) \cdot u.$$  

The boundary control $u = u_b(x)$ is determined from $s^{(q)}(x, u) = 0$ as

$$u = u_b(x) = -\frac{s_1(x)}{s_2(x)}.$$  

Junction Theorem for $q = 1$

Let $q = 1$ and let a bang-bang arc be joined with a boundary arc at $t_1 \in (0, T)$.

Claim: If the control is discontinuous at $t_1$, then the adjoint variable is continuous at $t_1$. 
Model of the immune response

Dynamic model of the immune response:


Optimal control:

Innate Immune Response: state and control variables

**State variables:**

\[ x_1(t) : \text{concentration of pathogen} \]
\[ (=\text{concentration of associated antigen}) \]
\[ x_2(t) : \text{concentration of plasma cells,} \]
\[ \text{which are carriers and producers of antibodies} \]
\[ x_3(t) : \text{concentration of antibodies, which kill the pathogen} \]
\[ (=\text{concentration of immunoglobulins}) \]
\[ x_4(t) : \text{relative characteristic of a damaged organ} \]
\[ (0 = \text{healthy}, 1 = \text{dead}) \]

**Control variables:**

\[ u_1(t) : \text{pathogen killer} \]
\[ u_2(t) : \text{plasma cell enhancer} \]
\[ u_3(t) : \text{antibody enhancer} \]
\[ u_4(t) : \text{organ healing factor} \]
Generic dynamical model of the immune response

\[ \begin{align*}
\dot{x}_1(t) &= (1 - x_3(t))x_1(t) - u_1(t), \\
\dot{x}_2(t) &= 3A(x_4(t))x_1(t - d)x_3(t - d) - (x_2(t) - 2) + u_2(t), \\
\dot{x}_3(t) &= x_2(t) - (1.5 + 0.5x_1(t))x_3(t) + u_3(t), \\
\dot{x}_4(t) &= x_1(t) - x_4(t) - u_4(t).
\end{align*} \]

Immune deficiency function triggered by target organ damage

\[ A(x_4) = \begin{cases} 
\cos(\pi x_4), & 0 \leq x_4 \leq 0.5 \\
0, & 0.5 \leq x_4
\end{cases}. \]

For \( 0.5 \leq x_4(t) \) the production of plasma cells stops.

State delay \( d \geq 0 \) in variables \( x_1 \) and \( x_3 \)

Initial conditions \((d = 0)\) : \( x_2(0) = 2, \ x_3(0) = 4/3, \ x_4(0) = 0 \)

Case 1 : \( x_1(0) = 1.5 \), decay, requires no therapy (control)
Case 2 : \( x_1(0) = 2.0 \), slower decay, requires no therapy
Case 3 : \( x_1(0) = 3.0 \), diverges without control (lethal case)
Optimal control model: cost functional

State \( x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \),  
Control \( u = (u_1, u_2, u_3, u_4) \in \mathbb{R}^4 \)

\( L^2 \)-functional quadratic in control: Stengel et al.

Minimize \( J_2(x, u) = x_1(t_f)^2 + x_4(t_f)^2 \)
\[ + \int_0^{t_f} (x_1^2 + x_4^2 + u_1^2 + u_2^2 + u_3^2 + u_4^2) \, dt \]

\( L^1 \)-functional linear in control

Minimize \( J_1(x, u) = x_1(t_f)^2 + x_4(t_f)^2 \)
\[ + \int_0^{t_f} (x_1^2 + x_4^2 + u_1 + u_2 + u_3 + u_4) \, dt \]

Control constraints: \( 0 \leq u_i(t) \leq u_{\text{max}}, \; i = 1, \ldots, 4 \)

Final time: \( t_f = 10 \)
Theory of Optimal Control Problems with Pure State Constraints

Academic Example: order $q = 1$ of the state constraint

Van der Pol Oscillator: order $q = 1$ of the state constraint

Example: Immune Response

Example: Optimal Control of a Model of Climate Change

$L^2$–functional, $d = 0$: optimal state and control variables

State variables $x_1, x_2, x_3, x_4$ and optimal controls $u_1, u_2, u_3, u_4$: second-order sufficient conditions via matrix Riccati equation

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Theory of Optimal Control Problems with Pure State Constraints

Academic Example: order $q = 1$ of the state constraint

Van der Pol Oscillator: order $q = 1$

Example: Immune Response

Example: Optimal Control of a Model of Climate Change

$L^2$–functional, $d = 0$: state constraint $x_4(t) \leq 0.2$

State and control variables for state constraint $x_4(t) \leq 0.2$.

