Elastic wave propagation through a random array of dislocations

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A number of unsolved issues in materials physics suggest there is a need for an improved quantitative understanding of the interaction between acoustic (more generally, elastic) waves and dislocations. In this paper we study the coherent propagation of elastic waves through a two dimensional solid filled with randomly placed dislocations, both edge and screw, in a multiple scattering formalism. Wavelengths are supposed to be large compared to a Burgers vector and dislocation density is supposed to be small, in a sense made precise in the body of the paper. Consequently, the basic mechanism for the scattering of an elastic wave by a line defect is quite simple ("fluttering"): An elastic wave will hit each individual dislocation, causing it to oscillate in response. The ensuing oscillatory motion will generate outgoing (from the dislocation position) elastic waves. When many dislocations are present, the resulting wave behavior can be quite involved because of multiple scattering. However, under some circumstances, there may exist a coherent wave propagating with an effective wave velocity, its amplitude being attenuated because of the energy scattered away from the direction of propagation. The present study concerns the determination of the coherent wavenumber of an elastic wave propagating through an elastic medium filled with randomly placed dislocations. The real part of the coherent wavenumber gives the effective wave velocity and its imaginary part gives the attenuation length (or elastic mean free path). The calculation is performed perturbatively, using a wave equation for the particle velocity with a right hand side term, valid both in two and three dimensions, that accounts for the dislocation motion when forced by an external stress. In two dimensions, the motion of a dislocation is that of a massive particle driven by the incident wave; both screw and edge dislocations are considered. The effective velocity of the coherent wave appears at first order in perturbation theory, while the attenuation length appears at second order.

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I. INTRODUCTION

A. Motivation

Dislocation dynamics is a problem at the root of a number of outstanding issues in materials physics. Mechanically excited phonons in interaction with dislocations appear in acoustics experiments, and the vibrating string model of dislocation damping[1–3] has been quite successful in explaining a wealth of data, such as measurements of damping, internal friction and modulus change of solids. Thermally excited phonons in interaction with dislocations appear in thermal conductivity measurements, where the situation at low temperatures seems to be less satisfactory than in the acoustics case. Kneezel and Granato[4] conducted a careful study of phonon damping with the vibrating string model; they considered many effects, including angular effects emphasized by Ninomiya,[5] and found no accord with the data[6] on thermal resistivity at low temperature in alkali halides when dislocations are assumed to vibrate independently. A fit could be obtained, however, with dislocation *dipoles* at long wavelengths (hence vibrating "optically," that is, in opposition), which would require many dislocations to be arranged in dipoles. More recent work by Anderson and collaborators[7] has confirmed the inability of the model to account for the data. Qualitative, but not quantitative agreement with measurements in high purity niobium and tantalum can be obtained with a picture in which the dominant effect is the interaction of phonons with kinks on dislocations.[8] The importance of radiation damping due to kink oscillations[9] was already noted by Hikata and Elbaum.[10] The review of Anderson[11] has highlighted the need for an improved theoretical understanding of the elastic wave-dislocation interaction in order to use the thermal conductivity measurements of deformed bodies as a diagnostic tool for studying defect structures in solids.

The vibrating string model is based on the formulation of Koehler[12] in which the dislocation is modeled as a scalar string driven by a scalar time dependent stress. This model is very simple, a fact that allows for many applications, and it certainly captures the essence of the physics of the elastic wave-dislocation interaction. However, it does not consider the many complexities of this interaction. For example, it does not differentiate between edge and screw dislocations, nor among the various polarizations available to an elastic wave, and a significant body of current literature addresses this issue through numerical computations, both in a continuum, mesoscopic, approximation and at the atomic scale. The vibrating string model also treats dislocations singly, and the effect of many dislocations is simply accounted for by multiplication. However, the presence of many obstacles upon the path of a wave has collective effects in addition to the sum of single body effects: for example, a random array of scatterers will attenuate a wave, even in the absence of an internal viscosity mechanism. Our purpose in the present paper is to offer results toward filling this gap: we consider first antiplane waves in interaction with screw dislocations, and then in plane (vector) waves in interaction with edge dislocations in which their vector nature is considered in full, and we get formulas describing the behavior of elastic waves in a continuum filled with randomly distributed dislocations.

There are further outstanding problems in materials physics, such as the brittle to ductile transition[13] and fatigue[14] where there is wide agreement that dislocations play a significant, possibly determinant, role. From an engineering design point of view the issue of brittle to ductile transition is well controlled in the sense that structures can be, and are, successfully constructed. From a basic physics point of view, however, those phenomena are very far from being understood, in the sense that current theoretical modeling has, to the best of our knowledge, no predictive power: If a new form of steel, say, is fabricated, current theory cannot make quantitative predictions concerning its mechanical properties as a function of temperature or cyclic loading. It is our opinion that one important cause for this lack of basic knowledge lies in the paucity of experimental measurements (as opposed to visualizations) concerning dislocations. This is so because dislocations need to be seen through electron microscopy of samples that must be specially prepared. It stands thus to reason that it would be very desirable to have quantitative measurements carried out with noninvasive probes. Is this possible? We believe acoustic, or ultrasonic, measurements may provide such quantitative data.[15] However, for this approach to be feasible, an improved theoretical understanding of the sound-dislocation interaction is needed, which is an additional reason for undertaking the calculations described in this paper.

B. Elastic waves in random media

The behavior of waves in random media has a long and distinguished history of scholarship and the literature is vast.[16–18] Current interest stems at least from two sources: the possibility that disorder will induce a change in wave behavior from transmission to diffusion to localization,[19–21] and the enhanced understanding of radiation transfer[22] their study has allowed.

There are many studies of elastic wave propagation in random media, from at least two domains: the geophysics literature seeks to understand the effect of inhomogeneities within the Earth's[1] crust on seismic waves,[23] and the nondestructive evaluation literature seeks to gauge the effect that flaws in elastic materials have on elastic waves.[24] In the case of an isotropic heterogeneous medium, the elastic wave equation takes the form

$$\rho(\mathbf{x})\ddot{u}_i - \frac{\partial}{\partial x_j} c_{ijkl}(\mathbf{x}) \frac{\partial u_l}{\partial x_k} = 0, \tag{1.1}$$

where $c_{ijkl} = \lambda \delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$ are the elastic constants. Heterogeneities can be treated as continuous or discontinuous. The case of continuous heterogeneities has been widely studied.[25–29] One way to solve this problem is to use a perturbative method assuming that the elastic constants are close to the values in the homogeneous medium. The case of discontinuous heterogeneities has been mainly studied to account for inclusions in the medium.[24,30–35] In this case, boundary conditions of force and displacement continuity at the inclusion surface have to be considered. In both cases, the problem of scattering by the random distribution of weak elastic heterogeneities can be solved starting from an integral representation for the scattered field and considering simplifications to reach a desired order of accuracy, as the Born approximation[28,32] or the averaged T-matrix approximation.[36] We mention also the work of Zhang and Gross,[37] who treat the problem of coherent propagation of an elastic wave though many cracks, defined as lines of finite length, free of traction, and with a displacement discontinuity. The authors calculate numerically the scattering functions for one scatterer and use Foldy's approach[17] to treat the multiple scattering configuration.

The heterogeneity we consider in the present paper is a dislocation. In two dimensions it is characterized as a point defect with a displacement discontinuity measured by its Burgers'vector **b**. The mechanism of wave scattering by a single dislocation is quite simple. An elastic wave will hit each individual dislocation, causing it to oscillate in response. The ensuing oscillatory motion will generate outgoing (from the dislocation position) elastic waves. Although the mechanism of scattering by a dislocation appears simple, it is unusual compared with the mechanism of scattering by inclusion type scatterers or by a continuous variation of the elastic constants. In these latter cases, the scattering results from a particular structure of the medium but no dynamical effect occurs. In contrast, the mechanism we are interested in does not contain any structural effect that would describe the structure of the dislocation core. This is a good approximation provided wavelengths are large compared to core size, a good approximation even at high ultrasonic frequencies.

An integral representation for the elastic wave generated by a moving dislocation has been known for some time;[38,39] however, an equation for the dislocation response to an incident wave is a more recent work.[40] In particular cases, Kiusalaas and Mura,[41] using Nabarro's results,[42] derived the total scattering cross section for the scattering of stress waves by a dislocation. The work of Lund[40] provided a theoretical underpinning, as well as a full tensor treatment, of the phenomenological string model of Koehler,[12] that was successfully implemented by Granato and Lücke[1–3] (hereafter GL) to explain internal friction measurements. They considered the response to an external stress wave of a damped, string-like dislocation segment endowed with mass and fine tension, subject to viscous loss, and pinned between two points. In a single scattering approach, the damping of the string leads to a damping of the wave, the inertia of the string leads to a modification of the wave speed of propagation, and the line tension leads to the possibility of having resonances.

As mentioned above, one reason for a study on the effect of dislocations on wave propagation lies in the desire to develop new probes to study phenomena such as the brittle to ductile transition,[13] where dislocation structure and dynamics appear to play a prominent role. In keeping with this motivation, we shall not consider the scattering of elastic waves by the effective elastic constant changes at the core of a dislocation since even at the highest frequencies likely to be generated by ultrasonic transducers (say, up to the GHz regime) the acoustic wavelengths will be much larger than dislocation core size. In this paper we compute the properties of a coherent elastic wave propagating through an elastic medium that is filled with randomly placed dislocations. It makes sense to study in some detail the properties of coherent elastic waves because in acoustics the phase can be measured. This is of course not the case for electromagnetic waves, nor for de Broglie waves such as electrons in solids.

This paper is organized as follows: In Sec. II we give a brief review of those aspects of dislocation dynamics that are needed to understand the scattering of elastic waves by dislocations. It also serves to unify notation. In Sec. III we present the calculation of the properties of a coherent antiplane wave traveling through a two-dimensional elastic medium filled with randomly located screw dislocations. This is done first with a simple minded approach whose limitations are pointed out and then through the perturbative calculation of a mass operator for Dyson's equation. In Sec. IV we present the calculation of the properties of a coherent inplane elastic wave traveling through a medium that is filled with randomly placed edge dislocations. This is also done at first with a simple minded approach and then through the calculation of a mass operator. Calculations here are more involved than in Sec. III due to the vector nature of the equations, and the matrix nature of the operators describing the wave-dislocation interaction. In Sec. V we present some concluding remarks. Some standard facts about multiple scattering, as adapted to the elastic wave-dislocation interaction, are collected in two appendices. Throughout the paper, Ω denotes the frequency of the incident wave while ω is the frequency variable in the Fourier transforms.

