MATHEMATICAL ANALYSIS OF ELASTIC SURFACE WAVES
IN TOPOGRAPHIC WAVEGUIDES

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We present here a theoretical study of the guided waves in an isotropic homogeneous
elastic half-space whose free surface has been deformed. The deformation is supposed
to be invariant in the propagation direction and localized in the transverse ones. We
show that finding guided waves amounts to solving a family of 2-D eigenvalue problems
set in the cross-section of the propagation medium. Then using the min-max principle
for non-compact self-adjoint operators, we prove the existence of guided waves for some
particular geometries of the free surface. These waves have a smaller speed than that of
the Rayleigh wave in the perfect half-space and a finite transverse energy. Moreover, we
prove that the existence results are valid for arbitrary high frequencies in the presence
of singularities of the free boundary. Finally, we prove that no guided mode can exist at
low frequency, except maybe the fundamental one.

1. Introduction

The mathematical domain of elastic waves propagation, governed by the elasto-
dynamics linearized equations, has been largely studied by a lot of authors, as
Achenbach,[2] Auld[3] and Miklowitz.[25] Surface waves in elastic media have been
pointed out for the first time by Lord Rayleigh,[27] in the case of a homogeneous
isotropic half-space. The anisotropic case has been studied more recently by
Guillot[15] and Chadwick.[10] In these two cases, the calculations can be handled expli-
citly. A more complicated case where analytic computations of surface waves are still
possible, is that of the exterior of an infinite hollow circular cylinder.[8,9,26] However,

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when analytic calculations are no longer possible, other mathematical techniques, based on the spectral theory for self-adjoint operators, have been employed to study surface waves in more general geometries. Let us quote the case of the exterior of an infinite cylinder with an arbitrary bounded cross-section (see Bamberger *et al.*[6] and Wilson and Morrison[33] for a high frequency analysis) or the case of infinite cracks.[1,16]

Of course, such techniques have been used to treat other problems of guided waves in unbounded domains (open waveguides) appearing in various areas of physics: in acoustics,[3,21] in electromagnetism,[4,22] in hydrodynamics[7,31] or in other applications in elasticity.[5] Let us simply emphasize the fact that, different from the applications in acoustics and electromagnetism where the existence of guided waves is due to the presence of heterogeneities, the guiding of elastic surface waves is due to the presence of a free boundary and linked to its geometry. One encounters the same type of phenomenon in the simpler example of trapping waves in hydrodynamics.

The problem that we consider here is the one of a topographic waveguide, evoked for instance by Auld, made of an elastic isotropic homogeneous medium, which is infinite and invariant under translation in one privileged direction, say x_3, and whose transverse cross-section Ω appears as the connected union of a half-plane and a bounded perturbation. The free surface condition is applied along the whole boundary of the domain of propagation. The guided waves are particular solutions propagating in the x_3 direction with a speed that is strictly smaller than the Rayleigh speed and whose transverse energy is essentially concentrated in the disturbed zone of the half-space.

Such a problem appears as considerably more difficult than those evoked above from both theoretical and numerical points of view, and the literature on the subject is rather poor, in spite of its evident practical interest (in microwave technology[20] and in geophysics). The only works we are aware of are those of Lagasse[19,20] and Maradudin.[21] In this paper, we consider more precisely the theoretical question of the existence of such surface waves. The question of the numerical computation of these waves is treated by Duterte and Joly.[13]

This paper is organized as follows: Sec. 2 is devoted to the mathematical setting of the problem. As we shall see, a guided mode will be characterized by its wave number β, its frequency ω and the distribution of the displacement field u in the cross-section Ω of the waveguide. One has to solve a family of self-adjoint eigenvalue problems involving some second-order partial differential operators derived from the linearized equations of elasticity. ω^2 plays the role of the eigenvalue, u plays the role of the eigenfunction, β appears as a parameter. In Sec. 3, we study the essential spectrum of these operators. This is a fundamental preliminary step to the analysis of their point spectrum which corresponds to the guided modes we are interested in. The general statements about this point spectrum are given in Sec. 4. Sections 5 and 6 are the two main sections of this paper. In Sec. 5, we establish the existence results. We point out the influence of the geometry of

the perturbation, and more precisely of its singularities. We prove in particular that, in the presence of particular singularities of the free surface, there are surface waves that propagate at arbitrary high frequencies. On the other hand, we prove that surface waves can also exist if the free surface is very regular, but only for a bounded range of frequencies. In Sec. 6, we treat the very delicate question of low frequency surface waves. We essentially obtain, with a proof that deeply relates with the tools introduced by Duterte and Joly,[13] a nonexistence-type result. We conclude in Sec. 7 by mentioning interesting open questions.

2. The Mathematical Framework

2.1. *The equations for the guided modes*

Let Ω be a connected open set in $\mathbb{R}^2$ such that:

$$\bar{\Omega} = \bar{\Omega}_0 \cup \bar{\mathcal{O}}, \tag{2.1}$$

where

$$\Omega_0 = \{x \in \mathbb{R}^2 : x_2 < 0\} \tag{2.2}$$

and $\mathcal{O}$ is a bounded connected open set such that

$$\mathcal{O} \subset \{x \in \mathbb{R}^2 : 0 < x_2\}\,.$$

In other words (cf. Fig. 1), the domain Ω differs from the perfect half-plane Ω_0 by a bounded connected part $\mathcal{O}$. We suppose that the domain $\mathcal{O}$ is locally Lipschitz so that:

(a) The two-dimensional Korn's inequality is valid in the domain $\mathcal{O}$ (see for example Kondratiev and Oleinik[18]).
(b) The embedding of $H^1(\mathcal{O})$ into $L^2(\mathcal{O})$ is compact (cf. Mazja[23]).

Let us now consider a propagation domain $\tilde{\Omega} \subset \mathbb{R}^3$ defined by:

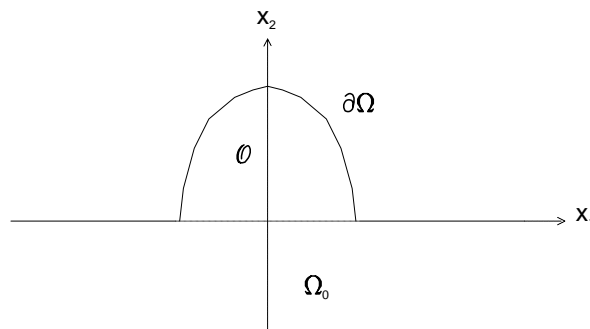$$\tilde{\Omega} = \{\tilde{x} = (x_1, x_2, x_3) \in \mathbb{R}^3 : \ x = (x_1, x_2) \in \Omega\}\,.$$

Fig. 1. The cross-section of the propagation domain.

In the following Ω is called the cross-section of $\tilde{\Omega}$. The boundary of $\tilde{\Omega}$ (resp. Ω) is denoted by $\partial\tilde{\Omega}$ (resp. $\partial\Omega$) and $\tilde{\nu}$ (resp. ν) is the unit outward normal to $\partial\tilde{\Omega}$ (resp. $\partial\Omega$). We suppose that the domain $\tilde{\Omega}$ is occupied by an elastic homogeneous isotropic solid, which is characterized by its Lamé coefficients $\tilde{\lambda}$ and $\tilde{\mu}$ and by its density $\tilde{\rho}$ ($\tilde{\lambda}$, $\tilde{\mu}$ and $\tilde{\rho}$ are strictly positive real numbers), and that the surface $\partial\tilde{\Omega}$ is stress-free.

For any displacement field

$$\tilde{U}(\tilde{x},t) = \begin{pmatrix} \tilde{U}_1(\tilde{x},t) \\ \tilde{U}_2(\tilde{x},t) \\ \tilde{U}_3(\tilde{x},t) \end{pmatrix},$$

we denote by $\tilde{\sigma}(\tilde{U})$ its stress tensor given by the Hooke's law:

$$\tilde{\sigma}_{ij}(\tilde{U}) = \tilde{\lambda}\left(\sum_{m=1}^{3}\tilde{U}_{m,m}\right)\delta_{ij} + \tilde{\mu}(\tilde{U}_{i,j}+\tilde{U}_{j,i}),$$

where δ_{ij} denotes the Kronecker's symbol and $\tilde{U}_{i,j} = \partial\tilde{U}_i/\partial x_j$. We are looking for particular solutions $\tilde{U}(\tilde{x},t)$ of the elastodynamic equations:

$$\begin{cases} \operatorname{div}\tilde{\sigma}(\tilde{U}) = \tilde{\rho}\dfrac{\partial^2\tilde{U}}{\partial t^2}, & \tilde{x}\in\tilde{\Omega}, t\in\mathbb{R} \\[2mm] \tilde{\sigma}(\tilde{U})\tilde{\nu} = 0, & \tilde{x}\in\partial\tilde{\Omega}, t\in\mathbb{R} \end{cases} \tag{2.3}$$

of the form:

$$\tilde{U}(\tilde{x},t) = \tilde{u}(x_1,x_2)e^{i(\beta x_3-\omega t)} \tag{2.4}$$

such that $\omega\in\mathbb{R}^+$, $\beta\in\mathbb{R}^+$ and $\tilde{u}\in H^1(\Omega)^3$. Such a solution is called a guided mode; it describes a wave which propagates in the direction x_3 with the velocity ω/β. We say that ω is the pulsation and β the propagation constant of the mode. The condition $\tilde{u}\in H^1(\Omega)^3$ means that the transverse energy of the mode is finite; numerical results (cf. Ref. 12) show that it is in fact localized in a bounded region of Ω containing $\mathcal{O}$. By substituting (2.4) into (2.3), we derive the two-dimensional problem satisfied by $\tilde{u}$. To obtain a system with real coefficients, we set:

$$u(x) = \begin{pmatrix} \tilde{u}_1(x) \\ \tilde{u}_2(x) \\ i\tilde{u}_3(x) \end{pmatrix}.$$

So we get the following equations:

$$\begin{cases} \operatorname{div}_{\beta}^{*}\sigma^{\beta}(u) = -\omega^2 u, & \text{in }\Omega \\ \sigma^{\beta}(u)\tilde{\nu} = 0, & \text{on }\partial\Omega \end{cases} \tag{2.5}$$

where

$$\sigma_{ij}^{\beta}(u) = \lambda(\mathrm{div}_{\beta}\, u)\delta_{ij} + 2\mu\varepsilon_{ij}^{\beta}(u) \quad \text{with } \lambda = \frac{\tilde{\lambda}}{\tilde{\rho}},\ \mu = \frac{\tilde{\mu}}{\tilde{\rho}} \tag{2.6}$$

$$\begin{cases} \varepsilon_{ij}^{\beta}(u) = \dfrac{1}{2}(u_{i,j} + u_{j,i}), & \text{if } i,j \in \{1,2\} \\[2mm] \varepsilon_{j3}^{\beta}(u) = \varepsilon_{3j}^{\beta}(u) = \dfrac{1}{2}(u_{3,j} - \beta u_j), & \text{if } j \in \{1,2\} \\[2mm] \varepsilon_{33}^{\beta}(u) = \beta u_3\,, \end{cases} \tag{2.7}$$

$$\mathrm{div}_{\beta}\, u = u_{1,1} + u_{2,2} + \beta u_3\,, \tag{2.8}$$

$$\mathrm{div}_{\beta}^{*}\, \sigma^{\beta}(u) = \begin{pmatrix} \sigma_{11,1}^{\beta}(u) + \sigma_{12,2}^{\beta}(u) + \beta\sigma_{13}^{\beta}(u) \\[1mm] \sigma_{21,1}^{\beta}(u) + \sigma_{22,2}^{\beta}(u) + \beta\sigma_{23}^{\beta}(u) \\[1mm] \sigma_{31,1}^{\beta}(u) + \sigma_{32,2}^{\beta}(u) - \beta\sigma_{33}^{\beta}(u) \end{pmatrix}, \tag{2.9}$$

$\sigma^{\beta}(u)$ and $\varepsilon^{\beta}(u)$ are called the reduced stress and deformation tensors.

We want to determine, for every value of β, the values of ω such that (2.5) has nontrivial solutions u in $L^2(\Omega)^3$. It is a two-dimensional parametrized eigenvalue problem: the parameter is β, the eigenvalue is ω^2 and the associated eigenvector is u. The relation between the eigenvalues ω^2 and the parameter β is called the dispersion relation of the guided modes.

2.2. *A mathematical formulation*

In the following, the usual scalar product and norm on $L^2(\Omega)^n$ for $n = 1, 2$ or 3 are denoted by $(\cdot, \cdot)_{\Omega}$ and $\| \cdot \|_{\Omega}$, respectively and the same notation is used if Ω is replaced by any other subdomain of $\mathbb{R}^2$. Let us set $V = H^1(\Omega)^3$ and consider the following bilinear form on $V \times V$:

$$a(\beta; u, v) = \int_{\Omega} \sum_{i,j} \sigma_{ij}^{\beta}(u)\varepsilon_{ij}^{\beta}(v)\, dx\,. \tag{2.10}$$

This bilinear form is continuous on $V \times V$. Moreover, it is symmetric and positive since:

$$a(\beta; u, v) = \lambda \int_{\Omega} \mathrm{div}_{\beta}\, u \, \mathrm{div}_{\beta}\, v\, dx + 2\mu \int_{\Omega} \sum_{ij} \varepsilon_{ij}^{\beta}(u)\varepsilon_{ij}^{\beta}(v)\, dx\,.$$

We denote by $A(\beta)$ the operator associated to $a(\beta; u, v)$. It is defined as follows:

$$D(A(\beta)) = \{u \in V : \ \mathrm{div}_{\beta}^{*}\, \sigma^{\beta}(u) \in L^2(\Omega)^3 \quad \text{and} \quad \sigma^{\beta}(u)\tilde{\nu} = 0 \text{ on } \partial\Omega\}$$

and

$$\forall\, u \in D(A(\beta)) \quad A(\beta)u = -\,\mathrm{div}_{\beta}^{*}\, \sigma^{\beta}(u)\,.$$

Then for every $u \in D(A(\beta))$ and $v \in V$

$$(A(\beta)u, v) = a(\beta; u, v)\,.$$

Notice that condition $\sigma^\beta(u)\tilde{\nu} = 0$ on $\partial\Omega$ has to be understood in the following weak sense:

$$\forall\, v \in V \quad \int_\Omega \operatorname{div}^*_\beta \sigma^\beta(u)v\, dx = -a(\beta; u, v)\,.$$

Now the problem of finding guided modes can be written:

For $\beta \in \mathbb{R}^+$, find $\omega \in \mathbb{R}^+$ such that there exists $u \in D(A(\beta))$, $u \neq 0$, satisfying:

$$A(\beta)u = \omega^2 u\,.$$

We therefore have to study the eigenvalues of the operator $A(\beta)$. First of all, we prove that it is self-adjoint.