Boundary arc $x_4(t) \equiv 0.2$ for $t_1 = 0.398 \leq t \leq t_2 = 1.35$
Compute multiplier $\eta$ as function of $(x, \lambda)$:

$$
\eta(x, \lambda) = \lambda_2 3\pi \sin(\pi x_4) x_1 x_3 - \lambda_1 + 2\lambda_4 - 2x_3 x_1 + 2x_4
$$

Scaled multiplier $0.1 \eta(t)$ and boundary arc $x_4(t) = 0.2$
Theory of Optimal Control Problems with Pure State Constraints

Academic Example: order \( q = 1 \) of the state constraint

Van der Pol Oscillator: order \( q = 1 \) of the state constraint

Example: Immune Response

Example: Optimal Control of a Model of Climate Change

\( L^2 \)-functional, delay \( d > 0 \), constraint \( x_4(t) \leq \alpha \)

Dynamics with state delay \( d > 0 \)

\[
\begin{align*}
\dot{x}_1(t) &= (1 - x_3(t))x_1(t) - u_1(t), \\
\dot{x}_2(t) &= 3 \cos(\pi x_4) x_1(t - d)x_3(t - d) - (x_2(t) - 2) + u_2(t), \\
\dot{x}_3(t) &= x_2(t) - (1.5 + 0.5x_1(t))x_3(t) + u_3(t), \\
\dot{x}_4(t) &= x_1(t) - x_4(t) - u_4(t) \\
x_4(t) &\leq \alpha \leq 0.5
\end{align*}
\]

Initial conditions

\[
\begin{align*}
x_1(t) &= 0, \quad -d \leq t < 0, \quad x_1(0) = 3, \\
x_3(t) &= 4/3, \quad -d \leq t \leq 0, \\
x_2(0) &= 2, \\
x_4(0) &= 0.
\end{align*}
\]
$L^2$–functional: delay $d = 1$ and $x_4(t) \leq 0.2$

State variables for $d = 0$ and $d = 1$
Theory of Optimal Control Problems with Pure State Constraints

Example: order $q = 1$ of the state constraint

\[ L^2 \text{–functional: delay } d = 1 \text{ and } x_4(t) \leq 0.2 \]

Optimal controls for $d = 0$ and $d = 1$
$L^2$–functional, $d = 1$: multiplier $\eta(t)$ for $x_4(t) \leq 0.2$

Compute multiplier $\eta$ as function of $(x, \lambda)$:

$$\eta(x, y, \lambda) = \lambda_2 3\pi \sin(\pi x_4) y_1 y_3 - \lambda_1 + 2\lambda_4 - 2x_3 x_1 + 2x_4$$

Scaled multiplier $0.1 \eta(t)$ and boundary arc $x_4(t) = 0.2$; $\eta(t)$ is discontinuous at $t = d = 1$
Theory of Optimal Control Problems with Pure State Constraints

**Academic Example:** order $q = 1$ of the state constraint

**Example:** Van der Pol Oscillator: order $q = 1$ of the state constraint

**Example:** Immune Response

**Example:** Optimal Control of a Model of Climate Change

**$L^1$-functional:** no delays

**Minimize**

$$J_1(x, u) = x_1(t_f)^2 + x_4(t_f)^2$$

$$+ \int_0^{t_f} (x_1^2 + x_4^2 + u_1 + u_2 + u_3 + u_4) \, dt$$

**Dynamics with delay $d$ and control constraints**

\[
\begin{align*}
\dot{x}_1(t) &= (1 - x_3(t))x_1(t) - u_1(t), \\
\dot{x}_2(t) &= 3A(x_4(t))x_1(t)x_3(t) - (x_2(t) - 2) + u_2(t), \\
\dot{x}_3(t) &= x_2(t) - (1.5 + 0.5x_1(t))x_3(t) + u_3(t), \\
\dot{x}_4(t) &= x_1(t) - x_4(t) - u_4(t),
\end{align*}
\]

$$0 \leq u_i(t) \leq u\text{max}, \quad 0 \leq t \leq t_f \quad (i = 1, \ldots, 4)$$
$L^1$–functional: $u_{\text{max}} = 2$
$L^1$–functional: non-delayed, time–optimal control for
$x_1(t_f) = x_4(t_f) = 0$, $x_3(t_f) = 4/3$

$u_{\text{max}} = 1$: minimal time $t_f = 2.2151$, singular arc for $u_4(t)$
Dynamical Model of Climate Change


State Variables:

- $K(t)$ : Capital (per capita)
- $M(t)$ : CO$_2$ concentration in the atmosphere
- $T(t)$ : Temperature (Kelvin)

Control Variables

- $C(t)$ : Consumption
- $A(t)$ : Abatement per capita
Dynamical Model of Climate Change

