About identification of defects in an elastic-plastic medium from boundary measurements in the antiplane case

L. Bourgeois[†], J. Dardé[†*]

[†] Laboratoire POEMS, 32, Boulevard Victor, 75739 Paris Cedex 15, France
[*] Laboratoire J.-L. Lions, Université Pierre et Marie Curie,
175, Rue du Chevaleret, 75252 Paris Cedex 05, France

Abstract

This paper is devoted to a new method to find defects in some elastic-plastic structure in the antiplane case from overdeterminated data on a subpart of its boundary. This iterative method merges the method of quasi-reversibility and a simple level set technique. The foundations of the method are justified from a theoretical point of view and its efficiency is shown with the help of numerical experiments.

1 Introduction

In the field of mechanics of materials, the problem of finding defects within some structure from measurements of both displacements and tractions on a subpart of the boundary of such structure is a classical inverse problem. Many authors have considered such problem in the simple case of elasticity (namely the behaviour of the medium is governed by Hooke's law), in particular isotropic elasticity. The reader will find in the review article [1] a large amount of contributions concerning identification of obstacles or cracks in elasticity. Among the most popular techniques that are used in this context, some are based for example on the reciprocity gap, the topological derivative or the linear sampling method. Linearity of the constitutive law is an essential ingredient of these techniques, and introducing some non-linearity in the constitutive law would make them fail.
Introducing some non-linearity is precisely what we do in the present contribution, which is to our knowledge one of the first step towards identification of defects for elastic-plastic constitutive law, though some ideas contained in our paper were already sketched in [19] and in [2]. The authors would like to emphasize the importance of elastic-plastic constitutive law in view of applications. If for example we think about the problem of finding cracks in a metallic structure, it is well known that the elasticity model is not exact in the vicinity of the crack tips. Actually, the elasticity model predicts that the stress field is singular near the crack tips, but physics says that in reality some plastic strains are created at points where the stress field reaches a certain threshold, the so-called yield criterion, so that such threshold cannot be exceeded. Hence the elastic-plastic regime, which is non-linear, should be considered instead of the elastic regime near the

crack tips. The reader will find in [19] a description of various material constitutive laws we have to deal with in real life.
Unfortunately, the elastic-plastic law raises many issues, as exposed in [2]. First of all, the forward problem is governed by evolution partial differential equations, while the problem is stationary for elasticity. Secondly, plastic strains are given by a differential inclusion, which implies that the constitutive law is not sufficiently smooth to be differentiated. In a view to transform the inverse problem into an optimal control problem, this means that the gradient of the cost function cannot be computed analytically. That the constitutive law is non-smooth also makes the solution of the forward problem not easy to compute, which is again a shortcoming in view of the optimal control problem. Some attempt to identify plastic zones from boundary measurements was done in [25], but the yield criterion was not used in the method, which obliged the authors to admit very strong a priori assumptions, for example the a priori knowledge of the (small) support of plastic strains. This is one of the novelty of our paper to take benefit of the yield criterion in order to get rid of such a priori assumptions.
The main idea of our contribution is the following: since the physical presence of a defect creates some plastic strains in the vicinity of the defect, we try to identify the plastic zone instead of the defect itself, by using the yield criterion. To do so we assume that measurements take place at the end of a so-called increasing loading, which amounts to say that the medium is divided into a region denoted the elastic zone where the yield criterion is not reached and which is free of plastic strains and a region denoted the plastic zone where the yield criterion is satisfied and plastic strains do exist. The elastic zone is bounded from outside by the boundary of the structure and from inside by the boundary of the plastic zone. The displacement field is given by the traditional Navier equation in the elastic zone, which gives us the opportunity to identify the plastic zone "from outside": we solve the ill-posed Cauchy problem for the Navier equation in the elastic zone from our overspecified boundary data in an increasing domain, the growth of which we stop when the obtained solution fulfills the yield criterion. For sake of simplicity, we only consider defects that are characterized by Dirichlet boundary condition and we content ourselves with considering the antiplane case, which strongly simplifies the equations, in particular the three dimensional problem is replaced by a two dimensional one and the Navier equation becomes the Laplace equation.
Concerning now our identification technique itself, we use an iterative approach which merges:

1. the method of quasi-reversibility in order to update an approximate solution of the Cauchy problem outside some given test subdomain which can be viewed as a current plastic zone,

2. a level set method in order to update such current plastic zone from our approximate solution.

In step (1), the method of quasi-reversibility [18] consists in transforming the initial second order ill-posed equation into a family of fourth order well-posed equations that depend on a small parameter. In step (2), our level set method consists in solving in the current plastic zone a Laplace equation with boundary condition based on the approximate displacement field outside the plastic zone, which is a simple alternative method to traditional methods based on eikonal equation. The same technique has been already successfully used in [5] in order to solve the inverse obstacle problem with Dirichlet condition.

The outline of our paper is the following. The second section is a brief description of the elastic-plastic constitutive law in the antiplane case: it is essentially intended to readers who are not familiar with mechanics of materials. The mathematical setting of the problem that results from section two is given in the third section. In section four we describe the iterative approach we use to solve the inverse problem and the corresponding justifications, while numerical experiments are given in section 5.

2 The elastic-plastic law in the antiplane case

The antiplane case is characterized by a displacement field u given by $u = w(x_1, x_2)e_3$ in a system of coordinates (x_1, x_2, x_3). As a result, the strain field $\varepsilon = (\nabla u + \nabla^t u)/2$ is given by

$$\varepsilon = \frac{1}{2}\frac{\partial w}{\partial x_1}(e_1 \otimes e_3 + e_3 \otimes e_1) + \frac{1}{2}\frac{\partial w}{\partial x_2}(e_2 \otimes e_3 + e_3 \otimes e_2).$$