Lemma 1.   *The operator $A(\beta)$ is self-adjoint and positive.*

Proof. $A(\beta)$ is clearly symmetric and positive. To prove that it is a self-adjoint operator, we will prove there exists $\Lambda > 0$ such that $a(\beta; u, v) + \Lambda(u, v)_\Omega$ is coercive on $V \times V$. Expanding the expression of $a(\beta; u, u)$ with respect to β, we get:

$$a(\beta; u, u) = a_0(u, u) + \beta a_1(u, u) + \beta^2 a_2(u, u)\,,$$

where

$$a_0(u, u) = \lambda \int_\Omega (u_{1,1} + u_{2,2})^2\, dx + 2\mu \int_\Omega (u_{1,1}^2 + u_{2,2}^2)\, dx$$

$$+ \mu \int_\Omega ((u_{1,2} + u_{2,1})^2 + u_{3,1}^2 + u_{3,2}^2)\, dx\,,$$

$$a_1(u, u) = 2\lambda \int_\Omega (u_{1,1} + u_{2,2})u_3\, dx - 2\mu \int_\Omega (u_1 u_{3,1} + u_2 u_{3,2})\, dx\,,$$

$$a_2(u, u) = (\lambda + 2\mu)\int_\Omega u_3^2\, dx + \mu \int_\Omega (u_1^2 + u_2^2)\, dx\,.$$

By Korn's inequality, there exist two positive constants C_1 and C_2 such that:

$$a_0(u, u) \geq C_1 \sum_{i=1}^3 \|\nabla u_i\|_\Omega^2 - C_2 \sum_{i=1}^2 \|u_i\|_\Omega^2\,, \quad \forall\, u \in V\,.$$

Moreover, by Young's inequality applied to $a_1(u, u)$, there exists a constant C_3 such that, for every $\varepsilon > 0$:

$$|a_1(u, u)| \leq \varepsilon \sum_{i=1}^3 \|\nabla u_i\|_\Omega^2 + \frac{C_3}{\varepsilon}\|u\|_\Omega^2, \quad \forall\, u \in V\,.$$

Finally, we get, for every $\varepsilon > 0$:

$$a(\beta; u, u) \geq (C_1 - \beta\varepsilon)\sum_{i=1}^3 \|\nabla u_i\|_\Omega^2 + \left(\beta^2\mu - C_3\frac{\beta}{\varepsilon} - C_2\right)\|u\|_\Omega^2, \quad \forall\, u \in V\,.$$

The coerciveness result follows by taking $\varepsilon < C_1/\beta$ and $\Lambda > C_2 + C_3(\beta/\varepsilon) - \beta^2\mu$. $\qquad\square$

3. The Essential Spectrum

Let us denote by $\sigma(\beta)$ the spectrum of the operator $A(\beta)$ and by $\sigma_{\text{ess}}(\beta)$ its essential spectrum. Our purpose now is to determine $\sigma_{\text{ess}}(\beta)$. At first we briefly recall some useful results concerning the Rayleigh wave in the perfect half-space.

3.1. *The Rayleigh wave*

The Rayleigh wave is well known: it is a wave which propagates at the free surface of a homogeneous half-space. Its velocity c_R is strictly smaller than the speeds c_S and c_P of the S and P waves, which are given respectively by:

$$c_S = \sqrt{\mu} \quad \text{and} \quad c_P = \sqrt{\lambda + 2\mu}\,.$$

To derive the Rayleigh wave, let us consider the following one-dimensional problem: find

$$u(x_2) = \begin{pmatrix} 0 \\ u_2(x_2) \\ u_3(x_2) \end{pmatrix},$$

such that

$$\begin{cases} \operatorname{div}^*_\beta \sigma^\beta(u) = -\omega^2 u & \text{if } x_2 < 0, \\ \sigma^\beta(u)e_2 = 0 & \text{if } x_2 = 0, \\ \displaystyle\int_{-\infty}^0 |u(x_2)|^2\, dx_2 < +\infty\,, \end{cases} \tag{3.11}$$

where $e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$. Similar to problem (2.5), it is a parametrized eigenvalue problem.

This problem can be solved explicitly (see, for example, Miklowitz[25]). For every value of $\beta > 0$, there is a unique eigenvalue ω^2 given by $\omega^2 = \beta^2 c_R^2$ where c_R is the unique solution $z \in \,]0, c_S[$ of the Rayleigh equation:

$$4\left(1 - \frac{z^2}{c_P^2}\right)^{1/2}\left(1 - \frac{z^2}{c_S^2}\right)^{1/2} - \left(2 - \frac{z^2}{c_S^2}\right)^2 = 0\,.$$

We therefore have:

$$0 < c_R < c_S < c_P\,.$$

The associated field is given by

$$u(x_2) = \begin{pmatrix} 0 \\ u_2^R(\beta x_2) \\ u_3^R(\beta x_2) \end{pmatrix},$$

where

$$\begin{cases} u_2^R(x_2) = \alpha_P\left(e^{\alpha_P x_2} - \left(1 - \frac{c_R^2}{2c_S^2}\right)^{-1} e^{\alpha_S x_2}\right), \\ u_3^R(x_2) = -e^{\alpha_P x_2} + \left(1 - \frac{c_R^2}{2c_S^2}\right)e^{\alpha_S x_2}, \end{cases} \tag{3.12}$$

with $\alpha_P^2 = 1 - c_R^2/c_P^2$, $\alpha_S^2 = 1 - c_R^2/c_S^2$, $\alpha_P > 0$, $\alpha_S > 0$.

Problem (3.11) can also be handled as a spectral problem for a family of unbounded operators $B(\beta)$ defined on $L^2(\mathbb{R}^-)^2$ and associated to the bilinear form:

$$b(\beta; (u_2, u_3), (v_2, v_3)) = \lambda \int_{-\infty}^0 (u_{2,2} + \beta u_3)(v_{2,2} + \beta v_3)\, dx_2$$

$$+ \mu \int_{-\infty}^0 \{2u_{2,2}v_{2,2} + 2\beta^2 u_3 v_3 + (u_{3,2} - \beta u_2)(v_{3,2} - \beta v_2)\}\, dx_2 \quad (3.13)$$

defined on $H^1(\mathbb{R}^-)^2 \times H^1(\mathbb{R}^-)^2$. $B(\beta)$ is a self-adjoint positive operator and one can establish the following results (cf. Appendix A):

Lemma 2.

(a) *The essential spectrum of $B(\beta)$ is the interval $[\beta^2 c_S^2, +\infty[$.*
(b) *$B(\beta)$ has a unique eigenvalue ω^2 below $\beta^2 c_S^2$.*

By a scaling argument, we show that this eigenvalue is necessarily of the form $\beta^2 c_R^2$ where c_R is independent of β. We notice finally that, since $(u_2^R(\beta x_2), u_3^R(\beta x_2))$ minimizes the Rayleigh quotient

$$\frac{b(\beta; (u_2, u_3), (u_2, u_3))}{\int_{-\infty}^0 \{u_2^2 + u_3^2\}\, dx_2},$$

we have:

$$\beta^2 c_R^2 \int_{-\infty}^0 (u_2^2 + u_3^2)\, dx_2 \leq \lambda \int_{-\infty}^0 (u_{2,2} + \beta u_3)^2\, dx_2$$

$$+ \mu \int_{-\infty}^0 \{2(u_{2,2})^2 + 2\beta^2 u_3^2 + (u_{3,2} - \beta u_2)^2\}\, dx_2 \quad (3.14)$$

for every (u_2, u_3) in $H^1(\mathbb{R}^-)^2$. This inequality, which becomes an equality if $(u_2, u_3) = (u_2^R(\beta x_2), u_3^R(\beta x_2))$, will be used in the following sections.

3.2. *A lower bound for the essential spectrum*

We set now $V_0 = H^1(\Omega_0)^3$ where Ω_0 is the perfect half-plane defined by (2.2), and for $(u, v) \in V_0 \times V_0$

$$a(\Omega_0; \beta; u, v) = \lambda \int_{\Omega_0} \operatorname{div}_\beta u \operatorname{div}_\beta v\, dx + 2\mu \int_{\Omega_0} \sum_{i,j} \varepsilon_{ij}^\beta(u)\varepsilon_{ij}^\beta(v)\, dx.$$

Lemma 3. *For every u in V_0:*

$$a(\Omega_0; \beta; u, u) \geq \beta^2 c_R^2 \int_{\Omega_0} |u|^2\, dx. \quad (3.15)$$

Proof. This inequality is evident if $\beta = 0$ by the positivity of $a(\Omega_0; \beta; u, u)$. Suppose now $\beta \neq 0$.

Let us denote by $\hat{u}(\xi, x_2)$ the partial Fourier transform of $u(x_1, x_2)$ with respect to x_1:

$$\hat{u}(\xi, x_2) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} u(x_1, x_2) e^{-i\xi x_1} \, dx_1 \, .$$

By Plancherel's theorem, we have for all u in V_0:

$$a(\Omega_0; \beta; u, u) = \lambda \int_{\Omega_0} |i\xi\hat{u}_1 + \hat{u}_{2,2} + \beta\hat{u}_3|^2 \, dx_2 \, d\xi$$

$$+ \mu \int_{\Omega_0} 2\{\xi^2|\hat{u}_1|^2 + |\hat{u}_{2,2}|^2 + \beta^2|\hat{u}_3|^2\} \, dx_2 \, d\xi$$

$$+ \mu \int_{\Omega_0} \{|\hat{u}_{1,2} + i\xi\hat{u}_2|^2 + |i\xi\hat{u}_3 - \beta\hat{u}_1|^2 + |\hat{u}_{3,2} - \beta\hat{u}_2|^2\} \, dx_2 \, d\xi \, .$$

If we set $k = \sqrt{\beta^2 + \xi^2}$ and:

$$\begin{cases} \hat{w}_1 = \dfrac{1}{k}(\beta\hat{u}_1 + i\xi\hat{u}_3) \, , \\ \hat{w}_2 = \hat{u}_2 \, , \\ \hat{w}_3 = \dfrac{1}{k}(\beta\hat{u}_3 + i\xi\hat{u}_1) \, , \end{cases}$$

this equality reads:

$$a(\Omega_0; \beta; u, u) = \lambda \int_{\Omega_0} |\hat{w}_{2,2} + k\hat{w}_3|^2 \, dx_2 \, d\xi$$

$$+ \mu \int_{\Omega_0} \{2k^2|\hat{w}_3|^2 + 2|\hat{w}_{2,2}|^2 + |\hat{w}_{3,2} - k\hat{w}_2|^2\} \, dx_2 \, d\xi$$

$$+ \mu \int_{\Omega_0} \{|\hat{w}_{1,2}|^2 + k^2|\hat{w}_1|^2\} \, dx_2 \, d\xi \, .$$

By inequality (3.14), we have:

$$k^2 c_R^2 \int_{-\infty}^{0} (|\hat{w}_2|^2 + |\hat{w}_3|^2) \, dx_2 \leq \lambda \int_{-\infty}^{0} |\hat{w}_{2,2} + k\hat{w}_3|^2 \, dx_2$$

$$+ \mu \int_{-\infty}^{0} \{2k^2|\hat{w}_3|^2 + 2|\hat{w}_{2,2}|^2 + |\hat{w}_{3,2} - k\hat{w}_2|^2\} \, dx_2 \, .$$

Integrating this inequality with respect to ξ, we obtain finally the following inequality:

$$a(\Omega_0; \beta; u, u) \geq \int_{\Omega_0} (\beta^2 + \xi^2) c_R^2 (|\hat{w}_2|^2 + |\hat{w}_3|^2) \, dx_2 \, d\xi$$

$$+ \mu \int_{\Omega_0} \{|\hat{w}_{1,2}|^2 + (\beta^2 + \xi^2)|\hat{w}_1|^2\} \, dx_2 \, d\xi \, .$$

Since $c_R^2 < \mu$, this implies:

$$a(\Omega_0; \beta; u, u) \geq \beta^2 c_R^2 \int_{\Omega_0} (|\hat{w}_1|^2 + |\hat{w}_2|^2 + |\hat{w}_3|^2)\, dx_2\, d\xi$$

which proves the lemma thanks the identity:

$$|\hat{w}_1|^2 + |\hat{w}_2|^2 + |\hat{w}_3|^2 = |\hat{u}_1|^2 + |\hat{u}_2|^2 + |\hat{u}_3|^2\,.$$

$\square$

Using the previous lemma, we can now prove the

Lemma 4.

$$\sigma_{\text{ess}}(\beta) \subset [\beta^2 c_R^2, +\infty[\,. \tag{3.16}$$

Proof. Suppose $\gamma \in \sigma_{\text{ess}}(\beta)$ and consider a singular sequence $u^{(n)} \in D(A(\beta))$ (as defined in p. 15 of Schechter[29]) such that:

$$\|u^{(n)}\|_\Omega = 1\,, \tag{3.17}$$

$$u^{(n)} \to 0 \text{ weakly in } L^2(\Omega)^3\,, \tag{3.18}$$

$$\|A(\beta)u^{(n)} - \gamma u^{(n)}\|_\Omega \to 0\,. \tag{3.19}$$

By (3.17) and (3.19):

$$\gamma = \lim_{n \to +\infty} a(\beta; u^{(n)}, u^{(n)})\,. \tag{3.20}$$

Then by (3.17) and by Korn's inequality, $(u^{(n)})$ is bounded in V. Consequently, by (3.18), $u^{(n)} \rightharpoonup 0$ in V. Then we notice that

$$a(\beta; u^{(n)}, u^{(n)}) \geq a(\Omega_0; \beta; u^{(n)}, u^{(n)})$$

and consequently, by (3.15):

$$a(\beta; u^{(n)}, u^{(n)}) \geq \beta^2 c_R^2 \|u^{(n)}\|_{\Omega_0}^2\,. \tag{3.21}$$

Moreover, by the compact injection of $H^1(\mathcal{O})$ into $L^2(\mathcal{O})$:

$$u^{(n)}_{|\mathcal{O}} \to 0 \quad \text{in } L^2(\mathcal{O})^3$$

and therefore:

$$\|u^{(n)}\|_{\Omega_0}^2 = \|u^{(n)}\|_\Omega^2 - \|u^{(n)}\|_\mathcal{O}^2 \to 1 \text{ as } n \to +\infty\,. \tag{3.22}$$

Finally, we deduce from (3.20)–(3.22) that $\gamma \geq \beta^2 c_R^2$. $\square$

3.3. *The singular sequences*

We will now construct singular sequences to prove that inclusion (3.16) is in fact an equality.

Theorem 1.

$$\sigma_{\mathrm{ess}}(\beta) = [\beta^2 c_R^2, +\infty[\,.$$

Proof. Let $\xi > 0$. We will prove that $\gamma = (\beta^2 + \xi^2)c_R^2 \in \sigma_{\mathrm{ess}}(\beta)$ by building a singular sequence $(u^{(n)})$ satisfying (3.17)–(3.19). We set $k = \sqrt{\xi^2 + \beta^2}$ and for $x \in \Omega_0$:

$$\begin{cases} u_1(x) = \dfrac{-i\xi}{k} u_3^R(kx_2)e^{i\xi x_1}\,, \\[2mm] u_2(x) = u_2^R(kx_2)e^{i\xi x_1}\,, \\[2mm] u_3(x) = \dfrac{\beta}{k} u_3^R(x_2)e^{i\xi x_1}\,. \end{cases}$$

Notice that $u(x)e^{i(\beta x_3 - \omega t)}$ describes a Rayleigh wave propagating in the direction $\mathbf{k} = (\xi/k, 0, \beta/k)$. The field u therefore satisfies the following equalities:

$$\begin{cases} \sigma^\beta(u)e_2 = 0 & \text{if } x_2 = 0\,, \\[2mm] \mathrm{div}_\beta^* \sigma^\beta(u) = -k^2 c_R^2 u & \text{if } x \in \Omega_0 \end{cases} \tag{3.23}$$

but is not in $H^1(\Omega)^3$. Consider now the sequence $v^{(n)}$ of $L^2(\Omega)^3$ defined as follows:

$$v^{(n)}(x) = \frac{1}{\sqrt{n}} u(x)\theta\left(\frac{x_1}{n}\right) \quad \text{if } x \in \Omega_0 \quad \text{and } v^{(n)}(x) = 0 \quad \text{if } x \in \mathcal{O}$$

where θ is a $\mathcal{C}^\infty$-function which is not identically equal to 0 and satisfies:

$$\theta(x_1) = 0 \quad \text{if } |x_1| \le a \quad \text{or } |x_1| > R$$

for some $R > a$. The sequence $v^{(n)}$ clearly belongs to $H^1(\Omega)^3$ and has the following properties:

(a) $\|v^{(n)}\|_\Omega$ is independent of n. Indeed:

$$\|v^{(n)}\|_\Omega^2 = \frac{1}{n} \int_{\Omega_0} |u(x)|^2 \left|\theta\left(\frac{x_1}{n}\right)\right|^2 dx$$

$$= \int_{\Omega_0} (|u_2^R(x_2)|^2 + |u_3^R(x_2)|^2)|\theta(z_1)|^2 \, dz_1\, dx_2\,.$$

(b) $v^{(n)}$ converges weakly to 0 in $L^2(\Omega)^3$. Indeed, $(v^{(n)}, \varphi)_\Omega \to 0$ for every $\varphi \in \mathcal{D}(\Omega)^3$ which remains true, by the density of $\mathcal{D}(\Omega)^3$ in $L^2(\Omega)^3$, for every $\varphi \in L^2(\Omega)^3$.

