RADIATION CONDITION FOR A NON SMOOTH INTERFACE BETWEEN DIELECTRIC AND METAMATERIAL

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Talk Abstract

We study a scalar harmonic wave transmission problem between a classical dielectric and a metamaterial (a medium with a real valued negative permittivity/permeability). When the interface between the two media has a corner, depending on the value of the contrast (ratio) of the physical constants, this non-coercive problem can be ill-posed (not Fredholm) in H^1 (see [1]). This is due to the degeneration of the two dual singularities which then behave like $r^{\pm i\eta} = e^{\pm i\eta \ln r}$ with $\eta \in \mathbb{R}^*$. In this work, we derive a functional framework by adding to a smaller space than H^1 one of these singularities. This phenomenon is very similar to what happens for scattering problems in unbounded domains. In the same manner, well-posedness of our problem is obtained by imposing a radiation condition in the neighbourhood of the geometrical singularity using a limiting absorption principle.

Setting of the problem

To simplify, let us consider the polygonal domain $\Omega \subset \mathbb{R}^2$ of Figure 1. $\overline{\Omega} = \overline{\Omega_1} \cup \overline{\Omega_2}$ and $\Omega_1 \cap \Omega_2 = \emptyset$. The interface Σ between the two media is straight and $\overline{\Sigma} \cap \partial\Omega = \{O, O'\}$. The domain Ω is locally symmetric in the neighbourhood of O' so the only relevant geometrical singularity is located at O.

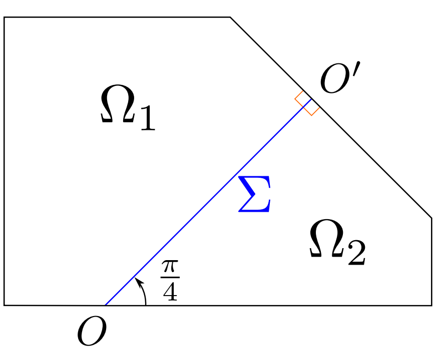

Figure 1: Geometry of the problem.

Let us denote (r, θ) the polar coordinates associated with O. For $k = 1, 2$, we note that the subset Ω_k coincides with an open cone in a neighbourhood $\mathcal{V}$ of O : $\Omega_k \cap \mathcal{V} = \mathcal{K}_k \cap \mathcal{V}$ with

$$\mathcal{K}_1 := \{(r\cos\theta, r\sin\theta)\,|\, r > 0,\ \pi/4 < \theta < \pi\}$$
$$\mathcal{K}_2 := \{(r\cos\theta, r\sin\theta)\,|\, r > 0,\ 0 < \theta < \pi/4\}\,.$$

We shall focus on the problem

$$-\text{div}\,(\sigma\,\nabla u) = f \ \text{ in } \Omega; \quad u = 0 \ \text{ on } \partial\Omega, \qquad (1)$$

set in some functional spaces to be defined, with $\sigma = \sigma_1$ on Ω_1 and $\sigma = \sigma_2$ on Ω_2, $\sigma_1, \sigma_2 \in \mathbb{C}$. The key parameter to study problem (1) is the contrast $\kappa_\sigma := \sigma_2/\sigma_1$. We would proceed exactly in the same way for the problem $-\text{div}\,(\sigma\,\nabla u) - \omega^2 u = f$ in Ω; $u = 0$ on $\partial\Omega$, with $\omega \neq 0$.

Using Lax-Milgram's theorem, one can easily prove that if $\kappa_\sigma \in \mathbb{C}\backslash\mathbb{R}_-$, for every $f \in H^{-1}(\Omega)$, there exists one and only one solution to problem (1) depending continuously on the data f. In the sequel, we shall concentrate on the case $\kappa_\sigma \in \mathbb{R}^*_-$.