Production: \[ Y = K^{0.18} \cdot D(T - T_o), \quad T_o = 288 \, (K) \]

Damage: \[ D(T - T_o) = (0.025 \, (T - T_o)^2 + 1)^{-0.025} \]

Dynamics of per-capita capital \( K \)

\[ \dot{K} = Y - C - A - (\delta + n)K, \quad K(0) = K_0. \]
\[ (\delta = 0.075, \quad n = 0.03) \]

Emission: \[ E = 3.5 \cdot 10^{-4} \cdot K / A \]

Dynamics of \( \text{CO}_2 \) concentration \( M \)

\[ \dot{M} = 0.49 \, E - 0.1 \, M, \quad M(0) = M_0. \]
Dynamical Model of Climate Change (continued)

**Albedo** (non-reflected energy) at temperature $T$ (Kelvin):

$$1 - \alpha_1(T) = k_1 \frac{2}{\pi} \arctan \left( \frac{\pi(T-293)}{2} \right) + k_2,$$

$$k_1 = 5.6 \cdot 10^{-3}, \quad k_2 = 0.1795.$$

**Radiative forcing**: $5.35 \ln(M)$.

**Outgoing radiative flux** (Stefan-Boltzmann-law): $\epsilon \sigma T^4$.

**Parameters**: $\epsilon = 0.95$, $\sigma T = 5.67 \cdot 10^{-8}$,

$$cth = 0.149707, \quad Q = 1367.$$

**Dynamics of temperature $T$ with delay $d \geq 0$**

$$\dot{T}(t) = cth \cdot \left[ \left( 1 - \alpha_1(T(t)) \right) \cdot \frac{Q}{4} - \frac{19}{116} \cdot \epsilon \cdot \sigma T \cdot T(t)^4 + 5.35 \cdot \ln(M(t - d)) \right],$$

$$T(0) = T_0.$$
Optimal Control Model of Climate Change

Control constraints for $0 \leq t \leq t_f = 200$:

$$0 < C(t) \leq C_{\text{max}} = 1, \quad 7 \cdot 10^{-4} \leq A(t) \leq 3 \cdot 10^{-3}$$

State constraints of order 2 and 3:

$$M(t) \leq M_{\text{max}}, \quad 0 \leq t \leq t_f = 200,$$

$$T(t) \leq T_{\text{max}}, \quad t_e = 20 \leq t \leq t_f = 200.$$

Maximize consumption

$$J(K, M, T, C, A) = \int_0^{t_f} e^{-(n-\rho)t} \ln(C(t)) \, dt$$
Stationary points of the canonical system

Constant Abatement $A = 1.21 \cdot 10^{-3}$ : 3 stationary points

- $T_s := 291.607, \quad M_s = 2.05196, \quad K_s = 1.44720,$
- $T_s := 294.258, \quad M_s = 1.90954, \quad K_s = 1.34726,$
- $T_s := 294.969, \quad M_s = 2.07792, \quad K_s = 1.46606.$

Control variable abatement $A(t)$ : Social Optimum

- $T_s := 288.286, \quad M_s = 1.28500, \quad K_s = 1.79647.$
\[ T(0) = 291, \ M(0) = 2.0, \ K(0) = 1.4 \]
Theory of Optimal Control Problems with Pure State Constraints

Academic Example: order $q = 1$ of the state constraint

Van der Pol Oscillator: order $q = 1$ of the state constraint

Example: Immune Response

Example: Optimal Control of a Model of Climate Change

$T(0) = 291$, $M(0) = 2.0$, $K(0) = 1.4$

$T(t_f) = 291$, $M(t_f) = 1.8$, $K(t_f) = 1.4$
\( T(0) = 291, \ M(0) = 2.0, \ K(0) = 1.4 \)
\( T(t_f) = 291, \ M(t_f) = 1.8, \ K(t) \geq 1.2, \ 0.85 \leq C(t) \leq 1 \)
\[ d = 10 : T(0) = 291, \ M(0) = 1.8, \ K(0) = 1.4 \]
\[ T(t_f) = 291, \ M(t_f) = 1.8, \ K(t) = 1.4 \]
$T(0) = 294.5, \ M(0) = 2.2, \ K(0) = 1.4$

$T(t_f) = 291, \ M(t_f) = 1.8, \ K(t) = 1.4, \ 0.85 \leq C(t) \leq 1$
\[ d = 10 : \quad T(0) = 294.5, \quad M(0) = 2.2, \quad K(0) = 1.4 \]
\[ T(t_f) = 291, \quad K(t) = 1.4, \quad 0.85 \leq C(t) \leq 1 \]

State constraint of order two: \( M(t) \leq 1.9 \) for \( 30 \leq t \leq t_f = 200 \)
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