(c) Finally, we have:

$$\sigma^\beta(v^{(n)}) = \begin{cases} 0 & \text{if } x \in \mathcal{O} \\[2mm] \dfrac{1}{\sqrt{n}}\theta\left(\dfrac{x_1}{n}\right)\sigma^\beta(u) + \dfrac{1}{\sqrt{n^3}}\theta'\left(\dfrac{x_1}{n}\right)\tau(u) & \text{if } x \in \Omega_0 \end{cases} \tag{3.24}$$

where $\tau(u)$ is the following tensor:

$$\tau(u) = \begin{pmatrix} (\lambda + 2\mu)u_1 \; \mu u_2 \; \mu u_3 \\ \mu u_2 \quad \lambda u_1 \quad 0 \\ \mu u_3 \quad\; 0 \quad \lambda u_1 \end{pmatrix}.$$

Consequently, by (3.23):

$$\operatorname{div}_\beta^* \sigma^\beta(v^{(n)}) = \begin{cases} 0 & \text{if } x \in \mathcal{O} \\ -\gamma v^{(n)} + \dfrac{1}{\sqrt{n}^3}\theta'\left(\dfrac{x_1}{n}\right)\sigma^\beta(u)e_1 \\ +\dfrac{1}{\sqrt{n}^3}\theta'\left(\dfrac{x_1}{n}\right)\operatorname{div}_\beta^* \tau(u) + \dfrac{1}{\sqrt{n}^5}\theta''\left(\dfrac{x_1}{n}\right)\tau(u)e_1 & \text{if } x \in \Omega_0 \end{cases}$$

where $e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. This proves that $\operatorname{div}_\beta^* \sigma^\beta(v^{(n)}) \in L^2(\Omega)^3$ and that

$$\|\operatorname{div}_\beta^* \sigma^\beta(v^{(n)}) + \gamma v^{(n)}\|_\Omega \to 0 \,.$$

However, $v^{(n)}$ is not a singular sequence because $v^{(n)} \notin D(A(\beta))$. Indeed, its normal stress $t^{(n)} = \sigma^\beta(v^{(n)})\tilde{\nu}_{|\partial\Omega}$ is not identically equal to 0 (cf. Fig. 2). However, by (3.23) and (3.24), we have:

$$t^{(n)} = \begin{cases} 0 & \text{if } x \in \partial\Omega \cap \partial\mathcal{O} \\ \dfrac{1}{\sqrt{n}^3}\theta'\left(\dfrac{x_1}{n}\right)\begin{pmatrix} \mu u_2 \\ \lambda u_1 \\ 0 \end{pmatrix} & \text{if } x \in \partial\Omega \cap \partial\Omega_0 \end{cases}$$
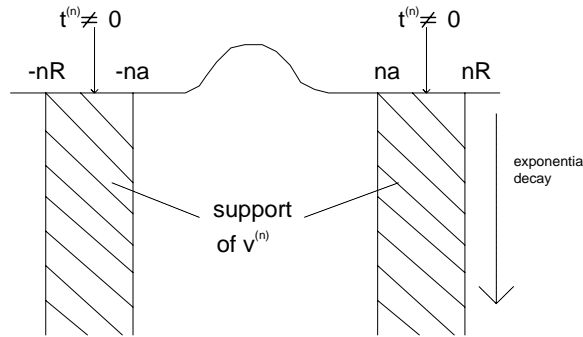

Fig. 2. The support of the singular sequence.

so that $\|t^{(n)}\|_{L^2(\partial\Omega)^3} \to 0$. To obtain a singular sequence, it suffices to add to $v^{(n)}$ a correcting term $w^{(n)}$ chosen as the unique solution of the following problem:

$$\begin{cases} -\operatorname{div}_\beta^* \sigma^\beta(w^{(n)}) + \Lambda w^{(n)} = 0 & \text{in } \Omega \\ \sigma^\beta(w^{(n)})\tilde{\nu} = -t^{(n)} & \text{on } \partial\Omega \end{cases}$$

where Λ is great enough to ensure the coerciveness of $a(\beta; u, u) + \Lambda(u, u)_\Omega$. Then by Lax–Milgram theorem, there is a constant C such that: $\|w^{(n)}\|_V \leq C\|t^{(n)}\|_{L^2(\partial\Omega)^3}$ and consequently:

$$\|w^{(n)}\|_V \to 0 \quad \text{and} \quad \|\operatorname{div}^*_\beta \sigma^\beta(w^{(n)})\|_V \to 0\,.$$

Therefore the sequence $u^{(n)} = v^{(n)} + w^{(n)}$ is a singular sequence associated to the value $\gamma = (\beta^2 + \xi^2)c_R^2$. $\qquad\qquad\square$

4. Study of the Point Spectrum

We are now interested in the point spectrum of the operator $A(\beta)$. Each eigenvalue γ of $A(\beta)$ corresponds to a guided wave such that $\omega^2 = \gamma$. The eigenvalues γ such that $\gamma < \beta^2 c_R^2$ form the discrete spectrum; they can be studied by means of the min-max principle as will be developed in the following. The study of the so-called *embedded eigenvalues* γ such that $\gamma > \beta^2 c_R^2$ is much more complicated. Examples of embedded eigenvalues have been exhibited for some related problems.[17,34] However, it seems that these examples are in some sense exceptions and that embedded eigenvalues generally not exist. For the problem we consider now, the question of the existence of embedded eigenvalues is still open.

4.1. *The min-max principle and the dispersion curves of the guided modes*

We want now to study the discrete spectrum of the operator $A(\beta)$, or, in other words, the eigenvalues smaller than $\beta^2 c_R^2$. To these eigenvalues are associated guided modes with a speed ω/β smaller than the Rayleigh speed c_R. Obviously, this discrete spectrum may be empty: it is the case for example if the elastic half-space is not perturbed. Therefore, our goal is to find out types of perturbation of the free surface for which guided waves exist. The mathematical tool we will use for that purpose is the min-max principle. Let us recall it. For every integer $m \geq 1$, we set:

$$s_m(\beta) = \inf_{F \in \mathcal{V}_m(V)} \sup_{u \in F, u \neq 0} \mathcal{R}(\beta; u)\,, \qquad (4.25)$$

where $\mathcal{V}_m(V)$ is the set of all m-dimensional subspaces of V and $\mathcal{R}(\beta; u)$ is the Rayleigh quotient defined as follows:

$$\mathcal{R}(\beta; u) = \frac{a(\beta; u, u)}{\|u\|_\Omega^2}\,. \qquad (4.26)$$

Another expression of the min-max values $s_m(\beta)$ is sometimes more convenient. It is the following:

$$s_m(\beta) = \sup_{F \in \mathcal{V}_{m-1}(L^2(\Omega)^3)} \inf_{u \in F^\perp \cap V, u \neq 0} \mathcal{R}(\beta; u)\,, \qquad (4.27)$$

where by definition $F^\perp = \{u \in L^2(\Omega)^3; \ (u,v)_\Omega = 0, \ \forall \, v \in F\}$. The equivalence between both expressions is not obvious but results from the proof of the min-max principle. For the problem we are interested here, this principle reads as follows:

Theorem 2.   *The sequence $(s_m(\beta))_{m \geq 1}$ is increasing and converges to $\beta^2 c_R^2$. Moreover, the following alternative holds*:

- $s_m(\beta) < \beta^2 c_R^2$. *Then $A(\beta)$ has at least m eigenvalues* (*counted with their multiplicity*) *below $\beta^2 c_R^2$ and these eigenvalues are the m first min-max values $s_1(\beta)$, $s_2(\beta), \dots, s_m(\beta)$.*
- $s_m(\beta) = \beta^2 c_R^2$ *and then $A(\beta)$ has at most $m-1$ eigenvalues strictly smaller than $\beta^2 c_R^2$.*

See for example Reed and Simon[28] for a proof. In the following, we denote by $N(\beta)$ the number of eigenvalues of $A(\beta)$ located strictly below $\beta^2 c_R^2$. By the min-max principle:

- if $s_m(\beta) < \beta^2 c_R^2$, then $N(\beta) \geq m$,
- if $s_m(\beta) = \beta^2 c_R^2$, then $N(\beta) \leq m - 1$

We are interested, if possible, in describing the variations of the eigenvalues ω^2 with respect to the parameter β, or in other words, the dispersion curves of the guided modes. Some results can be obtained directly by studying the min-max functions $\beta \to s_m(\beta)$. We can for example establish the following two properties:

- The function $\beta \to s_m(\beta)$ is positive continuous and almost everywhere differentiable.
- The function $\beta \to s_m(\beta) - \beta^2 c_S^2$ is negative and decreasing.

The first assertion can be proved by standard arguments (see for example Bonnet–Ben Dhia and Joly[7]). The second assertion has been proved by Bamberger *et al.*[6] In fact, the problem they have considered is different from ours, but the proof of this result can be applied to our case without any modification.

Notice that, in the context of the problem studied by Bamberger *et al.*,[6] the monotonicity of $s_m(\beta) - \beta^2 c_S^2$ allows one to prove that the number of guided modes $N(\beta)$ is an increasing function of β. Indeed, in that case, $\beta^2 c_S^2$ is the lower bound of the essential spectrum. In the case we consider here, this lower bound is $\beta^2 c_R^2$ and the monotonicity of $s_m(\beta) - \beta^2 c_S^2$ does not imply the monotonicity of $N(\beta)$. In fact, we conjecture that $N(\beta)$ is generally not monotonic, but we did not succeed in building counterexamples.

4.2. *Comparison results*

A classical tool[5,7] to obtain estimates on the eigenvalues of the operator $A(\beta)$ is to compare them with the eigenvalues of another operator, associated to a waveguide whose transverse section is bounded and included in the one of the initial waveguide. Such a waveguide is usually called a closed waveguide.

Suppose that Ω_b is a bounded domain which is locally Lipschitz and consider the unbounded operator $A_D(\beta)$ of $L^2(\Omega_b)^3$ defined as follows:

$$D(A_D(\beta)) = \{u \in V_D; \operatorname{div}_\beta^* \sigma^\beta(u) \in L^2(\Omega_b)^3 \quad \text{and} \quad \sigma^\beta(u)\tilde{\nu} = 0 \quad \text{on} \quad \Gamma_b^F\}$$

and

$$\forall\, u \in D(A_D(\beta)), \quad A_D(\beta)u = -\operatorname{div}_\beta^* \sigma^\beta(u)\,,$$

where

$$\partial\Omega_b = \Gamma_b^D \cup \Gamma_b^F\,,$$

$\tilde{\nu}$ denotes the unit outward normal to $\partial\Omega_b \times \mathbb{R}$ and

$$V_D = \{u \in H^1(\Omega_b)^3 : u = 0 \text{ on } \Gamma_b^D\}\,. \tag{4.28}$$

The part Γ_b^F of the boundary is a stress free surface while Γ_b^D is a Dirichlet boundary. The operator $A_D(\beta)$ is positive and self-adjoint. Moreover, since the embedding of $H^1(\Omega_b)$ into $L^2(\Omega_b)$ is compact, $A_D(\beta)$ has compact resolvent. Its spectrum is discrete and consists of an increasing positive sequence $s_m^D(\beta)$ tending to infinity with m. Moreover, the values $s_m^D(\beta)$ have the following min-max characterization:

$$s_m^D(\beta) = \inf_{F \in \mathcal{V}_m(V_D)} \sup_{u \in F, u \neq 0} \mathcal{R}(\Omega_b; \beta; u)\,, \tag{4.29}$$

where $\mathcal{R}(\Omega_b; \beta; u)$ is the Rayleigh quotient defined as follows:

$$\mathcal{R}(\Omega_b; \beta; u) = \frac{a(\Omega_b; \beta; u, u)}{\|u\|_{\Omega_b}^2} \tag{4.30}$$

with

$$a(\Omega_b; \beta; u, v) = \int_{\Omega_b} \sum_{i,j} \sigma_{ij}^\beta(u)\varepsilon_{ij}^\beta(v)\, dx\,.$$

As usual, we have the following *Dirichlet comparison principle* which results from the inclusion $V_D \subset V$, extending the functions by zero:

Lemma 5. *Suppose that $\Omega_b \subset \Omega$ and $\Gamma_b^F \subset \partial\Omega$. Then for every $m \geq 1$ and every $\beta \geq 0$:*

$$s_m(\beta) \leq s_m^D(\beta)$$

and therefore

$$N(\beta) \geq N_D(\beta)\,,$$

where $N_D(\beta) = \max\{m\colon s_m^D(\beta) < \beta^2 c_R^2\}$.

This will be extensively used in the following to prove the existence results for guided modes.
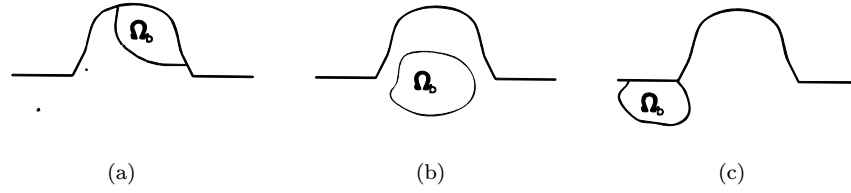
Fig. 3. The Dirichlet comparison principle.

Remark 1.

(a) $N_D(\beta)$ is the number of guided modes, in the closed waveguide, whose speed ω/β is less than c_R.

(b) If $\Gamma_b^F = \emptyset$ (cf. Fig. 3b), one can easily prove that $s_m^D(\beta) \geq \beta^2 c_S^2$ for every m. Indeed, in that case $V_D = H_0^1(\Omega_b)^3$ and one has, by first Korn's inequality, $\forall\, u \in H_0^1(\Omega_b)^3$:

$$\sum_{i,j} \|\varepsilon_{ij}^\beta(u)\|_{\Omega_b}^2 \geq \frac{1}{2}(\|\nabla u\|_{\Omega_b}^2 + \beta^2 \|u\|_{\Omega_b}^2)$$

so that

$$a(\Omega_b; \beta; u, u) \geq \beta^2 c_S^2\,.$$

We have therefore $N_D(\beta) = 0$ and the Dirichlet comparison principle does not provide existence results.

Likewise, assume that Γ_b^F is included in a straight line Σ (cf. Fig. 3c) and consider the half-plane Ω_Σ such that $\partial\Omega_\Sigma = \Sigma$ and $\Omega_b \subset \Omega_\Sigma$. Then, by (3.15):

$$\forall\, u \in H^1(\Omega_\Sigma), \quad a(\Omega_\Sigma; \beta; u, u) \geq \beta^2 c_R^2 \|u\|_{\Omega_\Sigma}^2\,.$$

Moreover, in that case, V_D can be considered (extending the functions by 0) as a subspace of $H^1(\Omega_\Sigma)$ and consequently:

$$\forall\, u \in V_D, \quad a(\Omega_b; \beta; u, u) \geq \beta^2 c_R^2 \|u\|_{\Omega_b}^2\,.$$

So one has $s_m^D(\beta) \geq \beta^2 c_R^2$ for every m and $N_D(\beta) = 0$. Again, the Dirichlet comparison principle does not work in that situation.

