Time Optimal Problems with Boundary Controls

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Abstract

We consider time optimal control problems governed by semilinear parabolic equations with pointwise state constraints and unbounded controls. We derive a Pontryagin's principle for boundary controls. We prove a regularity result for the gradient of the state variable and by this way we are able to define a Hamiltonian functional which intervenes in an optimality condition satisfied by the optimal terminal time.

Keywords. Time optimal problem, optimal control, nonlinear boundary controls, semilinear parabolic equations, state constraints, Pontryagin's minimum principle, parabolic equations with measure as data, unbounded controls.

1 Introduction

Let Ω be a bounded open subset in $\mathbb{R}^N$ $(N \geq 2)$, with a regular boundary Γ, and let A be a second order differential operator of the form $Ay(x) = \sum_{i,j1}^N D_i(a_{ij}(x)D_j y(x))$ (D_i denotes the partial derivative with respect to x_i). Consider a system described by the parabolic equation:

$$\begin{aligned} &\frac{\partial y}{\partial t} + Ay + f(\cdot, y) = 0 \text{ in } Q_T, \\ &\frac{\partial y}{\partial n_A} + g(\cdot, y, v) = 0 \text{ on } \Sigma_T, \qquad y(0) = y_0 \text{ in } \Omega, \end{aligned} \tag{1}$$

where $Q_T \Omega \times]0, T[$, $\Sigma_T \Gamma \times]0, T[$, $y_0 \in C(\overline{\Omega})$. The set of constraints on the control variable v is defined by :

$$V_{ad} = \{v \in L^\sigma_{loc}([0, \infty[; L^\sigma(\Gamma)) \mid v(s,t) \in K_V(s) \text{ for a.e. } (s,t) \in \Gamma \times \mathbb{R}\},$$

where $\sigma > N+1$, K_V is a measurable multimapping from Γ with nonempty and closed values into $\mathcal{P}(\mathbb{R})$ (the set of all subsets of $\mathbb{R}$). We also consider state constraints of the form:

$$\Phi(t, y(t)) \in \mathcal{C} \quad \text{for every } t \in [0, T], \tag{2}$$

where $\mathcal{C}$ is a closed convex subset with a nonempty interior in $C(\overline{\Omega})$. We study the control problem:

$$(\mathcal{P}) \qquad \inf\{J(y, v, T) \mid (v, T) \in V_{ad} \times \mathbb{R}^*_+, \quad (y, v, T) \text{ satisfies } (1), (2)\},$$

where the cost functional J is defined on $C(\overline{\Omega} \times \mathbb{R}_+) \times L^\sigma_{loc}([0, +\infty[; L^\sigma(\Gamma)) \times \mathbb{R}^*_+$ by

$$J(y, v, T) = \int_0^T \int_\Omega F(\cdot, y)\, dxdt + \int_0^T \int_\Gamma G(\cdot, y, v)\, dsdt + \int_\Omega L(\cdot, T, y(T))dx.$$

We are mainly interested in optimality conditions for $(\mathcal{P})$ and the existence of optimal solutions is a priori assumed. The control variables of $(\mathcal{P})$ are $v \in V_{ad}$ and $T \in \mathbb{R}^*_+$. For such problems, optimality conditions for v (in the form of a Pontryagin's principle) are obtained for bounded controls in [5, 17], and for unbounded controls in [28]. Since T is also a control variable, an other equation must be derived to obtain full first order necessary optimality conditions.

For control problems governed by Ordinary Differential Equations this additional condition on T is an equation satisfied by the Hamiltonian function of the corresponding problem at the terminal time T [32]. In the case of control problems governed by Partial Differential Equations some results are obtained in [9, 31, 20, 21, 18, 12, 13, 3, 33]. In particular, Fattorini [13] obtains an invariance principle for some Hamiltonian functional of autonomous control problems governed by semilinear parabolic equations (extending in this way some results well known for problems governed by Ordinary Differential Equations). In [29], we have obtained some generalizations for nonautonomous problems governed by the equation

$$\frac{\partial y}{\partial t} + Ay + \psi(\cdot, y, u) = 0 \text{ in } Q_T, \quad \frac{\partial y}{\partial n_A} = 0 \text{ on } \Sigma_T, \quad y(0) = y_0 \text{ in } \Omega,$$

with a target constraint: $\phi(T, y(T)) \in \mathcal{C}$ (where $\mathcal{C}$ is a closed convex subset of finite codimension in $C(\overline{\Omega})$), and a cost functional of the form:

$$\mathcal{J}(y, u, T) = \int_0^T \int_\Omega \mathcal{F}(\cdot, y, u)\, dxdt + \int_\Omega \mathcal{L}(\cdot, T, y(T))dx.$$

To obtain an optimality condition for T, we have introduced the Hamiltonian functional

$$\mathcal{H}(t, y, u, p, \nu) = \nu \int_\Omega \mathcal{F}(x, t, y, u)\, dx - \langle p, \psi(\cdot, y, u) + Ay\rangle_{L^{q'}(\Omega)\times L^q(\Omega)},$$

defined on $\mathbb{R}^*_+ \times W^{2,q}(\Omega) \times L^q(\Omega) \times L^{q'}(\Omega) \times \mathbb{R}$. We have proved that for an optimal solution $(\bar{y}, \bar{u}, \bar{T})$, the distributed control $\bar{u}$ satisfies a Pontryagin's principle and that the additional equation for $\bar{T}$ is

$$\begin{cases} \dfrac{d\mathcal{H}}{dt}(t, \bar{y}(t), \bar{u}(t), \bar{p}(t), \bar{\nu}) = \dfrac{\partial \mathcal{H}}{\partial t}(t, \bar{y}(t), \bar{u}(t), \bar{p}(t), \bar{\nu}) \quad \text{in }]0, \bar{T}[, \\ \mathcal{H}(\bar{T}, \bar{y}(\bar{T}), \bar{u}(\bar{T}), \bar{p}(\bar{T}), \bar{\nu}) = \bar{\nu} \int_\Omega \mathcal{L}'_t(x, \bar{T}, \bar{y}(\bar{T}))\, dx + \langle \phi'_t(\bar{T}, \bar{y}(\bar{T})), \bar{\mu}\rangle_{\overline{\Omega}}, \end{cases}$$

where $\bar{\nu}$ is the multiplier of the cost functional, the measure $\bar{\mu}$ is the multiplier corresponding to the state constraint $\phi(T, y(T)) \in \mathcal{C}$, $\bar{p}$ is the solution of the adjoint equation associated with $(\bar{y}, \bar{u}, \bar{\mu}, \bar{\nu})$, and $\langle \cdot, \cdot \rangle_{\overline{\Omega}}$ stands for the duality pairing between $C(\overline{\Omega})$ and $\mathcal{M}(\overline{\Omega})$ (see [29], Theorem 2.1).