In order to shorten notations, we simply view ε has a vector of $\mathbb{R}^2$ the components of which are the two non-vanishing components of the three dimensional above tensor, namely $(\varepsilon_{13}, \varepsilon_{23})$. The standard scalar product of two dimensional vector space is denoted $(.,.)$, the norm is denoted $|.|$. Using these notations, the above equation simply writes

$$\varepsilon = \frac{1}{2}\nabla w. \tag{2.1}$$

As mentioned in the introduction, the elastic-plastic law leads to an evolution problem, hence the different fields defined in this section depend also on time t in addition to the spatial point $x = (x_1, x_2)$. Again to shorten notations we denote by $\dot{a}$ the partial derivative of function a with respect to time t. In the case of elasticity, the constitutive law has the linear expression

$$\dot{\sigma} = 2\mu\dot{\varepsilon},$$

where σ is the stress field and $\mu > 0$ is the shear modulus. In the case of elasto-plasticity with kinematical hardening, we have to introduce the plastic strain ε^p such that the constitutive law writes

$$\dot{\sigma} = 2\mu(\dot{\varepsilon} - \dot{\varepsilon}^p) \tag{2.2}$$

$$\dot{\varepsilon}^p \in \partial I_K(A), \tag{2.3}$$

where $A = \sigma - H\varepsilon^p$, K is the closed ball $\{A \in \mathbb{R}^2,\ |A| \leq k\}$ and coincide with the Von Mises convex set of plasticity in the particular antiplane case, I_K is the indicator function of set K, and $\partial\phi(A)$ is the subdifferential of function ϕ at point A. Lastly, $k > 0$ and $H > 0$ denote the shear yield strength and the modulus of kinematical hardening respectively. Our elastic-plastic law is hence governed by 3 material constants, which are

supposed to be known in the sequel: μ, k and H.
Equation (2.3) is the so-called flow rule and consists of a differential inclusion. We prove by using the definition of the subdifferential that (2.3) is equivalent to

$$\begin{cases} & |A| \leq k & & \\ \text{if} & |A| < k & \text{then} & \dot{\varepsilon}^p = 0 \\ \text{if} & |A| = k & \text{then} & \dot{\varepsilon}^p = \lambda A, \quad \lambda \geq 0. \end{cases}$$

The above equations have to be complemented by the equilibrium equation, which in the absence of body forces is expressed as

$$\text{div}\sigma = 0. \tag{2.4}$$

From the above definitions, we are in a position to derive the pointwise equations that govern a forward evolution problem in elasto-plasticity in some interval of time (t_0, t_f). We assume that the medium occupies the domain $\Omega \subset \mathbb{R}^2$, the boundary of Ω is divided into a subpart Γ_u on which the displacement is prescribed and a subpart Γ_T on which the traction is prescribed. We denote by n the outward unit normal. From equations (2.1)-(2.4), the volumic equations in $\Omega \times (t_0, t_f)$ are given by

$$\begin{cases} \dot{\varepsilon} = \frac{1}{2}\nabla\dot{w} \\ \text{div}\dot{\sigma} = 0 \\ \dot{\sigma} = 2\mu(\dot{\varepsilon} - \dot{\varepsilon}^p) \\ \dot{\varepsilon}^p \in \partial I_K(\sigma - H\varepsilon^p), \end{cases} \tag{2.5}$$

the boundary conditions are governed on $\partial\Omega \times (t_0, t_f)$ by

$$\begin{cases} \dot{w}|_{\Gamma_u} = 0 \\ \dot{\sigma}.n|_{\Gamma_T} = \dot{T}, \end{cases} \tag{2.6}$$

and lastly the initial condition at t_0 is characterized by vanishing values of all fields w, ε, σ and ε^p.
The analysis of weak formulations of the above forward problem in appropriate functional spaces is well understood. The case of perfectly plastic materials, namely $H = 0$, was treated in [20, 22]. Note that in this case, one obtains an existence result for σ and w, but uniqueness holds only for w. The case of materials with kinematical hardening, namely $H > 0$, is treated in [23, 24] and is well-posed: this is why we consider this particular case in our paper.
From the flow rule (2.3), we obtain that at points x such that for all $t \in [t_0, t_f]$, $|\sigma(x,t)| < k$, then for all $t \in [t_0, t_f]$, $\varepsilon^p(x,t) = 0$, in particular $\varepsilon^p(x, t_f) = 0$ and $|A(x, t_f)| < k$. The set of such points x will be called the *elastic zone* $\mathcal{E}$ in the following, because the elastic regime applies to that points during the whole interval of time. The points x such that $\varepsilon^p(x, t_f) \neq 0$ form a set we call the *plastic zone* $\mathcal{P}$, because the elastic-plastic regime applies to that points during a certain interval of time. Now we introduce our increasing loading assumption, which can be expressed as

Assumption (H0) (increasing loading):

$$\forall x \in \overline{\Omega}, \quad \varepsilon^p(x, t_f) = 0 \Leftrightarrow |A(x, t_f)| < k.$$

From assumption $(H0)$ and flow rule (2.3), at point x such that $\varepsilon^p(x, t_f) \neq 0$ we have $|A(x, t_f)| = k$. Roughly speaking, the assumption $(H0)$ amounts to suppose that if the material starts deforming plastically at point x at time t_s, then it goes on deforming plastically at x during the rest of the interval of time $[t_s, t_f]$. This is what we observe numerically for reasonable geometries Ω by prescribing a proportional increasing traction T in (2.6), that is

$$T(x,t) = T_0(x)\Lambda(t), \quad \dot{\Lambda} > 0.$$

Now we integrate the forward problem (2.5) (2.6) over the interval of time $[t_0, t_f]$ and use the assumption $(H0)$ to obtain a problem satisfied by w at final time t_f. We obtain the following equation in Ω for function $u := w(., t_f)$:

$$\Delta u = 2\mathrm{div}(\varepsilon^p) \quad \text{in} \quad \Omega \tag{2.7}$$

with boundary conditions

$$\begin{cases} u|_{\Gamma_u} = 0 \\ \dfrac{\partial u}{\partial n}|_{\Gamma_T} = \dfrac{T}{\mu} + 2\varepsilon^p.n, \end{cases} \tag{2.8}$$

and the additional proprieties

$$\begin{cases} |\nabla u| < \dfrac{k}{\mu} & \text{in} \quad \mathcal{E} \\ |\nabla u - \dfrac{H + 2\mu}{\mu}\varepsilon^p| = \dfrac{k}{\mu} & \text{in} \quad \mathcal{P}. \end{cases} \tag{2.9}$$

From (2.7), we obtain that $\Delta u = 0$ in the elastic zone $\mathcal{E}$. Since the plastic zone develop near the defects, which lie within the structure, it is reasonable to assume

Assumption (H1) (confined plasticity):

$$\mathrm{supp}\,(\varepsilon^p(., t_f)) \Subset \mathcal{D}.$$

This implies in particular that $\varepsilon^p = 0$ on Γ_T in boundary conditions (2.8). Furthermore, by assuming some continuity for ε^p, we obtain from (2.9) that

$$|\nabla u| = \frac{k}{\mu} \quad \text{on} \quad \partial\mathcal{E} \cap \partial\mathcal{P}. \tag{2.10}$$

Such identity, which relies on the increasing loading assumption (H0), will enable us to identify the boundary of the plastic zone from measurements in the following section.