Finally, Dirichlet comparison principle is susceptible to provide existence results only if Γ_b^F is neither empty nor included in a straight line (cf. Fig. 3a).

To obtain upper estimates for $N(\beta)$, we shall now establish a *Neumann type comparison principle*. Contrary to the Dirichlet comparison principle, the bounded domain cannot be chosen as an arbitrary subdomain of Ω. We suppose here that $\Omega_b = \mathcal{O}$ where $\mathcal{O}$ denotes the perturbed part of the cross-section Ω and we introduce the operator $A_N(\beta)$ defined as follows:

$$D(A_N(\beta)) = \{u \in V_N : \mathrm{div}_\beta^* \sigma^\beta(u) \in L^2(\mathcal{O})^3 \quad \text{and} \quad \sigma^\beta(u)\tilde{\nu} = 0 \quad \text{on} \quad \partial\mathcal{O}\}\,,$$

where $V_N = H^1(\mathcal{O})^3$ and $\tilde{\nu}$ denotes the unit outward normal to $\partial\mathcal{O} \times \mathbb{R}$, and

$$\forall\, u \in D(A_N(\beta)), \quad A_N(\beta)u = -\mathrm{div}_\beta^* \,\sigma^\beta(u)\,.$$

As previously, the operator $A_N(\beta)$ is positive, self-adjoint and has compact resolvent. Its eigenvalues form an increasing positive sequence $s_m^N(\beta)$ tending to infinity with m and characterized by the following formulas:

$$s_m^N(\beta) = \inf_{F \in \mathcal{V}_m(V_N)} \sup_{u \in F, u \neq 0} \mathcal{R}(\mathcal{O}; \beta; u) \tag{4.31}$$

or

$$s_m^N(\beta) = \sup_{F \in \mathcal{V}_{m-1}(L^2(\mathcal{O})^3)} \inf_{u \in F^\perp \cap V_N, u \neq 0} \mathcal{R}(\mathcal{O}; \beta; u)\,. \tag{4.32}$$

We will prove now the following lemma:

Lemma 6. *For every $m \geq 1$ and every $\beta \geq 0$:*

$$s_m(\beta) \geq \min(s_m^N(\beta), \beta^2 c_R^2)$$

and therefore

$$N(\beta) \leq N_N(\beta)\,,$$

where $N_N(\beta) = \max\{m\colon s_m^N(\beta) < \beta^2 c_R^2\}$.

Proof. Consider first the case $m = 1$ and let $u \in V$. Then

$$a(\beta; u, u) = a(\mathcal{O}; \beta, u, u) + a(\Omega_0; \beta; u, u)\,.$$

By (3.15),

$$a(\Omega_0; \beta, u, u) \geq \beta^2 c_R^2 \|u\|_{\Omega_0}^2$$

and by (4.31),

$$a(\mathcal{O}; \beta; u, u) \geq s_1^N(\beta)\|u\|_{\mathcal{O}}^2\,.$$

Consequently:

$$a(\beta; u, u) \geq \min(\beta^2 c_R^2, s_1^N(\beta))\|u\|_\Omega^2$$

which proves the result for $m = 1$. Suppose now $m > 1$ and let us denote by $w_1, \ldots, w_{m-1}(m-1)$ linearly independent eigenfunctions associated to the first $(m-1)$ eigenvalues of $A_N(\beta)$. Then we define $(m-1)$ functions of $L^2(\Omega)^3$ as follows:

$$\tilde{w}_j(x) = \begin{cases} w_j(x) & \text{if } x \in \mathcal{O}\,, \\ 0 & \text{if } x \in \Omega_0\,. \end{cases}$$

We denote by F the $(m-1)$-dimensional subspace of $L^2(\Omega)^3$ spanned by the functions $\tilde{w}_j$, $j = 1, 2, \ldots, m-1$. By (4.27):

$$s_m(\beta) \geq \inf_{u \in V \cap F^\perp, u \neq 0} \mathcal{R}(\beta; u)\,.$$

Let $u \in V \cap F^{\perp}$, $u \neq 0$. Since $u \in F^{\perp}$, we have:

$$a(\mathcal{O}; \beta; u, u) \geq s_m^N(\beta) \|u\|_{\mathcal{O}}^2$$

and therefore:

$$a(\beta; u, u) = a(\mathcal{O}; \beta; u, u) + a(\Omega_0; \beta; u, u) \geq \min(s_m^N(\beta), \beta^2 c_R^2) \|u\|_{\Omega}^2$$

which can also be written:

$$\mathcal{R}(\beta; u) \geq \min(s_m^N(\beta), \beta^2 c_R^2) \,.$$

This proves the lemma. $\square$

Since $s_m^N(\beta)$ tends to $+\infty$ when $m \to +\infty$, we deduce immediately from this lemma the following important property:

Corollary 1.   *The number $N(\beta)$ is finite.*

In a lot of applications,[7,11] the values $s_m^D(\beta)$ (resp. $s_m^N(\beta)$) can be calculated analytically, when Ω_b (resp. $\mathcal{O}$) has a separable geometry. Unfortunately, it is not the case for the equations of elasticity (for instance, there is no image principle for such equations and separation of variables fails in a lot of cases). This is one of the reasons which makes this problem particularly difficult.

5. Existence Results for Guided Modes

We will now prove the existence of guided modes by using the Dirichlet comparison principle. As we have seen in the previous section, it consists of finding a subdomain Ω_b of Ω such that:

$$s_m^D(\beta) < \beta^2 c_R^2 \tag{5.33}$$

for some $m \geq 1$. The boundary of Ω_b is generally made of two parts Γ_b^F and Γ_b^D where $\Gamma_b^F = \partial \Omega_b \cap \partial \Omega$ and $\Gamma_b^D = \partial \Omega_b \backslash \Gamma_b^F$. The min-max $s_m^D(\beta)$ are the eigenvalues γ of the following problem:

$$\begin{cases} -\text{div}_{\beta}^* \sigma^{\beta}(u) = \gamma u & \text{in } \Omega_b \,, \\ u = 0 & \text{on } \Gamma_b^D \,, \\ \sigma^{\beta}(u)\tilde{\nu} = 0 & \text{on } \Gamma_b^F \,. \end{cases} \tag{5.34}$$

As one can expect from Remark 1, the existence of eigenvalues satisfying (5.33) relies mainly on the geometry of the free surface Γ_b^F.

5.1.   *Description of basic geometries*

Let us now define two particular classes of bounded domains which are respectively rectangular and triangular (cf. Fig. 4). In each case, we have to specify what part of the boundary corresponds to the free surface.
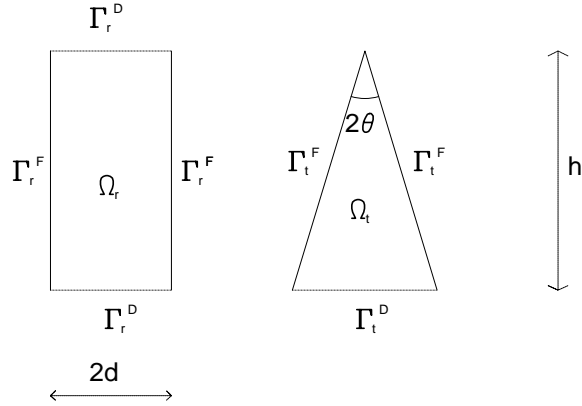
Fig. 4. The basic geometries.

▷ **The rectangle**

We denote by Ω_r a rectangular domain defined as follows:

$$\Omega_r = \{(x_1, x_2); \ -d < x_1 < d, \ 0 < x_2 < h\},$$

where d and h are two positive real numbers. The Dirichlet boundary Γ_r^D consists of the two "horizontal" sides, i.e.

$$\Gamma_r^D = \{(x_1, 0); \ -d < x_1 < d\} \cup \{(x_1, h); \ -d < x_1 < d\}$$

and the free surface Γ_r^F by the two "vertical" sides, i.e.

$$\Gamma_r^F = \{(-d, x_2); \ 0 < x_2 < h\} \cup \{(d, x_2); \ 0 < x_2 < h\}.$$

▷ **The triangle**

We define in the same way a triangular domain Ω_t such that:

$$\Omega_t = \{(x_1, x_2); \ |x_1| < (h - x_2)\tan\theta, \ 0 < x_2 < h\},$$

$$\Gamma_t^D = \{(x_1, 0); \ -h\tan\theta < x_1 < h\tan\theta\},$$

$$\Gamma_t^F = \{(x_1, x_2); \ |x_1| = (h - x_2)\tan\theta, \ 0 < x_2 < h\},$$

where h and θ are positive real numbers such that $0 < \theta < \pi/2$.

The first question we will consider is that of the existence of guided waves traveling more slowly than the Rayleigh wave in a closed waveguide whose cross-section is Ω_b and whose free surface is $\Gamma_b^F \times \mathbb{R}$ for $b = r, \ t$ or c. This amounts to proving that inequality (5.33) holds for some β, at least for a certain range of the geometrical parameters h, d and θ. For the rectangular domain Ω_r, we prove that

guided waves traveling more slowly than the Rayleigh wave exist if the ratio d/h is small enough and if β lies in some bounded interval $]\beta_{\min}, \beta_{\max}[$. For the case of the triangular domain, taking advantage of the singularity of the free surface, we obtain a stronger existence result: indeed, we prove that guided waves with a speed smaller than c_R exist as soon as the angle θ is small enough and β is greater than some minimal value $\beta_{\min}$, i.e. for arbitrary high frequency.

Then using the Dirichlet comparison principle we will deduce from the results for the closed waveguides, existence results of guided modes for a wide class of open waveguides. We refer to Duterte[12] for the extension of these existence results to more general geometries.

5.2. *The choice of test fields*

To prove (5.33) in a bounded domain Ω_b, we have to find some field $u \in V_D$ such that:

$$a(\Omega_b; \beta; u, u) < \beta^2 c_R^2 \|u\|_{\Omega_b}^2 \,. \tag{5.35}$$

This inequality can be written in expanded form:

$$\lambda \int_{\Omega_b} (u_{1,1} + u_{2,2} + \beta u_3)^2 \, dx + 2\mu \int_{\Omega_b} (u_{1,1}^2 + u_{2,2}^2 + \beta^2 u_3^2) \, dx$$

$$+ \mu \int_{\Omega_b} \{(u_{1,2} + u_{2,1})^2 + (u_{3,1} - \beta u_1)^2 + (u_{3,2} - \beta u_2)^2\} \, dx$$

$$< \beta^2 c_R^2 \int_{\Omega_b} (u_1^2 + u_2^2 + u_3^2) \, dx \,.$$

It seems attractive, in order to simplify this expression, to work with fields having a particular polarization. However, the two following estimates which result directly from the previous inequality show that a field satisfying (5.35) can neither be transverse ($u_3 = 0$) nor longitudinal ($u_1 = u_2 = 0$). Indeed, if (5.35) holds, then:

$$\int_{\Omega_b} u_3^2 \, dx \leq \frac{c_R^2}{2c_S^2 - c_R^2} \int_{\Omega_b} (u_1^2 + u_2^2) \, dx \tag{5.36}$$

and

$$\int_{\Omega_b} (u_1^2 + u_2^2) \, dx \leq \frac{2c_S^2}{\beta(c_S^2 - c_R^2)} \int_{\Omega_b} (u_{3,1} u_1 + u_{3,2} u_2) \, dx \tag{5.37}$$

so that $u_1 = u_2 = 0$ implies $u_3 = 0$, and $u_3 = 0$ implies $u_1 = u_2 = 0$. This is not surprising since the Rayleigh wave itself results from a coupling between S and P waves.

We got the idea for choosing good test fields by looking at numerical results. However this idea can be found by looking at the case of the rectangular perturbation when the height h is very large with respect to the width d. Imagine for a while that h goes to infinity, then the limit problem is the one of the propagation of elastic waves in an infinite plate of width d. In such a case, analytical calculations

of the so-called Rayleigh–Lamb waves are available.[25] These computations show in particular the existence of an antisymmetrical flexural wave whose speed tends to 0 at low frequency. This leads us to consider test fields of the form

$$u(x) = \begin{pmatrix} v(x_2) \\ 0 \\ \beta x_1 v(x_2) \end{pmatrix} \qquad (5.38)$$

so that:

$$\varepsilon_{11}^{\beta}(u) = \varepsilon_{13}^{\beta}(u) = \varepsilon_{22}^{\beta}(u) = 0 \,.$$

We therefore obtain

$$a(\Omega_b; \beta; u, u) = c_P^2 \beta^4 \int_{\Omega_b} x_1^2 v(x_2)^2 \, dx + c_S^2 \int_{\Omega_b} \left(\frac{dv}{dx_2}(x_2) \right)^2 (1 + \beta^2 x_1^2) \, dx \quad (5.39)$$

and

$$\|u\|_{\Omega_b}^2 = \int_{\Omega_b} (1 + \beta^2 x_1^2) v(x_2)^2 \, dx \,. \qquad (5.40)$$

Inequality (5.35) then becomes:

$$\mathcal{R}(\Omega_b; \beta; u) < \beta^2 c_R^2$$

with

$$\mathcal{R}(\Omega_b; \beta; u) = \frac{c_P^2 \beta^4 \displaystyle\int_{\Omega_b} x_1^2 v(x_2)^2 \, dx + c_S^2 \displaystyle\int_{\Omega_b} \left(\frac{dv}{dx_2}(x_2) \right)^2 (1 + \beta^2 x_1^2) \, dx}{\displaystyle\int_{\Omega_b} (1 + \beta^2 x_1^2) v(x_2)^2 \, dx} \,. \qquad (5.41)$$

For the sake of simplicity (and also on the basis of numerical results), the same form of test fields will be used for the triangular and cuspidal domains. Indeed, our goal here is to establish by quite simple arguments typical existence results which make clear what type of perturbation of the free surface produces a guiding phenomenon. We refer to Duterte[12] who derived weaker existence conditions for the case of the triangular perturbation by using more complicated test functions.

5.3. *The case of the rectangle and generalizations*

We consider first the case of the rectangle and we use the notations of the previous section.

The main result is the following:

Theorem 3. *Suppose that* $\Omega_b = \Omega_r$ *and* $\Gamma_b^D = \Gamma_r^D$ *where* Ω_r *is the rectangle defined in Sec.* 5.1. *Let* $m \geq 1$. *There exist two positive constants* $\alpha_{\min}$ *and* $\alpha_{\max}$, *depending only on* c_S *and* c_P, *such that, if*

$$\frac{d}{h} < \frac{\sqrt{3}}{m\pi} \left(\frac{c_P - \sqrt{c_P^2 - c_R^2}}{c_S} \right)$$

then for $j = 1, 2, \ldots, m$, $N_D(\beta) \geq j$ *if* $\beta^2 \in \,]j^2 \alpha_{\min}/h^2, \alpha_{\max}/d^2[$.