Although $(u, v) \mapsto (\sigma\,\nabla u, \nabla v)$ is not coercive on $H^1_0(\Omega) \times H^1_0(\Omega)$, using a suitable isomorphism T of $H^1_0(\Omega)$ and studying $(u, v) \mapsto (\sigma\,\nabla u, \nabla(T\,v))$ (see the T-coercivity method presented in [1]), we establish the

Proposition 1. *If $\kappa_\sigma \in \mathbb{R}^*_-\backslash[-1; -1/3]$ then the operator $A_0(\kappa_\sigma) : u \mapsto -div\,(\sigma\,\nabla u)$ is Fredholm of index 0 from $H^1_0(\Omega)$ to $H^{-1}(\Omega)$.*

In particular, when $\kappa_\sigma \in \mathbb{R}^*_-\backslash[-1; -1/3]$, if $u = 0$ is the only solution to problem (1) with $f = 0$, problem (1) is well-posed for all $f \in H^{-1}(\Omega)$. On the other hand, when $\kappa_\sigma = -1$ (i.e. when $\sigma_2 = -\sigma_1$), it is proved in [1] that the operator $A_0(\kappa_\sigma)$ is not Fredholm from $H^1_0(\Omega)$ to $H^{-1}(\Omega)$ because its range is not closed. Henceforth, the contrast κ_σ will be different from -1.

Now, consider $u \in H^1_0(\Omega)$ such that $\text{div}\,(\sigma\,\nabla u) \in L^2(\Omega) \subset H^{-1}(\Omega)$. When $\kappa_\sigma \notin [-1; -1/3]$, u can be decomposed (see [2]) under the form $u = u_{reg} + c\,r^\lambda \varphi(\theta)$, where u_{reg} is a piecewise

H^2-function and $(r,\theta) \mapsto r^\lambda\varphi(\theta)$, with $\Re e\,\lambda > 0$, is the singularity associated with O. Notice that with the help of a partition of unity, one can restrict the study of this problem of singularity in the neighbourhood of O. With the change of variables $z := \ln r$, the sector $\{(r\cos\theta, r\sin\theta)\,|\,0 < r < 1, 0 < \theta < \pi\}$ becomes the half strip $\mathbb{R}^*_- \times]0;\pi[$, the Helmholtz eq. (1), the Helmholtz eq. (2) below and the singularity $r^\lambda\varphi(\theta) = e^{\lambda z}\varphi(\theta)$, an evanescent mode in the neighbourhood of $-\infty$.

But when $\kappa_\sigma \in]-1;-1/3[$, the real part of the singular exponent λ vanishes. More precisely, $\lambda = \pm i\eta$ with $\eta \in \mathbb{R}^*$ and the two dual singularities $r^{\pm i\eta}\varphi(\theta) = e^{\pm i\eta z}\varphi(\theta) \notin H^1$ turn into propagative modes in the half strip : H^1 is no longer an appropriate framework.

A waveguide problem in the half strip

In this section, the domain considered is the half strip $\mathcal{S} := \{(z,\theta) \in \mathbb{R}^*_- \times]0;\pi[\}$. We denote $\mathcal{S}_1 := \{(z,\theta) \in \mathbb{R}^*_- \times]\pi/4;\pi[\}$ and $\mathcal{S}_2 := \{(z,\theta) \in \mathbb{R}^*_- \times]0;\pi/4[\}$. We study the problem

$$-(\sigma\partial_z^2 + \partial_\theta\sigma\partial_\theta)u = e^{2z}f \text{ in } \mathcal{S};\ u = 0 \text{ on } \partial\mathcal{S}, \quad (2)$$

which can be written $-\mathrm{div}\,(\sigma\,\nabla u) = e^{2z}f$ in $\mathcal{S}$; $u = 0$ on $\partial\mathcal{S}$. The function σ is such that $\sigma|_{\mathcal{S}_1} = \sigma_1$ and $\sigma|_{\mathcal{S}_2} = \sigma_2$.

Modes of the waveguide

Owing to the geometry, we would like to decompose solutions to (2) on the modes $u_k^\pm(z,\theta) = \varphi_k(\theta)e^{\pm\lambda_k z}$ (where $\Re e\,\lambda_k \geq 0$) in the neighbourhood of $-\infty$. To that aim, let us introduce the transverse operator $\mathcal{L}(\lambda)$:

$$\begin{array}{ccl} D(\mathcal{L}) & \to & L^2(]0;\pi[) \\ \varphi & \mapsto & \mathcal{L}(\lambda)\varphi = -(\sigma\,\lambda^2 + d_\theta\,\sigma\,d_\theta)\varphi \end{array} \quad (3)$$

with $D(\mathcal{L}) := \{\varphi \in H_0^1(]0;\pi[)\,|\,(d_\theta\,\sigma\,d_\theta)\varphi \in L^2(]0;\pi[)\}$.