The extension of such a result to the control problem $(\mathcal{P})$ considered in this paper is not completely obvious. Indeed contrary to [29], the state variable y (solution of (1)) does not belong to $L^q(0, T; W^{2,q}(\Omega))$ for some $q \geq 1$.

Therefore the Hamiltonian functional cannot be defined as above. Here we define $\mathcal{H}$ by setting:

$$\mathcal{H}(t,y,v,p,\nu) = \nu \int_\Gamma G(s,t,y(s),v(s))\,ds + \nu \int_\Omega F(x,t,y(x))\,dx$$

$$- \int_\Gamma p(s)g(s,t,y(s),v(s))\,ds - \int_\Omega \Big(\sum_{i,j} a_{ij} D_j y D_i p + f(x,t,y)p(x) \Big)\,dx$$

for every $(t,v,\nu) \in \mathbb{R}^*_+ \times L^\sigma(\Gamma) \times \mathbb{R}$, every $y \in W^{1,d}(\Omega) \cap L^\infty(\Omega)$ and every $p \in W^{1,d'}(\Omega)$ for some $d > 1$. For problem (P), because the state constraints are imposed on the whole cylinder, the adjoint state p belongs to $L^{\delta'}(0,T;W^{1,d'}(\Omega))$ only if (δ,d) satisfies $\frac{N}{2d} + \frac{1}{\delta} < \frac{1}{2}$. Thus, to well define the Hamiltonian functional $\mathcal{H}$, we have to prove that the solution of (1) belongs to $L^\delta(0,T;W^{1,d}(\Omega))$ for some (δ,d) satisfying $\frac{N}{2d} + \frac{1}{\delta} < \frac{1}{2}$ (see Theorem 3.1). Section 4 is devoted to the adjoint equation, and a Green formula, useful to derive an optimality condition for T, is stated in Theorem 4.2. To obtain an optimality condition for T, as in [33, 29], we transform the problem by making a change of variable in time and by introducing a new control variable. Optimality conditions are obtained with the Ekeland variational principle applied to some problems in which the state constraints are penalized. A Pontryagin principle for v is obtained as in [28, 29]. The delicate point is to calculate the derivative of the optimal cost with respect to T. In this analysis, Taylor expansions stated in Theorem 5.1 play a major role. The proof of the main result (Theorem 2.1) is given in §6.

Let us insist on that the optimality condition for the optimal time T obtained in (7), (9) of Theorem 2.1 is a completely new result. None of the previously mentioned papers deals with optimal time problems, boundary controls, pointwise state constraints on the full cylinder Q_T, nonautonomous problems. Moreover, contrary to the case when the state constraint is only imposed at the terminal time, if $(\bar{y},\bar{v},\bar{T})$ is an optimal solution, if $\bar{\nu}$ and $\bar{\lambda}$ are the associated multipliers and if $\bar{p}$ the corresponding adjoint state, then $t \to \mathcal{H}(t,\bar{y}(t),\bar{u}(t),\bar{p}(t),\bar{\nu})$ does not belong to $W^{1,1}(0,\bar{T})$. It only belongs to

$BV([0,\bar{T}])$. This function satisfies a differential equation with a right hand side measure. The jumps come from the measure $\Phi'_t(\cdot,\bar{y}(\cdot))^*\bar{\lambda}$ (which is the measure on $[0,\bar{T}]$ defined by $\varphi \to \langle\bar{\lambda}, \Phi'_t(\cdot,\bar{y}(\cdot))\varphi\rangle$ for every $\varphi \in C([0,\bar{T}])$). We notice that there is no jump (that is $t \to \mathcal{H}(t,\bar{y}(t),\bar{u}(t),\bar{p}(t),\bar{\nu})$ belongs to $W^{1,1}(0,\bar{T})$) if the mapping Φ in (2) does not depend on t.

Let us finally mention that results presented in this paper can be adapted to a terminal state constraint of the form $\Phi(T,y(T)) \in \mathcal{C}$, as the one considered in [29], where $\mathcal{C}$ is a closed convex subset with finite codimension in $C(\overline{\Omega})$ (see Remark 2.5).

2 Assumptions. Main results

For any $T > 0$, we denote by Q_T the cylinder $\Omega\times]0,T[$ and by Σ_T the lateral surface $\Gamma\times]0,T[$. We also set $\overline{\Omega}_0 = \overline{\Omega}\times\{0\}$ and $\overline{\Omega}_T = \overline{\Omega}\times\{T\}$. For every $1 \le m \le \infty$, the usual norms of the spaces $L^m(\Omega)$, $L^m(\Gamma)$, $L^m(Q_T)$, $L^m(\Sigma_T)$, $L^m(\Omega\times]0,\bar{t}[)$, $L^m(\Gamma\times]0,\bar{t}[)$ will be denoted by $\|\cdot\|_{m,\Omega}$, $\|\cdot\|_{m,\Gamma}$, $\|\cdot\|_{m,Q_T}$, $\|\cdot\|_{m,\Sigma_T}$, $\|\cdot\|_{m,\Omega\times]0,\bar{t}[}$, $\|\cdot\|_{m,\Gamma\times]0,\bar{t}[}$. The Hilbert space $W(0,\bar{t};H^1(\Omega),(H^1(\Omega))') = \{y \in L^2(0,\bar{t};H^1(\Omega)) \mid \frac{dy}{dt} \in L^2(0,\bar{t};(H^1(\Omega))')\}$ endowed with its usual norm, will be denoted by $W(0,\bar{t})$ (this notation will be used for $\bar{t}1$ and $\bar{t}T$). For shortening we write $L^2(0,T;H^1)$, $C([0,T];L^2)$ and $L^\delta(0,T;W^{1,d})$ in place of $L^2(0,T;H^1(\Omega))$, $C([0,T];L^2(\Omega))$ and $L^\delta(0,T;W^{1,d}(\Omega))$. When there is no confusion, we also write $f(t,y)$ and $g(t,y,v)$ for $f(x,t,y)$ and $g(s,t,y,v)$ (the same abuse of notation is made for other functions). We denote by C_i, for $i \in N^*$, constants which intervene in the estimates given in propositions, while the letters K or K_i, $i \in N$, throughout the proofs denote various constants depending on known quantities. If $\mathcal{O}$ is an open subset in $\overline{Q}_T$, if μ belongs to $\mathcal{M}_b(\mathcal{O})$, and if y belongs to $C_b(\mathcal{O})$, the integral $\int_{\mathcal{O}} y d\mu$ will be denoted by $\langle y,\mu\rangle_{b,\mathcal{O}}$. If λ belongs to $\mathcal{M}([0,T])$, and φ belongs to $C([0,T])$, the integral $\int_{[0,T]} \varphi d\lambda$ will be denoted by $\langle\lambda,\varphi\rangle_{[0,T]}$. In the same way, if y belongs to $C(\overline{Q}_T)$ and if μ belongs to $\mathcal{M}(\overline{Q}_T)$, we shall write $\langle \mu, y \rangle_{\overline{Q}_T}$ in place of $\int_{\overline{Q}_T} y d\mu$. The duality pairing between $H^1(\Omega)$ and $(H^1(\Omega))'$ will be denoted by $\langle \cdot, \cdot \rangle_1$.