II. DISLOCATION DYNAMICS—A BRIEF REVIEW

Consider three dimensional space with coordinates $\mathbf{x}=(x_1,x_2,x_3)$. The dynamical variables are the (small) displacements $\mathbf{u}$ of points $\mathbf{x}$ away from their equilibrium positions as a function of time t. A dislocation loop is described by a closed curve $\mathbf{X}(\sigma,t)$, where σ is a Lagrangian coordinate along the loop L. The dislocation loop is characterized by a Burgers vector $\mathbf{b}$, defined by a discontinuity of the displacement field $\mathbf{u}$:

$$\oint_{\mathcal{C}} d\mathbf{u} \equiv -\mathbf{b}.$$

The integral is taken along a closed curve $\mathcal{C}$ around the dislocation with a direct orientation respect to $\boldsymbol{\tau}=\partial\mathbf{X}/\partial\sigma$, the tangent to the loop. (Fig. 1). This means that the displacement field $\mathbf{u}$ is multivalued, with a jump equal to $\mathbf{b}$ when crossing a surface $\mathcal{S}$ bounded by the loop L. This condition can be formally written as

$$[\mathbf{u}]_{\mathcal{S}}=\mathbf{b}.$$

The surface $\mathcal{S}$ is arbitrary, and should not appear in expressions involving physically measurable quantities.

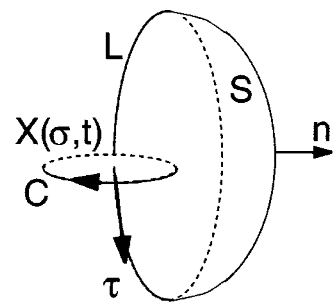

FIG. 1. Definition of the Burgers vector.

A. GENERATION OF ELASTIC WAVES BY A MOVING DISLOCATION

1. General case

The displacements $\mathbf{u}$ generated by a dislocation loop moving in an arbitrary but prescribed way through an homogeneous elastic medium are obtained[39] by solving the dynamic equations

$$\rho\frac{\partial^2}{\partial t^2}u_i(\mathbf{x},t) - c_{ijkl}\frac{\partial^2}{\partial x_j\ \partial x_k}u_l(\mathbf{x},t) = 0, \tag{2.1}$$

with boundary conditions

$$[u_i]_{\mathcal{S}} = b_i, \quad \left[c_{ijkl}\frac{\partial u_l}{\partial x_k}n_j\right]_{\mathcal{S}} = 0. \tag{2.2}$$

The first condition is the discontinuity of displacement already mentioned, and the second equation is the continuity of the stress across the surface $\mathcal{S}$.

In the isotropic case there are only two independent elastic constants,

$$c_{ijkl} = \lambda\,\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}),$$

where (λ,μ) are the Lamé coefficients. Using the Green's function for free space $G^{0(3D)}$, defined by

$$\rho\frac{\partial^2}{\partial t^2}G^{0(3D)}_{im}(\mathbf{x}-\mathbf{x}',t-t') - c_{ijkl}\frac{\partial^2}{\partial x_j\,\partial x_k}G^{0(3D)}_{lm}(\mathbf{x}-\mathbf{x}',t-t') = \delta(\mathbf{x}-\mathbf{x}')\,\delta(t-t')\,\delta_{im}, \tag{2.3}$$

the displacement u_m can be written as an integral representation:

$$u_m(\mathbf{x},t) = c_{ijkl}\int\int_{S(t')} dt'\ dS\ b_l n_k\frac{\partial}{\partial x_j}G^{0(3D)}_{im}(\mathbf{x}-\mathbf{x}',t-t'), \tag{2.4}$$

where $\mathbf{n}$ denotes the unit normal to $\mathcal{S}(t)$, the surface of discontinuity for the displacement. This surface is time dependent since the dislocation line $\mathbf{X}(\sigma,t)$ may change in time, and its change $\Delta\mathcal{S}$ during a small time interval Δt obeys

$$\int_{\Delta\mathcal{S}} dS\ n_k = \epsilon_{knh}\oint_L d\sigma\ \dot{X}_n\tau_h\ \Delta t,$$

where an overdot means time derivative and ϵ_{ijk} is the usual completely antisymmetric tensor. Equation (2.4) is a convo-

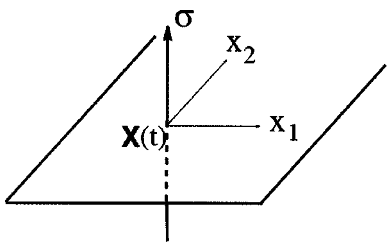

FIG. 2. Configuration of the 2D problem.

lution of the Green's function with a source that is localized along the surface of discontinuity $\mathcal{S}$. Since this surface does not have a special physical significance, it should be possible to express physically meaningful quantities in terms of a source that is localized along the loop L. Indeed, it is not displacements $\mathbf{u}$ but their time and space derivatives that have physical meaning, since it is they that appear in expressions for energy and momentum. From above, the velocity $v_m \equiv \dot{u}_m$ satisfies the integral representation

$$v_m(\mathbf{x},t) = \epsilon_{knh} c_{ijkl} \int \oint_L dt'\ d\sigma\ b_l \dot{X}_n(\sigma,t') \tau_h(\sigma) \times \frac{\partial}{\partial x_j} G_{im}^{0(3D)}(\mathbf{x} - \mathbf{X}(\sigma,t'),t-t'), \tag{2.5}$$

which is a convolution of the Green's function with a source localized along the loop and not on the surface $\mathcal{S}$:

$$v_m(\mathbf{x}) = \int_{\mathcal{V}} d\mathbf{x}'\ G_{im}^0(\mathbf{x}-\mathbf{x}') s_i(\mathbf{x}'),$$

with

$$s_i(\mathbf{x},t) = c_{ijkl} \epsilon_{mnk} \oint_L d\sigma\ \dot{X}_m \tau_n b \frac{\partial}{\partial x_j} \delta(\mathbf{x} - \mathbf{X}), \tag{2.6}$$

and where $\mathcal{V}$ denotes the volume of considered space. Consequently, an inhomogeneous wave equation for particle velocity can be written as

$$\rho \ddot{v}_i(\mathbf{x},t) - c_{ijkl} \frac{\partial^2 v_l(\mathbf{x},t)}{\partial x_j\, \partial x_k} = s_i(\mathbf{x},t). \tag{2.7}$$

We shall need this equation when discussing scattering.

2. Two dimensions

When a dislocation loop is an infinite straight line, say, along the x_3 direction,

$$\mathbf{X}(\sigma,t) = (X_1(t), X_2(t), \sigma),$$

and displacements are independent of x_3 (Fig. 2), the problem is two dimensional. In this case

$$v_m^s(\mathbf{x},t) = \epsilon_{cb} c_{iacl} \int dt'\ b_l \dot{X}_b(t') \frac{\partial}{\partial x_a} G_{im}^0(\mathbf{x} - \mathbf{X}(t'), t-t'), \tag{2.8}$$

where $a,b,c,\cdots = 1,2$, $\epsilon_{cb} \equiv \epsilon_{cb3}$ and

$$G_{im}^0(x_1,x_2,t) \equiv \int dx_3\ G_{im}^{0(3D)}(x_1,x_2,x_3,t)$$

is the Green's function in two dimensions. In other words, the source term becomes

$$s_i(\mathbf{x},t) = c_{iacl} \epsilon_{cb} \dot{X}_b b_l \frac{\partial}{\partial x_a} \delta(\mathbf{x} - \mathbf{X}) \tag{2.9}$$

and (2.8) can be written in the frequency domain as

$$v_m^s(\mathbf{x},\omega) = \epsilon_{cb} c_{iacl} b_l \dot{X}_n(\omega) \frac{\partial}{\partial x_a} G_{im}^0(\mathbf{x},\omega), \tag{2.10}$$

where $G_{im}^0(\mathbf{x} - \mathbf{X}(t'), t-t')$ has been taken, to leading order with respect to the small amplitude $X(t)$, equal to $G_{im}^0(\mathbf{x}, t-t')$, when the dislocation equilibrium position is at the origin.

B. Response of a dislocation to an incoming stress wave in two dimensions

In the following, we consider the two dimensional motion of a dislocation line, oriented along the x_3 axis, moving under the action of a stress wave Σ_{ib}. In this case, a method of finding an equation of motion for a dislocation loop can be found in Ref. 40, based on the observation that the equations of dynamic elasticity follow from a variational principle. When dislocation velocity is small compared to the speed of longitudinal and shear waves $\dot{X} \ll \alpha, \beta$, where $\alpha = \sqrt{(\lambda + 2\mu)/\rho}$ and $\beta = \sqrt{\mu/\rho}$, the equation of motion in the first Born approximation (i.e., neglecting the radiation reaction) is

$$\frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot{X}_a}\right) = \epsilon_{ab} b_i \Sigma_{ib}, \tag{2.11}$$

where $\Sigma_{ib} = c_{ibkl} \partial u_k / \partial x_l$ is the stress tensor evaluated at the dislocation position. $\mathcal{L}$ is the Lagrangian,

$$\mathcal{L} = -\frac{\mu}{4\pi} \ln\left(\frac{\delta}{\epsilon}\right) \left\{ b_\parallel^2 \left(1 - \frac{\dot{X}^2}{2\beta^2}\right) + b_\perp^2 \left[2(1-\gamma^{-2}) - \frac{\dot{X}^2}{2\beta^2}(1+\gamma^{-4}) \right] + \frac{(b_\perp \wedge \dot{\mathbf{X}})^2}{\beta^2}(1-\gamma^{-4}) \right\},$$

with $\gamma = \alpha/\beta$ and δ, ϵ are long- and short-distance cutoff lengths, respectively. $b_\parallel$ and $b_\perp$ are the components of the Burgers vector parallel and perpendicular to the dislocation line, and $\mu b^2/\beta^2$ has dimension of mass per unit length. Equation (2.11) is a second order in time ordinary differential equation for a point particle (the dislocation position in two dimensions) subject to the usual Peach-Koehler force.[43] In a more general case, say for oblique wave incidence, there are additional terms arising from the line tension associated with the dislocation line curvature.[5]

III. THE ANTIPLANE CASE

The antiplane case corresponds to the interaction of an antiplane shear wave $\mathbf{v} = (0,0,v)$ with a screw dislocation

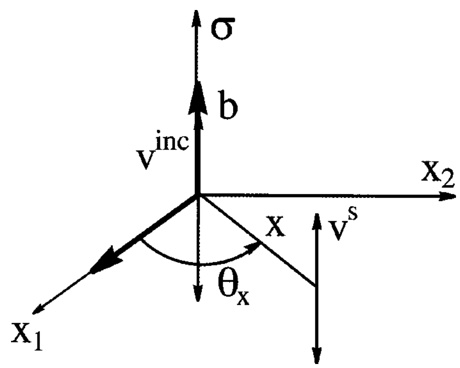

FIG. 3. Scalar problem of the shear wave interacting with a screw dislocation.

