Fast Multipole Method for 3-D elastodynamic boundary integral equations. Application to seismic wave propagation.

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1 Motivations and background

2 3-D elastodynamic FMM: single-region

3 3-D elastodynamic FMM: multi-region

4 Modelling simple 3-D seismic problems

5 Improvement: Preconditioning strategy

6 Conclusions and future work
Motivation

Modelling of elastic wave propagation in large/unbounded domains

- Site effects
- Soil-structure interaction
- Computational forward solution method for inverse problems
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Pros and cons of BEMs for seismic waves

Domain methods (FEM, SEM, ...)
- Domain mesh
- Approx. radiation conditions
- Sparse matrix

BEM
- Surface mesh (i.e. reduced dim.)
- Exact radiation conditions
- Fully-populated matrix
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**BEM**
- Surface mesh (i.e. reduced dim.)
- Exact radiation conditions
- Fully-populated matrix

→ BEM adequate for large (unbounded) media, simple (linear) prop
→ Fully-populated BEM influence matrix: severe limiting factor
Previous works

- Scattering of elastic waves  [Rizzo et al., Int. J. Num. Method Eng., 1985]
- Seismic wave amplification (3-D)
- Fundamental solutions in elastodynamics  
  [Kausel, Cambridge University Press, 2006]
BEM for wave propagation

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Previous works at LMS and LCPC

- Crack propagation [Koller et al., Wave Motion, 1992]
- Inverse problems for elastic waves [Guzina and Bonnet, QJMAM, 2004]
- Acoustics [Nemitz, PhD thesis (LMS), 2006]
- Seismic wave amplification (2-D and 3-D, frequency domain)
  [Dangla et al., BSSA, 2005], [Delépine, PhD thesis (LCPC), 2007]
- Influence of Site-City interaction [Semblat et al., BSSA, 2008]
Standard BEM (3-D elastodynamics, freq. domain)

Governing direct integral equation for boundary displ. and tractions

\[ c_{ik}(x)u_i(x) = \int_{\partial \Omega} \left[ t_i(y) U_{ik}^k(x, y; \omega) - u_i(y) T_{ik}^k(x, y; \omega) \right] dS_y, \quad (x \in \partial \Omega) \]

BEM discretization \( \Rightarrow \) fully-populated system of linear equations.
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BEM discretization ⇒ fully-populated system of linear equations.

Full-space elastodynamic fundamental solutions

$$U_i^k(x, y; \omega) = \frac{1}{k_S^2 \mu} \left( (\delta_{qs}\delta_{ik} - \delta_{qk}\delta_{is}) \frac{\partial}{\partial x_q} \frac{\partial}{\partial y_s} G_S(|x - y|) + \frac{\partial}{\partial x_i} \frac{\partial}{\partial y_k} G_P(|x - y|) \right)$$

$$T_i^k(x, y; \omega) = C_{ijh} \frac{\partial}{\partial y_\ell} U_i^k(x, y; \omega) n_j(y)$$

$$G_\alpha(z) = \frac{\exp(ik_\alpha z)}{4\pi z} \text{ (fund. sol. Helmholtz eqn., } \alpha = P, S)$$
Computational limitations of standard BEM

Solution of fully-populated matrix equation

- **Direct solvers** (LU factorization, ...)
  - **Pros**: robust, accurate;
  - **Cons**: $O(N^2)$ memory and $O(N^3)$ CPU

- **Iterative solvers** (GMRES, ...)
  - **Pros**: $O(N_{\text{iter}} \times N^2)$ CPU;
  - **Cons**: $O(N^2)$ memory; $N_{\text{iter}}$ may be large

($N$: number of BE DOFs)
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Limitations of standard BEM
- High memory cost: $O(N^2)$
  - ⇒ problem size limit $N = O(10^4)$ (PC, single-proc.)
- Limited geometric complexity, (piecewise) heterogeneity, frequency range
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Fast BEMs
- \(\mathcal{H}\)-matrix [Hackbusch, Computing, 1999];
- Adaptive Cross-Approximation [Bebendorf and Rjasanow, Comp., 2003];
Fast Multipole Method (FMM):

- Based on iterative linear equation solvers (GMRES)
- Fast, approximate method for evaluating the linear integral operator (matrix-vector product, called by iterative solver)
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A few milestones
- Laplace [Rokhlin, J. Comp. Phys., 1985];
- Electrostatics [Greengard, 1988];
- Electromagnetics [Darve, J. Comp. Phys., 2000];
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- Electromagnetics [Darve, J. Comp. Phys., 2000];
- Elastodynamics frequency domain [Fujiwara, Geophys. J. Int., 2000];
- Isotropic Elastostatics: [Yoshida, PhD thesis, 2001];
- Effective prop. of composite mater. [Liu et al., J. Appl. Mech., 2005];
Motivations and background

3-D elastodynamic FMM: single-region

3-D elastodynamic FMM: multi-region

Modelling simple 3-D seismic problems

Improvement: Preconditioning strategy

Conclusions and future work
Decomposition of Helmholtz fundamental solution


\[
\begin{align*}
\mathbf{r} &= \mathbf{x} - \mathbf{y} = (\mathbf{y}_0 - \mathbf{y}) + (\mathbf{x}_0 - \mathbf{y}_0) - (\mathbf{x}_0 - \mathbf{x}) \\
&= \lim_{L \to +\infty} \int_{\hat{s} \in S} e^{ik\hat{s} \cdot (\mathbf{y}_0 - \mathbf{y})} g_L(\hat{s}; \mathbf{r}_0; k) e^{-ik\hat{s} \cdot (\mathbf{x}_0 - \mathbf{x})} d\hat{s},
\end{align*}
\]

\( y \)
\( \mathbf{y}_0 \)
\( r \)
\( \mathbf{r}_0 \)
\( x \)
\( x_0 \)
Decomposition of Helmholtz fundamental solution


\[
\begin{align*}
    r &= x - y = (y_0 - y) + (x_0 - y_0) - (x_0 - x) \\
    \exp\left(i k |x - y| / |x - y| \right) &= \lim_{L \to +\infty} \int_{\mathbf{s} \in S} e^{i k \mathbf{s} \cdot (y_0 - y)} G_L(\mathbf{s}; r_0; k) e^{-i k \mathbf{s} \cdot (x - x_0)} d\mathbf{s},
\end{align*}
\]

Transfer function

\[
G_L(\mathbf{s}; r_0; k) = \frac{ik}{16\pi^2} \sum_{p=0}^{L} (2p + 1) i^p h_p^{(1)}(k |r_0|) P_p(\cos(\mathbf{s}, r_0))
\]
Decomposition of elastodynamic fundamental solution

Multipole expansion of $U^k$

\[
U^k_i(x, y; \omega) = \lim_{L \to +\infty} \int_{\hat{s} \in S} e^{ik_p \hat{s} \cdot (y - y_0)} U^k_{i, L} (\hat{s}; r_0) e^{-ik_p \hat{s} \cdot (x - x_0)} d\hat{s}
\]

\[
+ \lim_{L \to +\infty} \int_{\hat{s} \in S} e^{ik_s \hat{s} \cdot (y - y_0)} U^k_{i, L} (\hat{s}; r_0) e^{-ik_s \hat{s} \cdot (x - x_0)} d\hat{s}
\]

where

\[
U^k_{i, L} (\hat{s}; r_0) = \frac{1}{\mu} (\delta_{ik} - \hat{s}_k \hat{s}_i) G_L (\hat{s}; r_0; k_S),
\]

\[
U^k_{i, L} (\hat{s}; r_0) = \frac{\gamma^2}{\mu} \hat{s}_i \hat{s}_k G_L (\hat{s}; r_0; k_P)
\]

with $\gamma = k_P / k_S$. 