3 The mathematical setting of the inverse problem

We assume that $\mathcal{D}$ is an open, bounded, connected domain of $\mathbb{R}^2$ with Lipschitz boundary. We consider then a set $\mathcal{O} \Subset \mathcal{D}$ which may be either a Lipschitz open domain or a set of connected curves such that $\Omega := \mathcal{D} \setminus \overline{\mathcal{O}}$ is connected. The domains Ω and $\mathcal{O}$ are referred to as the structure and the defect respectively.
We consider a forward problem as described in the previous section by equations (2.5) (2.6) and trivial initial conditions, in the particular case $\Gamma_u = \partial\mathcal{O}$ and $\Gamma_T = \partial\mathcal{D}$. This means that displacement is fixed to 0 on the boundary of the defect and tractions T are prescribed on the exterior boundary of the structure. Assuming (H0) and (H1), that is the increasing loading assumption and the confined plasticity assumption, we now consider the structure Ω at final time t_f, which is our observation time. In particular, we assume that some measurements of the displacement field are available on a non-empty open part Γ of $\partial\mathcal{D}$, so that we know a pair of Cauchy data $(u|_\Gamma, \partial_n u|_\Gamma) = (u_0, g_0)$ on Γ. In order to guarantee uniqueness for the inverse problem, we define

$$\mathcal{P}_e = \mathcal{D} \setminus \overline{NB(\mathcal{D} \setminus \overline{\mathcal{P}})},$$

where $NB(\omega)$ denotes the connected component of ω which is in contact with Γ and we add the following

Assumption (H2) (invisibility of the obstacle): $\mathcal{O} \subset \mathcal{P}_e$.

This assumption amounts to consider that the plastic zone $\mathcal{P}$ completely surrounds the obstacle $\mathcal{O}$. In figure 1, some examples of admissible situation and non-admissible situation with respect to assumption (H2) are presented. By definition, $\partial\mathcal{P}_e$ is the exterior boundary of the plastic zone, and $\mathcal{P}_e$ will be called the exterior plastic zone in the following. It contains the plastic zone $\mathcal{P}$, the defect $\mathcal{O}$ and possibly some part of the elastic zone $\mathcal{E}$, like in the second configuration presented on figure 1. The set $\mathcal{D}\setminus\overline{\mathcal{P}_e}$ is connected and fully contained in the elastic zone $\mathcal{E}$.
The assumption (H2) together with equation (2.10) imply that

$$|\nabla u| = \frac{k}{\mu} \quad \text{on} \quad \partial\mathcal{P}_e.$$

The inverse problem we introduce now consists in finding the exterior plastic zone $\mathcal{P}_e$ from the Cauchy data (u_0, g_0) on Γ, provided assumptions (H0), (H1) and (H2) are satisfied. By using the conclusions of the previous section with $\mu = k = 1$, the inverse problem may be expressed as follows: find an open domain $P \Subset \mathcal{D}$ with Lipschitz boundary such that $\mathcal{D} \setminus \overline{P}$ is connected and a function $u \in C^1(\overline{D} \setminus P)$ such that:

$$\begin{cases} \Delta u = 0 & \text{in } \mathcal{D} \setminus \overline{P} \\ u = u_0 & \text{on } \Gamma \\ \partial_n u = g_0 & \text{on } \Gamma \\ |\nabla u| < 1 & \text{in } \mathcal{D} \setminus \overline{P} \\ |\nabla u| = 1 & \text{on } \partial P. \end{cases} \tag{3.1}$$

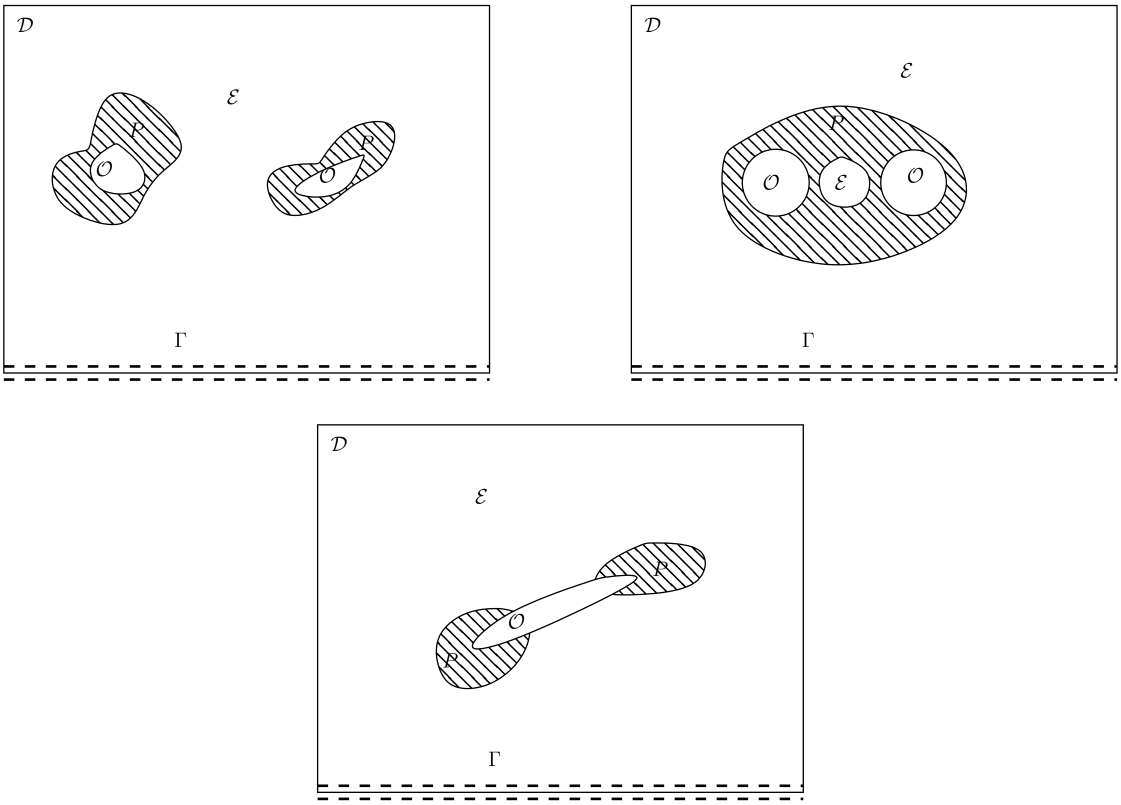

Figure 1: Top figures: admissible geometry. Bottom figure: non-admissible geometry.