$\mathbf{b}=(0,0,b)$ (Fig. 3). It is easy to see in (2.7) and (2.9) that no interaction can occur with an in-plane Burgers vector (see also Ref. 44). In this case the wave equation takes the form, using (2.9),

$$\rho\ddot{v}(\mathbf{x},t) - \mu\nabla^2 v(\mathbf{x},t) = \mu b \epsilon_{ab}\dot{X}_b \frac{\partial}{\partial x_a}\delta(\mathbf{x}-\mathbf{X}).$$

We consider now a distribution of N screw dislocations in the $\mathbf{x}$-plane. $\mathbf{X}^N(t)$-denotes the location of the Nth dislocation in a volume $\mathcal{V}$ (actually, here a surface) and the bulk limit is taken with $n=N/\mathcal{V}$, the density of dislocations, assumed uniform. b^N denotes the Burgers vector of the Nth dislocation, assumed small compared to any other lengths, such as $n^{-1/2}$ the typical distance between dislocations, and any elastic wavelength. In this case, and for wavelengths large compared to the dislocation core size, the wave equation keeps the same form as for one dislocation and the source term corresponds to the sum over all dislocations:

$$\rho\ddot{v}(\mathbf{x},t) - \mu\nabla^2 v(\mathbf{x},t) = \mu\sum_{n=1}^{N} b^n \epsilon_{ab}\dot{X}_b^n \frac{\partial}{\partial x_a}\delta(\mathbf{x}-\mathbf{X}^n). \tag{3.2}$$

As previously said, the wave equation has to be completed with the law for the dislocation motion $\dot{\mathbf{X}}(t)$in order to have a self consistent problem. This is given by Eq. (2.11), which, in the antiplane case becomes

$$M\ddot{X}_b(t) = -\mu b \epsilon_{bc}\frac{\partial}{\partial X_c}u(\mathbf{X},t), \tag{3.3}$$

where M is the classical mass per unit length,[40,45]

$$M = \frac{\mu b^2}{4\pi\beta^2}\ln\frac{\delta}{\epsilon}. \tag{3.4}$$

We consider in the following a plane wave of frequency Ω and wave vector $\mathbf{k}_\beta$ with $k_\beta=\Omega/\beta$ traveling though a medium filled with randomly distributed dislocations, as described by Eq. (3.2). We are interested in characterizing the effective medium in terms of a complex wave vector $\mathbf{K}_\beta$ parallel to $\mathbf{k}_\beta$, whose real part gives a renormalized speed of propagation, and whose imaginary part gives an attenuation length.

A. Simple—but incomplete—calculation

A classic result that goes back at least to Foldy[17] establishes that, if a scalar plane wave $v=\exp(ik_\beta x_1 - i\Omega t)$ is incident upon a slab with many, randomly placed, scatterers, for low densities n there will be a coherent wave $\langle v\rangle = \exp(iK_\beta x_1 - i\Omega t)$ characterized by an effective wave vector,

$$K_\beta = k_\beta + n\sqrt{\frac{2\pi}{k\beta}}\langle f(0)\rangle e^{-i\pi/4}, \tag{3.5}$$

where $f(0)$ is the forward scattering amplitude for a single scatterer, and the brackets denote an average over the internal variables that characterize the scatterers. In our case this would be an average over various Burgers vectors, if a distribution of Burgers vectors was allowed.

The scattering amplitude for an antiplane wave incident head-on on a screw dislocation is,[44] with $\theta_x=(\widehat{Ox_1},\mathbf{x})$,

$$f(\theta_x) = -\frac{\mu b^2}{2M}\frac{e^{i\pi/4}}{\sqrt{2\pi\Omega}\beta^{3/2}}\cos\,\theta_x, \tag{3.6}$$

so that the effective wave vector is, when all dislocations have the same Burgers vector,

$$K_\beta = k_\beta\left(1-\frac{\mu n b^2}{2M\Omega^2}\right). \tag{3.7}$$

This expression provides an effective speed of propagation for antiplane waves traveling through many, randomly placed, dislocations. However, being real, it does not provide an expression for the attenuation due to the energy that is taken away from the incident direction by the scatterers. In order to get this, we turn to a Green's function formalism, whose general ideas are reviewed in Appendix A.

The modification in wave velocity Ω/K_β implicit in Eq. (3.7) can be compared with the result of Granato and Lücke.[1] The authors performed calculations for one dislocation loop of total length L with fixed ends and submitted to an external periodic stress. The Koehler[12] equation of motion they use contains an inertial term, a drag force and a line tension. This equation is coupled with a wave equation, similar to our Eq. (3.2). In this model, the inertial term is responsible for the modification of the wave velocity, the drag force is responsible for the wave attenuation, and the line tension force leads to resonant phenomena. If the drag and line tension are neglected, the modified speed of propagation of GL is similar to (3.7). In the GL model, our dependence in n is replaced by a dependence in L. However, for many dislocations, L becomes the total length of dislocation per unit volume, similar to our n.

B. Green's function formalism—Antiplane case

The heart of the matter is the calculation of the mass operator Σ, given by Eq. (A7), in terms of which the effective wave number is given by (3.32). Using (3.2) and (3.3), we have the following equation in the frequency domain:

$$(\nabla^2 + k_\beta^2)v(\mathbf{x},\omega) = -V(\mathbf{x},\omega)v(\mathbf{x},\omega), \tag{3.8}$$

where the potential V is

$$V(\mathbf{x},\omega) = \sum_{n=1}^{N} \frac{\mu}{M}\left(\frac{b^n}{\omega}\right)^2 \frac{\partial}{\partial x_a}\delta(\mathbf{x}-\mathbf{X}_0^n) \left.\frac{\partial}{\partial x_a}\right|_{\mathbf{X}_0^n}. \quad (3.9)$$

In this expression, the positions $\mathbf{X}^n(t)$ of the screw dislocations have been replaced by their mean values (positions at rest) $\mathbf{X}_0^n$.

The Green's functions G^0 and G satisfy the equations

$$(\nabla^2 + k_\beta^2)G^0(\mathbf{x}-\mathbf{x}') = -\delta(\mathbf{x}-\mathbf{x}'), \quad (3.10)$$

$$(\nabla^2 + k_\beta^2 + V(\mathbf{x}))G(\mathbf{x},\mathbf{x}') = -\delta(\mathbf{x}-\mathbf{x}'), \quad (3.11)$$

with the potential V defined by (3.9). To proceed, we need the average of V:

$$\langle V\rangle(\mathbf{x}) = \frac{\mu}{\omega^2}\int \rho_{b^1}db^1\cdots\rho_{b^N}\,db^N \times\frac{d\mathbf{X}_0^1\cdots d\mathbf{X}_0^N}{\mathcal{V}^N}\sum_n \frac{(b^n)^2}{M}w_{ii}(\mathbf{x},\mathbf{X}_0^n), \quad (3.12)$$

where

$$w_{ij}(\mathbf{x},\mathbf{X}_0) \equiv \frac{\partial}{\partial x_i}\delta(\mathbf{x}-\mathbf{X}_0) \left.\frac{\partial}{\partial x_j}\right|_{\mathbf{X}_0}. \quad (3.13)$$

This average is taken over all realizations of V assuming no correlation between (uniformly distributed) scatterers and denoting ρ_b the probability distribution function of the Burgers vector. $\langle V\rangle$ is the sum of N identical terms so that

$$\langle V\rangle(\mathbf{x}) = \frac{\mu}{\omega^2}N\int \rho_b\, db\frac{b^2}{M}\int \frac{d\mathbf{X}_0}{\mathcal{V}}w_{ii}(\mathbf{x},\mathbf{X}_0). \quad (3.14)$$

Concerning the integral over $\mathbf{X}_0$, we use that

$$\int d\mathbf{X}_0\; w_{ij}(\mathbf{x},\mathbf{X}_0) = \frac{\partial}{\partial x_i}\int d\mathbf{X}_0\;\delta(\mathbf{x}-\mathbf{X}_0)\frac{\partial}{\partial X_{0j}} = \frac{\partial^2}{\partial x_i\,\partial x_j}. \quad (3.15)$$

To calculate the integral over b, we use the explicit relation[40] $M \simeq Cb^2\ln(L/b)$ with C a constant and, for the sake of clarity, we define M^* as

$$\left\langle \frac{b^2}{M}\right\rangle \equiv \frac{\langle b^2\rangle}{M^*}. \quad (3.16)$$

For instance, when all Burgers vectors are identical, $\rho_b = \delta(b-b_0)$, and we have $M^* = M(b_0)$. For a uniform distribution, $\rho_b = 1/b_M$ if $0 < b < b_M$ and zero otherwise, we have $M^* = M(b_M)/3$ using

$$\left\langle \frac{b^2}{M}\right\rangle = \int_0^{b_M}\frac{db}{b_M}\frac{1}{C\ln(L/b)} \simeq \frac{1}{C\ln(L/b_M)} = \frac{b_M^2}{M(b_M)}. \quad (3.17)$$

The calculation of $\langle V\rangle$ can be achieved and we finally obtain

$$\langle V\rangle(\mathbf{x}) = \frac{\mu}{M^*\omega^2}n\langle b^2\rangle\nabla^2. \quad (3.18)$$

In the frequency domain this is

$$\langle V\rangle(k) = -\frac{\mu}{M^*\omega^2}n\langle b^2\rangle k^2. \quad (3.19)$$

In order to derive the second order effects in Σ we write $V = \sum_{n=1}^N V^n$. Thus, we have

$$\langle VG^0V\rangle = \frac{(N-1)}{N}\langle V\rangle G^0\langle V\rangle + \left\langle \sum_{n=1}^N V^nG^0V^n\right\rangle, \quad (3.20)$$

when no correlation exists among different dislocations. For large values of N this means

$$\langle VG^0V\rangle - \langle V\rangle G^0\langle V\rangle \simeq N\langle V^1G^0V^1\rangle. \quad (3.21)$$

We now calculate $\langle V^1G^0V^1\rangle$ [using (3.9)]:

$$\langle V^1(\mathbf{x})G^0(\mathbf{x}-\mathbf{x}')V^1(\mathbf{x}')\rangle = \left(\frac{\mu}{\omega^2}\right)^2\int \rho_b\,db\frac{b^4}{M^2}\int \frac{d\mathbf{X}_0}{\mathcal{V}}w_{ii}(\mathbf{x},\mathbf{X}_0) \times G^0(\mathbf{x}-\mathbf{x}')w_{jj}(\mathbf{x}',\mathbf{X}_0). \quad (3.22)$$

This integral has a divergence at short distances. However, the continuum theory we are using does not make sense at short wavelengths, that is, wavelengths that are comparable to the atomic spacing (i.e., the Burgers vector). In order to face this issue, we go to Fourier space and introduce a cutoff function $f(k)$ to suppress the effect of short wavelengths $k \geqslant 1/b$ [the qualitative nature of our results does not depend on the detailed nature of $f(k)$]

$$G^0(\mathbf{x}) = \frac{1}{(2\pi)^2}\int d\mathbf{q}\;G^0(q)e^{i\mathbf{q}\cdot\mathbf{x}}f(q). \quad (3.23)$$

In the Fourier space, we have thus

$$\langle V^1G^0V^1\rangle(\mathbf{k}) = \left(\frac{\mu}{2\pi rM^*\omega^2}\right)^2\frac{\langle b^4\rangle}{V}\int d\mathbf{X}_0\,d\mathbf{q}\;G^0(q)f(q) \times\int d\mathbf{x}\;e^{-i\mathbf{k}\cdot\mathbf{x}}w_{ii}(\mathbf{x},\mathbf{X}_0)e^{i\mathbf{q}\cdot\mathbf{x}} \times\int d\mathbf{x}'\;e^{-i\mathbf{q}\cdot\mathbf{x}'}w_{jj}(\mathbf{x}',\mathbf{X}_0)e^{i\mathbf{k}\cdot\mathbf{x}'}. \quad (3.24)$$

We have introduced in this latter expression a coefficient r of order 1, such that

$$\left\langle\frac{b^4}{M^2}\right\rangle \equiv \frac{\langle b^4\rangle}{(rM^*)^2}. \quad (3.25)$$

For instance, with $\rho_b = \delta(b-b_0)$, we have $r=1$ and with $\rho_b = 1/b_M$, a uniform distribution from 0 to b_M, we have $r = 3/\sqrt{5}$.