S. Chaillat
Decomposition of elastodynamic fundamental solution

Multipole expansion of $U^k$

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U^k_i(x, y; \omega) = \lim_{L \to +\infty} \int_{\hat{s} \in S} e^{ikP\hat{s}.(y-y_0)} U^k_i, P(\hat{s}; r_0) e^{-ikP\hat{s}.(x-x_0)} d\hat{s}
+ \lim_{L \to +\infty} \int_{\hat{s} \in S} e^{ikS\hat{s}.(y-y_0)} U^k_i, S(\hat{s}; r_0) e^{-ikS\hat{s}.(x-x_0)} d\hat{s}
\]

where

\[
U^k_i, S(\hat{s}; r_0) = \frac{1}{\mu} (\delta_{ik} - \hat{s}_k \hat{s}_i) G_L(\hat{s}; r_0; k_S),
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U^k_i, P(\hat{s}; r_0) = \frac{\gamma^2}{\mu} \hat{s}_i \hat{s}_k G_L(\hat{s}; r_0; k_P)
\]

with $\gamma = k_P/k_S$.

Multipole expansion of same form for $T^k$
Single-level FM single region method

Boundary of interest enclosed in cubic grid

\[ \partial \Omega \]

\[ d \]
Single-level FM single region method

Boundary of interest enclosed in cubic grid

Interior problem

Exterior problem
Convergence of multipole expansion assured if $x$ and $y$ lie in non-adjacent cells.
Convergence of multipole expansion assured if $x$ and $y$ lie in non-adjacent cells.

$\Omega$ domain boundary

$C_x \notin A(C_x)$

$C_y \notin A(C_x)$

$C_y \in A(C_x)$

$\Rightarrow$ System matrix split into near and far parts.
Matrix-vector product ← evaluation of integral operator

- Must compute (for given $u_i(y)$):

$$[\mathcal{K}\{u\}](x) := c_{ik}(x)u_i(x) + \int_{\partial\Omega} u_i(y) T_i^k(x, y, \omega) dS_y$$

- Split integrals into near and FM contributions:

$$\int_{\partial\Omega} = \sum_{c_y \in \mathcal{A}(c_x)} \int_{\partial\Omega \cap c_y} + \sum_{c_y \notin \mathcal{A}(c_x)} \int_{\partial\Omega \cap c_y}$$

$$[\mathcal{K}\{u\}](x) = [\mathcal{K}\{u\}]^{\text{near}}(x) + [\mathcal{K}\{u\}]^{\text{FM}}(x)$$
Single-level FM single region method

Algorithm

Integration cell

<table>
<thead>
<tr>
<th>y_1</th>
<th>y_2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>y_0</td>
</tr>
<tr>
<td>y_3</td>
<td>y_4</td>
</tr>
</tbody>
</table>

\[ C_y \]

Collocation cell

<table>
<thead>
<tr>
<th>x_1</th>
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</tr>
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<tbody>
<tr>
<td></td>
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</tr>
<tr>
<td>x_4</td>
<td>x_2</td>
</tr>
</tbody>
</table>

\[ C_x \]
Algorithm

Integration cell

\[ C_y \]

Collocation cell

\[ C_x \]

Compute multipole moments for each cell \( C_y \) and quadrature point \( \hat{s} \in S \):

\[
R_k^S(\hat{s}; C_y) = -ik_S \left[ \delta_{ik} \hat{s}_j + \delta_{jk} \hat{s}_i - 2 \hat{s}_i \hat{s}_j \hat{s}_k \right] \int_{\partial \Omega \cap C_y} u_i(y) n_j(y) e^{ik_S \hat{s} \cdot (y - y_0)} dS_y
\]

\[
R^P(\hat{s}; C_y) = -ik_S \gamma^3 \left[ \frac{2\nu}{1 - 2\nu} \delta_{ij} + 2 \hat{s}_i \hat{s}_j \right] \int_{\partial \Omega \cap C_y} u_i(y) n_j(y) e^{ik_P \hat{s} \cdot (y - y_0)} dS_y
\]
Algorithm

Integration cell

Collocation cell

Compute multipole moments for each cell $C_y$ and quadrature point $\hat{s} \in S$:

Transfer from $C_y$ to non-adjacent $C_x$:

$$\mathcal{L}_k^S(\hat{s}; C_x) = \sum_{C_y \notin A(C_x)} G_L(\hat{s}; r_0; k_S) R_k^S(\hat{s}; C_y)$$

$$\mathcal{L}_k^P(\hat{s}; C_x) = \sum_{C_y \notin A(C_x)} G_L(\hat{s}; r_0; k_P) R_k^P(\hat{s}; C_y)$$
...Single-level FM single region method

Algorithm

Integration cell

\[ C_y \]

\[
\begin{array}{cccc}
  y_1 & y_2 \\
  & y_0 \\
  y_3 & y_4 \\
\end{array}
\]

Collocation cell

\[ C_x \]

\[
\begin{array}{cccc}
  x_1 & x_3 \\
  & x_0 \\
  x_4 & x_2 \\
\end{array}
\]

- Compute multipole moments for each cell \( C_y \) and quadrature point \( \hat{s} \in S \):
- Transfer from \( C_y \) to non-adjacent \( C_x \):
- Evaluate FM contribution \( (Ku)^{FM}(x) \) to matrix-vector product:

\[
(Ku)^{FM}_k(x) \approx \sum_q w_q [e^{-ikS\hat{s}_q \cdot (x-x_0)} L^S_k(\hat{s}_q; C_x) + e^{-ikP\hat{s}_q \cdot (x-x_0)} (\hat{s}_q)_k L^P(\hat{s}_q; C_x)]
\]
...Single-level FM single region method

Algorithm

- Compute multipole moments for each cell $C_y$ and quadrature point $\hat{s} \in S$:
- Transfer from $C_y$ to non-adjacent $C_x$:
- Evaluate FM contribution $(\mathcal{K}u)^{FM}(x)$ to matrix-vector product:
- Add near contribution $(\mathcal{K}u)^{near}(x)$ to matrix-vector product (computed using standard BEM techniques)
Single-region FMM (homogeneous domain)

Complexity of single-level FMM: $O(N^{3/2})$ instead of $O(N^2)$

![Diagram of FMM and BEM](attachment:diagram.png)

- **Standard BEM**
- **FM-BEM**

FMM for 3-D seismic wave computation
Complexity of single-level FMM: $O(N^{3/2})$ instead of $O(N^2)$

Improve FMM computational efficiency: multi-level FMM
Multi-level FMM

level \( \ell = 0 \)

level \( \ell = 1 \)

- computation organization based on recursive subdivision: octree
- non-FMM calculations confined to smallest spatial region;
- FMM calculations performed between large groups;
Multi-level FMM

- **level $\ell = 0$**
- **level $\ell = 1$**
- **level $\ell = 2$**

- Computation organization based on recursive subdivision: octree
- Non-FMM calculations confined to smallest spatial region;
- FMM calculations performed between large groups;
- Octree: start at level $\ell = 2$
Multi-level FMM

- level $\ell = 0$
- level $\ell = 1$
- level $\ell = 2$

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Multi-level FMM

level $\ell = 0$

level $\ell = 1$

level $\ell = 2$

level $\ell = 3$

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Multi-level FMM

level $\ell = 0$

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level $\ell = 2$

level $\ell = 3$

: level $\ell = \bar{\ell}$ (leaf)