We have the following uniqueness result.

Proposition 3.1 *The domain P and the function u that satisfy (3.1) are uniquely defined by data (u_0, g_0).*

PROOF. We assume that two domains P_1 and P_2 satisfy (3.1) with corresponding functions u_1 and u_2. Let denote $\tilde{P}$ the connected component of $\mathcal{D} \setminus (\overline{P_1} \cup \overline{P_2})$ which in contact with Γ. That P_1 is different from P_2 implies for example that $P_1 \not\subset P_2$, then there exists $x \in \partial\tilde{P} \cap \partial P_1$ with $x \notin \overline{P_2}$. Here we have used the fact that $\mathcal{D} \setminus \overline{P_2}$ is connected. Since $x \in \partial\tilde{P}$, unique continuation in the connected domain $\tilde{P}$ for function $u_1 - u_2$ and regularity of functions u_1 and u_2 imply that $|\nabla u_1(x)| = |\nabla u_2(x)|$. Since $x \in \partial P_1$ we have $|\nabla u_1(x)| = 1$, while since $x \notin \overline{P_2}$, we have $|\nabla u_2(x)| < 1$, which is a contradiction. It follows that $P_1 = P_2 = P$. By unique continuation, we have $u_1 = u_2$ in $\mathcal{D} \setminus \overline{P}$. ■

Such uniqueness result allows us to consider the problem of computing the pair (P, u) from the Cauchy data (u_0, g_0) on Γ, which is the aim of the following section.

Remark 3.2 *It should be noted that owing to the assumption (H2) concerning invisibility of the obstacle, the condition on the boundary of the obstacle $\mathcal{O}$ plays no role in problem (3.1). We could have considered any other boundary condition on $\partial\mathcal{O}$, for example Neumann condition.*

4 An iterative method to solve the inverse problem

We adapt the approach introduced in [5], which concerned an inverse problem involving an obstacle characterized by a Dirichlet condition $u = 0$, to the case of an obstacle characterized by the condition $|\nabla u| = 1$. Our approach consists in constructing iteratively a current domain P_m and the displacement field u_m in $\mathcal{D} \setminus \overline{P_m}$ by merging two different techniques (quasi-reversibility and level set techniques) as follows:

- Choose an initial guess P_0
- Step 1: find an approximation u_m of the displacement field in the current domain $\mathcal{D} \setminus \overline{P_m}$ from the Cauchy data on Γ by using the method of quasi-reversibility
- Step 2: update the current domain P_m by using an original level set technique based on the velocity $V_m = 1 - |\nabla u_m|^2$
- Iterate steps 1 and 2 until a certain stopping criterion is reached

Under some suitable assumptions, (P_m, u_m) provides a very good approximation of (P, u) for appropriate topology when m goes to infinity.
We first describe step 1 (method of quasi-reversibility), then describe step 2 (level set method).

4.1 The method of quasi-reversibility

Step 1 consists in solving the Cauchy problem for the Laplace equation in $\mathcal{D} \setminus \overline{P_m}$ from the Cauchy data (u_0, g_0) on Γ. In order to introduce such problem, let (P, u) be the solution to problem (3.1) with $u \in H^2(\mathcal{D} \setminus \overline{P}) \cap C^1(\overline{\mathcal{D}} \setminus P)$. We denote $\hat{P}$ an open domain such that $P \subset \hat{P} \Subset D$ and $\hat{E} := \mathcal{D} \setminus \overline{\hat{P}}$ is connected. The domain $\hat{P}$ could be any current domain P_m in step 1. In view of problem (3.1), the function u solves the following Cauchy problem:

$$\begin{cases} \Delta u = 0 & \text{in } \hat{E} \\ u = u_0 & \text{on } \Gamma \\ \partial_n u = g_0 & \text{on } \Gamma. \end{cases} \tag{4.1}$$

Such problem is well-known to be ill-posed in the sense of Hadamard: we have uniqueness but existence is not guaranteed for any Cauchy data (u_0, g_0), which also implies that some perturbation on the data may produce some large error on the solution. As a result, problem (4.1) requires a regularization technique. The method of quasi-reversibility is one of them and consists, following [18], in transforming the second order ill-posed equation (4.1) into a family of fourth order well-posed equations that depend on a small parameter $\varepsilon > 0$. We introduce the following sets

$$V = \{v \in H^2(\hat{E}) |\, v = u_0,\ \partial_n v = g_0 \text{ on } \Gamma\}$$

$$V_0 = \{v \in H^2(\hat{E}) |\, v = 0,\ \partial_n v = 0 \text{ on } \Gamma\}.$$

It can be easily proved that V_0, endowed with the classical scalar product of $H^2(\hat{E})$, is a Hilbert space. We introduce the following variational formulation of quasi-reversibility: find $u_\varepsilon \in V$ such that $\forall v \in V_0$, we have

$$(\Delta u_\varepsilon, \Delta v)_{L^2(\hat{E})} + \varepsilon (u_\varepsilon, v)_{H^2(\hat{E})} = 0. \tag{4.2}$$

The following proposition, which is a slightly different version of the one presented in [3], provides the justification of the method of quasi-reversibility.

Proposition 4.1 *The problem (4.2) has a unique solution $u_\varepsilon \in V$, such that $||u_\varepsilon - u||_{H^2(\hat{E})} \to 0$ when $\varepsilon \to 0$.*

Some alternative formulations of quasi-reversibility may be found in [3, 4, 7]. In practice, the Cauchy data (u_0, g_0) on Γ are not known exactly because they result from measurements. The only available data are noisy data (u_0^δ, g_0^δ), where δ denotes the amplitude of noise, and this remark is crucial since the Cauchy problem is unstable. The question of how to adapt problem (4.2) to such noisy data is detailed in [6]. In particular, [6] introduces in the case of non-smooth noisy data a procedure based on the Morozov's discrepancy principle in order to regularize such data and compute a relevant value of the regularization parameter ε with the help of duality in optimization. We do not reproduce such discussion here and we invite the reader to refer to such paper.