We say that $u_k^\pm(z,\theta) = \varphi_k(\theta)e^{\pm\lambda_k z}$ are modes of the waveguide when there exists a non trivial $\varphi_k \in D(\mathcal{L})$ such that $\mathcal{L}(\lambda_k)\varphi_k = 0$. λ_k is then called an eigenvalue of $\mathcal{L}$.

⋄ Is $\lambda = 0$ an eigenvalue of $\mathcal{L}$? The $\varphi \in D(\mathcal{L})$ which satisfy $\mathcal{L}(0)\varphi = 0$ write

$$\varphi(\theta) = A\,\theta \text{ on } [0;\pi/4]\,;\ \varphi(\theta) = B\,(\theta-\pi) \text{ on } [\pi/4;\pi].$$

Writing the transmission conditions, we prove that $\lambda = 0$ is an eigenvalue of $\mathcal{L}$ if and only if $\kappa_\sigma = \sigma_2/\sigma_1 = -1/3$.

⋄ Now, if $\varphi \in D(\mathcal{L})$ satisfies $\mathcal{L}(\lambda)\varphi = 0$ with $\lambda \neq 0$, then necessarily

$$\begin{array}{ll} \varphi(\theta) = A\,\sin\,\lambda\theta & \text{on } [0;\pi/4] \\ \varphi(\theta) = B\,\sin\,\lambda\,(\theta-\pi) & \text{on } [\pi/4;\pi]. \end{array}$$

Taking into account the transmission conditions, such a λ must satisfy

$$\sigma_1\,\sin\,\lambda\pi/4\,\cos\,3\lambda\pi/4 + \sigma_2\,\cos\,\lambda\pi/4\,\sin\,3\lambda\pi/4 = 0.$$

Thus, for $\kappa_\sigma \neq -1$, λ is an eigenvalue of $\mathcal{L}$ if and only if

$$\pm\lambda \in \Lambda(\kappa_\sigma) := 2\mathbb{N}^* \cup \{\xi(\kappa_\sigma) + 4\mathbb{N}\} \cup \{\overline{\xi(\kappa_\sigma)} + 4\mathbb{N}\} \quad (4)$$

with $\xi(\kappa_\sigma) := \frac{2}{\pi}$ arccos $\rho(\kappa_\sigma)$ and $\rho(\kappa_\sigma) := \frac{\sigma_1-\sigma_2}{2(\sigma_1+\sigma_2)}$.

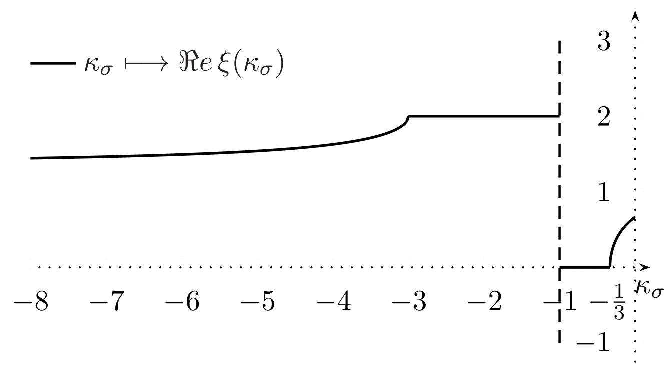

Figure 2: Real part of $\kappa_\sigma \mapsto \xi(\kappa_\sigma)$ for $\kappa_\sigma \in]-8;0[$.

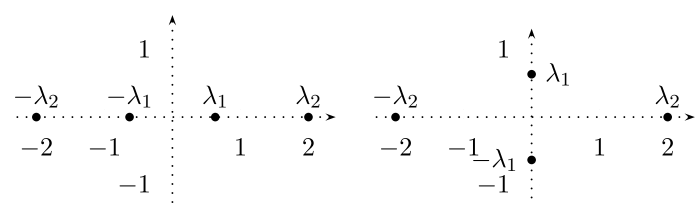

Figure 3: Spectrum of $\mathcal{L}$ in the complex plane - left, $\kappa_\sigma = -1/4$ - right, $\kappa_\sigma = -1/2$.

Consequently, we distinguish two situations :

• For $\kappa_\sigma \in]-\infty;-1[\cup]-1/3;0[$, $\Re e\,\xi(\kappa_\sigma) > 0$. Therefore, all the modes are exponentially increasing or decreasing.