2.1 Assumptions

Throughout the sequel, we make the following assumptions.

(A1) - Ω is a bounded open subset in $\mathbb{R}^N$ $(N \geq 2)$ of class $C^{2+\beta}$ for some $0 < \beta < 1$ (that is, the boundary Γ of Ω is an $(N-1)$-dimensional manifold of class $C^{2+\beta}$ such that Ω lies locally on one side of Γ. A function is of class $C^{2+\beta}$ if it is of class C^2 and if its partial derivatives of second order are Hölder continuous of order β). We denote by q, σ positive numbers satisfying:

$$q > \frac{N}{2} + 1, \qquad \sigma > N + 1 \qquad \text{and} \qquad q\sigma + q > qN + 2\sigma.$$

(A2) - The coefficients a_{ij} of the operator A belong to $C^{1+\beta}(\overline{\Omega})$, satisfy $a_{ij}(x) = a_{ji}(x)$ and the condition:

$$m_0 |\xi|^2 \leq \sum_{i,j1}^{N} a_{ij}(x) \xi_j \xi_i \tag{3}$$

for all $x \in \overline{\Omega}$ and all $\xi \in \mathbb{R}^N$, with $0 < m_0$, (D_i denotes the partial derivative with respect to x_i).

(A3) - For every $(t, y) \in \mathbb{R}^+ \times \mathbb{R}$, $f(\cdot, t, y)$ is measurable on Ω. For almost every $x \in \Omega$, $f(x, \cdot)$ is of class C^1 on $\mathbb{R}^+ \times \mathbb{R}$. For almost every $x \in \Omega$, $f(x, \cdot)$, $f'_t(x, \cdot)$ and $f'_y(x, \cdot)$ are continuous on $\mathbb{R}^+ \times \mathbb{R}$. The following estimates hold

$$|f(x, t, 0)| \leq M_1(x) \eta(t), \quad |f'_t(x, t, y)| \leq M_1(x) \eta(t) \eta(|y|),$$

$$C_0 \leq f'_y(x, t, y) \leq M_1(x) \eta(t) \eta(|y|),$$

where M_1 belongs to $L^q(\Omega)$, η is a nondecreasing function from $\mathbb{R}^+$ to $\mathbb{R}^+$, $m_1 \in \mathbb{R}^+$ and $C_0 \in \mathbb{R}$. (We have denoted by f'_t and by f'_y the partial derivatives of f with respect to t and to y, in all the sequel we adopt the same kind of notation for other functions.)

(A4) - For every $(t,y,v) \in \mathbb{R}^+ \times \mathbb{R}^2$, $g(\cdot,t,y,v)$ is measurable on Γ. For almost every $s \in \Gamma$, and for every $v \in \mathbb{R}$, $g(s,\cdot,v)$ is of class C^1 on $\mathbb{R}^+ \times \mathbb{R}$. For almost every $s \in \Gamma$, $g(s,\cdot)$, $g'_t(s,\cdot)$ and $g'_y(s,\cdot)$ are continuous on $\mathbb{R}^+ \times \mathbb{R}^2$. The following estimates hold

$$|g(s,t,0,v)| \leq (M_2(s) + |v|)\eta(t), \quad |g'_t(s,t,y,v)| \leq (M_2(s) + |v|)\eta(t)\eta(|y|),$$

$$C_0 \leq g'_y(s,t,y,v) \leq (M_2(s) + |v|)\eta(t)\eta(|y|),$$

where $M_2 \in L^\sigma(\Gamma)$, η , m_1 and C_0 are as in (A3).

(A5) - For every $(t,y) \in \mathbb{R}^+ \times \mathbb{R}$, $L(\cdot,t,y)$ is measurable on Ω. For almost every $x \in \Omega$, $L(x,\cdot)$ is of class C^1 on $\mathbb{R}^+ \times \mathbb{R}$. The following estimate holds

$$|L(x,t,y)| + |L'_t(x,t,y)| + |L'_y(x,t,y)| \leq M_3(x)\eta(t)\eta(|y|),$$

where $M_3 \in L^1(\Omega)$.

(A6) - For every $(t,y) \in \mathbb{R}^+ \times \mathbb{R}$, $F(\cdot,t,y)$ is measurable on Ω. For almost every $x \in \Omega$, $F(x,\cdot)$ is of class C^1 on $\mathbb{R}^+ \times \mathbb{R}$. For almost every $x \in \Omega$, $F(x,\cdot)$, $F'_t(x,\cdot)$ and $F'_y(x,\cdot)$ are continuous on $\mathbb{R}^+ \times \mathbb{R}$. The following estimate holds

$$|F(x,t,y)| + |F'_t(x,t,y)| + |F'_y(x,t,y)| \leq M_4(x)\eta(t)\eta(|y|),$$

where $M_4 \in L^1(\Omega)$.

(A7) - For every $(t,y,v) \in \mathbb{R}^+ \times \mathbb{R}^2$, $G(\cdot,t,y,v)$ is measurable on Γ. For almost every $s \in \Gamma$, for every $v \in \mathbb{R}$, $G(s,\cdot,v)$ is of class C^1 on $\mathbb{R}^+ \times \mathbb{R}$. For almost every $s \in \Gamma$, $G(s,\cdot)$, $G'_t(s,\cdot)$ and $G'_y(s,\cdot)$ are continuous on $\mathbb{R}^+ \times \mathbb{R}^2$. The following estimate holds

$$|G(s,t,y,v)| + |G'_t(s,t,y,v)| + |G'_y(s,t,y,v)| \leq (M_5(s) + m_1|v|)\eta(t)\eta(|y|),$$

where $M_5 \in L^1(\Gamma)$.

(A8) - There exists a measurable function $k_0 \; : \; \Gamma \to \mathbb{R}$ such that $k_0 \in L^\sigma(\Gamma)$

and $k_0(s) \in K_V(s)$ for almost every $s \in \Gamma$.

(A9) - $\Phi : \mathbb{R} \times C(\overline{\Omega}) \to C(\overline{\Omega})$ is of class C^1, and $\mathcal{C}$ is a closed convex subset with a nonempty interior in $C(\overline{\Omega})$.

(A10) - y_0 belongs to $C^2(\overline{\Omega}) \cap C_0(\Omega)$, and $\Phi(0, y_0) \in \operatorname{int} \mathcal{C}$.

Remark 2.1. For simplicity we have supposed that y_0 belongs to $C^2(\overline{\Omega})$. But the main result of this paper (Theorem 2.1) is still true if $y_0 \in C(\overline{\Omega})$ (see Remarks 3.1, 6.1 for the adaptation of the proof to this case). The justification of the condition $\Phi(0, y_0) \in \operatorname{int} \mathcal{C}$ is given in Remark 2.4.

