The theoretical and numerical analysis of high-order absorbing boundary conditions for Helmholtz and Maxwell equations

Supervisors
Théophile Chaumont-Frelet (theophile.chaumont@inria.fr)
Marion Darbas (marion.darbas@u-picardie.fr)
Serge Nicaise (serge.nicaise@uphf.fr)

Summary
The scattering of a time-harmonic wave by an obstacle is an important physical process in various applications, such as radar, telecommunications, or medical imaging. This phenomenon is accurately modeled by Maxwell’s equations for electromagnetic waves, and by the Helmholtz equation for acoustic waves. These mathematical models are precise, but the exact analytical solution is only available for very specific geometries. In the general case, the use of numerical methods to approximate the solution is mandatory. The computational and memory storage costs associated with these schemes are limiting factors, especially if high-frequency problems are considered. These considerations motivate the design of numerical schemes that are both efficient and accurate.

The scattered wave \( u_{\text{ sca}}^{\infty} \) (that we want to compute) is solution to a so-called “exterior” problem (see Figure 1). Its propagation into the unbounded domain \( \Omega_{\infty} \) (complement of the obstacle \( D \)) is modeled by the Helmholtz equation or Maxwell’s equations. A boundary condition related to the incident wave \( u_{\text{ inc}} \) is prescribed on the border \( \Gamma := \partial D \) of the obstacle. In addition, to ensure the well-posedness of the problem, a radiation condition (also known as outgoing condition) is imposed at infinity (the Sommerfeld condition in acoustics and the Silver-Müller condition in electromagnetics).

Most commonly used numerical methods for this problem can be sorted in two categories: volumic and surfacic methods. In this PhD project, we will focus on volumic discretization techniques, and in particular, on finite element methods. They are very versatile and can handle fairly complex geometries and material properties. On the other hand, they are not natively tailored to take into account the unbounded nature of the computational domain.

The key idea then consists in truncating the exterior domain \( \Omega_{\infty} \) by enclosing the obstacle in an artificial boundary \( \Sigma \). This leads to a boundary value problem in the truncated domain \( \Omega_{tr} \), bounded by the boundary of the obstacle \( \Gamma \), and the artificial boundary \( \Sigma \). The difficult point is then to design an “absorbing boundary condition” (or ABC) on \( \Sigma \) that models the outgoing character of the scattered wave, and minimizes as efficiently as possible spurious non-physical reflections. An “ideal” ABC is realized by the exact exterior Dirichlet-to-Neumann (DtN) operator, but its practical use is limited, since the discretization of this non-local operator leads to dense linear systems, that are costly to factorize.

Different families of local ABC have been proposed in the literature [7, 9]. In the context of this PhD, we will focus in particular on accurate local approximations of the DtN operator introduced in [2, 6]. These approximations have been successfully applied as robust analytic preconditioners for boundary integral equations [1] or as transmission conditions in the context...
Figure 1: Scattering problem: original problem in a unbounded domain (left) and truncated problem (right).

of domain decomposition techniques [3]. However, they have not been theoretically analyzed as a substitute to the exact radiation condition, which is what we plan to study in the context of this thesis. The main goal of this PhD is thus the design of novel high-order ABC, as well as a rigorous analysis of the error committed by finite element discretizations coupled to these ABC.

The PhD student will first focus on the scalar case of an acoustic wave modeled by the Helmholtz equation. He/She will then consider the vectorial case of Maxwell’s equations. In both settings, the student will follow three steps. First, he/she will analyze the modeling error due to the introduction of the ABC, compared to the original radiation condition. To do so, he/she will employ stability analysis tools known as “Morawetz multipliers” [4, 8]. Second, the approximation error due to the finite element discretization of the truncated problem will be finely taken into account. The PhD student will especially focus on high-order finite element methods, and will employ analysis techniques recently introduced for first-order ABC [5, 10], that he/she will have to adapt to the high-order ABC considered here. Finally, he/she will implement computational codes (in 2D, and if possible, in 3D) and carry out numerical experiments to assess the sharpness of his theoretical convergence results. For this, the student will use different simulation software packages such as Matlab or FreeFEM++.
Contact
Should you need further information, please contact one of the supervisors.

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Localization
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References