• For $\kappa_\sigma \in]-1;-1/3[$, $\Re e\,\xi(\kappa_\sigma) = 0$ and there are two propagative modes associated with $\pm\lambda_1 = \pm\xi(\kappa_\sigma) = \pm i\eta(\kappa_\sigma)$ with $\eta(\kappa_\sigma) \in \mathbb{R}^*$:

$$u_1^\pm(z,\theta) = \varphi_1(\theta)e^{\pm i\eta(\kappa_\sigma)z}.$$

To deal with the second case, let us make the analogy with a classical waveguide problem. For a source term f with a compact support, the well-posedness of our problem would be obtained by decomposing u in the neighbourhood of $-\infty$ on the outgoing modes : $u = \sum_{k\geq 1} c_k\, \varphi_k\, e^{\lambda_k\, z}$. The solution u then splits into a propagative component (first mode) and an evanescent part (modes for $k \geq 2$).

Unfortunately, in our situation, we can not handle such a decomposition. Indeed, due to the sign-changing of σ, we are not able to prove that the eigenvectors of the transverse operator define a basis of $L^2(]0;\pi[)$.

However, using cut off functions, weighted functional spaces and Fourier-Laplace transform in the infinite direction, we establish a similar decomposition thanks to two key results. More precisely, we prove there exists a unique solution u to problem (2) with

$$u = c_1\, \varphi_1\, e^{\lambda_1\, z} + u_e\,, \tag{5}$$

where the contribution u_e is such that $e^{-\beta z}\, u_e \in H^1(\mathcal{S})$ for some $\beta > 0$: this last condition expresses the evanescent behaviour in the neighbourhood of $-\infty$.

Isomorphism property

For $\beta \in \mathbb{R}$, we introduce the space W_β, completion of $\mathcal{C}_0^\infty(\mathcal{S})$ with respect to the norm

$$\|v\|_{W_\beta} := \left\|e^{\beta z} v\right\|_{H^1(\mathcal{S})}\,,$$

so that $W_0 = H_0^1(\mathcal{S})$ and, if $\beta > 0$, $W_{-\beta} \subset H_0^1(\mathcal{S}) \subset W_\beta$. We denote ${W_\beta}^*$ the dual space of W_β. Let us consider the bounded operator

$$B_\beta(\kappa_\sigma)\ :\ W_\beta \to {W_{-\beta}}^*$$

such that, for all $(u, v) \in W_\beta \times W_{-\beta}$,

$$\langle B_\beta(\kappa_\sigma)\, u,\, v\rangle = (\sigma\, \nabla u,\, \nabla v)\,.$$

Adapting the theory presented for elliptic problems in chapters 5-6 of [3], we prove the first key result :

Theorem 2. *Suppose $\kappa_\sigma \neq -1$. The operator $B_\beta(\kappa_\sigma)$ is Fredholm if and only if $\mathcal{L}$ has no eigenvalue on the line $\ell_\beta := \{\lambda \in \mathbb{C}\,|\, \Re e\, \lambda = \beta\}$.*

Let us fix $\kappa_\sigma \in]-1;-1/3[$ and $0 < \beta < 2$. $\mathcal{L}$ has two conjugate eigenvalues on ℓ_0 and no eigenvalue on ℓ_β (see Figure 3, right). Therefore, $B_0(\kappa_\sigma)$ is not Fredholm whereas $B_\beta(\kappa_\sigma)$ and $B_{-\beta}(\kappa_\sigma)$ are Fredholm. We have :

$$\begin{array}{lccc} B_{-\beta}(\kappa_\sigma): & W_{-\beta} & \to & {W_\beta}^* \\ & \bigcap & & \bigcap \\ B_\beta(\kappa_\sigma): & W_\beta & \to & {W_{-\beta}}^*. \end{array}$$

Since the adjoint of $B_\beta(\kappa_\sigma)$ is $B_{-\beta}(\kappa_\sigma)$, we have

$$\text{coker}\, B_\beta(\kappa_\sigma) = \ker B_{-\beta}(\kappa_\sigma). \tag{6}$$

Making an odd reflection, one can prove with Fourier transform in the whole strip that $B_{-\beta}(\kappa_\sigma)$ is injective (and so $B_\beta(\kappa_\sigma)$ is onto). Let us define $s(z,\theta) = \sin(\eta z)\varphi_1(\theta) = (u_1^+(z,\theta) - u_1^-(z,\theta))/2i \in W_\beta \backslash W_{-\beta}$. Obviously, $s \in \ker B_\beta(\kappa_\sigma)$ so $B_\beta(\kappa_\sigma)$ is not injective and $B_{-\beta}(\kappa_\sigma)$ is not onto. Hence in the space of exponentially decreasing functions, problem (1) is injective but not onto, whereas in the space of exponentially increasing functions, problem (1) is onto but not injective. Next, we build from $B_\beta(\kappa_\sigma)$ and $B_{-\beta}(\kappa_\sigma)$ an isomorphism.