- computation organization based on recursive subdivision: octree
- non-FMM calculations confined to smallest spatial region;
- FMM calculations performed between large groups;
- octree: start at level $\ell = 2$
Multi-level FMM

Multi-level FMM algorithm

- Multipole moments evaluated for leaf cells
- **Upward pass**: multipole moments for level-\(\ell\) cells by aggregation of those for level-(\(\ell + 1\)) (children) cells
- Transfers (multipole-to-local) at highest possible level
- **Downward pass**: local expansions for level-\(\ell\) cells evaluated from those for level-(\(\ell - 1\)) (parent) cells
- Aggregation of FM contributions to \((\mathcal{K}u)\) at leaf cell level
- Near contributions to \((\mathcal{K}u)\) added at leaf cell level

⇒ Complexity of multi-level FMM: \(O(N \log N)\) instead of \(O(N^2)\)
Previous works on FMM for elastodynamics

- 2-D frequency-domain [Chen et al., Comp. Mechanics, 1997]
- 3-D frequency-domain (low frequency, crack problems, $N = O(6 \times 10^4)$) [Yoshida, PhD thesis, 2001]
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- 3-D frequency-domain (diagonal form):  [Fujiwara, GJI, 2000]
  - level-independent value for the truncation parameter \(L\)
  - low-frequency seismic oriented examples; \(N = O(2 \times 10^4)\)
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Present work: incorporation of recent improvements for the Maxwell equations (diagonal form); multi-region problems; $N = O(7 \times 10^5 - 10^6)$
Definition of illustrative example

Pressurized spherical cavity embedded in infinite medium

- Exact solution [Eringen, Academic Press, 1975]
- Non-dimensional frequency: \( \eta_P = k_P R / \pi = 2R / \lambda_P \)
  (# of P-wavelengths spanned by cavity diameter)
Transfer function:

\[ G_L(\hat{s}, r_0; k) = \frac{ik}{16\pi^2} \sum_{p=0}^{L} (2p + 1) i^p h_p^{(1)}(k|r_0|) P_p(\cos(\hat{s}, r_0)) \]
Truncation of transfer function

Transfer function:

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Choice of truncation parameter:

- **L** too small: convergence not reached for multipole expansion;
- **L** too large: divergence of \( h_p^{(1)} \) (numerical instabilities);
Truncation of transfer function

Transfer function:

\[ G_L(\hat{s}; r_0; k) = \frac{ik}{16\pi^2} \sum_{p=0}^{L} (2p + 1) i^p h_p^{(1)}(k|r_0|) P_p(\cos(\hat{s}, r_0)) \]

Choice of truncation parameter:

- **L** too small: convergence not reached for multipole expansion;
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Empirical formula used:

\[ L = \sqrt{3}k_Sd + C_\varepsilon \log_{10}(\sqrt{3}k_Sd + \pi) \]

(see [Darve, JCP, 2000] and [Sylvand, PhD thesis, 2002] for Maxwell eqns)
Truncation of transfer function:
Adjustment of constant $C_\varepsilon$

$$L = \sqrt{3}k_Sd + C_\varepsilon \log_{10}(\sqrt{3}k_Sd + \pi)$$
... truncation of transfer function: 
Adjustment of constant $C_\epsilon$

\[ L = \sqrt{3}k_S d + C_\epsilon \log_{10}(\sqrt{3}k_S d + \pi) \]

\[
\begin{align*}
C_\epsilon & = 7.5 \text{ (consistent with } \text{[Sylvand, PhD thesis, 2002], Maxwell eqns)}
\end{align*}
\]
...Truncation of transfer function: 
Break down at low frequencies


There exist four constants $C_1, C_2, C_3, C_4$ such that

$$L = C_1 + C_2 k |r - r_0| + C_3 \ln(k |r - r_0|) + C_4 \ln \epsilon^{-1}$$

$$\implies \left| \frac{\exp(ik|r|)}{4\pi|r|} - \int_{\hat{s} \in S} e^{ik\hat{s} \cdot (y-y_0)} G_L(\hat{s}; r_0; k) e^{-ik\hat{s} \cdot (x-x_0)} d\hat{s} \right| < \epsilon$$

for any chosen error level $\epsilon < 1$, whenever

$$|r - r_0|/|r_0| = |(y - y_0) - (x - x_0)|/|r_0| \leq 2/\sqrt{5}.$$
There exist four constants $C_1, C_2, C_3, C_4$ such that

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$$\implies \left| \frac{\exp(ik|\mathbf{r}|)}{4\pi|\mathbf{r}|} - \int_{\mathbf{s} \in S} e^{ik\mathbf{\hat{s}} \cdot (\mathbf{y} - \mathbf{y}_0)} G_L(\mathbf{\hat{s}}; \mathbf{r}_0; k) e^{-ik\mathbf{\hat{s}} \cdot (\mathbf{x} - \mathbf{x}_0)} d\mathbf{\hat{s}} \right| < \epsilon$$

for any chosen error level $\epsilon < 1$, whenever

$$|\mathbf{r} - \mathbf{r}_0|/|\mathbf{r}_0| = |(\mathbf{y} - \mathbf{y}_0) - (\mathbf{x} - \mathbf{x}_0)|/|\mathbf{r}_0| \leq 2/\sqrt{5}.$$
Number of levels

\[ L = \sqrt{3}k_Sd + C_\epsilon \log_{10}(\sqrt{3}k_Sd + \pi) \]

<table>
<thead>
<tr>
<th>( \bar{\ell} ) (leaf level)</th>
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<tr>
<td>3</td>
<td>1.32</td>
</tr>
<tr>
<td>4</td>
<td>0.66</td>
</tr>
<tr>
<td>5</td>
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</tr>
<tr>
<td>6</td>
<td>0.17</td>
</tr>
<tr>
<td>7</td>
<td>0.083</td>
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\((k_p a = 6\pi \text{ and } N = 122,886 \text{ DOFs})\)
Number of levels

\[ L = \sqrt{3}k_Sd + C_\varepsilon \log_{10}(\sqrt{3}k_Sd + \pi) \]

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<td>4.7 \times 10^{-4}</td>
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<tr>
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<td>3.7 \times 10^{-3}</td>
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\((k_p a = 6\pi \text{ and } N = 122,886 \text{ DOFs})\)
Number of levels

\[ L = \sqrt{3} k_S d + C_\epsilon \log_{10}(\sqrt{3} k_S d + \pi) \]

<table>
<thead>
<tr>
<th>$\bar{\ell}$ (leaf level)</th>
<th>$d^{(\bar{\ell})} \times k_S / 2\pi$</th>
<th>FMM / BEM</th>
<th>CPU / iter (s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>1.32</td>
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<td>367</td>
</tr>
<tr>
<td>4</td>
<td>0.66</td>
<td>4.7 $10^{-4}$</td>
<td>134</td>
</tr>
<tr>
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</tr>
<tr>
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<td>5.1 $10^{-2}$</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>0.083</td>
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($k_p a = 6\pi$ and $N = 122,886$ DOFs)
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<tr>
<td>6</td>
<td>0.17</td>
<td>$5.1 \times 10^{-2}$</td>
<td>200</td>
</tr>
<tr>
<td>7</td>
<td>0.083</td>
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<td>380</td>
</tr>
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</table>

\((k_p a = 6\pi \text{ and } N = 122,886 \text{ DOFs})\)