We now use

$$\int d\mathbf{x}\; g(\mathbf{x}) w_{ii}(\mathbf{x},\mathbf{X}_0) e^{i\mathbf{k}.\mathbf{x}} = \int d\mathbf{x}\; g(\mathbf{x}) \frac{\partial}{\partial x_i} \delta(\mathbf{x}-\mathbf{X}_0) i k_i e^{i\mathbf{k}.\mathbf{X}_0} \tag{3.26}$$

$$= - i k_i e^{i\mathbf{k}.\mathbf{X}_0} \frac{\partial}{\partial X_{0i}} g(\mathbf{X}_0), \tag{3.27}$$

with $g(\mathbf{x}) = e^{-i\mathbf{k}.\mathbf{x}}$ or $g(\mathbf{x}') = e^{-i\mathbf{q}.\mathbf{x}'}$ in (3.24) to get

$$\langle V^1 G^0 V^1 \rangle(\mathbf{k}) = \left(\frac{\mu}{2\pi r M^* \omega^2}\right)^2 \langle b^4 \rangle n k_i k_j \int d\mathbf{q}\; q_i q_j G^0(\mathbf{q}) f(q)$$
$$= \left(\frac{\mu}{2 r M^* \omega^2}\right)^2 \frac{\langle b^4 \rangle n}{\pi} k^2 \int dq\; q^3\; f(q) G^0(q), \tag{3.28}$$

where

$$\int dq\; q^3 f(q) G^0(q) = \int dq \frac{q^3 f(q)}{q^2 - k_\beta^2} = \pi k_\beta^2 / 2 \left(i + \frac{\mathcal{C}}{\langle b^2 \rangle k_\beta^2} \right), \tag{3.29}$$

with $\mathcal{C} \simeq 1/\pi$ a numerical constant depending weakly on the cutoff function f. We have finally the mass operator,

$$\Sigma(k) = \frac{\mu n \langle b^2 \rangle}{M^* \omega^2} k^2 \left[-1 + \frac{\rho}{8 M^* r^2} \frac{\langle b^4 \rangle}{\langle b^2 \rangle} \left(i + \frac{\mathcal{C}}{k_\beta^2 \langle b^2 \rangle} \right) \right]. \tag{3.30}$$

The Dyson equation (algebraic in the Fourier space) can now be solved to determine the effective wave number K_β, pole of $\langle G \rangle(k) = [k^2 - k_\beta^2 - \Sigma(k)]^{-1}$; as K_β is expected to be close to k_β, we get

$$K_\beta \simeq k_\beta \left(1 + \frac{\Sigma(k_\beta)}{2 k_\beta^2} \right),$$
$$\simeq k_\beta \left\{ 1 + \frac{\mu n \langle b^2 \rangle}{2 M^* \Omega^2} \left[-1 + \frac{\rho}{8 r^2 M^*} \frac{\langle b^4 \rangle}{\langle b^2 \rangle} \left(i + \frac{\mathcal{C}}{k_\beta^2 \langle b^2 \rangle} \right) \right] \right\}. \tag{3.31}$$

To leading order, this expression reduces to the simple minded result (3.7). The effective group velocity is decreased from its value in the absence of scatterers while phase velocity is increased. As mentioned in Ref. 31, there is no particular tendency to be expected from this latter quantity as a function of the scatterer density: In the case of inclusion, the phase velocity is found to increase while it is found to decrease for cracks and cavities. This modification of speed of propagation appears at first order in V and it is actually the only effect at this order. From (3.32), we obtain the elastic mean free path $\Lambda^{-1} = 2\Im(K_\beta)$, that gives the distance scale over which the coherent wave is attenuated due to the energy that is taken away from the incident wave by the scattering from the randomly placed scatterers:

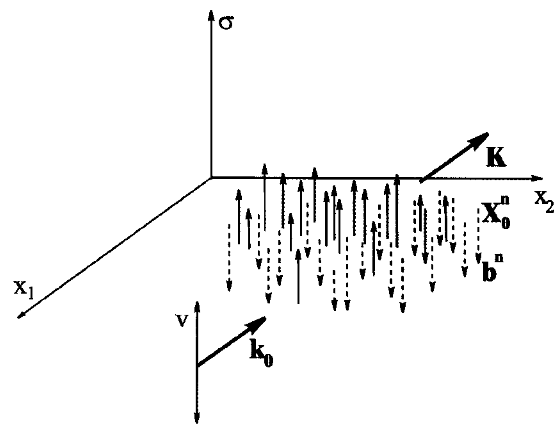

FIG. 4. Scalar problem of the shear wave interacting with a distribution of screw dislocations (solid arrows correspond to positive Burgers vectors and dotted arrows correspond to negative Burgers vectors).

$$\Lambda = 8 \frac{r^2 M^{*2}}{n \rho^2 \langle b^4 \rangle} k_\beta. \tag{3.32}$$

As it is expected, Λ decreases when increasing the density of scatterers n or the scatterer strength, in term of $\langle b^4 \rangle$. The increase of the attenuation $\sim 1/\Lambda$ with increasing n has been experimentally observed (see, for instance Ref. 14). A very interesting behavior would be the dependence of the attenuation on the frequency, that is here found to be surprisingly decreasing for increasing frequency (see the discussion in Sec. III C). An experimental verification of this behavior does not appear to be available, although it may become possible with the techniques described in Ref. 15.

The physical mechanism for the attenuation (3.32) lies in the multiple scattering of the wave by the randomly located dislocations that takes out energy from the wave propagating in the forward direction. This is different from the mechanism of Granato and Lücke,[1] where the attenuation is due to internal loss because of viscous damping of the string-like dislocation (see the discussion in Sec. III D).

C. Comments and discussion

An apparently surprising result has to be underlined at this point. The elastic mean free path (3.32) is found to decrease when increasing the wavelength [see also the calculation of Λ for edge dislocations below, Eq. (4.39)], although the scattering is usually expected to vanish for wavelengths long compared with the size of the obstacle. The explanation is found in the particular behavior of dislocations as scatterers, a fact that has been emphasized by Nabarro.[42] Indeed, two mechanisms of wave scattering by a dislocation have to be distinguished. The first mechanism is related to the microstructure of the dislocation core and has to vanish for long wavelengths; the second mechanism, which is the one considered in the present study, is that the dislocation moves under the influence of the incoming wave and reradiates a cylindrical (scattered) wave. Most previous studies consider scatterers such as inclusions and voids, involving a scattering mechanism of the first type. Thus, a scattering vanishing for long wavelength is not surprisingly found. In the present paper (see also Ref. 44), the investigated scattering mechanism is the second one. Thus, there is no reason to expect similar results here. Preferably, the scattering strength has to be found in the equation of motion of the dislocation in the presence of an incoming wave, whose amplitude increases with the wavelength in the dynamical models of Refs. 38, 40, and 42. (See Fig. 4).

Actually, the attenuation $1/\Lambda$ has an upper bound. This is because the weak scattering approximation implies that wavelength must be small compared to a cut-off length: $\lambda \ll L_c$. This can be seen from (3.19) and (3.8) [as well as the forthcoming Equations (4.29) with (4.16)], where weak scattering means

$$\frac{\mu n b^2}{M\Omega^2} = \left(\frac{\lambda}{L_c}\right)^2 \ll 1, \tag{3.33}$$

with

$$L_c = \sqrt{\frac{M}{n\rho b^2}} \simeq \frac{1}{\sqrt{n}}. \tag{3.34}$$

To obtain the equivalence in (3.34), we have used $M \simeq \rho b^2$ from (3.4). Indeed, $\delta \sim L$ the length of the dislocation line, $\epsilon \sim b$ and typically $L \simeq 10^6 - 10^8 b$, so that we have $\ln(\delta/\epsilon)/4\pi$ roughly equals unity.

A recent experiment on acoustic wave interaction with dislocations in $LiNbO_3$[15] provides a convenient system to estimate the parameters that appear in our formulas.

This experimental configuration is of particular interest in the framework of our study because, even if the measurements of the velocity and attenuation are not yet available, it shows that our calculations can be expected to have a reasonable region of validity (see also Sec. III D). Indeed, the wavelength is smaller than the cutoff length L_c: $\lambda \sim 10^{-1} L_c$. Since the ratio λ/L_c measures the scattering strength [see (3.33)], we also see that the scattering effect can be expected to be significant. Besides, this is also seen since visualizations of the scattered wave are possible in this experiment.

Finally, the static scattering by the dislocation core has been neglected. It has thus to be checked that the corresponding static mean free path Λ_{static} is large compared to all other characteristic lengths, L_c and Λ. As previously said, such scattering has a vanishing strength at long wavelengths. A discussion on this mechanism can be found in Ref. 42 where the author estimates the scattering cross section of order $b^2\Omega/\beta$, leading to

$$\Lambda_{static} \simeq \frac{1}{nb^2 k_\beta}. \tag{3.35}$$

It is easy to check that

$$\Lambda_{static} \simeq \Lambda (k_\beta\, b)^{-2} \gg \Lambda,$$

$$\Lambda_{static} \simeq L_c (k_\beta\, b)^{-1} (\sqrt{n}\, b)^{-1} \gg L_c, \tag{3.36}$$

since $b \ll k_\beta^{-1}$, $\sqrt{n}^{-1}$. A typical value of Λ_{static} in Ref. 15 is given in Table I. In this experiment, we have

TABLE I. Experimental values of λ, b, L (the length of the dislocation line) and n from Ref. 15, with $\omega = 580$ MHz. Corresponding values of L_c [from (3.34)], Λ [from (3.32)] and Λ_{static} [from (3.35)] are calculated [with $M^* = M$ and $r = 1$ in (3.4) and using $\rho = 4640$ kg m^{-3} for $LiNbO_3$].

λ (μm)	b (nm)	L (μm)	n (m^{-2})	L_c (μm)	Λ (cm)	Λ_{static} (m)
6	0.5	200	2.10^8	70	4	2.10^4

$$b \ll \lambda \ll 1/\sqrt{n} \simeq L_c \ll \Lambda \ll \Lambda_{static}, \tag{3.37}$$

so the assumptions needed in our calculations are satisfied by the parameters of the experimental setup described in Ref. 15.