Remark 2. Let us mention again a simple physical interpretation of this phenomenon, which shows the existence of very slow waves in a rectangular elastic waveguide. If d/h is small enough, the closed rectangular waveguide behaves locally like an infinite plate of thickness $2d$ with free surfaces. The computation of the so-called Rayleigh–Lamb waves[25] reveals the existence of a mode whose dispersion relation is equivalent, for small values of βd, to:

$$\omega^2 = \frac{4}{3}\beta^4 d^2 c_S^2 \left(1 - \frac{c_S^2}{c_P^2} \right) \,,$$

so that its velocity ω/β tends to 0 with βd. This interpretation is confirmed by comparisons between the dispersion curve of this fundamental Lamb wave and the measured dispersion curve of the fundamental mode of a ridge waveguide for different values of the ratio d/h.[20]

Proof. We consider test fields of the form (5.38) with $v \in W_D$ where

$$W_D = \{v \in H^1(]0,h[); v(0) = v(h) = 0\} \,.$$

If $v \in W_D$, then u defined by (5.38) belongs to V_D defined by (4.28). Integrating expression (5.41) on the rectangle Ω_r, we obtain:

$$\mathcal{R}(\Omega_r;\beta;u) = \frac{\beta^2 d^2}{3 + \beta^2 d^2}\beta^2 c_P^2 + \frac{\displaystyle\int_0^h \frac{dv}{dx_2}(x_2)^2\,dx_2}{\displaystyle\int_0^h v(x_2)^2\,dx_2}c_S^2 \,.$$

When v describes an m-dimensional subspace V_m of W_D, the corresponding field u given by (5.38) describes an m-dimensional subspace of V_D. It follows that:

$$s_m^D(\beta) \leq \frac{\beta^2 d^2}{3 + \beta^2 d^2}\beta^2 c_P^2 + \gamma_m(h)c_S^2 \,,$$

where

$$\gamma_m(h) = \min_{V_m \in \mathcal{V}_m(W_D)} \max_{v \in V_m; v \neq 0} \frac{\displaystyle\int_0^h \frac{dv}{dx_2}(x_2)^2\,dx_2}{\displaystyle\int_0^h v(x_2)^2\,dx_2} \,.$$

This value $\gamma_m(h)$ can be calculated explicitly as a function of h since it is the mth eigenvalue of the following 1-D problem:

$$\begin{cases} -\dfrac{d^2v}{dx_2^2} = \gamma v & \text{in }]0,h[\\ v(0) = v(h) = 0 \end{cases}$$

and we find:

$$\gamma_m(h) = m^2\frac{\pi^2}{h^2} \,,$$

so that finally:

$$s_m^D(\beta) \leq \frac{\beta^2 d^2}{3 + \beta^2 d^2}\beta^2 c_P^2 + m^2\frac{\pi^2}{h^2}c_S^2 \,.$$

Therefore, inequality (5.33) holds as soon as the following inequality holds:

$$\frac{\beta^2 d^2}{3 + \beta^2 d^2}\frac{c_P^2}{c_R^2} + m^2\frac{\pi^2}{\beta^2 h^2}\frac{c_S^2}{c_R^2} < 1 \,.$$

Let us set $r = \beta^2 h^2$ and $t = d^2/h^2$. Then the above condition reads:

$$F_m(r, t) < 1 \,, \tag{5.42}$$

where we have set:

$$F_m(r, t) = \frac{rt}{3 + rt}\frac{c_P^2}{c_R^2} + m^2\frac{\pi^2}{r}\frac{c_S^2}{c_R^2} \,.$$

We want to derive a sufficient condition on t such that (5.42) holds for some values of r. We notice first that the function $t \to F_m(r, t)$ is strictly increasing on $\mathbb{R}^+$. On the other hand, for any fixed t:

$$\lim_{r \to 0} F_m(r, t) = +\infty$$

and

$$\lim_{r \to +\infty} F_m(r, t) = \frac{c_P^2}{c_R^2} > 1 \,.$$

$\square$

Then an easy calculation shows that (see Fig. 5):

- If $t \geq t_m$ with

$$t_m = \frac{3}{m^2\pi^2}\frac{c_P^2}{c_S^2} \,,$$

  then the function $r \to F_m(r, t)$ is decreasing on $\mathbb{R}^+$. Therefore condition (5.42) cannot be satisfied for any value of r.

- If $0 < t < t_m$, then the function $r \to F_m(r, t)$ is minimal at $r = r_m(t)$ where

$$r_m(t) = \frac{3}{\sqrt{t}(\sqrt{t_m} - \sqrt{t})} \,.$$

Then we just have to calculate the greatest value t_m^* of t such that:

$$F_m(r_m(t), t) \leq 1 \,.$$

We find:

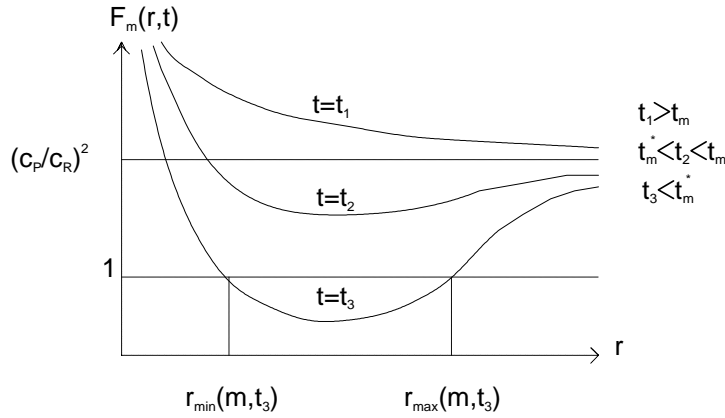$$t_m^* = \left(1 - \sqrt{1 - \frac{c_R^2}{c_P^2}}\right)^2 t_m \,.$$

Fig. 5. The function $F_m(r,t)$.

Summing up the previous results, we can state the following assertion: if $0 < t < t_m^*$, then $F_m(r,t)$ is strictly smaller than 1 for $r \in]r_{\min}(m,t), r_{\max}(m,t)[$ where $r_{\min}(m,t)$ and $r_{\max}(m,t)$ are the two positive solutions of the equation:

$$F_m(r,t) = 1 \,.$$

One can show that there exist two positive constants $\alpha_{\min}$ and $\alpha_{\max}$ depending only on the material such than:

$$\begin{aligned} r_{\min}(m,t) &\leq m^2 \alpha_{\min} \\ r_{\max}(m,t) &\geq \frac{\alpha_{\max}}{t} \end{aligned} \qquad \forall \, t \in [0, t_m^*] \,.$$

To prove the first inequality, just notice that

$$r_{\min}(m,t) \leq r_m(t_m^*) = \left(1 - \frac{c_R^2}{c_P^2}\right)^{-1/2} \left(1 - \sqrt{1 - \frac{c_R^2}{c_P^2}}\right)^{-1} \frac{c_S^2}{c_P^2} m^2 \pi^2 \,.$$

To prove the second one, we notice that

$$r_{\max}(m,t) = \frac{1}{t} r_{\max}(\tau, 1) \,,$$

where $\tau = m\sqrt{t}$. The function $\tau \to r_{\max}(\tau, 1)$ is positive continuous on $[0, \kappa]$ where

$$\kappa^2 = m^2 t_m^* = \left(1 - \sqrt{1 - \frac{c_R^2}{c_P^2}}\right)^2 \frac{3 c_P^2}{\pi^2 c_S^2} \,.$$

We obtain the inequality by setting $\alpha_{\max} = \min_{0 \leq \tau \leq \kappa} r_{\max}(\tau, 1)$.

This ends the proof.   □

Using the Dirichlet comparison principle, we can then establish the existence of guided modes for the initial problem.

Corollary 2. *Assume that there are two open sets Ω' and Ω_b of $\mathbb{R}^2$ such that*:

(a) $\bar{\Omega} = \bar{\Omega}' \cup \bar{\Omega}_b$.
(b) *There exists an isometric affine transformation $\mathcal{F} : \mathbb{R}^2 \to \mathbb{R}^2$ such that* (*cf. Fig.* 6b):

$$\Omega_b = \mathcal{F}(\Omega_r) \quad and \quad \mathcal{F}(\Gamma_r^F) \subset \partial\Omega \,.$$

Then, if

$$\frac{d}{h} < \frac{\sqrt{3}}{m\pi} \left(\frac{c_P - \sqrt{c_P^2 - c_R^2}}{c_S} \right)$$

the following assertion holds:

$$For \quad j = 1, 2, \ldots, m, \quad N(\beta) \geq j \ if \ \beta^2 \in \left] j^2 \frac{\alpha_{\min}}{h^2}, \frac{\alpha_{\max}}{d^2} \right[\,.$$
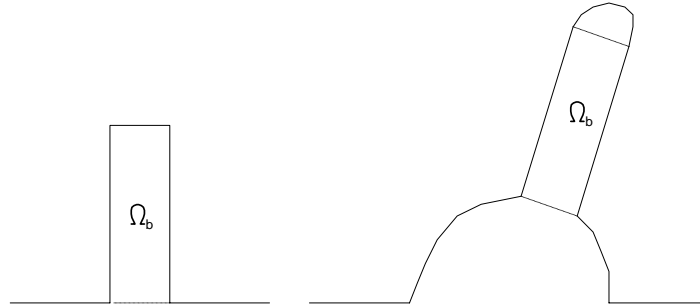

Fig. 6. The rectangular perturbation.

Remark 3. The previous result holds in particular if $\mathcal{O} = \Omega_r$ (cf. Fig. 6a). Such a guide is called a ridge waveguide. But this result also applies to cases where Γ_F is smooth.

Using very similar techniques, we can prove the existence of guided modes for a wide class of "elongated" perturbations.

Lemma 7. *Suppose that:*

$$\Omega_r' \subset \Omega_b \subset \Omega_r \,,$$

where Ω_r and Ω_r' are two rectangles, defined as in Sec. 5.1, *with respective widths $2d$ and $2d'$, and respective lengths h and h'. We suppose moreover (with obvious notations) that:*

$$\Gamma_r^{D'} \subset \Gamma_b^D \subset \Gamma_r^D \,.$$

There exist three positive constants η, $\alpha_{\min}$ and $\alpha_{\max}$, depending only on c_S, c_P and $\tau = d'/d$ such that, if

$$\frac{d}{h'} < \frac{\eta}{m}$$

then for $j = 1, 2, \ldots, m$, $N_D(\beta) \geq j$ if $\beta^2 \in]j^2\alpha_{\min}/h'^2, \alpha_{\max}/d^2[$.
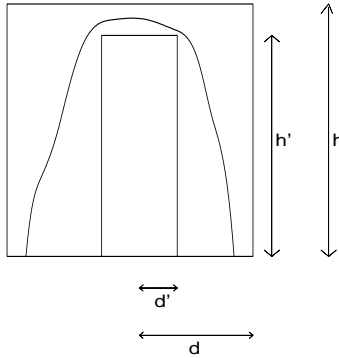

Fig. 7. Illustration of Lemma 7.

Proof. We consider again test fields u of the form (5.38) with $v \in H^1(]0, h[)$ satisfying $v(0) = 0$ and $v(x_2) = 0$ if $x_2 > h'$. Then we have:

$$\mathcal{R}(\Omega_b; \beta; u) \leq \frac{c_P^2 \beta^4 \displaystyle\int_{\Omega_r} x_1^2 v(x_2)^2 \, dx + c_S^2 \displaystyle\int_{\Omega_r} \left(\frac{dv}{dx_2}(x_2)\right)^2 (1 + \beta^2 x_1^2) \, dx}{\displaystyle\int_{\Omega'_r} (1 + \beta^2 x_1^2) v(x_2)^2 \, dx}$$

$$= \left\{ \frac{\beta^2 d^2}{3 + \beta^2 d'^2} \beta^2 c_P^2 + \frac{3 + \beta^2 d^2}{3 + \beta^2 d'^2} \frac{\displaystyle\int_0^{h'} \left(\frac{dv}{dx_2}(x_2)\right)^2 dx_2}{\displaystyle\int_0^{h'} v(x_2)^2 \, dx_2} c_S^2 \right\} \frac{d}{d'}.$$

So we obtain:

$$s_m^D(\beta) \leq \left\{ \frac{\beta^2 d^2}{3 + \beta^2 d'^2} \beta^2 c_P^2 + \frac{3 + \beta^2 d^2}{3 + \beta^2 d'^2} \frac{m^2 \pi^2}{h'^2} c_S^2 \right\} \frac{d}{d'}$$

since

$$\min_{V_m \in \mathcal{V}_m(H_0^1(]0,h'[))} \max_{v \in V_m; v \neq 0} \frac{\displaystyle\int_0^{h'} \frac{dv}{dx_2}(x_2)^2 \, dx_2}{\displaystyle\int_0^{h'} v(x_2)^2 \, dx_2} = \frac{m^2 \pi^2}{h'^2}.$$

Therefore, inequality $N_D(\beta) \geq m$ holds as soon as:

$$\left\{ \frac{\beta^2 d^2}{3 + \beta^2 d'^2} \frac{c_P^2}{c_R^2} + \frac{3 + \beta^2 d^2}{3 + \beta^2 d'^2} \frac{m^2 \pi^2}{\beta^2 h'^2} \frac{c_S^2}{c_R^2} \right\} \frac{d}{d'} < 1$$

and, *a fortiori*, as soon as

$$\left\{ \frac{\beta^2 d^2}{3 + \beta^2 d^2} \frac{c_P^2}{c_R^2} + \frac{m^2 \pi^2}{\beta^2 h'^2} \frac{c_S^2}{c_R^2} \right\} \frac{d^3}{d'^3} < 1 \,.$$

We set $r = \beta^2 h'^2$, $t = d^2/h'^2$ and $\tau = d'/d$. The previous inequality then reads:

$$F_m(\tau, r, t) < 1$$

with

$$F_m(\tau, r, t) = \frac{1}{\tau^3} \left(\frac{rt}{3 + rt} \frac{c_P^2}{c_R^2} + \frac{m^2 \pi^2}{r} \frac{c_S^2}{c_R^2} \right) \,.$$

A study of function F_m leads to the desired result, as in the proof of Theorem 3. We omit the details. $\qquad\square$

5.4. *The case of a triangular deformation*

Using again the notations of Sec. 5.2, we shall establish the following result:

Theorem 4. *Suppose that Ω_t is the triangle defined in Sec. 5.1. Let $m \geq 1$. There exist two positive constants τ and κ, depending only on c_S and c_P, such that, if $\tan \theta < \tau/m$ then $N_D(\beta) \geq m$ for $\beta^2 > \kappa^2 m^2/h^2$.*

Proof. For $\alpha > 0$, we define the following functional space:

$$X(\alpha) = \left\{ w; y^{1/2} w(y) \in L^2(0, \alpha), y^{1/2} \frac{dw}{dy}(y) \in L^2(0, \alpha), w(\alpha) = 0 \right\} \,.$$

Assume that α, the value of which will be fixed later, is given. For any function w in $X(\alpha)$ and $\beta > 0$, we can define for $x_2 < h$:

$$v_\beta(x_2) = \tilde{w}(\beta(h - x_2)) \,,$$

where $\tilde{w}$ is the extension of w by 0 for $y > \alpha$. Note that

$$\mathrm{supp}\, v_\beta \subset \left[h - \frac{\alpha}{\beta}, h \right] \,.$$

Now consider the test field in the form (5.38) given by (the main difference with the choice we made for the rectangular cross-section is the scaling by β):

$$u = \begin{pmatrix} v_\beta(x_2) \\ 0 \\ \beta x_1 v_\beta(x_2) \end{pmatrix} \,.$$

Note that if β is chosen large enough such that $\beta > \alpha/h$, then u belongs to V_D. Moreover, we have:

$$\int_{\Omega_t} x_1^2 v_\beta(x_2)^2 \, dx = \frac{2 \tan \theta^3}{3} \int_0^h (h - x_2)^3 \tilde{w}(\beta(h - x_2))^2 \, dx_2 \,,$$

that is to say, setting $y = \beta(h - x_2)$

$$\int_{\Omega_t} x_1^2 v_\beta(x_2)^2 \, dx = \frac{2 \tan \theta^3}{3\beta^4} \int_0^{\beta h} y^3 \tilde{w}(y)^2 \, dy$$

or equivalently, since $\beta h > \alpha$:

$$\int_{\Omega_t} x_1^2 v_\beta(x_2)^2 \, dx = \frac{2 \tan \theta^3}{3\beta^{3q+1}} \int_0^\alpha y^3 w(y)^2 \, dy \,.$$

