Valued constraint satisfaction against an adversary

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Abstract

We propose to initiate a study of valued constraint satisfaction problems when some variables are controlled by an adversary. In concrete terms this mean that some variables are universally quantified, and that one needs to build as efficiently as possible a strategy to assign values to variables such that the cost implied by the constraints is minimal.

The main task we propose is to investigate classes of instances that would remain tractable because one can reduce an instance to polynomially many (hopefully tractable) instances of the unquantified valued constraint satisfaction problem.

Keywords: Computational Complexity, Combinatorics, Constraint Satisfaction Problems, Linear Programming, Universal Algebra

Constraint Satisfaction and the dichotomy conjecture : a melting pot of Algebra, Combinatorics, Complexity and Logic

Constraint Satisfaction provides a generic framework for modeling in a natural way a large number of computational problems, whether academic like propositional satisfiability (SAT) or graph coloring, or more applied like the problem of conjunctive query containment from databases.

The class of Constraint Satisfaction Problems, or CSP for short, remains NP-complete, even when restricted to variables ranging over a two element domain (because it is an avatar of SAT in this case) or to binary constraints (because it is an avatar of graph coloring in this case). In order to conduct a meaningful complexity-theoretical study, one usually introduces some additional restrictions which may lead to tractability, whether structural (e.g. the variables are constrained in a tree-like fashion), or on the set of allowed constraints, the constraint language (e.g. only linear constraints). The latter approach was championed by Feder and Vardi together with their highly influential *dichotomy conjecture* positing that the resulting problems are all either in P or NP-complete depending on the constraint language [9].

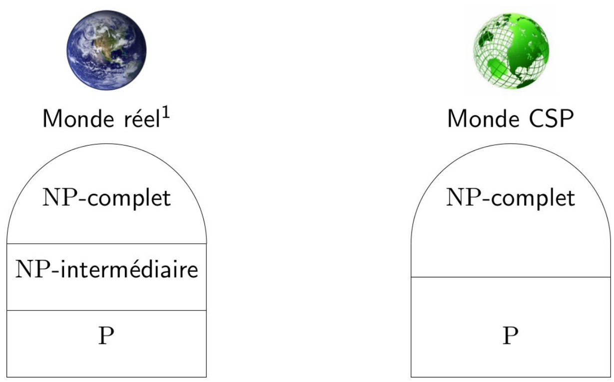

Figure 1: Dichotomy or no dichotomy? Left the world of NP according to Ladner's theorem (provisio $P \neq NP$), right the world of CSP according to Feder and Vardi's conjecture.

The quest for resolving this conjecture has lead to the involvement of a large number of fields and has transformed the CSP community into a veritable melting pot of Algebra, Combinatorics, Complexity and Logic [7, 10]

It has been long known that this conjecture holds in the Boolean case [15] and for (a single binary constraint that is an) undirected graph [11]. More recently, the careful interplay of the fields mentioned above has allowed for great progress and the conjecture has been established for a three element domain [3], and for arbitrary domain size in the conservative [1] case [4, 1]. Strong links have also been uncovered between universal algebra and the class of constraint satisfaction problems that can be solved by generalised consistency algorithms (this case corresponds in logical settings to expressivity in Datalog) [2]. For further details on this the reader is invited to check the short survey in French [12] which will in turn point him or her to several more complete surveys.

Larger applicability of the methodology

While there are has been some tremendous progress in the last decade, mainly via contributions from researcher from Universal Algebra, the dichotomy conjecture remains open. However, this algebraic approach, combined with Combinatorics or logic, has allowed to settle several other questions of interest, including several more refined complexity questions in the Boolean case, regarding counting and enumerating the solution, and other interesting problem from propositional logic such as abduction and circumscription [8]. Mostly, this is because the underlying object from universal algebra used to guide the complexity classification in the Boolean case, Post's lattice of clones, is very well understood and only countable. This is to be contrasted with larger domains where the clone lattice is uncountable and largely uncharted, which means that one needs more heavy machinery from Universal Algebra.

Universal quantification

However, when one extends the CSP framework to increase expressiveness, for example by allowing universal quantification, one obtains comparatively more benign underlying objects, that are more often than not amenable to a full complexity classification of the computational problem at hand. For example, a tetrachotomy between Pspace-complete, Co-NP-complete, NP-complete and P was obtained when one allows both universal quantification and disjunction [13, 14].

In concrete terms, these quantified problems allow to model the fact that some variables are controlled by a malicious opponent, or to model uncertainty, that is variables that are environment dependent.

QCSP and collapsibility

Extending CSP only with a universal quantifier (no disjunction this time), leads to the problem known as QCSP. For this problem, there are only partial classification results and a trichotomy is conjectured between P, NP-complete and Pspace-complete [6].

An important notion for QCSP is that of *collapsibility*: a language is collapsible whenever satisfiability of a quantified instance is equivalent to the satisfiability of all quantified instances obtained by fixing all universal but a fixed number of them to the same constant. This means that a collapsible restriction of the QCSP reduces to polynomially many instances of the CSP. Thus, for such a language, the QCSP is in NP. If moreover the language happens to be tractable for CSP then the QCSP problem is in fact tractable. For example, the Horn-Sat language enjoys this property and Quantified Horn-Sat is in NP (by collapsibility) and can be eventually seen to be in P (Horn-Sat is tractable). For further examples, consider reading the excellent survey [5].

[1] This case is very natural: it means that each variable is equipped with it own domain

Valued constraint satisfaction

Another very natural setting is to shift to the realm of optimization problems and if there are no solution, attempt to maximize the number of satisfied constraints (this is known as MaxCSP). Indeed, instances are more often than not overconstrained, and users of solvers are willing to relax some constraints, especially when one takes into account that modelling is a bit of a dark art.

MaxCSP is a restriction of a more general class of discrete optimization problems called *valued constraint satisfaction problems* (VCSP). By combining intuition and notions from the algebraic approach with linear programming, this framework has recently been completely classified from a complexity point of view [16, 17]. For example, given any constraint language, it is possible to determine (in polynomial time) whether the corresponding Max CSP is polynomial-time solvable or NP-hard.

Project proposal

The proposed project is an initial study of the quantified extension of the valued CSP. One possible avenue would be to investigate a suitable notion of collapsibility for this problem.

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