Proposition for M2 internship
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Primal linearization methods for fair assignment/matching problems

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1 Description of the proposition

The last decade has observed a rapidly growing academic and popular interest in algorithmic fairness. Fairness is a major concern in decision making processes, in data driven methods (machine learning). In this project, we consider the design of fairness solutions to combinatorial optimization problems. Specifically, we study the classic assignment/matching problems in which contrary to the classical version, our objective is not only to maximize the sum of the utilities but also to achieve some equity over all the agents.

Assignment/Matching problems. Given a set of n agents, a set of m tasks and a degree of utility (or satisfaction) to each task for each agent, the classical assignment problem (bipartite matching) consists in assigning every agent to a task while maximizing the total utility of the agents. The general (non-bipartite) matching problem can be stated briefly as follows: given a set of $2n$ agents with pairwise degrees of utility, one would like to form n teams of 2 agents while the total satisfaction of the agents is maximized. It is well known that the assignment and the matching problems can be formulated as the following linear programs (A) and (M), respectively.

$$
\begin{aligned}
&\max \sum_{i,j} w_{ij} z_{ij} && (A)\\
&\sum_{j=1}^{n} z_{ij} = 1 && \forall i\\
&\sum_{j=1}^{n} z_{ij} = 1 && \forall j\\
&z_{ij} \geq 0 && \forall i,j
\end{aligned}
$$

$$
\begin{aligned}
&\max \sum_{i,j} w_{ij} z_{ij} && (M)\\
&\sum_{i,j:i\neq j} z_{ij} \leq 1 && \forall j\\
&\sum_{i,j\in U:i\neq j} z_{ij} \leq \frac{|U|-1}{2} && \forall U \subset \{1,\dots,2n\}, |U| \text{ odd}\\
&z_{ij} \geq 0 && \forall i,j
\end{aligned}
$$

where $z_{ij} = 1$ means that the agent i is assigned to task j.

An illustrative example. Consider an example of the assignment problem depicted in Figure 1 where 3 agents needs to be assigned to 3 tasks. The optimal solution for the classical assignment is $S1 = (5, 15, 11)$, i.e. Agent 1 – Task 3 with utility 5, Agent 2 – Task 1 with utility 15 and Agent 3 – Task 2 with utility 11, and has a total utility of 31. However, although the solution S2=(10,11,9), i.e. Agent 1 – Task 1 with utility 10, Agent 2 – Task 2 with utility 11 and Agent 3 – Task 3 with utility 9, achieves a total utility slightly lower (30), it is much more equitable than the previous solution. An important topic in the fair optimization field is to quantify mathematically the notion of "equity" or "fairness" of a solution allowing to prefer solution S2

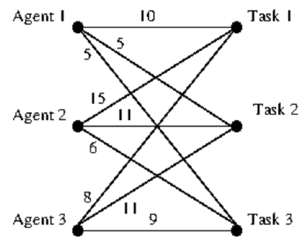

Figure 1: An example with 3 agents/tasks.

to solution S1 in our example. A simple criterion is the egalitarian criterion that identifies the value of a solution as its minimum utility component. In this regard, this criterion ranks S2 higher than S1. However, the egalitarian criterion does not discriminate between solutions having the same minimum utility but being very different in the other components. For example, two solutions with respectively (9, 9, 10) and (9, 15, 20) as utility vectors will be considered as equal with respect to the egalitarian criterion. To overcome this restriction, another possibility is the leximin criterion. Let x be an utility vector associated with a solution, let $x^{\uparrow}$ denote the vector x whose components have been sorted by increasing order. A vector x is preferred to y according to the leximin if and only if $x^{\uparrow} \geq_{\text{lex}} y^{\uparrow}$. We can see that this criterion ranks the solution (9, 9, 10) lower than (9, 15, 20). Both the egalitarian and leximin criteria do not take into account the total utility, which may be debatable. For instance, (1, 1, 1) is preferred to (0, 10, 10) for both criteria, however in some situations we may prefer the second vector as its total utility is much higher.

The two criteria are both special cases of the generalized Gini social-evaluation functions which are defined as follows: $W(x) = \sum_{i=1}^{n} w_i x^{\uparrow}$ where $w_i > w_{i+1}$ for $i = 1, \ldots, n-1$. Note that when $w_1 = 1$ and the other weights are sufficiently close to 0 then $W(x)$ becomes the egalitarian criterion. Whenever the differences $w_i - w_{i+1}$ tends to 0 (i.e. weights tend to be nearly equal) $W(x)$ tends to the utilitarian criterion (classical criterion). On the contrary, when all these differences tend to be arbitrary large, then $W(x)$ tends to the leximin criterion. Depending on the choices of the coefficients w_i's, $W(x)$ can offer many other possibilities and defines a family of functions which is known in multicriteria analysis under the name of ordered weighted averages (OWA) (we restrict here to the important special case of OWA with decreasing weights).

Objective/Approach The main aim of the internship is to provide exact solution methods for the assignment/matching problems when the objective is to maximize $W(x)$. Depending on the choices of w_i's, the problem could be more or less difficult. Various polynomial special cases are known, for instance, the Max-Min case. But the general case has recently been shown to be NP-hard [4]. A common choice for w_i's is $\frac{1}{n^2} \sum_{i=1}^{n} (2(n-i)+1)$ which is the original Gini social-evaluation function.

In fact, $W(x)$ can be viewed as a pseudo-boolean quadratic function over 0/1 variables of two assignment problems: one for the permutation representing the order of OWA weight vector and the other the assignement itself . Existing linearization methods [8], [2] are based on a trick of dualization of the permutation variables to obtain a Mixed Integer Problem (MIP) model. However it can be observed that (Nguyen and Weng [7]), there exists hard instances with small value of n for which generic solvers such as CPLEX or GUROBI can take hours for solving (MIP) with very slow improvements of the upper-bound in branch-and-bound search trees. To remedy this inconvenience, in Nguyen and Weng [7], the authors propose an efficient heuristic that makes use of the decomposable structure of (MIP). In spite of the fact that the heuristic is very fast and gives rather good solutions comparing with generic MIP solvers, it is still an heuristic without guarantee of performance.

The internship will experiment new methods of linearization in order to improve the exact solution of the fair assignment problem. In particular, it is expected to reformulate the problem of maximization of $W(x)$ to a contrained 0/1 quadratic minimization problem. Then several linearization methods can be experimented directly without dualizing permutation variables such as the classical method [3], the QCR method [1], the bilinear methods [9] and in particular recent methods [5], [6] that take into account the assignment constraints in the linearization. The objective of the internship is to adapt the above linearization methods for the fair assignment problems. It is also required to experiment the proposed method for various instances and to compare their performance to each other and to the performance of the "dual" linearization methods in [8], [2]. Extensions of the work are expected to other fair combinatorial optimization problems such as fair matching, fair TSP, fair MST,...

2 Timeline and remuneration

The internship is to be realized within 6 months from mid-February to mid-August 2018 at LIP6.

- First month: theoretical part of the linearization methods: description and proof of validity.
- Second and third months : implementation and experiments of the methods, comparisions with "dual" linearization methods, improvements.
- Fourth month: extension to other problems such as fair matching, fair TSP, fair MST,...
- Fifth and sixth months: writing thesis, possibly conference paper.

The remuneration is 554.40 euros monthly with (+ possibly at most 35 euros of transport compensation).

3 Perspectives

The work done during the internship is expected to be pursued in a PhD thesis in France or in Shanghai, China.

References

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