D. Discussion: Influence of dislocation drag and line tension

Here we give a brief discussion on the influence of the dislocation drag term and line tension term, introduced in the Granato–Lücke (GL) model.[1,2] In this model, the equation for the motion of a dislocation line (with length L) corresponds to a mathematical model introduced by Koehler,[12] where the line tension is defined by $C = 2\mu b^2/\pi(1-\nu)$ (with ν the Poisson ratio) and the viscous drag coefficient B is a free parameter. The expressions for the velocity v_{GL} and the attenuation Λ_{GL}^{-1} in the GL model are as follows:

$$1/v_{GL} = 1/\beta \left[1 + \frac{4}{\pi^2}\frac{\mu n b^2}{M}\frac{\Omega_0^2 - \Omega^2}{(\Omega_0^2 - \Omega^2)^2 + (\Omega d)^2} \right],$$

$$1/\Lambda_{GL} = \frac{4}{\pi^2}\frac{n\rho b^2}{M}\frac{\beta\Omega^2 d}{(\Omega_0^2 - \Omega^2)^2 + (\Omega d)^2}, \tag{3.38}$$

where $d = B/M$ and $\Omega_0 = (\pi/L)\sqrt{C/M}$. In applying these formulas, it is generally assumed that $\Omega_0 \gg d \gg \Omega$, leading to simplifications in the previous expressions. These simplified expressions are used, notably in Ref. 14, to interpret experimental results. Note, however, that an estimation of Ω_0 using the experimental data of Ref. 15 (from Table I, with $v = 0.3$) gives $\Omega_0 = 50$ MHz which is one order of magnitude smaller than the frequency used in that experiment (580 MHz).

A two dimensional situation as is considered in the present paper means there is no dislocation line curvature, and hence no effect of line tension. If Ω_0 is neglected, one obtains

$$1/v_{GL} = 1/\beta \left(1 - \frac{4}{\pi^2}\frac{\mu n b^2}{M}\frac{1}{\Omega^2 + d^2} \right),$$

$$1/\Lambda_{GL} = \frac{4}{\pi^2}\frac{n\rho b^2}{M}\frac{\beta d}{\Omega^2 + d^2}, \tag{3.39}$$

and these expressions can be compared with our expressions (3.7) [with $\Omega/v = \Re(K_\beta)$] and (3.32). Defining $R = d/\Omega$ and using $M = \rho b^2$ in the GL model, we get the following relation between our scheme (v, Λ) and GL(v_{GL}, Λ_{GL}):

$$v_{GL} - \beta \simeq \frac{v - \beta}{1 + R^2},$$

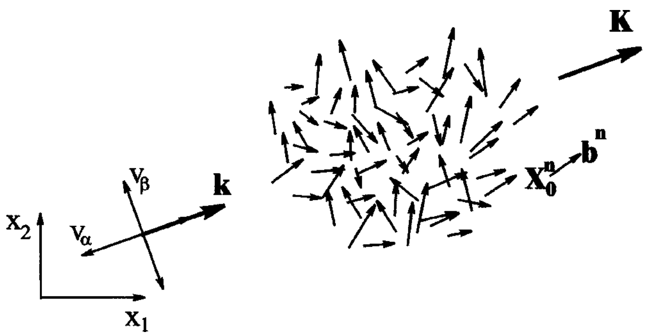

FIG. 5. Acoustic wave (v_α velocity) and in-place shear-wave (v_β velocity) interacting with a 2D gliding edge dislocation.

$$\Lambda_{GL}^{-1} \simeq \Lambda^{-1}\frac{R}{1+R^2}. \tag{3.40}$$

In the GL model, the viscosity is the unique source of damping so vanishing viscosity implies vanishing attenuation. This would be the case when $R \ll 1$, that is, for high enough frequencies. However, the values of B usually quoted as good fits to experiment (see, for instance Ref. 2) suggest, for the system studied in Ref. 15 ($\rho b^2\Omega \sim 10^{-6}$ kg m^{-1} s^{-1}) a value of $R \sim 1$ so that the effect of viscous drag and multiple scattering would be comparable.

IV. THE IN-PLANE CASE

The in-plane case corresponds to the interaction of edge dislocations with the in-plane waves, propagating at velocities α (longitudinal) and β (transverse) (Fig. 5); again, it is easy to see that an in-plane wave can only interact with an edge dislocation. We restrict ourselves to the case of gliding edge dislocations, for which the line dislocation moves only along its Burgers vector. Again, we assume that the Burgers vector is small compared with any length scale in the problem.

A. The simplified approach for the in-plane case

The simplified approach to the effect of random scatterers on a scalar wave can also be used in the in-plane case introducing two scalar potentials: Particle velocity is described in terms of a longitudinal (φ) and shear (ψ) potential:

$$\mathbf{v} = \boldsymbol{\nabla}\varphi + \boldsymbol{\nabla}\times\boldsymbol{\psi} \tag{4.1}$$

with $\boldsymbol{\psi}=(0,0,\psi)$. Each one obeys a scalar wave equation with

$$\left(\frac{\partial^2}{\partial t^2} - \alpha^2\nabla^2\right)\varphi = 0, \tag{4.2}$$

$$\left(\frac{\partial^2}{\partial t^2} - \beta^2\nabla^2\right)\psi = 0. \tag{4.3}$$

We use a previous result,[44] where the scattered potentials φ^s and ψ^s have been established (Fig. 6). With an incident wave of the form

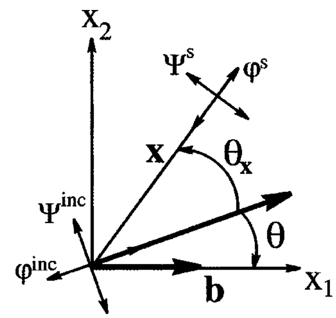

FIG. 6. Acoustic wave (v_α velocity) and in-plane shear-wave (v_β velocity) interacting with a distribution of gliding edge dislocations. The nth dislocation is characterized by its mean position X_0^n and its in plane Burgers vector b^n.

$$\varphi^{inc}(\mathbf{x},t) = A_\alpha e^{ik_\alpha x_1 - i\Omega t}, \quad \psi^{inc}(\mathbf{x},t) = A_\beta e^{ik_\beta x_1 - i\Omega t}, \tag{4.4}$$

we find that scattered potentials φ^s and ψ^s are given by

$$\varphi^s(\mathbf{x},t) = [f_{\alpha\alpha}(\theta_x)A_\alpha + f_{\alpha\beta}(\theta_x)A_\beta]\frac{e^{ik_\alpha x - i\Omega t}}{\sqrt{x}},$$

$$\psi^s(\mathbf{x},t) = [f_{\beta\alpha}(\theta_x)A_\alpha + f_{\beta\beta}(\theta_x)A_\beta]\frac{e^{ik_\beta x - i\Omega t}}{\sqrt{x}}, \tag{4.5}$$

with (see Ref. 44)

$$f_{\alpha\alpha}(\theta_x) = \frac{\mu b^2}{2M}\frac{e^{i\pi/4}}{\sqrt{2\pi k_\alpha}}\left(\frac{\beta}{\alpha}\right)^2\frac{\sin 2\theta}{\alpha^2}\sin(2\theta_x - 2\theta),$$

$$f_{\alpha\beta}(\theta_x) = \frac{\mu b^2}{2M}\frac{e^{i\pi/4}}{\sqrt{2\pi k_\alpha}}\left(\frac{\beta}{\alpha}\right)^2\frac{\cos 2\theta}{\beta^2}\sin(2\theta_x - 2\theta),$$

$$f_{\beta\alpha}(\theta_x) = -\frac{\mu b^2}{2M}\frac{e^{i\pi/4}}{\sqrt{2\pi k_\beta}}\frac{\sin 2\theta}{\alpha^2}\cos(2\theta_x - 2\theta),$$

$$f_{\beta\beta}(\theta_x) = -\frac{\mu b^2}{2M}\frac{e^{i\pi/4}}{\sqrt{2\pi k_\beta}}\frac{\cos 2\theta}{\beta^2}\cos(2\theta_x - 2\theta), \tag{4.6}$$

where $\theta_x = (\widehat{Ox_1,\mathbf{x}})$ and $\theta = (\widehat{Ox_1,\mathbf{b}})$. In order to get a relation between effective wave vector and forward scattering amplitude similar to (3.5), a slight generalization of Foldy's simplified approach is used, taking into account the possibility of mode conversion due to scattering:

$$\varphi(\mathbf{x}) = \varphi^{inc}(\mathbf{x}) + \sum_{i=1}^{N}[F_{\alpha\alpha}(\mathbf{x},\mathbf{X}_i)\varphi(\mathbf{X}^i) + F_{\alpha\beta}(\mathbf{x},\mathbf{X}_i)\psi(\mathbf{X}^i)],$$

$$\psi(\mathbf{x}) = \psi^{inc}(\mathbf{x}) + \sum_{i=1}^{N}[F_{\beta\alpha}(\mathbf{x},\mathbf{X}_i)\varphi(\mathbf{X}^i) + F_{\beta\beta}(\mathbf{x},\mathbf{X}_i)\psi(\mathbf{X}^i)]. \tag{4.7}$$

In this expression, for instance, $F_{\alpha\beta}(\mathbf{x},\mathbf{X}_i)\psi(\mathbf{X}^i)$ is the contribution to the longitudinal potential $\varphi(\mathbf{x})$ at $\mathbf{x}$ due to the ith scatterer receiving the shear potential $\psi(\mathbf{X}^i)$. Taking the averages over all realizations of the random distribution of scatterers, we get

$$\langle\varphi\rangle(\mathbf{x}) = \varphi^{inc}(x) + n\int d\mathbf{X}[\langle F_{\alpha\alpha}\rangle_{b,\theta}(\mathbf{x},\mathbf{X})\langle\varphi\rangle(\mathbf{X}) + \langle F_{\alpha\beta}\rangle_{b,\theta}(\mathbf{x},\mathbf{X})\langle\psi\rangle(\mathbf{X})],$$

$$\langle\psi\rangle(\mathbf{x}) = \psi^{inc}(\mathbf{x}) + n\int d\mathbf{X}[\langle F_{\beta\alpha}\rangle_{b,\theta}(\mathbf{x},\mathbf{X})\langle\varphi\rangle(\mathbf{X}) + \langle F_{\beta\beta}\rangle_{b,\theta}(\mathbf{x},\mathbf{X})\langle\psi\rangle(\mathbf{X})], \quad (4.8)$$

where $\langle\cdot\rangle_{b,\theta}$ indicates the average over all possible Burgers vector orientations (θ) and magnitudes (b). We assume that no correlation exists among the dislocation positions and Burgers vectors. As in the scalar case, we look for a solution of $\langle\varphi\rangle$ and $\langle\psi\rangle$ as plane waves and the functions $F_{ab}(\mathbf{x},\mathbf{X})$ are related to the scattering functions $f_{ab}(\theta_x)$ of a single scatterer [see Eq. (B10)],

$$F_{ab}(\mathbf{x},\mathbf{X}) = f_{ab}(\theta_x)\frac{e^{ik_a|\mathbf{x}-\mathbf{X}|}}{\sqrt{|\mathbf{x}-\mathbf{X}|}}. \quad (4.9)$$