Choice of leaf cell size (\(\Leftrightarrow\) choice of # of levels):
- influence on CPU
- influence on accuracy
Number of levels

\[ L = \sqrt{3} k_S d + C_\varepsilon \log_{10}(\sqrt{3} k_S d + \pi) \]

<table>
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<tr>
<td>3</td>
<td>1.32</td>
<td>1.110^{-5}</td>
<td>367</td>
</tr>
<tr>
<td>4</td>
<td>0.66</td>
<td>4.710^{-4}</td>
<td>134</td>
</tr>
<tr>
<td>5</td>
<td>0.33</td>
<td>3.710^{-3}</td>
<td>104</td>
</tr>
<tr>
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<td>200</td>
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\((k_p a = 6\pi \text{ and } N = 122,886 \text{ DOFs})\)

Choice of leaf cell size (\(\Leftrightarrow\) choice of # of levels):
- influence on CPU
- influence on accuracy

\[ \Rightarrow d(\bar{\ell}) \geq 0.3 \times \lambda_S \]
Other formulations

Low-frequency FMM

  [Darve and Havé, J. Comp. Physics, 2004]
  [Cheng et al., J. Comp. Physics, 2006]

Other formulations

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  [Darve and Havé, J. Comp. Physics, 2004]
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Combination of low- and mid-frequency FM techniques

  [Otani and Nishimura, J. Comp. Physics, 2008]

- So far not applied to elastodynamics
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Combination of low- and mid-frequency FM techniques

  [Otani and Nishimura, J. Comp. Physics, 2008]

- So far not applied to elastodynamics

⇒ Present work limited to mid-frequencies but extension to low frequencies and low-mid combination feasible
Quadrature over unit sphere

\[
\frac{\exp(i k |x - y|)}{|x - y|} = \lim_{L \to +\infty} \int_{\hat{s} \in S} e^{i k \hat{s} \cdot (y_0 - y)} G_L(\hat{s}; r_0, k) e^{-i k \hat{s} \cdot (x - x_0)} d\hat{s}
\]

Product rule in angular spherical coordinates \((\phi, \theta)\):

- \(\phi\): 2\(L\) + 1-point uniform rule over \([0, 2\pi]\)
- \(\theta\): \(L\) + 1-point Gauss-Legendre rule over \([0, \pi]\)
Quadrature over unit sphere

\[ \frac{\exp(ik|x - y|)}{|x - y|} = \lim_{L \to +\infty} \int_{\hat{s} \in S} e^{ik\hat{s}.(y_0 - y)} G_L(\hat{s}; r_0; k) e^{-ik\hat{s}.(x - x_0)} d\hat{s} \]

Product rule in angular spherical coordinates \((\phi, \theta)\):

- \(\phi\): 2L + 1-point uniform rule over \([0, 2\pi]\)
- \(\theta\): \(L + 1\)-point Gauss-Legendre rule over \([0, \pi]\)

- \(Q = (L + 1)(2L + 1)\) quadrature points overall
- \(L(\ell) \implies Q(\ell)\)
Extrapolation procedure

Nb quadrature points level-dependent $\rightarrow$ direct (resp. inverse) extrapolation procedure after upward (resp. downward) pass

Three-step algorithm (Maxwell eqns [Darve, JCP, 2000]) $\mathcal{F}_{i'j'} = \mathcal{F}(\theta_{i'}, \phi_{j'})$
Extrapolation procedure

Nb quadrature points level-dependent $\rightarrow$ direct (resp. inverse) extrapolation procedure after upward (resp. downward) pass

Three-step algorithm (Maxwell eqns [Darve, JCP, 2000]) $\mathcal{F}_{ij'p'} = \mathcal{F}(\theta_{ij'}, \phi_{ij'}$)

- forward Fast Fourier Transform:

$$\tilde{\mathcal{F}}_{im}^{l+1} = \sum_{j=0}^{2L^{l+1}} e^{-im\phi_{ij}^{l+1}} \mathcal{F}_{ij}^{l+1} \quad (|m| \leq L^{l+1})$$
Extrapolation procedure

Nb quadrature points level-dependent $\implies$ direct (resp. inverse) extrapolation procedure after upward (resp. downward) pass

Three-step algorithm (Maxwell eqns [Darve, JCP, 2000]) $\mathcal{F}_{i'j'} = \mathcal{F}(\theta_{i'}, \phi_{j'})$

- forward Fast Fourier Transform:
- dense matrix-vector product:

$$\tilde{\mathcal{F}}_\ell^{i' m} = \sum_{i=0}^{L^{\ell+1}} B_{i'i}^{m,\ell} \tilde{\mathcal{F}}_{im}^{\ell+1}$$

$$B_{i'i}^{m,\ell} = \sum_{p=|m|}^{L^{\ell+1}} Q_p^m (\cos \theta_i^{\ell+1}) Q_p^m (\cos \theta_{i'}^\ell), \quad Q_p^m (u) = \sqrt{\frac{2p+1}{4\pi}} \frac{(p-m)!}{(p+m)!} P_p^m (u)$$
Extrapolation procedure

Nb quadrature points level-dependent $\implies$ direct (resp. inverse) extrapolation procedure after upward (resp. downward) pass

Three-step algorithm (Maxwell eqns [Darve, JCP, 2000]) $F_{i'j'} = \mathcal{F}(\theta_{i'}, \phi_{j'})$

- forward Fast Fourier Transform:
- dense matrix-vector product:
- backward Fast Fourier Transform:

$$F_{i'j'}^\ell = \sum_{m=-L_{\ell+1}}^{L_{\ell+1}} e^{im\phi_{i'}} \tilde{F}_{i'm}^\ell$$
Extrapolation procedure

Nb quadrature points level-dependent $\implies$ direct (resp. inverse) extrapolation procedure after upward (resp. downward) pass

Three-step algorithm (Maxwell eqns [Darve, JCP, 2000]) $\mathcal{F}_{i'j'} = \mathcal{F}(\theta_{i'}, \phi_{j'})$

- forward Fast Fourier Transform:
- dense matrix-vector product:
- backward Fast Fourier Transform:

Advantages
- exact method
- only modest fraction of CPU time in our implementation
Implementation based on 3-noded linear triangular BEs and GMRES

<table>
<thead>
<tr>
<th>$k_p a/\pi$</th>
<th>0.1</th>
<th>0.50</th>
<th>1.00</th>
<th>2.00</th>
</tr>
</thead>
<tbody>
<tr>
<td># nodes per $\lambda_S$</td>
<td>80</td>
<td>16</td>
<td>8</td>
<td>4</td>
</tr>
<tr>
<td>(a) $r = R$</td>
<td>$2 \times 10^{-2}$</td>
<td>$6 \times 10^{-3}$</td>
<td>$6 \times 10^{-3}$</td>
<td>$2 \times 10^{-2}$</td>
</tr>
<tr>
<td>(b) $R &lt; r \leq 3R$</td>
<td>$1 \times 10^{-2}$</td>
<td>$6 \times 10^{-3}$</td>
<td>$8 \times 10^{-3}$</td>
<td>$3 \times 10^{-2}$</td>
</tr>
</tbody>
</table>

RMS relative solution error
(a) on cavity surface
(b) at internal observation points
\[
\frac{\exp(ik|x - y|)}{|x - y|} = \lim_{L \to +\infty} \int_{\hat{s} \in S} e^{ik\hat{s} \cdot (y_0 - y)} G_L(\hat{s}; r_0; k) e^{-ik\hat{s} \cdot (x - x_0)} d\hat{s}
\]

