Geometry of risk-trading with Arrow-Debreu securities

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This internship will be the opportunity to discover some advanced concept in optimization under uncertainty (coherent risk measure, perfect risked equilibrium, complementarity conditions...) but mainly requires **good mathematical and geometrical sense** with some knowledge of **basic probability** theory and convex and/or linear **optimization**: we prefer a bright student to a knowledgeable one.

Subject

We consider a market (inspired by energy markets) where agents can trade so-called Arrow-Debreu (AD) securities to trade risks. An AD security can be seen has some kind of insurance that costs μ, and returns 1 if an elementary event ω occurs. In recent work it has been shown that AD securities can complete a market against risk, and that a perfect equilibrium in this market is equivalent to a social optimum.

More precisely, we assume that each agents $a \in \mathcal{A}$ has a (random) cost Z_a and can trade AD W_a for a price μ. Further we assume that they are risk averse with a coherent risk measure $\rho_a := \max_{\mathbb{P}\in\mathcal{P}_a} \mathbb{E}_{\mathbb{P}}$ such that she wants to minimize

$$\min_{W_a} \underbrace{\mu^\top W_a}_{\text{AD cost}} + \max_{\mathbb{P}\in\mathcal{P}_a} \mathbb{E}_{\mathbb{P}}[\underbrace{Z_a}_{\text{cost}} - \underbrace{W_a}_{\text{AD pay-off}}].$$

In this case, when $\bigcap_{a\in\mathcal{A}} \mathcal{P}_a \neq \emptyset$ we can show (see [2] that the optimal price μ satisfies

$$\mu \in \underset{\tilde{\mu}\in\bigcap_{a\in\mathcal{A}}\mathcal{P}_a}{\arg\max}\ \mathbb{E}_{\mu}[\sum_{a\in\mathcal{A}} Z_a]$$

and further that

$$Z_a - W_a \in \mathcal{N}_\mu(\mathcal{P}_a)$$
$$\sum_{a\in\mathcal{A}} W_a = 0$$

where $\mathcal{N}_\mu(\mathcal{P}_a)$ stands for the normal cone at μ of the set $(\mathcal{P}_a)$.

We are especially interested in the case where all agents have polyhedral risk measures as discussed in [1], and particularly given as $\rho_a = (1-\lambda)\mathbb{E} + \lambda AVaR_\alpha$ where AVaR is average value at risk (also known as conditional value at risk [3].) This widely used risk measure has a simple representation of its risk sets as

$$\mathcal{P}_{\lambda,\alpha} := (1-\lambda)\Delta + \lambda\{p \in \Delta \mid p_s \leq \alpha/S, \forall s\}.$$

The objective of the internship is to identify the risk sets based on observed trades (*i.e.,* find information on λ_a and α_a based on observed Z, W and μ). More precisely you are going to study the following (assuming first equiprobability over 3 scenarios):

1. Assume that we observe Z, W and μ. For a given α, what is the (generally unique) associated λ? Then deduce bounds on α.
2. Find conditions on Z under which 2 or more trades would perfectly identify the risk coefficients.
3. Increase the number of agents/scenarios. Drop the equiprobability assumptions.
4. Turn to estimation consideration if we have more trades than necessary, maybe leveraging symmetries. We could compare with the estimation of convex combination of more than one $AVaR_\alpha$.

Material Conditions

- Duration: 4-6 months starting as soon as possible
- Location: CERMICS, Ecole Nationale des Ponts et Chaussées, Noisy sur Marnes
- Potential visit to New Zealand
- Stipend: legal gratification (≈600€ per month)

References

[1] H. Gérard, V. Leclère, and A. Philpott. On risk averse competitive equilibrium. *Operations Research Letters*, 46(1):19–26, 2018.

[2] D. Ralph and Y. Smeers. Risk trading and endogenous probabilities in investment equilibria. *SIAM Journal on Optimization*, 25(4):2589–2611, 2015.

[3] R.T. Rockafellar and S. Uryasev. Optimization of conditional value-at-risk. *Journal of risk*, 2:21–42, 2000.