Since $\langle f_{\alpha,\beta}\rangle_{b,\theta}(0)=\langle f_{\beta,\alpha}\rangle_{b,\theta}(0)=0$, the result is that a plane wave (longitudinal or transversal) with undisturbed wave number k_c ($c=\alpha,\beta$) will propagate coherently with an effective wave number,

$$K_c = k_c + n\sqrt{\frac{2\pi}{k_c}}\langle f_{cc}\rangle_{b,\theta}(0)e^{-i\pi/4},$$

$$=k_c\left(1-\frac{n\mu\langle b^2\rangle\mathcal{A}_c}{4M\Omega^2}\right), \quad (4.10)$$

with

$$\mathcal{A}_\alpha = \beta^2/\alpha^2,$$

$$\mathcal{A}_\beta = 1. \quad (4.11)$$

It can be noticed that there is no cross-coupling for the resulting multiple scattered coherent wave. This has been also observed in Ref. 28. In that case, this results from a particular behavior of the cross-coupled waves scattered by a unique scatterer, that remain always apart from the incident wave. Thus, the mode conversion is simply neglected in the pure forward scattering problem that is involved in the Foldy approach. In our case, there is no particular behavior of the cross-coupled scattered waves [Eq. (4.6)] in the incident direction. To determine the coherent wave characteristics, the full vectorial problem has to be considered. However, it is found that cross-coupled waves for the coherent waves vanish because the averaged cross-coupled scattering functions vanish.

As in the antiplane case, (4.10) is real and this approach cannot describe the attenuation of the wave. A Green's function approach is needed for this.

B. Green's function approach for the in-plane case

The wave equation (2.7) with (2.9) is now vectorial:

$$\rho\ddot{v}_i(\mathbf{x},t) - c_{ijkl}\frac{\partial^2 v_l(\mathbf{x},t)}{\partial x_j\,\partial x_k} = \mu\sum_{n=1}^{N}(\epsilon_{ib}b_a^n + \epsilon_{ab}b_i^n)\dot{X}_b^n\frac{\partial}{\partial x_a}\delta(\mathbf{x} - \mathbf{X}^n). \quad (4.12)$$

In order to describe the motion $\dot{\mathbf{X}}(t)$ of a gliding edge submitted to an externally generated wave displacement field $\mathbf{u}$ we introduce coordinates $(\tilde{x}_1,\tilde{x}_2)$, with $\tilde{x}_1$ along $\mathbf{b}$. The equation of motion (2.11) becomes

$$M\ddot{\mathbf{X}}(t) = \sigma_{\tilde{1}\tilde{2}}\mathbf{b}, \quad (4.13)$$

where

$$\sigma_{\tilde{1}\tilde{2}} = c_{12kl}\frac{\partial}{\partial\tilde{x}_l}\tilde{u}_k(\tilde{\mathbf{x}},t) \quad (4.14)$$

is the stress tensor and M is the effective mass for an edge dislocation:

$$M = \frac{\mu b^2}{4\pi\beta^2}\left(1+\frac{\beta^4}{\alpha^4}\right)\ln\frac{\delta}{\epsilon}, \quad (4.15)$$

where δ and ϵ have the same definition as in (3.4).

We consider now a plane wave of frequency Ω traveling through an elastic medium filled with randomly located and oriented edge dislocations, described by (4.12). The acoustic component has a wave vector $\mathbf{k}_\alpha$ ($k_a=\Omega/\alpha$) and the shear component a wave vector $\mathbf{k}_\beta$ ($k_\beta=\Omega/\beta$). The task is to see if we can define two effective wave vectors $\mathbf{K}_c(c=\alpha,\beta)$ parallel to $\mathbf{k}_c$ to describe the medium as an effective medium (Fig. 5). The procedure using the modified Green's formalism is similar to the procedure we used for the antiplane case although calculations are more involved because of the vector nature of the wave equation. The homogeneous wave equation for particle velocity is now a vector equation,

$$[\nabla^2 + k_\beta^2 + (\gamma^2-1)\,\nabla\,\nabla\,.\,]\mathbf{v}(\mathbf{x},\omega) = -\mathbf{V}(\mathbf{x},\omega)\mathbf{v}(\mathbf{x},\omega), \quad (4.16)$$

where the interaction operator $\mathbf{V}$ now has a matrix structure,

$$V_{ij}(\mathbf{x},\omega) = \sum_{n=1}^{N}\frac{\mu(b^n)^2}{M\omega^2}V_i^n(\mathbf{x})\delta(\mathbf{x}-\mathbf{X_0^n})V_j^n(\mathbf{x})\Big|_{\mathbf{X_0^n}}, \quad (4.17)$$

with V_1^n, V_2^n scalar operators describing the interaction of the nth dislocation with the stress wave:

$$V_1^n(\mathbf{x}) = \left(-\sin 2\theta^n\frac{\partial}{\partial x_1} + \cos 2\theta^n\frac{\partial}{\partial x_2}\right),$$

$$V_2^n(\mathbf{x}) = \left(\cos 2\theta^n\frac{\partial}{\partial x_1} + \sin 2\theta^n\frac{\partial}{\partial x_2}\right). \quad (4.18)$$

$\mathbf{X}_0^n$, b^n denote, respectively, for the nth dislocation, the position vector and the Burgers vector and $\theta^n=(\widehat{Ox_1,\mathbf{b}^n})$. As previously, the Green's function G_{ab} for the propagation in the presence of dislocations is related to the Green's function G^0 for the free propagation by the integral representation,

$$G_{ab}(\mathbf{x},\mathbf{x}') = G^0_{ab}(\mathbf{x}-\mathbf{x}') + \int d\mathbf{x}''\, G^0_{ac}(\mathbf{x}-\mathbf{x}'')V_{cd}(\mathbf{x}'')G_{db}(\mathbf{x}'',\mathbf{x}'), \tag{4.19}$$

where G_{ab} and G^0_{ab} satisfy

$$k_\beta^2 G^0_{ij}(\mathbf{x}-\mathbf{x}') + \frac{c_{iklm}}{\mu}\nabla_k\nabla_l G^0_{mj}(\mathbf{x}-\mathbf{x}') = -\delta_{ij}\delta(\mathbf{x}-\mathbf{x}'), \tag{4.20}$$

$$k_\beta^2 G_{ij}(\mathbf{x},\mathbf{x}') + \left(\frac{c_{iklm}}{\mu}\nabla_k\nabla_l + V_{im}(\mathbf{x})\right)G_{mj}(\mathbf{x},\mathbf{x}') = -\delta_{ij}\delta(\mathbf{x}-\mathbf{x}'). \tag{4.21}$$

Let us now calculate the mass operator Σ (also, this time, a two by two matrix) to order 2 in $V=\Sigma_{n=1}^N V^n$, as defined in (4.17). V^n is rewritten as (where superscript n for b and $\mathbf{X}_0$ are suppressed for clarity),

$$V^n(\mathbf{x},\omega) = \frac{\mu b^2}{M\omega^2}[\cos^2 2\theta\, A(\mathbf{x},\mathbf{X}_0) + \sin^2 2\theta\, B(\mathbf{x},\mathbf{X}_0) + \cos 2\theta \sin 2\theta\, C(\mathbf{x},\mathbf{X}_0)], \tag{4.22}$$

where

$$A(\mathbf{x},\mathbf{X}_0) = \begin{pmatrix} w_{22} & w_{21} \\ w_{12} & w_{11}\end{pmatrix}, \tag{4.23}$$

$$B(\mathbf{x},\mathbf{X}_0) = \begin{pmatrix} w_{11} & -w_{12} \\ -w_{21} & w_{22}\end{pmatrix}, \tag{4.24}$$

$$C(\mathbf{x},\mathbf{X}_0) = \begin{pmatrix} -w_{22}-w_{21} & -w_{11}+w_{22} \\ -w_{11}+w_{22} & w_{12}+w_{21}\end{pmatrix}, \tag{4.25}$$

with w_{ij} defined in (3.13).

As it was the case for screw dislocations, $\langle V\rangle(\mathbf{x})$ is a sum of N identical terms:

$$\langle V\rangle(k) = \frac{1}{\mathcal{V}}\int \rho_b db \frac{d\theta}{2\pi}\frac{d\mathbf{X}_0}{\mathcal{V}} d\mathbf{x}\, e^{-i\mathbf{k}.\mathbf{x}}\sum_n V^n(\mathbf{x})e^{i\mathbf{k}.\mathbf{x}} = n\int \rho_b db \frac{d\theta}{2\pi}\frac{d\mathbf{X}_0}{\mathcal{V}} d\mathbf{x}\, e^{-i\mathbf{k}.\mathbf{x}}V^1(\mathbf{x})e^{i\mathbf{k}.\mathbf{x}} \tag{4.26}$$

The calculation is simplified by use of the following procedure: In the integral defining $\langle V\rangle(k)$, we change the spatial coordinates from (O,x_1,x_2) to $(\mathbf{X}_0,\tilde{x}_1,\tilde{x}_2)$ corresponding to a translation of vector $\mathbf{X}_0$ and a rotation of angle θ. In this transformation, $\mathbf{k}=(k,\theta_k)$ becomes $\tilde{\mathbf{k}}=(k,\tilde{\theta}=\theta_k-\theta)$. Thus, $\langle V\rangle(k)$ takes the form

$$\langle V\rangle(k) = n\int \rho_b db \frac{d\tilde{\theta}}{2\pi} d\tilde{\mathbf{x}} e^{-i\tilde{\mathbf{k}}.\tilde{\mathbf{x}}}\tilde{V}^1(\tilde{\mathbf{x}})e^{i\tilde{\mathbf{k}}.\tilde{\mathbf{x}}}. \tag{4.27}$$

Using (4.22) we see that in the new coordinates V^1 becomes

$$\tilde{V}^1(\tilde{\mathbf{x}}) = \frac{\mu b^2}{M\omega^2}A(\tilde{\mathbf{x}},0), \tag{4.28}$$

and using (3.27) we get

$$\langle V\rangle(k) = -\frac{\mu}{2M^*\omega^2} n\langle b^2\rangle k^2 \begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}, \tag{4.29}$$

where M^* is defined as in (3.16).