- Truncation parameter \( L \) chosen according to empirical formula \( L = \sqrt{3}k_Sd + C \log_{10}(\sqrt{3}k_Sd + \pi) \) ([Darve, JCP, 2000], Maxwell FMM)
- \( \bar{\ell} \) determined such that \( d^{(\bar{\ell})} \approx 0.3\lambda_S \) for smallest ("leaf") cell;
- Numerical quadrature over \( S \): \( Q = O(L^2) \) quadrature points \( Q \) is level-dependent (grows approx. quadratically with cell size)
- Upward (downward) passes feature direct (inverse) extrapolation step
\[
\frac{\exp(ik|x - y|)}{|x - y|} = \lim_{L \to +\infty} \int_{\hat{s} \in S} e^{i\hat{s} \cdot (y_0 - y)} \mathcal{G}_L(\hat{s}; r_0; k) e^{-i\hat{s} \cdot (x - x_0)} d\hat{s}
\]

- Truncation parameter $L$ chosen according to empirical formula $L = \sqrt{3}k_S d + C \log_{10}(\sqrt{3}k_S d + \pi)$ ([Darve, JCP, 2000], Maxwell FMM)
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- Upward (downward) passes feature direct (inverse) extrapolation step

**Complexity analysis:** $O(N \log N)$ per iter. (instead of $O(N^2)$ per iter. for standard BEM)
Numerical verification of theoretical complexity (CPU)

The diagram shows the CPU time per iteration (s) as a function of the number of nodes (N). The complexity of different methods is indicated:

- Standard BEM: $O(N^2)$
- Single-level FMM: $O(N^{3/2})$
- Multi-level FMM: $O(N \log N)$

The data points from a single-proc. 3 GHz PC match the theoretical complexity for multi-level FMM.
Numerical verification of theoretical complexity (CPU)

- Standard BEM: $O(N^2)$
- Single-level FMM: $O(N^{3/2})$
- Multi-level FMM: $O(N \log N)$

Limit standard BEM
Limit multi-level FMM

(Single-Proc. 3 GHz PC)
Numerical verification of theoretical complexity

Size of BEM matrix (MB)

\begin{align*}
\text{standard BEM} & : O(N^2) \\
\text{single-level FMM} & : O(N^{3/2}) \\
\text{multi-level FMM} & : O(N \log N)
\end{align*}

30 S. Chaillat
Numerical verification of theoretical complexity (memory)

Size of BEM matrix (MB)

- Standard BEM: $O(N^2)$
- Single-level FMM: $O(N^{3/2})$
- Multi-level FMM: $O(N \log N)$

Limit standard BEM
Limit multi-level FMM
Semispherical canyon (3-D configuration)

Diffraction plane P wave by semispherical canyon

Plane P wave

D = 3R

Free surface

Elastic half-space

R

x, y

z
Semispherical canyon (3-D configuration)

$(N = 23,382)$
Comparison with earlier results (low frequency)

Comparison with (semi-analytical) results of [Sánchez-Sesma, BSSA, 1983], $k_P a / \pi = 0.25$

$s/a$: normalized arc-length coordinate along ABC

![Graph showing comparison between present FMM and Sanchez-Sesma's results for displacement modulus $|u_z|$ and $|u_y|$]
Results for higher frequency $k_{Pa}/\pi = 5$

$N = 287,946$ (86 iter., 162 s per iter., single-proc. 3 GHz PC)

$s/a$: normalized arc-length coordinate along ABC

Motivations and background

3-D elastodynamic FMM: single-region

3-D elastodynamic FMM: multi-region

Modelling simple 3-D seismic problems

Improvement: Preconditioning strategy

Conclusions and future work
All examples concern geometrical configurations involving:
- a semi-infinite homogeneous reference medium,
- a finite number of finite homogeneous regions
Use of three-noded triangular BE

Interpolation:

- piecewise-linear interpolation of displacements
- piecewise-constant interpolation of tractions
Use of three-noded triangular BE

Interpolation:
- piecewise-linear interpolation of displacements
- piecewise-constant interpolation of tractions

Single region FMM applied independently in each sub-domain
- definition of an octree in each sub-domain
Continuous BEM formulations for seismic wave propagation (total field)

- in the semi-infinite homogeneous medium $\Omega_1$:

$$
\begin{align*}
& c_{ik}(x)u_i(x) + \int_{\partial\Omega_1} u_i(y) T^k_i \, dS_y - \int_{\Gamma_1} t_i(y) U^k_i \, dS_y \\
& = c_{ik}^F(x)u_i^F(x) + \int_{\Gamma_F} u_i^F(y) T^k_i \, dS_y, \quad \forall x \in \partial\Omega_1
\end{align*}
$$

$u^F$ : free field (incident and reflected waves from free surface $\Gamma^F$)

- in the finite region $\Omega_\ell$ ($2 \leq \ell \leq n$):

$$
\begin{align*}
& c_{ik}(x)u_{i\ell m}(x) + \int_{\Gamma_\ell} u_i^\ell(y) T^{k(\ell)}_i \, dS_y + \sum_{m=2}^{\ell-1} \int_{\Gamma_{\ell m}} \left( u_{i\ell m}^m(y) T^{k(\ell)}_i + t_i^{m\ell}(y) U^{k(\ell)}_i \right) dS_y \\
& + \sum_{m=\ell+1}^{n} \int_{\Gamma_{\ell m}} \left( u_{i\ell m}^{\ell m}(y) T^{k(\ell)}_i - t_i^{\ell m}(y) U^{k(\ell)}_i \right) dS_y = 0, \quad \forall x \in \Gamma_{\ell m}
\end{align*}
$$

Formulation in terms of total fields (perfect bonding conditions)
Linear combinations of collocation on interfaces

Single region FMM applied independently in each sub-domain
(collocation: nodes + element centers)

\[
\Omega_1: \begin{bmatrix}
C_1^1 + H_1^1 & H_1^{12} & H_1^{13} & -G_1^{12} & -G_1^{13} \\
H_1^{12} & C_1^{12} + H_1^{12} & H_1^{13} & -G_1^{12} & -G_1^{13} \\
H_1^{13} & H_1^{12} & C_1^{13} + H_1^{13} & -G_1^{12} & -G_1^{13} \\
H_1^{12} & C_1^{12} + H_1^{12} & H_1^{13} & -G_1^{12} & -G_1^{13} \\
H_1^{13} & H_1^{12} & C_1^{13} + H_1^{13} & -G_1^{12} & -G_1^{13}
\end{bmatrix} \begin{bmatrix}
u_1^1 \\
u_1^{12} \\
u_1^{13} \\
u_2^1 \\
u_2^{12} \\
u_2^{13} \\
u_3^1 \\
u_3^{12} \\
u_3^{13} \\
t_1^1 \\
t_1^{12} \\
t_1^{13} \\
t_2^1 \\
t_2^{12} \\
t_2^{13} \\
t_3^1 \\
t_3^{12} \\
t_3^{13}
\end{bmatrix} = \begin{bmatrix} b_1^1 \\
b_1^{12} \\
b_1^{13} \\
b_2^1 \\
b_2^{12} \\
b_2^{13} \\
b_3^1 \\
b_3^{12} \\
b_3^{13} \end{bmatrix}
\]