Remark 2.2. Under assumption (A8), V_{ad} is nonempty.

Let us give two simple examples of state constraints described by (2).

Example 2.1. If $b \in C(\overline{Q}_T)$, the state constraint $y(x, t) \leq b(x, t)$ on $\overline{Q}_T$, may be written in the form (2) by setting $\Phi(t, y) = y - b(t)$, $\mathcal{C} = \{z \in C(\overline{\Omega}) \mid z \leq 0\}$, where $b(t)$ is the function $x \to b(t, x)$. Even in this simple case, the measure $\Phi'_t(\cdot, \bar{y}(\cdot))^* \bar{\lambda}$, which intervenes in the adjoint differential equation (7), may have an atomic part.

Example 2.2. Suppose that a and b are two functions in $C(\overline{Q}_T)$ such that $a(x, t) < b(x, t)$ on $\overline{Q}_T$. The state constraint $a(x, t) \leq y(x, t) \leq b(x, t)$ on $\overline{Q}_T$ may be written in the form (2) by setting $\Phi(t, y) = \phi(y, a(t), b(t))$, $\mathcal{C} = \{z \in C(\bar{\Omega}) \mid z \leq 0\}$, where $\phi(y, a, b) = |y - (b + a)/2| - (b - a)/2$ if $y \notin [(b + 3a)/4, (3b + a)/4]$, $\phi(y, a, b) = \frac{2}{b-a}(y - \frac{b+3a}{4})(y - \frac{a+3b}{4}) + \frac{a-b}{4}$ if $y \in [(b + 3a)/4, (3b + a)/4]$, $a(t)$ is the function $x \to a(t, x)$ and $b(t)$ is the function $x \to b(t, x)$.

2.2 Statement of the main result

To derive optimality conditions for the optimal control problem $(\mathcal{P})$, we introduce the Hamiltonian function:

$$H(s, t, y, v, p, \nu) = \nu G(s, t, y, v) - p g(s, t, y, v)$$

for all $(s,t,y,v,p,\nu) \in \Omega \times \mathbb{R}^*_+ \times \mathbb{R}^4$, and the functional

$$\mathcal{H}(t,y,v,p,\nu) = \nu \int_\Gamma G(s,t,y(s),v(s))\,ds + \nu \int_\Omega F(x,t,y(x))\,dx$$
$$- \int_\Gamma p(s)g(s,t,y(s),v(s))\,ds - \int_\Omega \Big(\sum_{i,j} a_{ij} D_j y D_i p + f(x,t,y)p(x) \Big)\,dx$$

for all $(t,v,\nu) \in \mathbb{R}^*_+ \times L^\sigma(\Gamma) \times \mathbb{R}$, all $y \in W^{1,d}(\Omega) \cap L^\infty(\Omega)$, and all $p \in W^{1,d'}(\Omega)$ (for some $d > 2$).

Theorem 2.1. *If (A1)-(A10) are fulfilled and if* $(\bar{y},\bar{v},\bar{T})$ *is a solution of* $(\mathcal{P})$*, then there exist* $(\bar{p},\bar{q}) \in L^1(0,\bar{T};W^{1,1}(\Omega)) \times BV([0,\bar{T}])$*,* $\bar{\nu} \in \mathbb{R}^+$*, and* $\bar{\lambda} \in \mathcal{M}(\overline{Q}_{\bar{T}})$ *(the space of Radon measures on* $\overline{\Omega} \times [0,\bar{T}]$*), such that*

$$(\bar{\nu},\bar{\lambda}) \neq 0, \quad \langle \bar{\lambda}, z - \Phi(\cdot,\bar{y}(\cdot)) \rangle_{\overline{Q}_{\bar{T}}} \leq 0 \quad \forall z \in \{ z \in C(\overline{Q}_{\bar{T}}) \mid z(t) \in \mathcal{C} \}, \tag{4}$$

$$\begin{cases} -\dfrac{\partial \bar{p}}{\partial t} + A\bar{p} + f'_y(\cdot,\bar{y})\bar{p} = \bar{\nu} F'_y(\cdot,\bar{y}) + [\Phi'_y(\cdot,\bar{y}(\cdot))^* \bar{\lambda}]|_{Q_{\bar{T}}} & \textit{in } Q_{\bar{T}}, \\ \dfrac{\partial \bar{p}}{\partial n_A} + g'_y(\cdot,\bar{y},\bar{v})\bar{p} = \bar{\nu} G'_y(\cdot,\bar{y},\bar{v}) + [\Phi'_y(\cdot,\bar{y}(\cdot))^* \bar{\lambda}]|_{\Sigma_{\bar{T}}} & \textit{on } \Sigma_{\bar{T}}, \\ \bar{p}(\bar{T}) = \bar{\nu} L'_y(\cdot,\bar{T},\bar{y}(\bar{T})) + [\Phi'_y(\cdot,\bar{y}(\cdot))^* \bar{\lambda}]|_{\overline{\Omega}_{\bar{T}}} & \textit{in } \Omega, \end{cases} \tag{5}$$

$$\bar{p} \in L^{\delta'}(0,\bar{T};W^{1,d'}(\Omega)) \qquad \textit{for every } (\delta,d) \textit{ satisfying } \frac{N}{2d} + \frac{1}{\delta} < \frac{1}{2}, \tag{6}$$

$$\begin{cases} d\bar{q} = [\Phi'_t(\cdot,\bar{y}(\cdot))^* \bar{\lambda}] + \Big[\displaystyle\int_\Gamma [\bar{p}\, g'_t(\cdot,t,\bar{y},\bar{v}) - \bar{\nu} G'_t(\cdot,t,\bar{y},\bar{v})]\,ds \\ \qquad + \displaystyle\int_\Omega [\bar{p}\, f'_t(\cdot,t,\bar{y}) - \bar{\nu} F'_t(\cdot,t,\bar{y})]\,dx \Big]\,dt \\ \bar{q}(\bar{T}) = \bar{\nu} \displaystyle\int_\Omega L'_t(x,\bar{T},\bar{y}(x,\bar{T}))\,dx, \end{cases} \tag{7}$$

$$H(\cdot,\bar{y}(\cdot),\bar{v}(\cdot),\bar{p}(\cdot),\bar{\nu})(s,t) = \min_{v \in K_V(s)} H(\cdot,\bar{y}(\cdot),v,\bar{p}(\cdot),\bar{\nu})(s,t) \tag{8}$$

for a.e. $(s,t) \in \Sigma_{\bar{T}}$,

$$\mathcal{H}(t,\bar{y}(t),\bar{v}(t),\bar{p}(t),\bar{\nu}) - \bar{q}(t) = 0 \qquad \textit{for a.e. } t \in]0,\bar{T}[, \tag{9}$$