We follow the procedure of Nazarov-Plamenevsky (§3, chapter 5 of [4]). Let $\zeta \in \mathcal{C}_0^\infty(\mathbb{R}_-)$ denote a cut off function such that $\zeta(z) = 1$ for $z < -2$ and $\zeta(z) = 0$ for $z > -1$. We define the two truncated propagative modes $s^\pm(z,\theta) := \zeta(z)\, u_1^\pm(z,\theta) \in W_\beta \backslash W_{-\beta}$. In the same spirit as (5), consider the space

$$W^+ := \text{span}\ (s^+) \oplus W_{-\beta}.$$

Theorem 3. *Let $\kappa_\sigma \in]-1;-1/3[$ and $0 < \beta < 2$. The operator div $(\sigma\, \nabla\cdot)$ is an isomorphism from W^+ to ${W_\beta}^*$.*

Proof : • **Uniqueness :** Let $u = c\, s^+ + u_{-\beta} \in W^+$ be such that div $(\sigma\, \nabla u) = 0$. Integrating by parts on $]-L;0[\times]0;\pi[$, one finds

$$\begin{aligned} &\forall L > 0, \quad \Im m \int_{z=-L} \sigma \partial_z u\, \overline{u}\, d\theta = 0 \\ \Rightarrow\ &\lim_{L\to\infty} |c|^2\, \Im m\, \lambda_1 \int_{z=-L} \sigma\, |\varphi_1|^2\, d\theta = 0. \end{aligned}$$

Consequently, $c = 0$ and $u = u_{-\beta} \in W_{-\beta}$. Since $B_{-\beta}(\kappa_\sigma)$ is injective, $u = 0$.

• **Existence :** Let $f \in {W_\beta}^*$. Since $B_\beta(\kappa_\sigma)$ is onto, there exists $u_\beta \in W_\beta$ such that div $(\sigma\, \nabla u_\beta) = f$. With the help of the residual theorem, we prove that u_β admits the decomposition $u_\beta = c^+\, s^+ + c^-\, s^- + u_{-\beta}$, with $u_{-\beta} \in W_{-\beta}$ (our second key result). Now, define $u := u_\beta + 2ic^-\, s$. This element belongs to W^+ and satisfies div $(\sigma\, \nabla u) = f$ (remember that div $(\sigma\, \nabla s) = 0$). That ends the proof. □

Transposition in the initial bounded domain Ω

Let V_β be the completion of $\mathcal{C}_0^\infty(\Omega)$ with respect to the norm

$$\|v\|_{V_\beta} := (\|r^\beta \nabla v\|^2_{L^2(\Omega)^2} + \|r^{\beta-1} v\|^2_{L^2(\Omega)})^{1/2}.$$

For $\beta \in \mathbb{R}$, let us consider the bounded operator

$$A_\beta(\kappa_\sigma) \,:\, V_\beta \to {V_{-\beta}}^*$$

such that, for all $(u, v) \in V_\beta \times V_{-\beta}$,

$$\langle A_\beta(\kappa_\sigma)\, u,\, v\rangle = (\sigma\, \nabla u,\, \nabla v)\,.$$

Let us recall that, according to **Proposition 1**, the operator $A_0(\kappa_\sigma)$ is Fredholm of index 0 from $H^1_0(\Omega)$ to $H^{-1}(\Omega)$ when $\kappa_\sigma \in \mathbb{R}^*_-\backslash[-1; -1/3]$. Using **Theorem 2**, one can prove that $A_0(\kappa_\sigma)$ is not Fredholm from $H^1_0(\Omega)$ to $H^{-1}(\Omega)$ when $\kappa_\sigma \in]-1; -1/3]$.