\[
\Omega_2: \begin{bmatrix}
C_1^{21} + H_1^{21} & G_1^{21} & H_1^{23} & -G_1^{23} \\
G_1^{23} & C_1^{23} + H_1^{23} & -G_1^{23} \\
G_1^{21} & H_1^{23} & -G_1^{23} \\
G_1^{23} & C_1^{23} + H_1^{23} & -G_1^{23}
\end{bmatrix} \begin{bmatrix}
u_1^{12} \\
u_1^{23} \\
u_2^{12} \\
u_2^{23} \\
u_3^{12} \\
u_3^{23} \end{bmatrix} = \begin{bmatrix} b_n^{21} \\
b_n^{23} \\
b_2^{21} \\
b_2^{23} \\
b_3^{21} \\
b_3^{23} \end{bmatrix}
\]

\[
\Omega_3: \begin{bmatrix}
\ldots \\
\ldots \\
\ldots 
\end{bmatrix} = \begin{bmatrix} \ldots \end{bmatrix}
\]
Single region FMM applied independently in each sub-domain
(collocation: nodes + element centers)

\[ \Omega_1 : \begin{bmatrix} C_1^1 + H_1^1 & H_{12}^1 & H_{13}^1 & -G_{12}^1 & -G_{13}^1 \\ H_{12}^1 & C_{12}^{12} + H_{12}^{12} & H_{13}^{12} & -G_{12}^{12} & -G_{13}^{12} \\ H_{13}^1 & H_{12}^{13} & C_{13}^{13} + H_{13}^{13} & -G_{12}^{13} & -G_{13}^{13} \\ H_{12}^{12} & C_{12}^{12} + H_{12}^{12} & H_{13}^{12} & -G_{12}^{12} & -G_{13}^{12} \\ H_{13}^{13} & H_{12}^{13} & C_{13}^{13} + H_{13}^{13} & -G_{12}^{13} & -G_{13}^{13} \end{bmatrix} \begin{bmatrix} u_1^1 \\ u_{12}^1 \\ t_{12}^1 \\ u_{13}^1 \\ t_{13}^1 \end{bmatrix} = \begin{bmatrix} b_1^1 \\ b_{n12}^1 \\ b_{n13}^1 \\ b_e^{12} \\ b_e^{13} \end{bmatrix} + \begin{bmatrix} b_2^1 \\ b_{n23}^1 \\ b_{n21}^1 \\ b_{e21}^1 \\ b_{e23}^1 \end{bmatrix} \]

\[ \Omega_2 : \begin{bmatrix} C_{12}^{21} + H_{12}^{21} & G_{12}^{21} & H_{23}^{21} & -G_{23}^{21} \\ H_{12}^{23} & G_{12}^{23} & C_{23}^{23} + H_{23}^{23} & -G_{23}^{23} \\ C_{12}^{21} + H_{12}^{21} & G_{12}^{21} & H_{23}^{21} & -G_{23}^{21} \\ H_{12}^{23} & G_{12}^{23} & C_{23}^{23} + H_{23}^{23} & -G_{23}^{23} \end{bmatrix} \begin{bmatrix} u_{12}^1 \\ t_{12}^1 \\ u_{23}^1 \\ t_{23}^1 \end{bmatrix} = \begin{bmatrix} b_{n21}^1 \\ b_{n23}^1 \\ b_{e21}^1 \\ b_{e23}^1 \end{bmatrix} - \begin{bmatrix} b_3^1 \\ b_{n3}^1 \\ b_{n32}^1 \\ b_{e32}^1 \end{bmatrix} \]

\[ \Omega_3 : \begin{bmatrix} \ldots \\ \ldots \\ \ldots \end{bmatrix} \]

- Over-determined system;
- Contribution of \( \Omega_i \) and of \( \Omega_j \) to collocation on interface \( \Gamma_{ij} \) are linearly combined (optimal coefficients chosen using numerical experiments)
Determination of optimal weightings

Test problem: spherical cavity subjected to an internal time-harmonic uniform pressure, surrounded by a spherical shell embedded in an unbounded elastic medium
Determination of optimal weightings

**Test problem:** spherical cavity subjected to an internal time-harmonic uniform pressure, surrounded by a spherical shell embedded in an unbounded elastic medium

![Diagram of spherical cavity and shell](image)

**Set of coefficients**

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha_u^{12} )</td>
<td>0.5</td>
<td>0.5</td>
<td>0.5</td>
<td>-0.5</td>
<td>-0.5</td>
<td>-0.5</td>
</tr>
<tr>
<td>( \alpha_t^{12} )</td>
<td>0.5</td>
<td>-0.5</td>
<td>-0.5</td>
<td>0.5</td>
<td>-0.5</td>
<td>-0.5</td>
</tr>
<tr>
<td>( \alpha_u^{21} )</td>
<td>0.5</td>
<td>0.5</td>
<td>0.5</td>
<td>-0.5</td>
<td>0.5</td>
<td>-0.5</td>
</tr>
<tr>
<td>( \alpha_t^{21} )</td>
<td>-0.5</td>
<td>0.5</td>
<td>-0.5</td>
<td>-0.5</td>
<td>0.5</td>
<td>0.5</td>
</tr>
</tbody>
</table>
Determination of optimal weightings

<table>
<thead>
<tr>
<th>test problem</th>
<th>coeff. set</th>
<th>$E(u^1)$</th>
<th>$E(u^{12})$</th>
<th>$E(t^{12})$</th>
<th>nb iter.</th>
</tr>
</thead>
<tbody>
<tr>
<td>homogeneous</td>
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<td>/</td>
<td>/</td>
<td>/</td>
<td>$\geq$ 300</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>$3.2 \times 10^{-3}$</td>
<td>$2.5 \times 10^{-3}$</td>
<td>$1.6 \times 10^{-2}$</td>
<td>64</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>$8.8 \times 10^{-1}$</td>
<td>$8.8 \times 10^{-1}$</td>
<td>$1.6 \times 10^0$</td>
<td>90</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>/</td>
<td>/</td>
<td>/</td>
<td>$\geq$ 300</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>/</td>
<td>/</td>
<td>/</td>
<td>$\geq$ 300</td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>/</td>
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<td>/</td>
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</tr>
<tr>
<td>non homogeneous</td>
<td>1</td>
<td>$2.4 \times 10^{-2}$</td>
<td>$1.7 \times 10^{-2}$</td>
<td>$3.5 \times 10^{-2}$</td>
<td>94</td>
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<tr>
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<td>$2.4 \times 10^{-2}$</td>
<td>$1.8 \times 10^{-2}$</td>
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<td>$3.5 \times 10^{-2}$</td>
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<tr>
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<td>$1.7 \times 10^{-2}$</td>
<td>$3.5 \times 10^{-2}$</td>
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FMM for 3-D seismic wave computation
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<tr>
<th>test problem</th>
<th>coeff. set</th>
<th>$E(u^1)$</th>
<th>$E(u^{12})$</th>
<th>$E(t^{12})$</th>
<th>nb iter.</th>
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<td>$8.8 \times 10^{-1}$</td>
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<td>182</td>
</tr>
</tbody>
</table>

- Effects on accuracy (singular matrix system) and convergence properties.
- $\alpha_{ij}^u = \alpha_{ji}^u = +0.5$ and $\alpha_{ij}^t = -\alpha_{ji}^t = -0.5$ ($i < j$).
- Linear combinations of $p > 2$ block equations: equal weight $1/p$ to each contributing block equation.
Modelling simple 3-D seismic problems