The second order term $\langle VG^0V\rangle - \langle V\rangle G^0\langle V\rangle$ is calculated, as in the antiplane case, using $\langle VG^0V\rangle - \langle V\rangle G^0\langle V\rangle \simeq N\langle V^1G^0V^1\rangle$. With the same change of coordinates as in the previous paragraph, we get

$$\begin{aligned}\langle V^1G^0V^1\rangle(k) &= \frac{1}{\mathcal{V}}\int \rho_b db \frac{d\theta}{2\pi}\frac{d\mathbf{X}_0}{\mathcal{V}} d\mathbf{x}\, d\mathbf{x}' \times e^{-i\mathbf{k}.\mathbf{x}}V^1(\mathbf{x})G^0(\mathbf{x}-\mathbf{x}')V^1(\mathbf{x}')e^{i\mathbf{k}.\mathbf{x}'} \\ &= \frac{1}{\mathcal{V}}\int \rho_b db \frac{d\tilde{\theta}}{2\pi} d\tilde{\mathbf{x}} d\tilde{\mathbf{x}}' e^{-i\tilde{\mathbf{k}}.\tilde{\mathbf{x}}}\tilde{V}^1(\tilde{\mathbf{x}}) \times G^0(\tilde{\mathbf{x}}-\tilde{\mathbf{x}}')\tilde{V}^1(\tilde{\mathbf{x}}')e^{i\tilde{\mathbf{k}}.\tilde{\mathbf{x}}'}.\end{aligned} \tag{4.30}$$

The calculation of the matrix operators is tedious but straightforward, and uses the same procedure as for the antiplane case [each element is of the form (3.22)]. The final result in Fourier space is

$$\langle V^1G^0V^1\rangle(k) = \left(\frac{\mu}{4rM^*\omega^2}\right)^2 \frac{\langle b^4\rangle}{\mathcal{V}} k_\beta^2 k^2 \frac{1+\gamma^4}{\gamma^4}\left(i + \frac{\mathcal{C}}{k_\beta^2\langle b^2\rangle}\right) \times \begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}, \tag{4.31}$$

where r is defined as in (3.25). We finally obtain the mass operator $\Sigma(k)$,

$$\Sigma(k) = Sk^2\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}, \tag{4.32}$$

with

$$S = \frac{\mu n\langle b^2\rangle}{2M^*\omega^2}\left[-1 + \frac{\rho}{8r^2M^*}\frac{\langle b^4\rangle}{\langle b^2\rangle}\frac{1+\gamma^4}{\gamma^4}\left(i + \frac{\mathcal{C}}{k_\beta^2\langle b^2\rangle}\right)\right]. \tag{4.33}$$

The modified Green's function in Fourier space is given by (A4), with

$$G^0(k) = \frac{1}{\gamma^2(k^2-k_\alpha^2)(k^2-k_\beta^2)} \times \begin{pmatrix} k^2-k_\beta^2+(\gamma^2-1)k_2^2 & -(\gamma^2-1)k_1k_2 \\ -(\gamma^2-1)k_1k_2 & k^2-k_\beta^2+(\gamma^2-1)k_1^2\end{pmatrix}. \tag{4.34}$$

Using (A5) we have

$$\langle G\rangle^{-1}(k) = \begin{pmatrix} [1-S]k^2-k_\beta^2+(\gamma^2-1)k_1^2 & (\gamma^2-1)k_1k_2 \\ (\gamma^2-1)k_1k_2 & [1-S]k^2-k_\beta^2+(\gamma^2-1)k_2^2\end{pmatrix}. \tag{4.35}$$

The modified Green's function formalism is developed as previously and $\Sigma(k)$ is found, to second order in V, to be

$$\Sigma(k) = \frac{\mu n\langle b^2\rangle}{2M^*\omega^2}k^2\left[-1+\left(1+\frac{1}{\gamma^4}\right)\frac{\rho}{8r^2M^*}\frac{\langle b^4\rangle}{\langle b^2\rangle}\left(i+\frac{\mathcal{C}}{k_\beta^2\langle b^2\rangle}\right)\right]\times\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \quad (4.36)$$

where $\mathcal{C}$ is a numerical constant of order $1/\pi$; M^* is again a mean effective mass per unit length and r a numerical constant close to 1 and depending on the distribution function for b. The resulting modified wave numbers K_α and K_β correspond to the in-plane wave solutions of $\langle G\rangle$, given by the roots of the determinant of $G^0(k)^{-1}-\Sigma(k)$:

$$K_c = k_c\left\{1-\frac{\mu n\langle b^2\rangle\mathcal{A}_c}{4M^*\Omega^2}\left[1-\left(1+\frac{1}{\gamma^4}\right)\frac{\rho}{8r^2M^*}\frac{\langle b^4\rangle}{\langle b^2\rangle}\times\left(i+\frac{\mathcal{C}}{k_c^2\langle b^2\rangle}\right)\right]\right\}, \quad (4.37)$$

where

$$\mathcal{A}_\alpha = \beta^2/\alpha^2,$$

$$\mathcal{A}_\beta = 1. \quad (4.38)$$

The qualitative behavior of the effective wave number is the same as in the antiplane case; the effective phase velocities are increased, and the effective group velocities are decreased, from their values in the absence of scatterers. This result can be read off the leading order development of Sec. IV B, and it coincides with the result obtained using a simple-minded approach in Sec. III A. The imaginary part of the second order term in Sec. IV B gives the elastic mean free path for both waves:

$$\Lambda_\alpha = 16\frac{r^2M^{*2}\gamma^8}{n\rho^2\langle b^4\rangle\gamma^4+1}k_\alpha,$$

$$\Lambda_\beta = \frac{1}{\gamma^4}\Lambda_\alpha. \quad (4.39)$$

V. CONCLUSIONS

We have determined that elastic waves traveling through a two dimensional elastic material filled with randomly placed dislocations behave as an effective medium that allows the propagation of a coherent wave with effective velocities for longitudinal and transverse waves, and their respective attenuations. In the case of screw dislocations this leads to a scalar problem involving the antiplane shear wave. In the case of edge dislocations this leads to a vector problem involving the coupled in-plane shear and acoustical waves. In contrast to the mechanism usually studied in the multiple scattering of stress waves by static inhomogeneities or inclusions, the scattering mechanism considered in this paper involves a dynamic response by the dislocation that is responsible for the scattering. The effective medium approach allows us to determine modified wave numbers whose real part provides the change of wave speeds due to the presence of scatterers and whose imaginary part corresponds to the attenuation of the waves in the forward direction. The calculations have been performed in both cases, assuming that the scattering strength was small. This strength is measured by the potentials V appearing in the wave equations (3.8) and (4.16). Calculations at second order in V are then performed using a Green's function approach. Second order gives the attenuation length while first order gives the wave speed modification. This leading order behavior can also be obtained with a simpler, Foldy-Twersky, calculation.[17]

Calculations have been performed distinguishing edge and screw dislocation configurations. Many real materials (e.g., in silver) involve mixed dislocations whose Burgers vector is the sum of an edge and a screw dislocation Burgers vectors. Since all phenomena discussed in this paper are linear, the case of mixed dislocations can be simply obtained by superposition. Also, since we use continuum elasticity, there is no restriction on the value of the Burgers vector, which does not need to be a lattice vector and our results apply without change to partials.

While ultrasonic waves are routinely used in the nondestructive evaluation of materials because of their (well studied) interaction with flaws,[45–50] they do not appear to have been considered as probes to explore the characteristics of phenomena, such as the brittle-to-ductile transition, where dislocations are believed to play a prominent role. A very recent publication, however,[14] describes an experiment of ultrasound propagation during fatigue of pearlitic rail steel, showing that the attenuation and velocity of ultrasound are very sensitive to the presence of dislocations while they appear to be unaffected by the onset and growth of microcracks. Such experiments and the calculations presented in the present paper work suggest this may be a fruitful road to undertake. Work along these lines is in progress.

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APPENDIX A: THE MODIFIED GREEN'S FUNCTION FORMALISM

In the medium in the absence of any scatterer, let G^0 denote the usual Green's function characterized by the linear operator $\mathcal{L}$,

$$\mathcal{L}(\mathbf{x},\omega)G^0(\mathbf{x}-\mathbf{x}',\omega) = -\delta(\mathbf{x}-\mathbf{x}'). \quad (A1)$$

In the presence of scatterers, let G now designate the Green's function associated with the modified operator $\mathcal{L}+V$:

$$\mathcal{L}(\mathbf{x},\omega)G(\mathbf{x},\mathbf{x}',\omega) = -\delta(\mathbf{x}-\mathbf{x}') - V(\mathbf{x})G(\mathbf{x},\mathbf{x}',\omega). \quad (A2)$$

The modified Green's function can be expressed as

$$G(\mathbf{x},\mathbf{x}') = G^0(\mathbf{x}-\mathbf{x}') + \int d\mathbf{x}'' G^0(\mathbf{x}-\mathbf{x}'')V(\mathbf{x}'')G(\mathbf{x}'',\mathbf{x}'), \tag{A3}$$

formally written $G=G^0+G^0VG$. The scattering matrix $T=V+VG^0V+VG^0VG^0V+\cdots$ is introduced and verifies $G=G^0+G^0TG^0$. The integral representation of the scattering matrix is $T=V+VG^0T$.

Consider now the averaged Green's function $\langle G\rangle$ defined as the impulse response of the effective medium, defined as the average of the media over all realizations of V. We obtain $\langle G\rangle=G^0+G^0\langle VG\rangle$, which can be written as a Dyson equation,

$$\langle G\rangle = G^0 + G^0\Sigma\langle G\rangle, \tag{A4}$$

where $\Sigma=\langle T\rangle-\langle T\rangle G^0\Sigma$ is the mass operator. At this point, the averaged Green's function $\langle G\rangle$ can be determined simply by solving the algebraic equation (A4) in Fourier space. Formally this is written as

$$\langle G\rangle = ((G^0)^{-1}-\Sigma)^{-1} \tag{A5}$$

and the coherent wave vector is given by the pole (or poles) of $\langle G\rangle$. In a general case, $\langle T\rangle$ is hard or even impossible to determine. In our case, we consider V as a small correction of the operator of free propagation, and thus we have

$$\langle T\rangle = \langle V\rangle, \quad \text{at order } 1,$$

$$\langle T\rangle = \langle V\rangle + \langle VG^0V\rangle, \quad \text{at order } 2. \tag{A6}$$

At second order in V, Σ takes the form

$$\Sigma = \langle V\rangle + \langle VG^0V\rangle - \langle V\rangle G^0\langle V\rangle. \tag{A7}$$

We can now calculate the mass operator $\Sigma(\mathbf{k})$ in the Fourier space to solve (A5). Strictly speaking, (A5) gives $\Sigma(\mathbf{k},\mathbf{k}')$. Using the invariance under translations of both Green's functions $\langle G\rangle$ and G^0, we have, in the Fourier space, for $f=\langle G\rangle$ or G^0: $f(\mathbf{k},\mathbf{k}')=f(\mathbf{k})\delta(\mathbf{k}-\mathbf{k}')$. Equation (A5) becomes

$$\langle G\rangle(\mathbf{k}) = G^0(\mathbf{k}) + \int d\mathbf{k}'\ \Sigma(\mathbf{k},\mathbf{k}')\langle G\rangle(\mathbf{k}'), \tag{A8}$$

with

$$\Sigma(\mathbf{k},\mathbf{k}') = \int d\mathbf{x}\ d\mathbf{x}'\ e^{-i\mathbf{k}.\mathbf{x}}\Sigma(\mathbf{x},\mathbf{x}')e^{i\mathbf{k}'.\mathbf{x}'}. \tag{A9}$$