where $[\Phi'_y(\cdot,\bar y)^*\bar\lambda]|_{Q_{\bar T}}$ *is the restriction of* $[\Phi'_y(\cdot,\bar y)^*\bar\lambda]$ *to* $Q_{\bar T}$*,* $[\Phi'_y(\cdot,\bar y)^*\bar\lambda]|_{\Sigma_{\bar T}}$ *is the restriction of* $[\Phi'_y(\cdot,\bar y)^*\bar\lambda]$ *to* $\Sigma_{\bar T}$*,* $[\Phi'_y(\cdot,\bar y)^*\bar\lambda]|_{\overline{\Omega}_{\bar T}}$ *is the restriction of* $[\Phi'_y(\cdot,\bar y)^*\bar\lambda]$ *to* $\overline{\Omega}_{\bar T}$*,* $[\Phi'_y(\cdot,\bar y)^*\bar\lambda]$ *is the Radon measure on* $\overline{Q}_{\bar T}$ *defined by* $z\mapsto\langle\bar\lambda,\Phi'_y(\cdot,\bar y)z\rangle_{\overline{Q}_{\bar T}}$ *for* $z\in C(\overline{Q}_{\bar T})$*, and* $[\Phi'_t(\cdot,\bar y)^*\bar\lambda]$ *denotes the Radon measure on* $[0,\bar T]$ *defined by* $\varphi\mapsto\langle\bar\lambda,\,\Phi'_t(\cdot,\bar y)\varphi\rangle_{[0,\bar T]}$ *for* $\varphi\in C(\overline{Q}_{\bar T})$*.*

The meaning of weak solution for (5) is given in Section 4. Thanks to the regularity results for the state variable and for the adjoint state established in Theorems 3.1, 4.2, the integral and the duality product which intervene in the definition of the Hamiltonian $\mathcal{H}$ are well defined.

Recall that if $\xi\in BV([0,\bar T])$ and if ξ is identified with its representative right side continuous on $]0,\bar T[$, there then exists a measure $d\xi\in\mathcal{M}([0,\bar T])$ such that

$$\xi(t)=\xi(0)+d\xi([0,t])\quad\text{for every } t\in]0,\bar T].$$

(see [26].) Therefore the solution of the differential equation $\xi(T)\xi_T$, $d\xi\mu$, where $\mu\in\mathcal{M}([0,\bar T])$, is the function $\xi\in BV([0,\bar T])$ defined by $\xi(t)\xi_T-\mu(]t,T])$ if $t>0$ and $\xi(0)\xi_T-\mu([0,T])$. In equation (7), the term

$$\left[\int_\Gamma[\bar p g'_t(\cdot,t,\bar y,\bar v)-\bar\nu G'_t(\cdot,t,\bar y,\bar v)]\,ds+\int_\Omega[\bar p f'_t(\cdot,t,\bar y)-\bar\nu F'_t(\cdot,t,\bar y)]\,dx\right]dt$$

represents the measure whose density with respect to the one-dimensional Lebesgue measure dt is

$$\int_\Gamma[\bar p g'_t(\cdot,t,\bar y,\bar v)-\bar\nu G'_t(\cdot,t,\bar y,\bar v)]\,ds+\int_\Omega[\bar p f'_t(\cdot,t,\bar y)-\bar\nu F'_t(\cdot,t,\bar y)]\,dx.$$

Remark 2.3. As mentioned in Remark 2.1, the statement of Theorem 2.1 is still true if y_0 belongs to $C(\overline{\Omega})$. In Remark 6.1, we indicate how the proof of Theorem 2.1 may be extended to the case when y_0 does not belong to $C^2(\overline{\Omega})\cap C_0(\Omega)$, but only belongs to $C(\overline{\Omega})$.

Remark 2.4. Since $\Phi(0,y_0)\in\mathrm{int}\,\mathcal{C}$, from (4), we can deduce that $\bar\lambda(\overline{\Omega}_0)0$. Therefore, if $\mathrm{supp}\bar\lambda\subset\overline{\Omega}\times\cup_{i1}^k[t_i,t_{i+1}]$ (with $0<t_1<...<t_{k+1}\le\bar T$), and if for every $i1,...,k$ the mapping $z\to\Phi'_y(\cdot,\bar y(\cdot))z(\cdot)$ is a bijective mapping

from $C(\overline{\Omega}\times[t_i,t_{i+1}])$ onto $C(\overline{\Omega}\times[t_i,t_{i+1}])$, then we cannot have $\bar\nu=0$ and $\bar p=0$. Notice that this condition is satisfied by Φ in Examples 2.1 and 2.2.

Remark 2.5. The proof of Theorem 2.1 may be adapted to a terminal state constraint of the form $\Phi(T,y(T))\in\mathcal{C}$, where $\mathcal{C}$ is a closed convex subset with finite codimension in $C(\overline{\Omega})$, and Φ satisfies (A9). In this case the multiplier $\bar\lambda$ associated with this state constraint belongs to $\mathcal{M}(\overline{\Omega})$. The adjoint equation is

$$\begin{cases} -\dfrac{\partial\bar p}{\partial t}+A\bar p+f'_y(x,t,\bar y)\bar p=\bar\nu F'_y(x,t,\bar y) & \text{in } Q_{\bar T},\\[2mm] \dfrac{\partial\bar p}{\partial n_A}+g'_y(s,t,\bar y,\bar v)\bar p=\bar\nu G'_y(s,t,\bar y,\bar v) & \text{on } \Sigma_{\bar T},\\[2mm] \bar p(\bar T)=\bar\nu L'_y(x,\bar T,\bar y(\bar T))+[\Phi'_y(\bar T,\bar y(\bar T))^*\bar\lambda] & \text{on } \overline{\Omega},\end{cases}$$

the function $\bar q$ belongs to $W^{1,1}(0,\bar T)$, and it satisfies

$$\begin{cases} \bar q'(t)=\displaystyle\int_\Gamma[\bar p(s,t)g'_t(s,t,\bar y(s,t),\bar v(s,t))-\bar\nu G'_t(s,t,\bar y(s,t),\bar v(s,t))]\,ds\\ \qquad\qquad+\displaystyle\int_\Omega[\bar p(x,t)f'_t(x,t,\bar y(x,t))-\bar\nu F'_t(x,t,\bar y(x,t))]\,dx,\\[2mm] \bar q(\bar T)=\bar\nu\displaystyle\int_\Omega L'_t(x,\bar T,\bar y(x,\bar T))\,dx+\langle\bar\lambda,\Phi'_t(\bar T,\bar y(\bar T))\rangle_{\overline{\Omega}},\end{cases}$$

and the complementary condition is written in the form

$$\langle\bar\lambda,z-\Phi(\bar T,\bar y(\bar T))\rangle_{\mathcal{M}(\overline{\Omega})\times C(\overline{\Omega})}\le 0\quad\text{for all } z\in\{z\in C(\overline{\Omega})\mid z\in\mathcal{C}\}.$$

3 Linear parabolic equations. State equation

The main purpose of this section is to prove a higher integrability result for the gradient of the weak solution of equation (1). This result is obtained in Theorem 3.1. We first establish some regularity results for linear equations by using estimates on analytic semigroups (Propositions 3.1, 3.2, 3.3).