1. Motivations and background

2. 3-D elastodynamic FMM: single-region

3. 3-D elastodynamic FMM: multi-region

4. Modelling simple 3-D seismic problems

5. Improvement: Preconditioning strategy

6. Conclusions and future work
Diffraction plane P-wave by semispherical basin

(3-D configuration)

free surface

$\Omega_1$

$\Omega_2$

semi-infinite medium

$D = 5a$

plane P wave

$\frac{\mu_2}{\mu_1} = 0.3; \frac{\rho_2}{\rho_1} = 0.6; \nu_1 = 0.25; \nu_2 = 0.30$

QSHA project → http://qsha.unice.fr/
Semispherical sedimentary basin (3-D configuration)

\[ N = 17,502 \]
Comparison with earlier results (low frequency)

Comparison with results of [Sánchez-Sesma, BSSA, 1983] and [Delavaud, PhD thesis, 2007], \( k_{P}^{(1)} \frac{a}{\pi} = 0.5 \)

\[ \Omega_1 \quad \Omega_2 \]

\[ D = 5a \]

\[ N = 17,502 \text{ (39 iter., 8s / iter, single-proc. 3 GHz PC)} \]
Results for higher frequency $k_p^{(1)} a / \pi = 2$

$D = 5a$

$N = 190,299$ (627 iter., 79s / iter, single-proc. 3 GHz PC)
Results for higher frequency $k_P^{(1)} a / \pi = 2$

$P \Omega_2 / \pi = 2$

\[ |u_z| \text{ (present FMM)} \]
\[ |u_y| \text{ (present FMM)} \]

\[ N = 190, 299 \text{ (627 iter., 79s / iter, single-proc. 3 GHz PC)} \]

⇒ A good preconditioning strategy is required
Diffraction of a plane P-wave by a semispherical two-layered sedimentary basin

(3-D configuration)

\[ \frac{\mu_2}{\mu_1} = \frac{\mu_3}{\mu_2} = 0.3; \quad \frac{\rho_2}{\rho_1} = \frac{\rho_3}{\rho_2} = 0.6; \quad \nu_1 = 0.25; \quad \nu_2 = \nu_3 = 0.30 \]

\[ h_2 = \frac{\sqrt{2}a}{(1 + \sqrt{2})}; \quad h_3 = a/(1 + \sqrt{2}) \]
Diffraction of a plane P-wave by a semispherical two-layered sedimentary basin: mesh

Semispherical two-layered sedimentary basin (3-D configuration)

Basin and free-surface $N = 91,893$

Close-up on the two-layered basin
Validation with two-layers of the same material

Comparison with results for a single-layered basin, $k_p^{(1)} a/\pi = 1$

One layer

Two layers with same material
Validation with two-layers of the same material

Comparison with results for a single-layered basin, $k_P^{(1)} a/\pi = 1$

$\Omega_1 \quad \Omega_2 \quad \Omega_3$

$D = 5a$

One layer

Two layers with same material

$N = 91,893$ (240 iter., 48s / iter, single-proc. 3 GHz PC)
Results with two layers of different materials

Comparison with results for homogeneous basin, $k_P^{(1)} a/\pi = 1$

$\Omega_1$ $\Omega_2$ $\Omega_3$

One layer

$D = 5a$

Two layers

$D = 5a$
Results with two layers of different materials

Comparison with results for homogeneous basin, $k^{(1)}_P a / \pi = 1$

\[ a \]

\[ \Omega_1 \]

\[ \Omega_2 \]

\[ \Omega_3 \]

\[ D = 5a \]

\[ r \]

\[ 15 \]

\[ 10 \]

\[ 5 \]

\[ 0 \]

\[ 0 \]

\[ 1 \]

\[ 2 \]

\[ N = 91,893 \) (248 iter., 48s / iter, single-proc. 3 GHz PC) \]

⇒ High amplification at the basin center;

⇒ Different wavelengths;

N = 91, 893 (248 iter., 48s / iter, single-proc. 3 GHz PC)
⇒ High amplification at the basin center;
⇒ Different wavelengths;
Time-domain results


\[
\begin{align*}
\Omega_1 & \quad \Omega_2 \\
\text{free surface} & \quad a \\
D = 5a & \\
\text{semi-infinite medium} & \\
\text{plane SV wave} & \\
\theta = 30^\circ & \\
\end{align*}
\]

\[
\begin{align*}
\mu_1 &= 1; \rho_1 = 1; \nu_1 = 1/3; \\
\mu_2 &= 1/6; \rho_2 = 2/3; \nu_2 = 1/3; \\
N &= 36,033
\end{align*}
\]
Time-domain results


\[ \mu_1 = 1; \rho_1 = 1; \nu_1 = 1/3; \]
\[ \mu_2 = 1/6; \rho_2 = 2/3; \nu_2 = 1/3; \]
\[ N = 36,033 \]

Fast Fourier synthesis of frequency-domain responses

- Incident signal: Ricker wavelet, predominant period \( t_p = 4 \);
- \( 0 < \omega < 3.5 \times 2\pi/t_p \), 32 sample frequencies;
Time-domain results


\[ \mu_1 = 1; \rho_1 = 1; \nu_1 = 1/3; \]
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Time domain surface response: \( u_x \) [\( u_x \) (truncated scale)]

\[ \Omega_1 \quad a \quad \Omega_2 \]

\[ D = 5a \]

plane SV wave \[ \theta = 30^\circ \]

\[ \mu_1 = 1; \rho_1 = 1; \nu_1 = 1/3; \]
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Time domain surface response: \[ u_x \]
\[ u_x \text{ (truncated scale)} \]

[Chaillat S., Bonnet M., Semblat J.F., Geophys. J. Int., 2008 (accepted)]
Improvement: Preconditioning strategy

1. Motivations and background
2. 3-D elastodynamic FMM: single-region
3. 3-D elastodynamic FMM: multi-region
4. Modelling simple 3-D seismic problems
5. Improvement: Preconditioning strategy
6. Conclusions and future work
Preconditioning strategy

- No preconditioning method used in results shown so far.

- Major limitation of the present FM-BEM: number of iterations required to achieve convergence
Preconditioning strategy

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  - Iteration count increases with $N$ (fixed $\omega$)
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  - Multi-region problems are more badly conditioned than single-region problems
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$\Rightarrow$ Preconditioning strategy needed to improve convergence properties.
Preconditioning strategy

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  - iteration count increases with $N$ (fixed $\omega$)
  - iteration count increases with $\omega$ (fixed $N$)
  - multi-region problems are more badly conditioned than single-region problems

$\Rightarrow$ Preconditioning strategy needed to improve convergence properties.

$\Rightarrow$ When using FMM: complete system matrix not explicitly assembled.
Preconditioning strategy

Use of the near contributions matrix

- nested GMRES solvers
- inner GMRES applied for the choice of preconditioning matrix \( M = K_{\text{near}} \)
- outer solver: FGMRES (allow to vary the preconditioner at each step)

⇒ Not CPU-consuming (\( K_{\text{near}} \) already computed and not inverted)
Preconditioning strategy

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- outer solver: FGMRES (allow to vary the preconditioner at each step)

⇒ Not CPU-consuming ($K^{\text{near}}$ already computed and not inverted)

Algorithm

**Outer solver** (FGMRES)

for $k = 1, \ldots$ do

- Matrix-vector product: FMM
- Preconditioning: **Inner solver** (GMRES)

for $i = 1, \ldots$ do

- Matrix-vector product: multiply by the sparse matrix $K^{\text{near}}$
- No preconditioning

end for

end for
Efficiency of the preconditioning strategy

Diffraction of plane P- or SV-wave by a semi-ellipsoidal basin ($b = 2a$)

$$\nu^{(1)} = \nu^{(2)} = 1/3, \mu^{(2)} = 1/4\mu^{(1)}, \rho^{(2)} = \rho^{(1)}, \theta = 0^\circ, 30^\circ$$