4.2 The level set technique

In step 2, our level set method consists in solving in the current domain P_m a Poisson equation with Dirichlet boundary value V_m based on the previous quasi-reversibility solution u_ε computed in domain $\hat{E} = \mathcal{D} \setminus \overline{P_m}$ and with sufficiently large second member, then in using the level set 0 of the solution of such equation to update P_m.
In order to describe and justify such level set method we introduce an exact velocity function V, that is a function in $H^1(\mathcal{D})$ which satisfies:

$$\begin{cases} V = 1 - |\nabla u|^2 \quad \text{in} \quad \mathcal{D} \setminus \overline{P} \\ V|_P \in H^1_0(P) \\ V \leq 0 \quad \text{in} \quad P, \end{cases} \tag{4.3}$$

where (P, u) is the solution to problem (3.1). Such a function V exists if u is sufficiently smooth, as shown by following proposition:

Proposition 4.2 *If $u \in H^2(\mathcal{D} \setminus \overline{P}) \cap C^1(\overline{\mathcal{D}} \setminus P)$, then there exists $V \in H^1(\mathcal{D})$ satisfying (4.3).*

PROOF. We define V by $V = 1-|\nabla u|^2$ in $\mathcal{D}\setminus\overline{P}$ and $V = 0$ in P. Clearly, such V satisfies (4.3). It remains to prove that $V \in H^1(\mathcal{D})$. Since $u \in C^1(\overline{\mathcal{D}} \setminus P)$ and $1 - |\nabla u|^2 = 0$ on ∂P, we immediately have $V \in C^0(\overline{D})$. To prove that $V \in H^1(\mathcal{D})$ we just have to prove that $V \in H^1(\mathcal{D} \setminus \overline{P})$. That $V \in L^2(\mathcal{D} \setminus \overline{P})$ is trivial since $0 \leq V(x) \leq 1$ for $x \in \overline{\mathcal{D}} \setminus P$. Then $u \in C^1(\overline{\mathcal{D}} \setminus P)$ implies there exists $M > 0$ such that $|\nabla u(x)| \leq M$ for $x \in \overline{\mathcal{D}} \setminus P$. For $i = 1, 2$, we have

$$\frac{\partial V}{\partial x_i} = -2 \sum_{j=1,2} \frac{\partial u}{\partial x_j} \frac{\partial^2 u}{\partial x_i \partial x_j},$$

then

$$\int_{\mathcal{D}\setminus\overline{P}} \left| \frac{\partial V}{\partial x_i} \right|^2 dx \leq 8 \int_{\mathcal{D}\setminus\overline{P}} \sum_{j=1,2} \left| \frac{\partial u}{\partial x_j} \right|^2 \left| \frac{\partial^2 u}{\partial x_i \partial x_j} \right|^2 \leq 8M^2 ||u||^2_{H^2(\mathcal{D}\setminus\overline{P})},$$

which completes the proof. ■

Let $V \in H^1(\mathcal{D})$ satisfy (4.3). As a consequence we have $\Delta V \in H^{-1}(\mathcal{D})$. Now we consider $f \in H^{-1}(\mathcal{D})$ satisfying

$$f \geq \Delta V \tag{4.4}$$

in the sense of $H^{-1}(\mathcal{D})$ and for some open domain $\omega \Subset \mathcal{D}$, the well-posed Poisson problem: find $v \in H^1(\omega)$ such that

$$\begin{cases} \Delta v = f \quad \text{in} \quad H^{-1}(\omega) \\ v - V \in H^1_0(\omega). \end{cases} \tag{4.5}$$

From a given initial guess P_0 with

$$P \subset P_0 \Subset \mathcal{D}, \tag{4.6}$$

we define a sequence of open domains P_m by induction:

$$P_{m+1} = P_m \setminus \mathrm{supp}(\sup(v_m, 0)), \tag{4.7}$$

where v_m is the solution to problem (4.5) with $\omega = P_m$.
We have the following proposition, which is proved in [5] (proposition 2) and results from the weak maximum principle applied to problem (4.5).

Proposition 4.3 *The sequence of domains P_m satisfies: for all $m \in \mathbb{N}$, $P \subset P_{m+1} \subset P_m \Subset \mathcal{D}$.*

The above proposition and the results in [17] (paragraph 2.2.3) imply

Proposition 4.4 *The sequence of open domains P_m converges, in the sense of the Hausdorff distance for open domains, to the set*

$$\tilde{P} = \overbrace{\bigcap_m P_m}^{\circ},$$

such that $P \subset \tilde{P} \Subset \mathcal{D}$.

We are now in a position to state the main theorem that justifies our level set method, in the sense that we have defined a sequence of domains P_m that converge for appropriate topology to the exact exterior plastic zone P we wanted to retrieve.

Theorem 4.5 *Let (P, u) satisfy the problem (3.1). Let $V \in H^1(\mathcal{D})$ and $f \in H^{-1}(\mathcal{D})$ satisfy (4.3) and (4.4) respectively. Lastly we introduce the sequence of open domains P_m defined by some P_0 which satisfies (4.6) and by the induction formula (4.7).*
If additionally we assume that all the domains P_m are uniformly Lipschitz in the sense of definition 2.4.1 in [17], then we have

$$\overbrace{\bigcap_m P_m}^{\circ} = P,$$

with convergence in the sense of Hausdorff distance for open domains.