Throughout the sequel, we say that $((a_{ij})_{i,j},k_1,k_2)$ (where k_1,k_2 belong to

$\mathbb{R}$) satisfies the ellipticity condition (E_{m_0}) if and only if:

$$\int_\Omega \sum_{i,j} a_{ij}(x) D_j\varphi D_i\varphi\, dx + \int_\Omega k_1\varphi^2\, dx + \int_\Gamma k_2\varphi^2\, ds \geq \frac{m_0}{2}\|\varphi\|^2_{H^1(\Omega)} \quad (10)$$

for every $\varphi \in H^1(\Omega)$, where m_0 is the constant in (3). We denote by $\tilde{A}$ the operator defined by

$$\mathcal{D}(\tilde{A}) = \{y \in C^2(\overline{\Omega}) \mid \frac{\partial y}{\partial n_A} + k_2 y = 0 \quad \text{on } \Gamma\}, \qquad \tilde{A}y = Ay + k_1 y,$$

where $k_1, k_2 \in \mathbb{R}$ and $((a_{ij})_{i,j}, k_1, k_2)$ obeys the ellipticity condition (E_{m_0}). For $1 \leq l < \infty$, we denote by A_l the closure of $\tilde{A}$ in $L^l(\Omega)$. Since the spectrum of A_l does not depend on $1 \leq l < \infty$ ([1], p. 240 and Corollary 9.3), because of (10), $(-\infty, 0]$ is included in the resolvent set of $(-A_l)$. The operator $(-A_l)$ is the infinitesimal generator of a strongly continuous semigroup $(S_l(t))_{t\geq 0}$ and this semigroup is analytic in $L^l(\Omega)$ [1]. Therefore, for $0 < \gamma < 1$, we can define A_l^γ (the γ-power of A_l). Its domain, denoted by X_γ^l in the sequel, can be characterized by interpolation. From Theorems 1.15.2 and 4.3.3 in [30], we know that

$$W^{2\gamma+\epsilon,l}(\Omega) \hookrightarrow X_\gamma^l \hookrightarrow W^{2\gamma-\epsilon,l}(\Omega) \quad \text{if } 0 < \gamma < \frac{1}{2}(1+\frac{1}{l}) \text{ and } 0 < \epsilon < 2\gamma.$$

For every $1 \leq l < \lambda \leq \infty$ and every $\alpha > 0$, there exist constants $C_1 C_1(N, \Omega, k_1, k_2, l, \lambda)$, $C_2 C_2(N, \Omega, k_1, k_2, l, \lambda, \alpha)$ such that :

$$\begin{aligned} \|S_l(t)\varphi\|_{\lambda,\Omega} &\leq C_1\, t^{-\frac{N}{2}(\frac{1}{l}-\frac{1}{\lambda})}\|\varphi\|_{l,\Omega}, \\ \|A_l^\alpha S_l(t)\varphi\|_{\lambda,\Omega} &\leq C_2\, t^{-\frac{N}{2}(\frac{1}{l}-\frac{1}{\lambda})-\alpha}\|\varphi\|_{l,\Omega} \end{aligned} \quad (11)$$

for every $\varphi \in L^l(\Omega)$ and every $t > 0$ (see [1] and [27] Lemma 3.1).

Proposition 3.1. *Let $k_1 \in \mathbb{R}$, $k_2 \in \mathbb{R}$ be such that $((a_{ij})_{i,j}, k_1, k_2)$ obeys the ellipticity condition (E_{m_0}). Let ϕ be in $L^q(Q_T)$ and let $y \in W(0,T) \cap C(\overline{Q}_T)$ be the weak solution of*

$$\frac{\partial y}{\partial t} + Ay + k_1 y = \phi \quad \textit{in } Q_T, \qquad \frac{\partial y}{\partial n_A} + k_2 y = 0 \quad \textit{on } \Sigma_T, \qquad y(0) = 0 \quad \textit{in } \Omega.$$

Then the function y belongs to $L^\delta(0,T;W^{1,d}(\Omega))$ for every (δ,d) satisfying

$$q < d < \infty, \qquad q < \delta < \infty, \qquad \frac{N+2}{2q} < \frac{1}{2} + \frac{1}{\delta} + \frac{N}{2d}, \tag{12}$$

and $\|y\|_{L^\delta(0,T;W^{1,d}(\Omega))} \le C_3 \|\phi\|_{q,Q_T}$, *where C_3 depends on k_1, k_2, q, d, δ.*

Proof. We prove the estimate in $L^\delta(0,T;W^{1,d}(\Omega))$ for $\phi \in \mathcal{D}(Q_T)$, the general result follows from denseness arguments. Let (δ,d) be a pair satisfying (12). Let φ be in $\mathcal{D}(\Omega)$, and let z be the solution of the Cauchy problem

$$\frac{\partial z}{\partial t} + Az + k_1 z = 0 \quad \text{in } Q_T, \qquad \frac{\partial z}{\partial n_A} + k_2 z = 0 \quad \text{on } \Sigma_T, \qquad z(0) = A^\alpha_{d'}\varphi \quad \text{in } \Omega,$$

where α satisfies

$$\frac{1}{2} < \alpha < \frac{1}{2}(1 + \frac{1}{d}), \quad \frac{N}{2}(\frac{1}{q} - \frac{1}{d}) + \frac{1}{2} < \frac{N}{2}(\frac{1}{q} - \frac{1}{d}) + \alpha < 1 + \frac{1}{\delta} - \frac{1}{q}, \tag{13}$$

(let us notice that α exists because (δ,d) satisfies (12)). Since y belongs to $C^{2,1}(\overline{Q}_T) \subset C([0,T];X^d_\alpha)$, for every $t \in [0,T]$, we have:

$$\int_\Omega y(t) A^\alpha_{d'}\varphi \, dx = \int_\Omega A^\alpha_d y(t)\varphi \, dx.$$