Let $\chi \in \mathcal{C}^\infty_0(\mathbb{R})$ denote a function equal to 1 in the neighbourhood of 0 with sufficiently small support. Let us redefine the two singularities, now expressed in the coordinates (r, θ),

$$s^\pm(r, \theta) = \chi(r)\, u^\pm_1(\ln\, r, \theta) = \chi(r)\, r^{\pm i\,\eta(\kappa_\sigma)}\, \varphi_1(\theta),$$

the two spaces

$$\begin{array}{l} V^+ := \text{span}\,(s^+) \oplus V_{-\beta} \\ V^- := \text{span}\,(s^-) \oplus V_{-\beta} \end{array}$$

and the two restrictions

$$\begin{array}{rccc} A^+_\beta(\kappa_\sigma): & V^+ & \to & {V_\beta}^* \\ A^-_\beta(\kappa_\sigma): & V^- & \to & {V_\beta}^* \end{array} \qquad (7)$$

of the operator $A_\beta(\kappa_\sigma)$. Thanks to **Theorem 3**, we can express the

Theorem 4. *Let $\kappa_\sigma \in]-1; -1/3[$ and $0 < \beta < 2$. The operators $A^+_\beta(\kappa_\sigma)$ and $A^-_\beta(\kappa_\sigma)$ defined in (7) are Fredholm of index 0. Besides, ker $A^+_\beta(\kappa_\sigma)$ = ker $A^-_\beta(\kappa_\sigma)$ = ker $A_{-\beta}(\kappa_\sigma)$ and coker $A^+_\beta(\kappa_\sigma)$ = coker $A^-_\beta(\kappa_\sigma)$ = coker $A_\beta(\kappa_\sigma)$.*

We want now to determine which functional framework matches best with the physical reality.

Limiting absorption principle

Again, we fix $\kappa_\sigma \in]-1; -1/3[$ and $0 < \beta < 2$. Let us add some absorption to the medium which leads us to consider the problem

$$-\text{div}\,(\sigma^\gamma\, \nabla u^\gamma) = f \;\text{ in } \Omega; \quad u^\gamma = 0 \;\text{ on } \partial\Omega, \qquad (8)$$

with $\sigma^\gamma := \sigma\,(1 + i\,\text{sign}(\sigma)\,\gamma)$ and $\gamma > 0$. Since $\kappa_{\sigma^\gamma} \in \mathbb{C}\backslash\mathbb{R}_-$, this problem is well-posed in $H^1_0(\Omega)$. Consider $f \in {V_\beta}^* \subset H^{-1}(\Omega)$ such that $\langle f, v\rangle = 0$ for all $v \in$ coker $A_\beta(\kappa_\sigma)$. According to **Theorem 4**, one can define two non dissipative solutions $u^\pm$ such that $A^\pm_\beta(\kappa_\sigma)\, u^\pm = f$. The physical solution u^{rad} (either equal to u^+ or u^-) is then the limit of the dissipative solution u^γ when γ tends to 0. Using respectively the decompositions $u^{rad} = c^{rad}\, s^{rad} + u^{rad}_{-\beta}$, with $rad \in \{+, -\}$, and $u^\gamma = c^\gamma\, s^\gamma + u^\gamma_{-\beta}$, we choose rad such that $s^\gamma \to s^{rad}$ when $\gamma \to 0$. The proof of the convergence of u^γ toward u^{rad} in V_β is in progress.

To solve numerically this non-standard problem, that is, to approximate u^{rad}, one can use PMLs or Dirichlet-To-Neumann in the strip.

Physical interpretation

In our configuration, the selected propagative singularity is $s^-(r, \theta) = \chi(r)\, r^{-i\,\eta(\kappa_\sigma)}\, \varphi_1(\theta)$. Going back to the time-domain (for $\omega \neq 0$), one can write that the wave "located" at distance r_0 from O at time t_0 will be at distance $r(t)$ at time t with the relation

$$\begin{array}{lrcl} & -i\eta(\kappa_\sigma)\ln r(t) - i\omega t & = & -i\eta(\kappa_\sigma)\ln r_0 - i\omega t_0 \\ \Leftrightarrow & r(t) & = & r_0\, e^{-\frac{\omega}{\eta(\kappa_\sigma)}\,(t - t_0)}. \end{array}$$

Thus, the wave requires an infinite time to reach the origin. When $\kappa_\sigma \to -1$, we have $\eta \to +\infty$. Consequently, the closer κ_σ is to -1, the faster the wave propagates.

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