In the same way we have:

$$\int_{\Omega_t} v_\beta(x_2)^2 \, dx = \frac{2 \tan \theta}{\beta^2} \int_0^\alpha y w(y)^2 \, dy \,,$$

$$\int_{\Omega_t} x_1^2 \frac{dv_\beta}{dx_2}(x_2)^2 \, dx = \frac{2(\tan \theta)^3}{3\beta^2} \int_0^\alpha y^3 \frac{dw}{dy}(y)^2 \, dy \,,$$

$$\int_{\Omega_t} \frac{dv_\beta}{dx_2}(x_2)^2 \, dx = 2 \tan \theta \int_0^\alpha y \frac{dw}{dy}(y)^2 \, dy \,.$$

Expression (5.41) of the Rayleigh quotient $\mathcal{R}(\Omega_t; \beta; u)$ becomes:

$$\mathcal{R}(\Omega_t; \beta; u) = \beta^2 c_R^2 \tilde{\mathcal{R}}(\beta, w) \,,$$

where

$$\tilde{\mathcal{R}}(\beta, w) = \frac{\dfrac{c_P^2}{c_R^2} \dfrac{(\tan \theta)^2}{3} \displaystyle\int_0^\alpha y^3 w(y)^2 \, dy + \dfrac{c_S^2}{c_R^2} \displaystyle\int_0^\alpha \left(y \dfrac{dw}{dy}(y)^2 + \dfrac{(\tan \theta)^2}{3} y^3 \dfrac{dw}{dy}(y)^2 \right) dy}{\displaystyle\int_0^\alpha y w(y)^2 \, dy + \dfrac{(\tan \theta)^2}{3} \displaystyle\int_0^\alpha y^3 w(y)^2 \, dy}$$

and therefore:

$$\tilde{\mathcal{R}}(\beta, w) \leq \frac{\dfrac{c_P^2}{c_R^2} \dfrac{(\tan \theta)^2}{3} \alpha^2 \displaystyle\int_0^\alpha y w(y)^2 \, dy + \dfrac{c_S^2}{c_R^2} \left(1 + \dfrac{(\tan \theta)^2}{3} \alpha^2 \right) \displaystyle\int_0^\alpha y \dfrac{dw}{dy}(y)^2 \, dy}{\displaystyle\int_0^\alpha y w(y)^2 \, dy} \,.$$

Once again, if w describes an m-dimensional subspace V_m of $X(\alpha)$, the corresponding u describes an m-dimensional subspace of V_D. It follows that:

$$s_m^D(\beta) \leq \beta^2 c_R^2 \left(\frac{c_P^2}{c_R^2} \frac{(\tan \theta)^2}{3} \alpha^2 + \frac{c_S^2}{c_R^2} \left(1 + \frac{(\tan \theta)^2}{3} \alpha^2 \right) \zeta_m(\alpha) \right) \,,$$

where

$$\zeta_m(\alpha) = \min_{V_m \in \mathcal{V}_m(X(\alpha))} \max_{w \in V_m; w \neq 0} \frac{\displaystyle\int_0^\alpha y \frac{dw}{dy}(y)^2 \, dy}{\displaystyle\int_0^\alpha y w(y)^2 \, dy} \,.$$

Obviously, we have $\zeta_m(1) = \alpha^2 \zeta_m(\alpha)$ and a sufficient condition for having $N_D(\beta) \geq m$ for every β such that $\beta \geq \alpha/h$ is finally:

$$P\left((\tan\theta)^2 \delta_m, \frac{\alpha^2}{\delta_m}\right) < 0\,, \qquad (5.43)$$

where

$$P(T, Y) = \frac{c_P^2}{c_R^2} \frac{T}{3} Y^2 + \left(\frac{c_S^2}{c_R^2} \frac{T}{3} - 1\right) Y + \frac{c_S^2}{c_R^2}\,.$$

Inequality (5.43) admits solutions α as soon as the following two inequalities hold:

(i) $$\qquad\qquad (\tan\theta)^2 \delta_m < \frac{3c_R^2}{c_S^2}\,,$$

(ii) $$\qquad\qquad D((\tan\theta)^2 \delta_m) > 0\,,$$

where

$$D(T) = \frac{c_S^4}{9c_4^R} T^2 - 2\frac{c_S^2}{3c_R^2}\left(1 + 2\frac{c_P^2}{c_R^2}\right) T + 1\,.$$

One can easily prove that $D(T)$ has two strictly positive roots $T_{\min}$ and $T_{\max}$ which depend only on c_S and c_P and (ii) holds as soon as:

$$(\tan\theta)^2 < \frac{T_{\min}}{\delta_m}\,. \qquad (5.44)$$

Moreover, since $T_{\min}^2 \leq T_{\min} T_{\max} = 9c_R^4/c_S^4$, condition (5.44) implies (i).

Suppose now that the angle θ satisfies (5.44) and let us denote by $\alpha_{\min}^2(m, \theta)$ the smallest root α^2 of equation

$$P\left((\tan\theta)^2 \delta_m, \frac{\alpha^2}{\delta_m}\right) = 0\,.$$

Then $N_D(\beta) \geq m$ for every $\beta > \alpha_{\min}(m, \theta)/h$. Clearly, $\alpha_{\min}^2(m, \theta)/\delta_m$ is a continuous function of $T = (\tan\theta)^2 \delta_m$ for $T \in [0, T_{\min}]$ so that there exists a constant κ, depending only on c_S and c_P, such that:

$$\alpha_{\min}^2(m, \theta) \leq \kappa^2 \delta_m\,.$$

To conclude, we notice that the values δ_m are the eigenvalues of the following 1-D problem:

$$\begin{cases} -\dfrac{d}{dy}\left(y\dfrac{dw}{dy}\right) = \gamma y w \text{ in }]0, 1[\\ w(1) = 0 \\ \dfrac{dw}{dy}(0) = 0 \end{cases}$$

which can be solved explicitly by using the theory of Bessel functions. One finds:

$$\delta_m = j_m^2\,,$$

where j_m is the mth positive zero of the Bessel function J_0, so that

$$\delta_m \sim \left(m - \frac{1}{4}\right)^2 \pi^2 \ \text{ for large } m \,.$$

$\square$

Once again, we can use the Dirichlet comparison principle to prove the existence of guided modes for the initial problem, set in Ω.

Corollary 3. *Suppose there are two subdomains Ω' and Ω_b of $\mathbb{R}^2$ such that*:

(a) $$\bar{\Omega} = \bar{\Omega}' \cup \bar{\Omega}_b$$
(b) *There exists an isometric affine transformation $\mathcal{F} : \mathbb{R}^2 \to \mathbb{R}^2$ such that*:

$$\Omega_b = \mathcal{F}(\Omega_t) \ \text{ and } \ \mathcal{F}(\Gamma_t^F) \subset \partial\Omega \,,$$

then there exist two positive constants α and T depending only on c_S and c_P such that, if $\theta < T/m$ then $N(\beta) \geq m$ for $\beta^2 \geq \alpha^2 m^2/h^2$.

Remark 4.

1. The previous result holds in particular if $\mathcal{O} = \mathcal{F}(\Omega_t)$ (such a guide is called a wedge waveguide).
2. Notice that the phase velocity of the mth guided mode is such that

$$\limsup_{\beta \to +\infty} c_m(\beta) \leq \nu_m c_R \,,$$

where ν_m is a real constant which is strictly smaller than 1. Indeed

$$c_m(\beta)^2 = \frac{s_m(\beta)}{\beta^2} \leq c_R^2 \left(\frac{c_P^2}{c_R^2}\frac{a^2}{3}\alpha^2 + \frac{c_S^2}{c_R^2}\left(1 + \frac{a^2}{3}\alpha^2\right)\frac{\zeta_m(1)}{\alpha^2}\right) \,.$$

As in the case of rectangular perturbations, we can extend these existence results to perturbations which take place "between" two triangles (see Ref. 12).

6. An Upper Estimate for the Number of Guided Modes at Low Frequency

We are concerned now with the existence of guided modes for small values of β. The existence results we proved in the last paragraph are valid only for great enough values of β. Moreover, numerical computations achieved for various geometries confirm that the fundamental mode (i.e. the mode associated to the smallest eigenvalue $s_1(\beta)$) is always cutoff at low frequency.[12] It is therefore natural to conjecture that the following assertion holds:

Conjecture. There exists $\beta^* > 0$ such that $N(\beta) = 0$ if $0 < \beta \leq \beta^*$.

We did not succeed in proving it. However, we obtained the following result: there exists $\beta^* > 0$ such that $N(\beta) \leq 1$ if $0 < \beta \leq \beta^*$. Proving it is the object of this section.

In the following we will denote by Γ the boundary between $\mathcal{O}$ and Ω_0:

$$\Gamma = \partial \mathcal{O} \cap \partial \Omega_0 \,.$$

6.1. *Application of comparison principles*

The simplest way to estimate $N(\beta)$ for small β consists of using the comparison principles of Sec. 4. Indeed, taking $\Omega_b = \mathcal{O}$ and $\Gamma_b^D = \Gamma$, we have:

$$N_D(\beta) \leq N(\beta) \leq N_N(\beta), \quad \forall \beta > 0 \,.$$

We give in the next lemma estimates on $N_D(\beta)$ and $N_N(\beta)$ for small values of β.

Lemma 8. *There exists $\beta^* > 0$ such that*:

- $N_D(\beta) = 0$ *if* $\beta < \beta^*$,
- $2 \leq N_N(\beta) \leq 3$ *if* $\beta < \beta^*$.

Proof. Let us prove first by contradiction that $N_D(\beta) = 0$ for small β. Suppose that there are a sequence $\beta_p \to 0$ and a sequence $(u^{(p)})_{p \in \mathbb{N}}$, $u^{(p)} \in H^1(\mathcal{O})^3$, such that $u^{(p)} = 0$ on Γ,

$$a(\mathcal{O}; \beta_p; u^{(p)}, u^{(p)}) < \beta_p^2 c_R^2$$

and $\|u^{(p)}\|_{\mathcal{O}} = 1$ for every $p \in \mathbb{N}$. Then by Korn's inequality, $(u^{(p)})$ is bounded in $H^1(\mathcal{O})^3$ and there is a subsequence, still denoted by $(u^{(p)})$, which converges weakly to u in $H^1(\mathcal{O})^3$ and strongly in $L^2(\mathcal{O})^3$. Therefore:

$$\|u\|_{\mathcal{O}} = \lim \|u^{(p)}\|_{\mathcal{O}} = 1 \,.$$

Moreover (as in Lemma 1):

$$a(\mathcal{O}; \beta_p; u^{(p)}, u^{(p)}) = a_0(\mathcal{O}; u^{(p)}, u^{(p)}) + \beta_p a_1(\mathcal{O}; u^{(p)}, u^{(p)}) + \beta_p^2 a_2(\mathcal{O}; u^{(p)}, u^{(p)}) \,.$$

Since the sequences $a_j(\mathcal{O}; u^{(p)}, u^{(p)})$ are bounded for $j = 1$ and $j = 2$, one has:

$$\lim_{p \to +\infty} a_0(\mathcal{O}; u^{(p)}, u^{(p)}) = \lim_{p \to +\infty} a(\mathcal{O}; \beta_p; u^{(p)}, u^{(p)}) = 0 \,.$$

Now by the positivity of a_0,

$$a_0(\mathcal{O}; u, u) \leq \lim_{p \to +\infty} a_0(\mathcal{O}; u^{(p)}, u^{(p)})$$

so that eventually

$$a_0(\mathcal{O}; u, u) = 0 \,. \tag{6.45}$$

But

$$a_0(\mathcal{O}; u, u) = \lambda \int_{\mathcal{O}} (u_{1,1} + u_{2,2})^2 \, dx$$

$$+ \mu \int_{\mathcal{O}} \{2u_{1,1}^2 + 2u_{2,2}^2 + (u_{1,2} + u_{2,1})^2 + u_{3,1}^2 + u_{3,2}^2\} \, dx$$

so that (6.45) implies that u_3 is constant and $\binom{u_1}{u_2}$ is a rigid displacement in the plane. In other words, there are real constants t_1, t_2, t_3 and r such that:

$$u = \begin{pmatrix} t_1 - rx_2 \\ t_2 + rx_1 \\ t_3 \end{pmatrix} .$$

The contradiction results finally from the Dirichlet condition on Γ.

To prove that $N_N(\beta) \leq 3$ for small β, we will prove that:

$$\inf_{u \in H^1(\mathcal{O})^3 \cap RD^\perp, u \neq 0} a(\mathcal{O}; \beta; u, u) \geq \beta^2 c_R^2 \quad \text{for small } \beta,$$

where we have denoted by RD the three-dimensional space of transverse rigid displacements:

$$RD = \left\{ u = \begin{pmatrix} t_1 - rx_2 \\ t_2 + rx_1 \\ 0 \end{pmatrix} ; r, t_1, t_2 \in \mathbb{R} \right\}$$

and by $RD^\perp$ its orthogonal in $L^2(\mathcal{O})^3$. Suppose by contradiction that there exist a sequence $\beta_p \to 0$ and a sequence $(u^{(p)})_{p \in \mathbb{N}}$, $u^{(p)} \in H^1(\mathcal{O})^3$, such that

$$\begin{cases} a(\mathcal{O}; \beta_p; u^{(p)}, u^{(p)}) < \beta_p^2 c_R^2, \\ \|u^{(p)}\|_{\mathcal{O}} = 1, \\ (u^{(p)}, v)_{\mathcal{O}} = 0 \quad \forall v \in RD, \end{cases}$$

for every $p \in \mathbb{N}$. As in the previous case, we deduce the existence of $u \in H^1(\mathcal{O})^3$ such that $\|u\|_{\mathcal{O}} = 1$, u_3 is constant and $\binom{u_1}{u_2}$ is a rigid displacement in the plane. Moreover $u \in RD^\perp$ so that $u_1 = u_2 = 0$. Then by (5.36),

$$\int_{\mathcal{O}} u_3^2 \, dx \leq \frac{c_R^2}{2c_S^2 - c_R^2} \int_{\mathcal{O}} (u_1^2 + u_2^2) \, dx \tag{6.46}$$

so that $u_3 = 0$. This contradicts $\|u\|_{\mathcal{O}} = 1$.

To prove that $N_N(\beta) \geq 2$ for small β, we consider finally the following two-dimensional space:

$$V_2(\beta) = \left\{ u = \begin{pmatrix} t_1 \\ t_2 \\ \beta(t_1 x_1 + t_2 x_2) \end{pmatrix} ; t_1, t_2 \in \mathbb{R} \right\} .$$

For $u \in V_2(\beta)$, $u \neq 0$, one has:

$$\mathcal{R}(\mathcal{O}; \beta; u) = \frac{a(\mathcal{O}; \beta; u, u)}{\|u\|_{\mathcal{O}}^2} = \frac{\beta^4 c_P^2 \displaystyle\int_{\mathcal{O}} (t_1 x_1 + t_2 x_2)^2 \, dx}{\displaystyle\int_{\mathcal{O}} (t_1^2 + t_2^2 + \beta^2 (t_1 x_1 + t_2 x_2)^2) \, dx}$$

so that:

$$\sup_{u \in V_2(\beta), u \neq 0} \mathcal{R}(\mathcal{O}; \beta; u) \leq \beta^4 c_P^2 \sup_{(t_1, t_2) \in \mathbb{R}^2, (t_1, t_2) \neq 0} \frac{\displaystyle\int_{\mathcal{O}} (t_1 x_1 + t_2 x_2)^2 \, dx}{\displaystyle\int_{\mathcal{O}} (t_1^2 + t_2^2) \, dx} \, .$$

Consequently $s_2^N(\beta) < \beta^2 c_R^2$ for small β. $\qquad\qquad\qquad\qquad\qquad \square$

Remark 5. The estimates on the Neumann problem have the following interpretation: in a closed waveguide whose boundary is stress-free, there are at low frequency at least two guided modes (and at most three) which propagate slowlier than the Rayleigh wave.