Note that this transformation, similar but not identical to the Fourier transform of a function of two vector variables, does not have (2π) factors in its definition. We use now the invariance under the translation of $\Sigma(\mathbf{x},\mathbf{x}')$, in the sense that $\Sigma(\mathbf{x},\mathbf{x}')=\Sigma(\mathbf{u},\mathbf{u}')$ if $(\mathbf{x}-\mathbf{x}')=(\mathbf{u}-\mathbf{u}')$. Thus, we obtain

$$\Sigma(\mathbf{k},\mathbf{k}') = \delta(\mathbf{k}-\mathbf{k}')\Sigma(\mathbf{k}), \tag{A10}$$

with

$$\Sigma(\mathbf{k}) = \frac{1}{\mathcal{V}}\int d\mathbf{u}\ d\mathbf{u}'\ e^{-i\mathbf{k}.\mathbf{u}}\Sigma(\mathbf{u},\mathbf{u}')e^{i\mathbf{k}.\mathbf{u}'}. \tag{A11}$$

In coordinate space (A7) reads as

$$\Sigma(\mathbf{x},\mathbf{x}') = \langle V(\mathbf{x})\rangle\delta(\mathbf{x}-\mathbf{x}') + \langle V(\mathbf{x})G^0(\mathbf{x}-\mathbf{x}')V(\mathbf{x}')\rangle - \langle V(\mathbf{x})\rangle G^0(\mathbf{x}-\mathbf{x}')\langle V(\mathbf{x}')\rangle. \tag{A12}$$

As already mentioned, the coherent wave number K, corresponding to a wave propagation in the form $e^{i\mathbf{K}.\mathbf{x}}$, is simply determined by the poles of $\langle G\rangle$. In the antiplane case, (A4) is a scalar equation and we have to determine the pole of $\langle G\rangle$ while in the in-plane case, (A4) has a matrix structure and the task is to calculate the root of the determinant of $\langle G\rangle$.

APPENDIX B: SCATTERING FUNCTION AND FOLDY'S APPROACH, THE ANTIPLANE CASE

1. Foldy's approach

Before presenting the calculations using Foldy's approach, we summarize here the analogy of this approach with the usual potential approach we have used until now. The integral representation,

$$G = G^0 + \sum_i G^0V^iG, \tag{B1}$$

can be written for the velocity field $v(\mathbf{x})=v^{inc}(\mathbf{x})+\sum_i\int d\mathbf{x}'\ G^0(\mathbf{x}-\mathbf{x}')V^i(\mathbf{x}')v(\mathbf{x}')$. In the simplest case of isotropic punctual scatterers located in $\mathbf{X}^\mathbf{i}$, the operator V^i can be written as $V^i(\mathbf{x})=V^i\delta(\mathbf{x}-\mathbf{X}^i)$. Then, with $H(\mathbf{x},\mathbf{X}^i)=V^iG^0(\mathbf{x}-\mathbf{X}^i)$, the integral representation takes the form

$$v(\mathbf{x}) = v^{inc}(\mathbf{x}) + \sum_i H(\mathbf{x},\mathbf{X}^i)v(\mathbf{X}^i). \tag{B2}$$

However, this relation is hard to exploit since there is no physical interpretation of $V^i(\mathbf{x})$, notably, it is not related to the operator of scattering for a unique scatterer, denoted T^i in the following. Moreover, (B2) makes appear a term of self-irradiation $H(\mathbf{x},\mathbf{X}^i)H(\mathbf{X}^i,\mathbf{X}^i)$ (or, equivalently, the integral representation of G makes appear a term $G^0V^iG^0V^i$). Foldy[17] chose an alternative way where the operator T^i for a single scatterer is introduced. From Ref. 20, T^i satisfies the integral representation $G^i=G^0+G^0T^iG^0$ for one scatterer and for a set of scatterers,

$$G = G^0 + \sum_i G^0Q^iG^0, \tag{B3}$$

where Q^i can be related to T^i through $Q^i=T^i+T^iG^0\sum_{j\neq i}Q^j$. Note that $\sum_i Q_i$ denotes the exact total scattering operator including all the multiple scattering. Finally, we have $G=G^0+\sum_i G^0T^iG^0+\sum_i G^0T^i\sum_{j\neq i}G^0T^jG^0+\cdots$. We recognize in the first sum the single scattering, in the second the double scattering, etc., $\cdots$. Again, we suppose the simplest case of isotropic δ-function potentials $T^i(\mathbf{x})=T^i\delta(\mathbf{x}-\mathbf{X}^i)$. Introducing $F(\mathbf{x},\mathbf{X}^i)=T^iG^0(\mathbf{x}-\mathbf{X}^i)$, the representation for the velocity field is thus

$$v(\mathbf{x}) = v^{inc}(\mathbf{x}) + \sum_i F(\mathbf{x},\mathbf{X}^i)v^{inc}(\mathbf{X}^i) + \sum_i\sum_{j\neq i} F(\mathbf{x},\mathbf{X}^i)F(\mathbf{X}^i,\mathbf{X}^j)v^{inc}(\mathbf{X}^j) + \cdots. \tag{B4}$$

This is equivalent to the usual notation in Foldy's formalism,

$$v(\mathbf{x}) = v^{inc}(\mathbf{x}) + \sum_i F(\mathbf{x},\mathbf{X}^i)v^i(\mathbf{X}^i),$$

$$v^i(\mathbf{x}) = v^{inc}(\mathbf{x}) + \sum_{j\neq i} F(\mathbf{x},\mathbf{X}^j)v^j(\mathbf{X}^j). \tag{B5}$$

At this point, it is clear that $F(\mathbf{x},\mathbf{X}^i)$ is directly proportional to the Green function $G^0(\mathbf{x}-\mathbf{X}^i)$ when punctual isotropic scatterers are considered. If the scatterer is punctual but not isotropic, we have to account for the direction of the incident wave: nonisotropic means that the response of the scatterer depends on the direction of the wave incident on the scatterer[16] using $F(\mathbf{x},\mathbf{X}^i)=T^i(\mathbf{k}^{\mathbf{i}})G^0(\mathbf{x}-\mathbf{X}^i)$, where $\mathbf{k}^{inc}$ is a unitary vector indicating the direction of the incident wave.

2. Calculation of the modified wavenumber, the antiplane case

For an incident wave of unit amplitude, $v^{inc}(\mathbf{x},t)=e^{ikx_1-i\omega t}$, it has been established in a previous paper[44] that the scattered wave in polar coordinates $\mathbf{x}=(x,\theta_x)$, far from the scatterer reads as

$$v^s(x,\theta_x,t) = Af(\theta_x)\frac{e^{ikx-i\omega t}}{\sqrt{x}}, \tag{B6}$$

with f as the scattering function,

$$f(\theta_x) = -\frac{\mu b^2}{2M}\frac{e^{i\pi/4}}{\sqrt{2\pi\omega}\beta^{3/2}}\cos\,\theta_x. \tag{B7}$$

We adopt the usual Foldy equations (B4) averaged over all configurations of scatterers (for the sake of clarity, the average over b is omitted),

$$\langle v\rangle(\mathbf{x}) = v^{inc}(\mathbf{x}) + \int \frac{d\mathbf{X}^1\cdots d\mathbf{X}^N}{\mathcal{V}^N}\sum_{i=1}^N F(\mathbf{x},\mathbf{X}^i)v^i(\mathbf{X}^i)=v^{inc}(\mathbf{x}) + \frac{N}{\mathcal{V}}\int d\mathbf{X}^1\ F(\mathbf{x},\mathbf{X}^1)v^{inc}(\mathbf{X}^1)$$
$$+\frac{N(N-1)}{\mathcal{V}^2}\int d\mathbf{X}^1\ d\mathbf{X}^2 F(\mathbf{x},\mathbf{X}^1)F(\mathbf{X}^1,\mathbf{X}^2)v^{inc}(\mathbf{X}^2)$$
$$+\frac{N(N-1)(N-2)}{\mathcal{V}^3}\int d\mathbf{X}^1\ d\mathbf{X}^2\ d\mathbf{X}^3 F(\mathbf{x},\mathbf{X}^1)F(\mathbf{X}^1,\mathbf{X}^2)F(\mathbf{X}^2,\mathbf{X}^3)v^{inc}(\mathbf{X}^3)$$
$$+\frac{N(N-1)}{\mathcal{V}^2}\int d\mathbf{X}^1\ d\mathbf{X}^2 F(\mathbf{x},\mathbf{X}^1)F(\mathbf{X}^1,\mathbf{X}^2)F(\mathbf{X}^2,\mathbf{X}^1)v^{inc}(\mathbf{X}^1) + \cdots. \tag{B8}$$

The last integral and all others involving a scattering process that goes through the same scatterer more than once are neglected. Thus, for large value of N, one gets, with $n=N/\mathcal{V}$

$$\langle v\rangle(\mathbf{x}) = v^{inc}(\mathbf{x}) + n\int d\mathbf{X}^1\ F(\mathbf{x},\mathbf{X}^1)\Big(v^{inc}(\mathbf{X}^1) + n\int d\mathbf{X}^2\ F(\mathbf{X}^1,\mathbf{X}^2)(v^{inc}(\mathbf{X}^2)+\cdots\Big)$$
$$=v^{inc}(\mathbf{x}) + n\int d\mathbf{X}\ F(\mathbf{x},\mathbf{X})\langle v\rangle(\mathbf{X}). \tag{B9}$$

Looking for a solution as a plane wave $\langle v\rangle(\mathbf{x})=V_0e^{iKx_1-i\Omega t}$ for an incident plane wave $e^{ikx_1-i\Omega t}$, F is identified to the response of a unique scatterer,

$$F(\mathbf{x},\mathbf{X}) = f(\theta_x)\frac{e^{ik|\mathbf{x}-\mathbf{X}|}}{\sqrt{|\mathbf{x}-\mathbf{X}|}}. \tag{B10}$$

The integral can be now calculated on a slab of infinite size along x_2 and of width x_1, using

$$\int_0^{x_1}\int d\mathbf{X}\ e^{iK\ X_1}\langle f\rangle_b(\theta_x)\frac{e^{ik|\mathbf{x}-\mathbf{X}|}}{\sqrt{|\mathbf{x}-\mathbf{X}|}} = \frac{2\pi}{k}\langle f\rangle_b(0)e^{-i\pi/4}\frac{(e^{iKx_1}-e^{ikx_1})}{K-k}, \tag{B11}$$

where $\langle f\rangle_b$ indicates the average over b. Equation (B9) can be solved to find V_0 and K. This leads to $V_0=1$ and

$$K = k + n\sqrt{\frac{2\pi}{k}}\langle f\rangle_b(0)e^{-i\pi/4},$$
$$=k\left(1-\frac{\mu n\langle b^2\rangle}{2M^*\Omega^2}\right), \tag{B12}$$

where M^* is the mean square root of M [M^* appears as in (3.16) taking the mean value of f over b].

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