Proof. In view of proposition 4.4, the result to prove amounts to $\tilde{P} \subset P$. Let us assume that $R = \tilde{P} \setminus \overline{P}$ is not empty. Since the domains P_m are uniformly Lipschitz, we have from proposition 4.4 and from theorem 3.2.13 in [17] that the sequence v_m converges in $H_0^1(\mathcal{D})$ to the solution v of (3.1) with $\omega = \tilde{P}$. Furthermore $\tilde{P}$ is Lipschitz. By using the same arguments as in the proof of theorem 2.5 in [5], we obtain that $V \in H_0^1(R)$.
Now take some $x \in \partial\tilde{P}$ with $x \notin \overline{P}$. Since $\tilde{P}$ is Lipschitz continuous, R is locally Lipschitz continuous at point x. But $V \in C^0(\overline{\mathcal{D}} \setminus P)$, we hence have $V \in H_0^1(R) \cap C^0(\overline{R})$. This implies that $V = 0$ on ∂R in a vicinity of x (see [14], theorem IX.17), in particular $V(x) = 0$, this is $|\nabla u(x)| = 1$. But $x \in \mathcal{D} \setminus \overline{P}$, then $|\nabla u(x)| < 1$. This is a contradiction, hence R is empty, that is $P \subset \tilde{P} \subset \overline{P}$. Since P has Lipschitz boundary, $\tilde{P} = P$. ■

Remark 4.6 *The uniform Lipschitz assumption on domains P_m could have been replaced by an assumption of type: all domains $\mathcal{D} \setminus \overline{P_m}$ are connected, with the help of Šverak's theorem as in [5]. We would have the same convergence result for the functions v_m. However, the Lipschitz regularity of domain $\tilde{P}$ is required to complete the proof of theorem 4.5.*

4.3 The iterative algorithm

Our iterative algorithm computes an approximate solution (P, u) of problem (3.1) from the Cauchy data (u_0, g_0) on Γ, by coupling the method of quasi-reversibility and the level set method we have presented in the two last subsections.

1. Choose an initial guess P_0 such that $P \subset P_0 \Subset \mathcal{D}$ and $\mathcal{D} \setminus \overline{P_0}$ is connected.
2. Step 1: the domain P_m being given, solve the quasi-reversibility problem (4.2) in $\mathcal{D} \setminus \overline{P_m}$ with the Cauchy data (u_0, g_0) on Γ, for some selected small $\varepsilon > 0$. The solution is denoted u_m, and the correspondent velocity is $V_m = 1 - |\nabla u_m|^2$.
3. Step 2: the function V_m being given in $\mathcal{D} \setminus \overline{P_m}$, solve the non-homogeneous Dirichlet problem
$$\begin{cases} \Delta v &= f \quad \text{in} \quad P_m \\ v &= V_m \quad \text{in} \quad \partial P_m \end{cases} \tag{4.8}$$
for some sufficiently large f. With the help of the solution denoted v_m, define
$$P_{m+1} = \{x \in P_m |\, v_m(x) < 0\}.$$
4. Go back to the step 1 until the Hausdorff distance between P_{m-1} and P_m is sufficiently small.

By applying the above algorithm, we have no guarantee that the sequence of P_m converges to the true exterior plastic zone P. Indeed theorem 4.5 is not applicable because V_m (unlike V) is not based on the true displacement field u but on the approximation u_m of u. Nevertheless, we expect that if ε is chosen well (according the procedure described in [6]), then u_m is close to u in $\mathcal{D} \setminus \overline{P_m}$, and then the sequence P_m converges to a domain that is close to P.
The above algorithm is very similar to the one proposed in [5] for the inverse obstacle problem with boundary condition $u = 0$, the only difference is that $V_m = |u_m|$ is replaced by $V_m = 1 - |\nabla u_m|^2$. Our algorithm involves a few parameters to choose, essentially ε, f and the stopping criteria. We refer to [5, 6] for such discussion. More precisely, the choice of ε is discussed in [6], while the influence of f (chosen as a constant) and the stopping criterion are discussed in [5].

5 Numerical experiments

This section is devoted to numerical experiments in the square $\mathcal{D} = (-0.5, 0.5) \times (0.5, 0.5)$. The obstacle $\mathcal{O}$ is either an open surfacic domain or the union of segments. The support

Γ of the Cauchy data is the union of some edges of the square $\mathcal{D}$ and will be defined in the sequel with the following notations:

$$\begin{cases} \Gamma_1 = \{-0.5\} \times (-0.5, 0.5), & \Gamma_2 = (-0.5, 0.5) \times \{0.5\}, \\ \Gamma_3 = \{0.5\} \times (-0.5, 0.5), & \Gamma_4 = (-0.5, 0.5) \times \{-0.5\}. \end{cases}$$

The artificial Cauchy data (u_0, g_0) are obtained by solving the forward problem (2.5) (2.6) with vanishing initial fields at t_0 in the time interval $(t_0, t_f) = (0, 1)$, the prescribed displacement is $w = 0$ on $\Gamma_u = \partial\mathcal{O}$ and the prescribed traction T on $\Gamma_T = \partial\mathcal{D}$ is $T = \alpha_i\, t$ on Γ_i, with $\alpha_1 = \alpha_3 = 0$ and $\alpha_2 = \alpha_4 = \alpha > 0$, where α is a constant. We have imposed $\alpha = 0.25$ in subsections 5.1 and 5.2, while $\alpha = 0.5$ in subsections 5.3 and 5.4. The material constants are chosen as $\mu = 1$, $k = 1$ and $H = 0.5$.
The forward problem is solved by using a finite element method based on classical Lagrange $P1$ triangles and on a classical iterative algorithm for elasto-plasticity : we refer to [21] for a general presentation and to [26] (paragraph 3.4) for a presentation of the classical two loops algorithm we have used in the present paper.
The obtained data $u_0 = w(., t_f)|_\Gamma$ and $g_0 = \sigma(., t_f).n|_\Gamma/\mu$ at the end of our increasing loading are then prescribed on Γ to approximately solve the inverse problem (3.1). The iterative algorithm described in subsection 4.3 is then used. The initial guess P_0 is the open disc centered at $(0, 0)$ and of radius 0.45. The quasi-reversibility problem in step 1 is solved by using a non-conforming finite element method based on the Fraeijs de Veubeke's triangular element, exactly like in [5], where such element is presented and analyzed. The problem (4.5) in step 2 is solved by using a finite element method based on the classical Lagrange $P1$ triangles. The second member f in (4.5) is chosen as the constant function $f = 100$ (except in section 5.3), a practical procedure for choosing f is described in [5]. Note that the triangular mesh is the same in steps 1 and 2.
In order to test the robustness of our method, we use some artificial noisy Cauchy data (u_0^δ, g_0^δ) with amplitude of noise δ, namely

$$||u_0^\delta - u_0||_{L^2(\Gamma)} = \delta\, ||u_0||_{L^2(\Gamma)}, \quad ||g_0^\delta - g_0||_{L^2(\Gamma)} = \delta\, ||g_0||_{L^2(\Gamma)}.$$

These data are obtained by using a pointwise gaussian noise, the amplitude of which is rescaled to obtained the above equalities. As already mentioned at the end of subsection 4.1, in order to smooth our noisy data and select a relevant parameter ε as a function of δ in step 1, we use the procedure described in [6], that is an approach based on the Morozov's discrepancy principle and duality in optimization. This procedure replaces the quasi-reversibility problem in step 1 during the first iteration of the algorithm. Then the regularized data and the computed parameter ε that result from this procedure are used to solve the quasi-reversibility problem during the second iteration of the algorithm and all the iterations after. In the special case when data are uncontaminated by noise ($\delta = 0$), then the parameter ε is arbitrarily chosen as 10^{-6}.
In the remainder of this section, we test the method in different configurations. Note that except in subsection 5.3, the obstacles $\mathcal{O}$ are segments and assumptions $(H0)$, $(H1)$ and $(H2)$ are satisfied.