Computational data

<table>
<thead>
<tr>
<th>$k_p^{(1)} a/\pi$</th>
<th>$N$</th>
<th>$\bar{l}_1; \bar{l}_2$</th>
<th>CPU time (s) per iter.</th>
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<tbody>
<tr>
<td>1</td>
<td>278,304</td>
<td>4; 3</td>
<td>111</td>
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<tr>
<td>1.5</td>
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<td>6; 5</td>
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### Efficiency of the preconditioning strategy

<table>
<thead>
<tr>
<th>Wave Type</th>
<th>Formula</th>
</tr>
</thead>
</table>
| Vertical P–wave | $k_P^{(1)} a / \pi = 1$  
                 | $k_P^{(1)} a / \pi = 1.5$               |
| Oblique P–wave | $k_P^{(1)} a / \pi = 1$  
                | $k_P^{(1)} a / \pi = 1.5$               |
| Vertical SV–wave | $k_P^{(1)} a / \pi = 1$  
                | $k_P^{(1)} a / \pi = 1.5$               |
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                | $k_P^{(1)} a / \pi = 1.5$               |
### Efficiency of the preconditioning strategy

<table>
<thead>
<tr>
<th>Wave Type</th>
<th>Without prec.</th>
<th>With prec.</th>
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<tbody>
<tr>
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<td>nb iter.</td>
<td>nb iter.</td>
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<tr>
<td></td>
<td></td>
<td>outer</td>
</tr>
<tr>
<td><strong>vertical P–wave</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$k^P_1 a / \pi = 1$</td>
<td>734</td>
<td>44</td>
</tr>
<tr>
<td>$k^P_1 a / \pi = 1.5$</td>
<td>/</td>
<td>128</td>
</tr>
<tr>
<td><strong>oblique P–wave</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$k^P_1 a / \pi = 1$</td>
<td>681</td>
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<tr>
<td>$k^P_1 a / \pi = 1$</td>
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<td>$k^P_1 a / \pi = 1.5$</td>
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<tr>
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<td>608</td>
<td>46</td>
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<tr>
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<tr>
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<td>133</td>
<td>1077</td>
</tr>
</tbody>
</table>

⇒ Simple preconditioning already very efficient
⇒ Comparative study with other preconditioning strategies used in electromagnetic FMM (under way)
Mechanical parameters

- **Bedrock Ω₁:**
  \[ c_{p}^{(1)} = 5,600 \text{ m.s}^{-1}, \quad c_{s}^{(1)} = 3,200 \text{ m.s}^{-1} \quad \text{and} \quad \rho^{(1)} = 2,720 \text{ kg.m}^{-3} \]

- **Homogeneous layer Ω₂:**
  \[ c_{p}^{(2)} = 1,988 \text{ m.s}^{-1}, \quad c_{s}^{(2)} = 526 \text{ m.s}^{-1} \quad \text{and} \quad \rho^{(2)} = 2,206 \text{ kg.m}^{-3} \]
Diffraction of an incident plane wave by an Alpine valley (Grenoble)

Mechanical parameters

- **Bedrock \( \Omega_1 \):**
  \[ c_p^{(1)} = 5,600 \text{ m.s}^{-1}, \ c_s^{(1)} = 3,200 \text{ m.s}^{-1} \text{ and } \rho^{(1)} = 2,720 \text{ kg.m}^{-3} \]

- **Homogeneous layer \( \Omega_2 \):**
  \[ c_p^{(2)} = 1,988 \text{ m.s}^{-1}, \ c_s^{(2)} = 526 \text{ m.s}^{-1} \text{ and } \rho^{(2)} = 2,206 \text{ kg.m}^{-3} \]

Mesh
Mechanical parameters

- Bedrock $\Omega_1$:
  
  \[ c_p^{(1)} = 5,600 \text{ m.s}^{-1}, \quad c_s^{(1)} = 3,200 \text{ m.s}^{-1} \quad \text{and} \quad \rho^{(1)} = 2,720 \text{ kg.m}^{-3} \]

- Homogeneous layer $\Omega_2$:
  
  \[ c_p^{(2)} = 1,988 \text{ m.s}^{-1}, \quad c_s^{(2)} = 526 \text{ m.s}^{-1} \quad \text{and} \quad \rho^{(2)} = 2,206 \text{ kg.m}^{-3} \]

Tentative model, with several simplifications:

- Topography outside the sedimentary basin not considered (reduction DOFs)
- Only one layer (reduction DOFs+pb mesh generation)
- North ends of the valley closed artificially (pb mesh generation)
- Simplified incident motion (incident plane wave)
Diffraction of a vertical incident plane P–wave by the Alpine valley:

Modulus of the z-component of displacement, \( f = 0.6 \) Hz.

Solution: \( N = 141,288,747 \) iter. (with prec.), 75 h.
Diffraction of an incident plane wave by an Alpine valley (Grenoble)

Diffraction of a vertical incident plane P–wave by the Alpine valley:

Modulus of the z-component of displacement, $f = 0.6$ Hz.

Solution: $N = 141, 288, 747$ iter. (with prec.), 75 h.

Limitation of the present FM-BEM to deal with basin problems with high velocity contrast between two layers (non-uniform mesh)
Conclusions and future work

1. Motivations and background
2. 3-D elastodynamic FMM: single-region
3. 3-D elastodynamic FMM: multi-region
4. Modelling simple 3-D seismic problems
5. Improvement: Preconditioning strategy
6. Conclusions and future work
Summary and conclusions

- **Multi-level fast multipole method** successfully extended to single-region 3-D frequency-domain elastodynamics

- **Multi-region** FMM implemented for 3-D frequency-domain elastodynamics (based on single-region FMM)

- Simple **preconditioning** strategy implemented and efficiency demonstrated on seismic-oriented problems
Summary and conclusions

- **Multi-level fast multipole method** successfully extended to single-region 3-D frequency-domain elastodynamics.
- **Multi-region** FMM implemented for 3-D frequency-domain elastodynamics (based on single-region FMM).
- Simple **preconditioning** strategy implemented and efficiency demonstrated on seismic-oriented problems.
- Code **COFFEE** (single- and multi-region elastodynamic ML-FMM fully implemented, about 30,000 Fortran 90 instructions).
Multi-level fast multipole method successfully extended to single-region 3-D frequency-domain elastodynamics

Multi-region FMM implemented for 3-D frequency-domain elastodynamics (based on single-region FMM)

Simple preconditioning strategy implemented and efficiency demonstrated on seismic-oriented problems

Code COFFEE (single- and multi-region elastodynamic ML-FMM fully implemented, about 30,000 Fortran 90 instructions)

Implementation consistent with theoretical complexity

Implementation tested against published results for low frequencies, and run for higher frequencies

BE models of size $N = O(10^6)$ on a single-processor PC

Use of a Fourier transform to deal with time-domain problems

Simplified seismic wave amplification studied
Future work

- Multipole expansions for half-space elastodynamic fundamental solutions (in progress)
- Other preconditioning strategies (in progress)
- Extension to visco-elastic media (use of complex wavenumber), thesis E. Grasso 2008-2011: application to soil-structure interaction and vibration induced waves
- Parallel implementation
- Other coupling techniques (weak formulation, BETI, ...)
- Application to defect identification problems
Fast Multipole Method for 3-D elastodynamic boundary integral equations. Application to seismic wave propagation.

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