From the comparison principles and the previous lemma, we deduce the

Corollary 4. *There exists $\beta^* > 0$ such that $N(\beta) \leq 3$ for $0 \leq \beta \leq \beta^*$.*

6.2. *Equivalent formulations of the problem in a bounded domain*

To improve the results obtained in the previous paragraph, we have to take into account the behavior of the elastic field in the perfect half-plane Ω_0. This will be done by writing an equivalent formulation of the initial problem set in $\mathcal{O}$. We follow Duterte and Joly.[13] First we have (see Duterte and Joly[13] for a proof) the following result (the crucial point is the inequality $\omega < \beta c_R$):

Lemma 9. *For every $\omega \in [0, \beta c_R[$ and for every $\varphi \in H^{-1/2}(\Gamma)^3$, there exists a unique $u_\varphi \in H^1(\Omega_0)^3$ such that:*

$$b(\omega; \beta; u_\varphi, v) = \langle \varphi, v_{|\Gamma} \rangle, \quad \forall \, v \in H^1(\Omega_0)^3 \, ,$$

where:

$$b(\omega; \beta; u, v) = a(\Omega_0; \beta; u, v) - \omega^2 (u, v)_{\Omega_0} \, .$$

Moreover, the application

$$T(\omega, \beta) \colon \begin{cases} H^{-1/2}(\Gamma)^3 \to H^{1/2}(\Gamma)^3 \\ \varphi \to u_{\varphi|\Gamma} \end{cases}$$

is an isomorphism and satisfies:

(a) $\qquad \forall \, \varphi, \psi \in H^{-1/2}(\Gamma)^3, \quad \langle \varphi, T(\omega, \beta) \psi \rangle = \langle \psi, T(\omega, \beta) \varphi \rangle = b(\omega; \beta; u_\varphi, u_\psi)$

(b) $$\langle \varphi, T(\omega, \beta)\varphi \rangle \geq 0, \quad \forall\, \varphi \in H^{-1/2}(\Gamma)^3 \,.$$

Let us introduce the following two bilinear forms:

$$\forall\, \varphi, \psi \in H^{-1/2}(\Gamma)^3, \quad t(\omega; \beta; \varphi, \psi) = \langle \varphi, T(\omega, \beta)\psi \rangle = b(\omega; \beta; u_\varphi, u_\psi), \quad (6.47)$$

$$\forall\, u, v \in H^{1/2}(\Gamma)^3, \quad s(\omega; \beta; u, v) = \langle T(\omega, \beta)^{-1} u, v \rangle \,. \quad (6.48)$$

By the previous lemma, the bilinear forms $t(\omega; \beta; \varphi, \psi)$ and $s(\omega; \beta; u, v)$ are symmetric and positive. Moreover, one has the following identity:

$$t(\omega; \beta; \varphi, \varphi) = \langle \varphi, u_{\varphi|\Gamma} \rangle = s(\omega, \beta; u_{\varphi|\Gamma}, u_{\varphi|\Gamma}) \quad \forall\, \varphi \in H^{-1/2}(\Gamma)^3 \,. \quad (6.49)$$

Now we can write three equivalent formulations of the initial eigenvalue problem:

Theorem 5. *For $\beta > 0$, the following three problems are equivalent*:

(a) *Find $\omega \in [0, \beta c_R)$ such that there exists $u \in V$, $u \neq 0$, satisfying*:

$$a(\beta; u, v) = \omega^2 (u, v)_\Omega \quad \forall\, v \in V \,.$$

(b) *Find $\omega \in [0, \beta c_R)$ such that there exists $u \in H^1(\mathcal{O})^3$, $u \neq 0$, and $\varphi \in H^{-1/2}(\Gamma)^3$ satisfying*:

$$\begin{cases} a(\mathcal{O}; \beta, u, v) + \langle \varphi, v_{|\Gamma} \rangle = \omega^2 (u, v)_{\mathcal{O}} & \forall\, v \in H^1(\mathcal{O})^3 \,, \\ t(\omega; \beta; \varphi, \psi) = \langle \psi, u_{|\Gamma} \rangle & \forall\, \psi \in H^{-1/2}(\Gamma)^3 \,. \end{cases}$$

(c) *Find $\omega \in [0, \beta c_R)$ such that there exists $u \in H^1(\mathcal{O})^3$, $u \neq 0$ satisfying*:

$$a(\mathcal{O}; \beta, u, v) + s(\omega; \beta; u_{|\Gamma}, v_{|\Gamma}) = \omega^2 (u, v)_{\mathcal{O}} \quad \forall\, v \in (H^1(\mathcal{O}))^3 \,.$$

We notice that formulations (b) and (c) are set in a bounded domain but are nonlinear with respect to the eigenvalue ω^2. To solve them, we can consider for a given ω in the interval $[0, \beta c_R)$ the following linear eigenvalue problem:

Find $\gamma \in \mathbb{R}$ such that there exists $u \in H^1(\mathcal{O})^3$, $u \neq 0$ satisfying:

$$a(\mathcal{O}; \beta, u, v) + s(\omega; \beta; u_{|\Gamma}, v_{|\Gamma}) = \gamma(u, v)_{\mathcal{O}} \quad \forall\, v \in H^1(\mathcal{O})^3 \,.$$

By the positivity of $s(\omega; \beta; ., .)$, the bilinear form $a(\mathcal{O}; \beta; u, v) + s(\omega; \beta; u_{|\Gamma}, v_{|\Gamma}) + \Lambda(u, v)_{\mathcal{O}}$ is coercive on $H^1(\mathcal{O})^3$ for Λ large enough. Therefore, by the compact injection of $H^1(\mathcal{O})$ into $L^2(\mathcal{O})$, this problem admits an increasing sequence of eigenvalues $(\gamma_m(\beta, \omega))_{m \geq 1}$ tending to infinity with m and which have the following characterization:

$$\gamma_m(\beta, \omega) = \min_{V_m \in \mathcal{V}_m(H^1(\mathcal{O})^3)} \max_{u \in V_m; u \neq 0} \mathcal{R}(\beta, \omega, u)$$

where

$$\mathcal{R}(\beta; \omega; u) = \frac{a(\mathcal{O}; \beta, u, u) + s(\omega; \beta; u_{|\Gamma}, u_{|\Gamma})}{\|u\|_{\mathcal{O}}^2} \,.$$

Then the solutions ω of problem 3) are the solutions of the following fixed-point equations:

$$\begin{cases} 0 \leq \omega < \beta c_R \\ \gamma_m(\beta, \omega) = \omega^2 \end{cases} \tag{6.50}$$

for $m \geq 1$.

6.3. *Proof of non-existence for the second mode at low frequency*

We are now able to prove the following result:

Theorem 6. *There exists $\beta^* > 0$ such that*:

$$\forall \beta \in [0, \beta^*] \quad N(\beta) \leq 1 \,.$$

This theorem is a direct consequence of the following lemma which will be established by contradiction. Indeed, this lemma proves that fixed-point equation (6.50) has no solution for $m \geq 2$ and $\beta \in]0, \beta^*]$.

Lemma 10. *There exists $\beta^* > 0$ such that*:

$$\forall \beta \in]0, \beta^*] \ \ \forall \omega \in]0, \beta c_R[, \ \ \gamma_2(\beta, \omega) \geq \beta^2 c_R^2 \,.$$

Proof. Suppose this assertion is false. Then there exist two sequences β_n and ω_n such that:

$$\begin{cases} \beta_n > 0 \\ 0 < \omega_n < \beta_n c_R \\ \beta_n \to 0 \\ \gamma_2(\beta_n, \omega_n) < \beta_n^2 c_R^2 \,. \end{cases}$$

Moreover, by the min-max characterization of $\gamma_2(\beta, \omega)$:

$$\gamma_2(\beta_n, \omega_n) \geq \min_{u \in H^1(\mathcal{O})^3, \int_\mathcal{O} u_2 dx = 0, u \neq 0} \mathcal{R}(\beta_n; \omega_n; u) \,. \tag{6.51}$$

Consequently, there exists a sequence $(u^{(n)})_{n \in \mathbb{N}}$ such that $u^{(n)} \in H^1(\mathcal{O})^3$, $\|u^{(n)}\|_\mathcal{O} = 1$, $\int_\mathcal{O} u_2^{(n)} \, dx = 0$ and

$$a(\mathcal{O}; \beta_n; u^{(n)}, u^{(n)}) + s\left(\omega_n; \beta_n; u^{(n)}_{|\Gamma}, u^{(n)}_{|\Gamma}\right) < \beta_n^2 c_R^2 \,.$$

By the positivity of the bilinear forms a and s, this last inequality implies that

$$a(\mathcal{O}; \beta; u^{(n)}, u^{(n)}) < \beta_n^2 c_R^2 \tag{6.52}$$

and

$$s\left(\omega_n; \beta_n; u^{(n)}_{|\Gamma}, u^{(n)}_{|\Gamma}\right) < \beta_n^2 c_R^2 \,.$$

If we set

$$\varphi^{(n)} = T(\omega_n, \beta_n)^{-1} u_{|\Gamma}^{(n)} \,,$$

we have by (6.49):

$$t(\omega_n; \beta_n; \varphi^{(n)}, \varphi^{(n)}) < \beta_n^2 c_R^2 \,. \tag{6.53}$$

The various steps of the proof are as follows:

- First step:
  Proceeding as in the proof of Lemma 8, we deduce from (6.52) that sequence $u^{(n)}$ is bounded in $H^1(\mathcal{O})^3$ and admits a subsequence, converging weakly in $H^1(\mathcal{O})^3$ and strongly in $L^2(\mathcal{O})^3$. Its limit u is a rigid displacement, so that it is necessarily of the following form:

$$u = \begin{pmatrix} a_1 - b_0 x_2 \\ a_2 + b_0 x_1 \\ a_3 \end{pmatrix} \,,$$

  where a_1, a_2, a_3 and b_0 are real constants and it satisfies:

$$\|u\|_{\mathcal{O}} = 1 \tag{6.54}$$

  and

$$\int_{\mathcal{O}} u_2 \, dx = 0 \,. \tag{6.55}$$

- Second step:
  By definition of $\varphi^{(n)}$, we have for every $\psi \in H^{-1/2}(\Gamma)^3$:

$$t(\omega_n; \beta_n; \varphi^{(n)}, \psi) = \langle \psi, u_{|\Gamma}^{(n)} \rangle \,.$$

Consequently, by the Cauchy–Schwarz inequality:

$$|\langle \psi, u_{|\Gamma}^{(n)} \rangle|^2 \le t(\omega_n; \beta_n; \varphi^{(n)}, \varphi^{(n)}) t(\omega_n; \beta_n; \psi, \psi) \,,$$

which gives, by (6.53), the following estimate:

$$|\langle \psi, u_{|\Gamma}^{(n)} \rangle|^2 \le \beta_n^2 c_R^2 \, t(\omega_n; \beta_n; \psi, \psi) \,. \tag{6.56}$$

Using the explicit Fourier expression of the bilinear form $t(\omega; \beta; \varphi, \varphi)$, we prove in Appendix B that:

$$\beta_n^2 \, t(\omega_n; \beta_n; \psi, \psi) \to 0 \quad \text{as } n \to +\infty \tag{6.57}$$

if $\psi = \begin{pmatrix} \psi_1 \\ 0 \\ 0 \end{pmatrix}$ with $\psi_1 \in H^{-1/2}(\Gamma)$ or $\psi = \begin{pmatrix} 0 \\ \psi_2 \\ 0 \end{pmatrix}$ with $\psi_2 \in H^{-1/2}(\Gamma)$ and $\langle \psi_2, 1 \rangle = 0$. Consequently:

$$\begin{cases} \lim \langle \psi_1, u_{1|\Gamma}^{(n)} \rangle = \langle \psi_1, u_{1|\Gamma} \rangle = 0 \,, \\ \lim \langle \psi_2, u_{2|\Gamma}^{(n)} \rangle = \langle \psi_2, u_{2|\Gamma} \rangle = 0 \,, \end{cases}$$

for every $\psi_1 \in H^{-1/2}(\Gamma)$ and for every $\psi_2 \in H^{-1/2}(\Gamma)$ such that $\langle \psi_2, 1 \rangle = 0$. Using the expression of u and the second equality, we obtain:

$$b_0 \langle \psi_2, x_1 \rangle = 0$$

which gives $b_0 = 0$. Then the first equality reads

$$a_1 \langle \psi_1, 1 \rangle = 0$$

and gives $a_1 = 0$.

• Third step:

We conclude by using (6.46) and (6.55). From (6.55), it results that $a_2 = 0$ so that $u_1 = u_2 = 0$. Using (6.46) we find $u_3 = 0$ and (6.54) gives the contradiction.

$\square$

7. Open Questions

Our contribution in this paper answers in a rather precise way several questions which mainly concern the existence of surface waves in topographic waveguides. In that sense, we feel that this work fills the gap in the literature in this area. However, the theory is far from complete, in particular in comparison with the theory of guided waves in other domains of physics, and some interesting questions remain open. Let us mention some of them in this section, listed by decreasing order of generality:

 (i) The first very general question concerns guided waves which could propagate with speeds higher than the Rayleigh speed c_R. We already saw (Sec. 4) that such waves would correspond from a pure mathematical point of view to eigenvalues embedded in the essential spectrum. Notice that analogous phenomena have been put in evidence in other applications. In fact, we think that there does not exist any mode propagating with a speed greater than c_P, the speed of P waves and this could be proved by following the proof of Weder.[32] The case of waves propagating with a speed between c_S, the speed of S waves, and c_P and even more the one of modes propagating with a speed between c_R and c_S appears to be much more delicate.

 (ii) Does the fundamental mode exist at arbitrary low frequencies? As we already said, this question is naturally raised by our analysis of Sec. 6. We already know that only the fundamental mode may exist for such frequencies and our conjecture is that there exists a strictly positive cutoff frequency below which no guided mode can propagate.

(iii) Another rather general question concerns the number of guided modes at high frequencies. In the case where the geometry of the cross-section of the free boundary presents an angle, we have obtained (Sec. 5.4) a lower bound for this number. What is missing is an opposite result in the sense: the number of guided modes remains bounded. This is, like questions (i) and (ii), a non-existence type result.

(iv) A question is directly related to (iii) and concerns more precisely the case of a smooth boundary. In such a case, we have only been able to prove that, under adequate geometrical assumptions, guided waves existed in a finite range of frequencies (Sec. 5.3). The question is thus: are there some situations where guided modes exist at arbitrary high frequencies, as in the presence of singularities? Moreover, if such waves existed, it should be possible to prove that their propagation speed tends to the Rayleigh speed when the frequency goes to infinity.

(v) As a dual question to (iv), does there exist some situations where a given mode only exists for a finite range of frequencies, i.e. a mode which does not exist for low frequencies, appears at intermediate frequencies and disappears at high frequency? Such modes have been pointed out in other applications, in particular in the case of perturbed stratified waveguides in optics.[11] Some of the numerical results we have obtained make us think that a similar situation could occur in topographic waveguides.