By a straightforward calculation, we obtain :

$$\int_\Omega A^\alpha_d y(t)\varphi \, dx = \int_\Omega y(t) A^\alpha_{d'}\varphi \, dx = \int_0^t \left\{ \frac{d}{d\tau} \int_\Omega z(x,t-\tau) y(x,\tau) \, dx \right\} d\tau$$

$$= \int_0^t \left\{ \int_\Omega -\frac{\partial z}{\partial t}(x,t-\tau) y(x,\tau) + z(x,t-\tau)\frac{\partial y}{\partial t}(x,\tau) \, dx \right\} d\tau$$

$$= \int_0^t \left\{ \int_\Omega \left[Az(t-\tau)y(\tau) - z(t-\tau)Ay(\tau) + \phi(\tau)z(t-\tau) \right] \, dx \right\} d\tau$$

$$= \int_0^t \int_\Omega \phi(\tau) z(t-\tau) \, dx d\tau. \tag{14}$$

Observe that, for every $t > 0$, we have $z(t) S_{d'}(t) A^\alpha_{d'}\varphi A^\alpha_{d'} S_{d'}(t)\varphi$. Since $d' < q'$, (11) gives:

$$\|z(t)\|_{q',\Omega} \le C_2 \, t^{-\frac{N}{2}(\frac{1}{d'} - \frac{1}{q'}) - \alpha} \|\varphi\|_{d',\Omega} C_2 \, t^{-\frac{N}{2}(\frac{1}{q} - \frac{1}{d}) - \alpha} \|\varphi\|_{d',\Omega}, \tag{15}$$

for every $t > 0$. Taking (14) and (15) into account, we obtain

$$\|A_d^\alpha y(t)\|_{d,\Omega} \leq C_2 \int_0^t \|\phi(\tau)\|_{q,\Omega}(t-\tau)^{-\frac{N}{2}(\frac{1}{q}-\frac{1}{d})-\alpha}\, d\tau,$$

for every $t > 0$. Since $\frac{1}{2} < \alpha < \frac{1}{2}(1+\frac{1}{d})$, the imbedding $X_\alpha^d \hookrightarrow W^{1,d}(\Omega)$ is continuous and we have:

$$\|y(t)\|_{W^{1,d}(\Omega)} \leq K \int_0^t \|\phi(\tau)\|_{q,\Omega}(t-\tau)^{-\frac{N}{2}(\frac{1}{q}-\frac{1}{d})-\alpha}\, d\tau. \tag{16}$$

Since $t \mapsto \|\phi(t)\|_{q,\Omega}$ belongs to $L^q(0,T)$, and $t \mapsto t^{-\frac{N}{2}(\frac{1}{q}-\frac{1}{d})-\alpha} \in L^i(0,T)$ for every $i \geq 1$ satisfying:

$$1 > (\frac{N}{2}(\frac{1}{q} - \frac{1}{d}) + \alpha)i, \tag{17}$$

from estimates on convolution, it follows that

$$t \longmapsto \int_0^t \|\phi(\tau)\|_{q,\Omega}(t-\tau)^{-\frac{N}{2}(\frac{1}{q}-\frac{1}{d})-\alpha}\, d\tau$$

belongs to $L^\delta(0,T)$ if

$$1 + \frac{1}{\delta} = \frac{1}{q} + \frac{1}{i}, \quad i \geq 1. \tag{18}$$

If (δ, d) satisfies (12), for every α obeying (13), there exists $i \geq 1$ fulfilling conditions (17) and (18). Due to (16), we deduce that y belongs to $L^\delta(0,T;W^{1,d}(\Omega))$ and that

$$\|y\|_{L^\delta(0,T;W^{1,d}(\Omega))} \leq K\|\phi\|_{q,Q_T},$$

for every (δ, d) satisfying (12). □

Proposition 3.2. *Let $k_1 \in \mathbb{R}$, $k_2 \in \mathbb{R}$ be such that $((a_{ij})_{i,j}, k_1, k_2)$ obeys the ellipticity condition (E_{m_0}). Let ψ be in $L^\sigma(\Sigma_T)$ and let $y \in W(0,T) \cap C(\overline{Q}_T)$ be the weak solution of*

$$\frac{\partial y}{\partial t} + Ay + k_1 y = 0 \quad \text{in } Q_T, \qquad \frac{\partial y}{\partial n_A} + k_2 y = \psi \quad \text{on } \Sigma_T, \qquad y(0) = 0 \quad \text{in } \Omega.$$

Then the function y belongs to $L^\delta(0,T;W^{1,d}(\Omega))$ for every (δ,d) satisfying

$$\sigma < d < \infty, \quad \sigma < \delta < \infty, \quad \frac{N+1}{2\sigma} < \frac{1}{\delta} + \frac{N}{2d}, \tag{19}$$

and $\|y\|_{L^\delta(0,T;W^{1,d})} \le C_4 \|\psi\|_{\sigma,\Sigma_T}$*, where* C_4 *depends on* k_1*,* k_2*,* σ*,* d*, and* δ*.*

Proof. We suppose that $\psi \in C^1_c(\Sigma_T)$ (the set of C^1-functions with compact support in Σ_T). Therefore $y \in C^{2,1}(\overline{Q}_T)$. Let (δ, d) be a pair satisfying (19). Let $\varphi \in \mathcal{D}(\Omega)$, we consider the solution z of the Cauchy problem

$$\frac{\partial z}{\partial t} + Az + k_1 z = 0 \quad \text{in } Q_T, \qquad \frac{\partial z}{\partial n_A} + k_2 z = 0 \quad \text{on } \Sigma_T, \qquad z(0) = A^\alpha_{d'}\varphi \quad \text{in } \Omega,$$

where α satisfies

$$\frac{1}{2} < \alpha < \frac{1}{2}(1 + \frac{1}{d}) \quad \text{and} \quad \frac{N+1}{2\sigma} < \frac{N+1}{2\sigma} + \alpha - \frac{1}{2} < \frac{1}{\delta} + \frac{N}{2d} \tag{20}$$

(notice that α exists because (δ, d) obeys (19)). By a straightforward calculation, we obtain :

$$\int_\Omega A^\alpha_d y(t)\varphi\, dx = \int_\Omega y(t) A^\alpha_{d'}\varphi\, dx = \int_0^t \left\{ \frac{d}{d\tau} = \int_\Omega z(x, t-\tau) y(x,\tau)\, dx \right\} d\tau$$

$$= \int_0^t \int_\Gamma \psi(\tau) z(t-\tau)\, ds d\tau. \tag{21}$$

Let ϵ, k be positive numbers obeying $\frac{1}{\sigma'} < k < 2\epsilon$, $k\sigma' < N$, and

$$\frac{1}{\delta} + \frac{N}{2d} > \frac{N+1}{2\sigma} + \epsilon - \frac{1}{2\sigma'} + \alpha - \frac{1}{2} > \frac{N+1}{2\sigma} + \alpha - \frac{1}{2}.$$

With such a choice for ϵ, k, the following imbeddings are continuous:

$$X^{\sigma'}_\epsilon \hookrightarrow W^{k,\sigma'}(\Omega), \quad W^{k-\frac{1}{\sigma'},\sigma'}(\Gamma) \hookrightarrow L^{\frac{(N-1)\sigma'}{N-k\sigma'}}(\Gamma), \quad L^{\frac{(N-1)\sigma'}{N-k\sigma'}}(\Gamma) \hookrightarrow L^{\sigma'}(\Gamma).$$

