# Master 2 Research internship - 2021

# TROPICAL GEOMETRY APPLIED TO THE ANALYSIS OF INTERIOR POINTS METHODS

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## Context

A long standing problem in optimization is the existence of an algorithm solving linear programming by making a number of arithmetic operations bounded polynomially in terms of the number of constraints and variables. When restricted to linear feasibility, this is known as the 9th problem of Smale for the 21st century [Sma98]. A polynomial-time algorithm that satisfies the previous requirement is said to be strongly polynomial. There have been a huge interest to find a strongly polynomial algorithm in linear programming, since the invention of the simplex method. Connections have been recently established [ABGJ14] between this problem and the existence of a polynomial-time algorithm for zero-sum repeated games with mean payoff, which is another well studied open problem [GKK88, CJK$^+$17].

The interior points method replaces the linear programming problem

$$\min_{x\in\mathbb{R}^n} c\cdot x; \quad A_i x \leq b_i, 1\leq i\leq m \enspace ,$$

by the so called *barrier problem*

$$\min_{x\in\mathbb{R}^n} c\cdot x + \sum_{1\leq i\leq n} \mu\pi(b_i - A_i x) \enspace ,$$

where $\pi$ is a function known as a barrier (which, in particular is smooth, strictly convex, and infinite at the boundary of the orthant) and $\mu$ is a homotopy parameter. Such methods follow a trajectory called the central path, whose geometric properties govern the number of iterations. In [ABGJ18], we gave a counter example, showing that the most commonly used barrier, the logarithmic barrier $\pi(t) = -\log t$, leads to a central path with an exponential number of turns, and so to a non strongly polynomial behavior. The counter example originates from game theory. The analysis of its central path is done by working with linear programs over non-archimedean fields, i.e. classical linear programs with large coefficients, with combinatorial methods from tropical geometry.

## Projected work

There has been lately an intensive research on finding alternative barrier functions that lead to improved complexity estimates [LS14, BE15, LY18, LS19]. A natural question is whether our approach can be extended to these barriers. A first result in this direction is given in [AAGH20], where it is shown that the tropicalization of the central path associated with the entropic barrier is the same as the one of the logarithmic barrier. A first goal of the internship is to investigate whether there is a "universality result" showing that a larger class of self-concordant barrier functions are subject to the same pathology. As a first step, remarkable concrete barriers, such that the universal barrier of Nesterov and Nemirovski [NN94], or the weighted barrier of Lee and Sidford [LS19], will be analysed using tropical / nonarchimedean geometry methods. A broader objective is to understand how such barriers can be modified to get improved algorithms, and possibly strongly polynomial bounds for new classes of linear programs, including ones arising from the encoding of mean payoff games.

This Master 2 internship may be continued by a PhD thesis, concerning the application of tropical geometry to optimization and game theory.

## Supervision

The internship will be co-supervised by Xavier Allamigeon[1]) (INRIA and École polytechnique) and Stéphane Gaubert[2] (INRIA and École polytechnique). It will be performed in the "Tropical" research

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