(vi) This point is much more particular and concerns the wedge waveguide, more precisely the maximum angle below which there exists a surface wave (at least at high frequencies). 20 years ago, on the basis of finite element computations, Lagasse *et al.* conjectured that this angle was exactly $90°$.[20] On the basis of other numerical calculations and semi-analytical considerations,[12] we now think that this conjecture fails: this maximum angle can be strictly greater (although slightly) than $90°$, at least for some values of the Poisson's ratio. We are still currently working on this question which is of course closely related to the delicate problem of the existence of surface waves in an infinite wedge (which no longer corresponds to a compact perturbation of the half space!). In such a geometry one could naturally wonder if explicit or quasi-explicit solutions exist and in particular if the techniques of complex variables used by Merzon[24] for solving the famous problem of trapped water waves in an infinite sloping beach (see Ursell[31] or Bonnet-Ben Dhia and Joly[7]) could be applied here.

(vii) The last question concerns the case of a cuspidal perturbation of the free surface, which can be seen as a limit of the wedge waveguide when the angle tends to 0. Our approach does not work for such geometry since Korn's inequality fails in cuspidal domains.[14] Our conjecture is that the number of guided modes in a cuspidal waveguide tends to infinity with the frequency.

Appendix A

Let us consider the operator $B(\beta)$ defined in Sec. 3.1. This operator is associated with the bilinear form $b(\beta; ,\cdot, \cdot)$ defined by (3.13). Spectral properties of the operator $B(\beta)$ can be derived from the following lemma:

Lemma A.1. $\forall\, (u_2, u_3) \in H^1(\mathbb{R}^-)^2$

$$b(\beta; (u_2, u_3), (u_2, u_3)) = (\lambda + \mu) \int_{\mathbb{R}^-} (u_{2,2} + \beta u_3)^2 \, dx_2$$

$$+ \mu \int_{\mathbb{R}^-} (u_{2,2}^2 + u_{3,2}^2 + \beta^2 u_2^2 + \beta^2 u_3^2) \, dx_2 - 2\beta\mu u_2(0) u_3(0) \,. \qquad \text{(A.1)}$$

Proof. This results simply from the following integration by parts:

$$\int_{-\infty}^0 u_2 u_{3,2} \, dx_2 = -\int_{-\infty}^0 u_{2,2} u_3 \, dx_2 + u_2(0) u_3(0) \,.$$

$\square$

At first we determine the essential spectrum of $B(\beta)$.

Theorem A.1.

$$\sigma_{\text{ess}}(B(\beta)) = [\beta^2 c_S^2, +\infty[\,.$$

Proof. We proceed as in Sec. 3. Let $(u_2^{(n)}, u_3^{(n)})$ be a singular sequence associated to the spectral value γ:

$$\begin{cases} \int_{\mathbb{R}^-} ((u_2^{(n)})^2 + (u_3^{(n)})^2) \, dx_2 = 1 \\ u_2^{(n)} \to 0, \ u_3^{(n)} \to 0 \text{ weakly in } L^2(\mathbb{R}^-) \\ \|B(\beta)(u_2^{(n)}, u_3^{(n)}) - \gamma(u_2^{(n)}, u_3^{(n)})\|_{\mathbb{R}^-} \to 0 \,. \end{cases}$$

The sequence $(u_2^{(n)}, u_3^{(n)})$ is bounded in $H^1(\mathbb{R}^-)^2$ and converges weakly to 0. Consequently:

$$u_2^{(n)}(0) \to 0 \quad \text{and} \quad u_3^{(n)}(0) \to 0$$

and therefore, by (A.1):

$$\gamma = \lim b(\beta; (u_2^{(n)}, u_3^{(n)}), (u_2^{(n)}, u_3^{(n)})) \geq \beta^2 \mu \,.$$

This proves the inclusion:

$$\sigma_{\text{ess}}(B(\beta)) \subset [\beta^2 c_S^2, +\infty[\,.$$

The converse inclusion can be easily established by using the following singular sequence:

$$u_2^{(n)}(x_2) = \frac{\beta}{\sqrt{n}} \theta\left(\frac{x_2}{n}\right) e^{i\xi x_2} \,,$$

$$u_3^{(n)}(x_2) = \frac{-i\xi}{\sqrt{n}} \theta\left(\frac{x_2}{n}\right) e^{i\xi x_2} \,,$$

which is associated to the spectral value $\gamma = (\beta^2 + \xi^2) c_S^2$. $\square$

The description of the discrete spectrum of $B(\beta)$ results finally from:

Theorem A.2. $B(\beta)$ *has exactly one simple eigenvalue γ located below the essential spectrum.*

Proof. We apply the min-max principle.

1. To see that $B(\beta)$ has at least one eigenvalue, we show that:

$$\inf_{(u_2,u_3)\in H^1(\mathbb{R}^-)^2, (u_2,u_3)\neq 0} \frac{b(\beta;(u_2,u_3),(u_2,u_3))}{\displaystyle\int_{\mathbb{R}^-}(u_2^2+u_3^2)\,dx} < \beta^2 c_S^2\,.$$

To do that, we use for example a test function (u_2,u_3) defined as follows: u_3 is an arbitrary function of $H^1(\mathbb{R}^-)$ such that $u_3(0)=1$ and $u_2(x)=t\varphi(x/n)$ where $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$ and $\varphi(0)=1$. The required inequality then holds for n large enough and for suitable values of t.

2. To see that $B(\beta)$ has at most one eigenvalue, we show that

$$\sup_{(u_2,u_3)\in V_2, (u_2,u_3)\neq 0} \frac{b(\beta;(u_2,u_3),(u_2,u_3))}{\displaystyle\int_{\mathbb{R}^-}(u_2^2+u_3^2)\,dx} \geq \beta^2 c_S^2$$

for every two-dimensional subspace V_2 of $H^1(\mathbb{R}^-)^2$. Indeed, V_2 necessarily contains an element (u_2,u_3) such that $u_2(0)=0$.

$\square$

Appendix B

Lemma B.1.   *Suppose β_n and ω_n are two real sequences satisfying*:

$$\begin{cases} 0 < \omega_n < \beta_n c_R\,, \\ \beta_n \to 0\,. \end{cases}$$

Then for the bilinear form t defined by (6.47) *we have*:

$$\beta_n^2\, t(\omega_n;\beta_n;\psi,\psi) \to 0 \quad \text{as } n \to +\infty$$

for every $\psi = \begin{pmatrix} \psi_1 \\ 0 \\ 0 \end{pmatrix}$ with $\psi_1 \in H^{-1/2}(\Gamma)$ or $\psi = \begin{pmatrix} 0 \\ \psi_2 \\ 0 \end{pmatrix}$ with $\psi_2 \in H^{-1/2}(\Gamma)$ and $\langle \psi_2, 1\rangle = 0$.

Proof. For $\psi \in H^{-1/2}(\Gamma)^3$ we denote by $\hat\psi$ its Fourier transform defined by:

$$\hat\psi_j(\xi) = \left\langle \psi_j, \frac{1}{\sqrt{2\pi}}e^{-ix_1\xi}{}_{|\Gamma} \right\rangle \quad \text{for } \xi \in \mathbb{R} \text{ and } j=1,2,3\,.$$

Since ψ has compact support and belongs to $H^{-1/2}(\Gamma)^3$, $\hat{\psi}$ is analytic in ξ and

$$\int_{\mathbb{R}} \frac{|\hat{\psi}_j(\xi)|^2}{\sqrt{1+\xi^2}} d\xi < +\infty \quad \text{for } j = 1, 2, 3 \,. \tag{B.1}$$

If moreover $\langle \psi_2, 1 \rangle = 0$, then $\hat{\psi}_2(0) = 0$ and there exists an analytic function θ such that $\hat{\psi}_2(\xi) = \xi \theta(\xi)$. Therefore:

$$\int_{\mathbb{R}} \frac{\sqrt{1+\xi^2}|\hat{\psi}_2(\xi)|^2}{\xi^2} d\xi < +\infty \,. \tag{B.2}$$

Following Duterte and Joly,[13] we can write the coupling bilinear form $t(\omega; \beta; \psi, \psi)$ as follows:

$$t(\omega; \beta; \psi, \psi) = \int_{-\infty}^{+\infty} (\mathbf{T}(\omega, \beta, \xi)\hat{\psi}(\xi), \hat{\psi}(\xi)) \, d\xi \,,$$

where $\mathbf{T}(\omega, \beta, \xi)$ is a 3×3 symmetric matrix:

$$\mathbf{T}(\omega, \beta, \xi) = \begin{pmatrix} t_1(\omega, \beta, \xi) & t_{12}(\omega, \beta, \xi) & t_{13}(\omega, \beta, \xi) \\ t_{12}(\omega, \beta, \xi) & t_2(\omega, \beta, \xi) & t_{23}(\omega, \beta, \xi) \\ t_{13}(\omega, \beta, \xi) & t_{23}(\omega, \beta, \xi) & t_3(\omega, \beta, \xi) \end{pmatrix} \,.$$

The diagonal coefficients $t_1(\omega, \beta, \xi)$ and $t_2(\omega, \beta, \xi)$ have the following expression:

$$t_1(\omega, \beta, \xi) = \frac{1}{c_S^2\sqrt{\beta^2+\xi^2}} \frac{1}{\sqrt{1-\eta(\omega,\beta,\xi)^2}} \left(1 + \frac{\xi^2 g(\eta(\omega,\beta,\xi))}{(\beta^2+\xi^2)f_R(\eta(\omega,\beta,\xi))} \right) \,,$$

$$t_2(\omega, \beta, \xi) = \frac{\sqrt{1 - \frac{c_S^2}{c_P^2}\eta(\omega,\beta,\xi)^2}}{c_S^2\sqrt{\beta^2+\xi^2}} \frac{\eta(\omega,\beta,\xi)^2}{f_R(\eta(\omega,\beta,\xi))} \,,$$

where we have set:

$$\eta(\omega, \beta, \xi) = \frac{\omega}{c_S\sqrt{\beta^2+\xi^2}} \,,$$

$$f_R(\eta) = 4\sqrt{1-\eta^2}\sqrt{1 - \frac{c_S^2}{c_P^2}\eta^2} - (2-\eta^2)^2 \,,$$

$$g(\eta) = \eta^2(1-\eta^2) - f_R(\eta) \,.$$

For every $n \in \mathbb{N}$ and $\xi \in \mathbb{R}$, $0 < \eta(\omega_n, \beta_n, \xi) < c_R/c_S$. Consequently

$$t_1(\omega, \beta, \xi) \leq \frac{1}{c_S^2\sqrt{\beta^2+\xi^2}} \frac{1}{\sqrt{1 - \frac{c_R^2}{c_S^2}}} \left(1 + \frac{\xi^2 g(\eta(\omega,\beta,\xi))}{(\beta^2+\xi^2)\,f_R(\eta(\omega,\beta,\xi))} \right) \,,$$

$$t_2(\omega, \beta, \xi) \leq \frac{1}{c_S^2\sqrt{\beta^2+\xi^2}} \frac{\eta(\omega,\beta,\xi)^2}{f_R(\eta(\omega,\beta,\xi))} \,.$$

Let us consider now the variations of the functions f_R and g in the interval $[0, c_R/c_S]$. By classical properties of the Rayleigh function f_R, we have

$$\begin{cases} f_R(0) = f_R'(0) = g(0) = g'(0) = 0 \quad \text{and} \quad f_R''(0) > 0 \,, \\[2mm] f_R\left(\dfrac{c_R}{c_S}\right) = 0 \quad \text{and} \quad f_R'\left(\dfrac{c_R}{c_S}\right) < 0 \,, \\[2mm] f_R(\eta) > 0 \quad \text{for } \eta \in \left]0, \dfrac{c_R}{c_S}\right[\,. \end{cases}$$

Consequently, there are two positive constants (all constants are denoted by C) depending only on c_S and c_P such that:

$$f_R(\eta) \geq C\eta^2 \left(\frac{c_R}{c_S} - \eta\right) \quad \forall \, \eta \in \left[0, \frac{c_R}{c_S}\right] \tag{B.3}$$

and

$$g(\eta) \leq C\eta^2 \quad \forall \, \eta \in \left[0, \frac{c_R}{c_S}\right] \,. \tag{B.4}$$

Moreover,

$$\frac{c_R}{c_S} - \eta = \left(\frac{c_R^2}{c_S^2} - \eta^2\right)\left(\frac{c_R}{c_S} + \eta\right)^{-1} \geq \frac{c_S}{2c_R}\left(\frac{c_R^2}{c_S^2} - \eta^2\right)$$

and

$$\frac{c_R^2}{c_S^2} - \eta(\omega, \beta, \xi)^2 = \frac{\beta^2 c_R^2 - \omega^2}{(\beta^2 + \xi^2)\,c_S^2} + \frac{\xi^2 c_R^2}{(\beta^2 + \xi^2)\,c_S^2} \geq \frac{\xi^2 c_R^2}{(\beta^2 + \xi^2)\,c_S^2} \,.$$

Consequently, we have for every $n \in \mathbb{N}$:

$$\frac{c_R}{c_S} - \eta(\omega_n, \beta_n, \xi) \geq \frac{\xi^2 c_R}{2(\beta_n^2 + \xi^2)c_S} \,. \tag{B.5}$$

Finally using (B.3) and (B.5) we obtain:

$$f_R(\eta(\omega_n, \beta_n, \xi)) \geq C\eta(\omega_n, \beta_n, \xi)^2 \frac{\xi^2}{\beta_n^2 + \xi^2} \,.$$

Using the previous inequality and (B.4), we derive the following upper estimates for every $\xi \in \mathbb{R}$:

$$t_1(\omega_n, \beta_n, \xi) \leq \frac{C}{\sqrt{\beta_n^2 + \xi^2}} \,,$$

$$t_2(\omega_n, \beta_n, \xi) \leq C\frac{\sqrt{\beta_n^2 + \xi^2}}{\xi^2} \,.$$

By (B.2), if $\langle \psi_2, 1 \rangle = 0$ then, for n large enough,

$$\int_{\mathbb{R}} \frac{\sqrt{\beta_n^2 + \xi^2}}{\xi^2} |\hat{\psi}_2(\xi)|^2 \, d\xi \leq \int_{\mathbb{R}} \frac{\sqrt{1 + \xi^2}}{\xi^2} |\hat{\psi}_2(\xi)|^2 \, d\xi < +\infty$$

and consequently

$$\beta_n^2 \int_{\mathbb{R}} t_2(\omega_n, \beta_n, \xi) |\hat{\psi}_2(\xi)|^2 \, d\xi \to 0 \, .$$

Finally

$$\int_{\mathbb{R}} \frac{1}{\sqrt{\beta_n^2 + \xi^2}} |\hat{\psi}_1(\xi)|^2 \, d\xi \leq \left(\sup_{|\xi| \leq 1} |\hat{\psi}_1(\xi)|^2 \right) \int_{|\xi| \leq 1} \frac{1}{\sqrt{\beta_n^2 + \xi^2}} \, d\xi + \int_{|\xi| > 1} \frac{|\hat{\psi}_1(\xi)|^2}{|\xi|} \, d\xi \, .$$

The first term can be calculated and gives:

$$\int_{|\xi| \leq 1} \frac{1}{\sqrt{\beta_n^2 + \xi^2}} d\xi = 2 \operatorname{arcsinh}\left(\frac{1}{\beta_n} \right) \sim -2 \log(\beta_n) \quad \text{as} \quad n \to +\infty$$

and by (B.1) the second integral is finite. Consequently:

$$\beta_n^2 \int_{\mathbb{R}} t_1(\omega_n, \beta_n, \xi) |\hat{\psi}_1(\xi)|^2 \, d\xi \to 0$$

which ends the proof. $\qquad\square$

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