5.1 Case of partial data

We first test the method when $\mathcal{O}$ is the segment between points $(-0.2, -0.15)$ and $(-0.05, -0.1)$. Here, the Cauchy data are available on the whole boundary $\Gamma = \partial\mathcal{D}$ and are free of noise. The result of identification of the plastic zone created by such obstacle is shown on figure 2, the number of iterations to obtain convergence is about 50. The impact of the amount of data on the quality of the reconstruction is shown on figure 3.

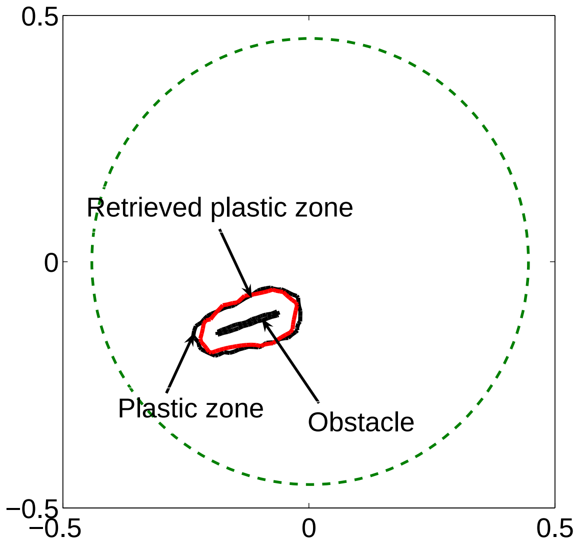

Figure 2: Identification with $\Gamma = \partial\mathcal{D}$

5.2 Case of noisy data

We now test the impact of the amplitude of noise on the data, which are here given on $\partial\mathcal{D}$. The obstacle $\mathcal{O}$ is unchanged. The reconstructions of the plastic zones with increasing relative noise δ are shown in figure 4.

5.3 Case of visible obstacle

We now test a configuration where the assumption $(H2)$ is not fulfilled, that is the obstacle $\mathcal{O}$ is not included in the exterior plastic zone $\mathcal{P}_e$. In other words, the plastic zone does not completely surrounds the obstacle, like in the last configuration of figure 1. As as result, the method presented above cannot be applied to identify the boundary of the exterior plastic zone $\partial\mathcal{P}_e$ because the domain $\mathcal{D} \setminus \overline{\mathcal{P}_e}$ is not fully contained in the elastic zone $\mathcal{E}$. Actually, it also contains the obstacle $\mathcal{O}$.
However, we can adapt our method to identify the boundary of the union of $\mathcal{P}_e$ and of

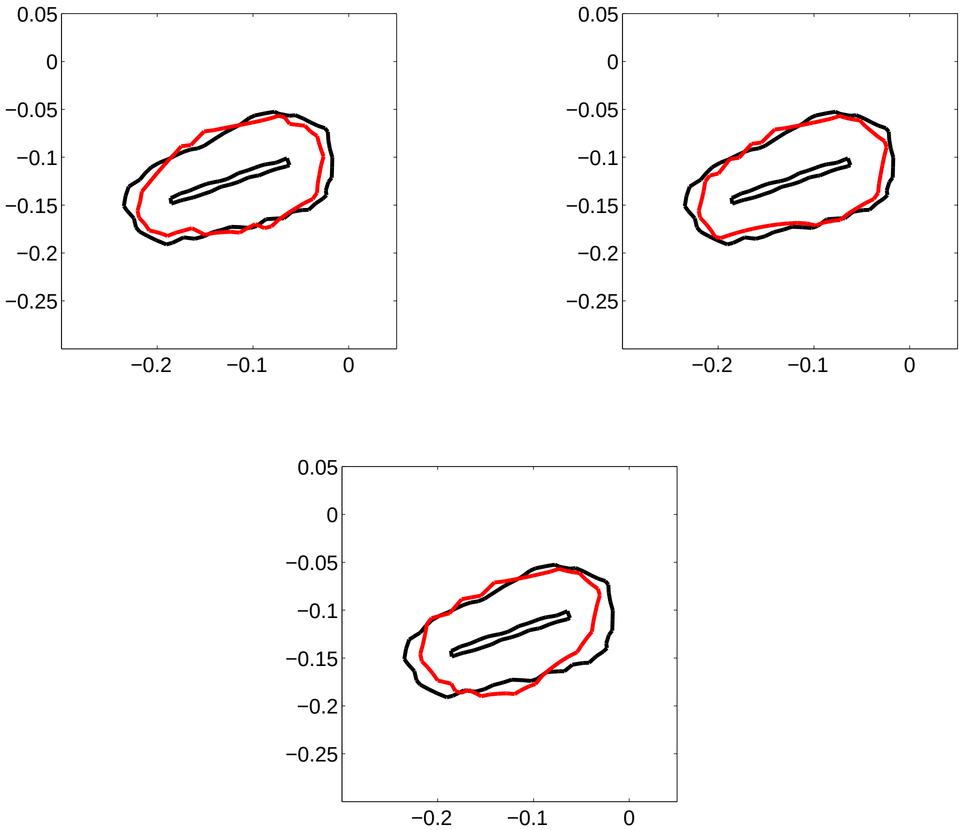

Figure 3: Identification with $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_4$ (top left), $\Gamma = \Gamma_1 \cup \Gamma_3 \cup \Gamma_4$ (top right) and $\Gamma = \Gamma_1 \cup \Gamma_3$ (bottom)

$\mathcal{O}$, which is characterized by either $|\nabla u| = 1$ or $u = 0$. Our aim is hence to retrieve the domain $P = \mathcal{P}_e \cup \mathcal{O}$ by using the velocity

$$V = \inf(|u|, 1 - |\nabla u|^2)$$

in definition (4.3) instead of $V = 1 - |\nabla u|^2$ and $V_m = \inf(|u_m|, 1 - |\nabla u_m|^2)$ in (4.8) instead of $V_m = 1 - |\nabla u_m|^2$. The result of identification is shown on figure 5, with obstacle $\mathcal{O}$ defined as a thin profile ended by points $(-0.2, 0)$ and $(0.25, 0.25)$, $\Gamma = \partial\mathcal{D}$ and noise free data. Note that in such case, the constant second member of equation (4.8) had to be decreased to $f = 40$.

5.4 Case of several obstacles

We complete this numerical section with an example of obstacle $\mathcal{O}$ which consists of three segments delimited respectively by points $(-0.3, -0.15)$ and $(-0.25, -0.2)$, points

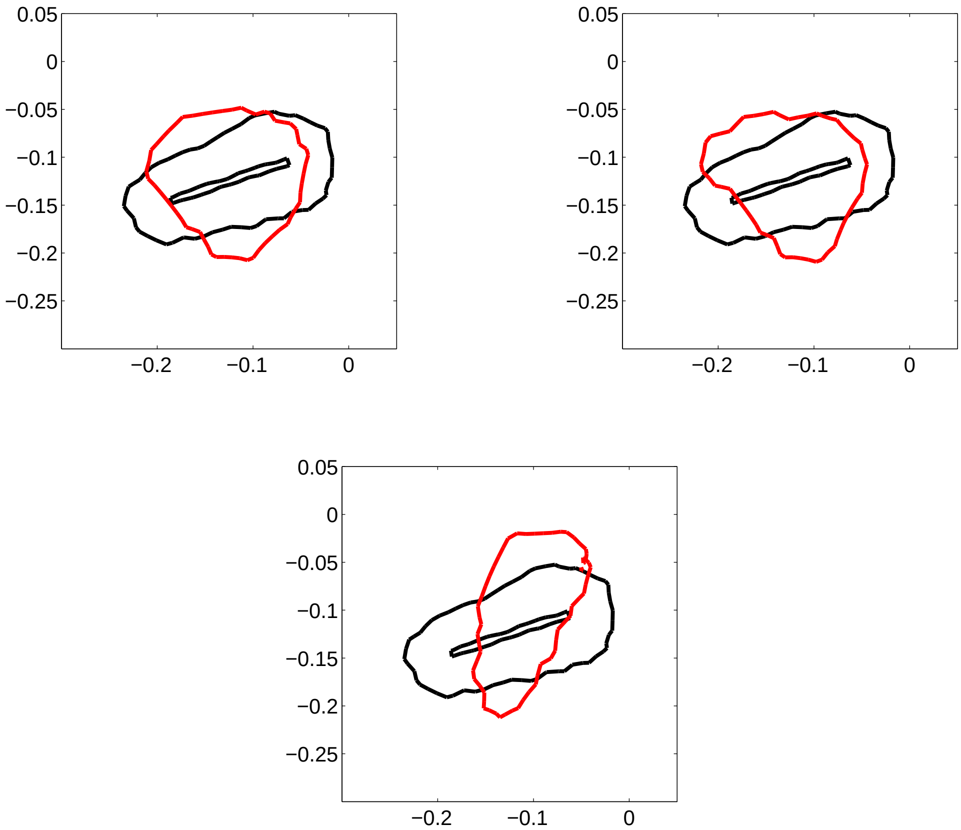

Figure 4: Identification with $\delta = 0.5\%$ (top left), $\delta = 1\%$ (top right) and $\delta = 5\%$ (bottom)

$(-0.1, 0.2)$ and $(0, 0.25)$, and lastly points $(0.25, -0.25)$ and $(0.3, -0.2)$. The result is shown on figure 6, with again $\Gamma = \partial\mathcal{D}$ and noise free data.
All the obtained results in the present numerical section tend to show that provided the plastic zone be somehow confined in the vicinity of the defects, our method is successful to localize those defects. Our approach consists in finding the plastic zones created by the defects rather than the defects themselves. It may be promising for non-destructive testing in elastic-plastic materials, in particular in case of multiple cracks.

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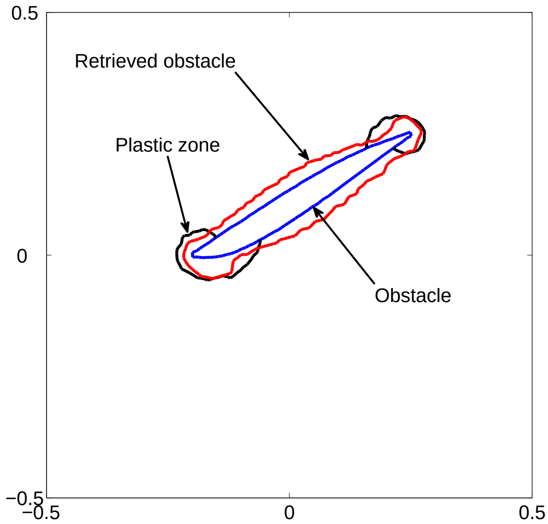

Figure 5: Identification of visible obstacle

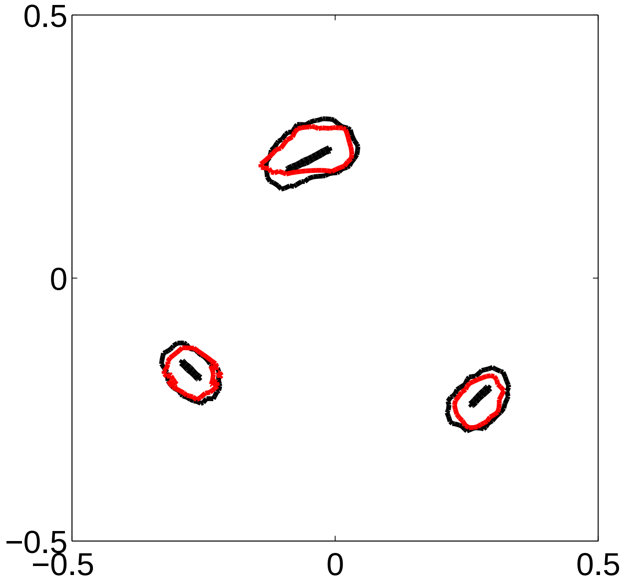

Figure 6: Identifications of three defects

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