Since $d' < \sigma'$, from (11) and from the previous imbeddings we have :

$$\|z(t)\|_{\sigma',\Gamma} \le K \|A^\epsilon_{\sigma'} z(t)\|_{\sigma',\Omega} \le K \|A^\epsilon_{\sigma'} S_{\sigma'}(t) A^\alpha_{d'}\varphi\|_{\sigma',\Omega}$$

$$\leq K \|A_{\sigma'}^{\alpha+\epsilon} S_{\sigma'}(t)\varphi\|_{\sigma',\Omega} \leq K\, t^{-\frac{N}{2}(\frac{1}{\sigma}-\frac{1}{d})-\alpha-\epsilon} \|\varphi\|_{d',\Omega}. \quad (22)$$

Taking (21) and (22) into account, we obtain

$$\|A_d^\alpha y(t)\|_{d,\Omega} \leq K \int_0^t \|\psi(\tau)\|_{\sigma,\Gamma} (t-\tau)^{-\frac{N}{2}(\frac{1}{\sigma}-\frac{1}{d})-\epsilon-\alpha}\, d\tau,$$

for every $t > 0$. Since $1/2 < \alpha < \frac{1}{2}(1+\frac{1}{d})$, the imbedding $X_\alpha^d \hookrightarrow W^{1,d}(\Omega)$ is continuous and we have:

$$\|y(t)\|_{W^{1,d}(\Omega)} \leq K_3 \int_0^t \|\psi(\tau)\|_{\sigma,\Gamma} (t-\tau)^{-\frac{N}{2}(\frac{1}{\sigma}-\frac{1}{d})-\epsilon-\alpha}\, d\tau.$$

By using the same arguments as in the proof of Proposition 3.1, we obtain

$$\|y\|_{L^\delta(0,T;W^{1,d}(\Omega))} \leq K\, \|\psi\|_{\sigma,\Sigma_T},$$

for every (δ, d) satisfying (19). □

Proposition 3.3. *Let a, ϕ be in $L^q(Q_T)$, let b, ψ be in $L^\sigma(\Sigma_T)$, such that $a \geq C_0$, $b \geq C_0$, $\|a\|_{q,Q_T} \leq M$, $\|b\|_{\sigma,\Sigma_T} \leq M$, for some $M > 0$. The weak solution $y \in W(0,T) \cap C(\overline{Q}_T)$ of*

$$\frac{\partial y}{\partial t} + Ay + ay = \phi \quad in\ Q_T, \qquad \frac{\partial y}{\partial n_A} + by = \psi \quad on\ \Sigma_T, \qquad y(0) = 0 \quad in\ \Omega,$$

belongs to $L^\delta(0,T;W^{1,d}(\Omega))$ for every (δ, d) satisfying

$$\begin{cases} 1 \leq \delta, d < \frac{(N+2)\sigma}{N+1} & if\ \ q \geq N+2, \\ 1 \leq \delta, d < \inf(\frac{(N+2)\sigma}{N+1}, \frac{(N+2)q}{N+2-q}) & if\ \ q < N+2, \end{cases} \quad (23)$$

and we have $\|y\|_{L^\delta(0,T;W^{1,d}(\Omega))} \leq C_5(\|\phi\|_{q,Q_T} + \|\psi\|_{\sigma,\Sigma_T})$, where C_5 depends on C_0, q, σ, d, δ, M, but is independent of a and b.

Proof. Following [27] (see the proof of Proposition 3.2 in [27]), there exists $\theta \in \mathbb{R}^+$ (only depending on C_0) such that $((a_{ij})_{i,j}, C_0+\theta, C_0)$ obeys the ellipticity condition (E_{m_0}). Notice that y can be written as $yy_1 + y_2$, where y_1 and y_2 are the weak solutions in Q_T of the linear equations:

$$\frac{\partial y_1}{\partial t} + Ay_1 + (C_0+\theta)y_1 = (C_0+\theta-a)y + \phi, \quad \frac{\partial y_1}{\partial n_A} + C_0 y_1 = 0, \quad y_1(0) = 0,$$

$$\frac{\partial y_2}{\partial t}+Ay_2+(C_0+\theta)y_2=0,\quad \frac{\partial y_2}{\partial n_A}+C_0y_2=(C_0-b)y+\psi,\quad \Sigma_T,\quad y_2(0)=0.$$

First notice that y belongs to $C(\overline{Q}_T)$ (see [27]), and that

$$\|y\|_{C(\overline{Q}_T)} \le K(\|\phi\|_{q,Q_T}+\|\psi\|_{\sigma,\Sigma_T}),$$

where K is independent of a and b. For every pair (δ, d) satisfying (23), there exist (δ_1, d_1) obeying (12) and (δ_2, d_2) obeying (19) such that $\delta \le \delta_1$, $\delta \le \delta_2$, $d \le d_1$ and $d \le d_2$. Propositions 3.1 and 3.2 give

$$\|y\|_{L^\delta(0,T;W^{1,d}(\Omega))} \le K(\|y_1\|_{L^{\delta_1}(0,T;W^{1,d_1}(\Omega))}+\|y_2\|_{L^{\delta_2}(0,T;W^{1,d_2}(\Omega))})$$

$$\le K(\{|C_0+\theta|+\|a\|_{q,Q_T}+|C_0|+\|b\|_{\sigma,\Sigma_T}\}\|y\|_{C(\overline{Q}_T)}+\|\phi\|_{q,Q_T}+\|\psi\|_{\sigma,\Sigma_T})$$

$$\le C_5(\|\phi\|_{q,Q_T}+\|\psi\|_{\sigma,\Sigma_T}),$$

where $C_5 = C_5(T, N, \Omega, C_0, q, \sigma, M, d, \delta)$ is independent of a and b. □

Theorem 3.1. *For every $v \in L^\sigma(\Sigma_T)$, the weak solution $y_v \in W(0,T) \cap C(\overline{Q}_T)$ of equation (1) belongs to $L^\delta(0,T;W^{1,d}(\Omega))$ for every (δ, d) obeying (23), and it satisfies :*

$$\|y_v\|_{L^\delta(0,T;W^{1,d}(\Omega))}+\|y_v\|_{C(\overline{Q}_T)} \le C_6(1+\|v\|_{\sigma,\Sigma_T}+\|y_0\|_{C^2(\overline{\Omega})}),$$

where $C_6 = C_6(T, \Omega, N, C_0, q, \sigma, d, \delta)$. In particular, there exist $\delta > 2$, $d > 2$ satisfying $\frac{1}{\delta}+\frac{N}{2d}<\frac{1}{2}$, such that y_v belongs to $L^\delta(0,T;W^{1,d}(\Omega))$ for every $v \in L^\sigma(\Sigma_T)$.

Proof. The estimate of y_v in $C(\overline{Q}_T)$ is in [27]. To prove the estimate in $L^\delta(0,T;W^{1,d}(\Omega))$, we notice that y_v can be written as $y_v = y_1 + y_2$, where y_1 is the weak solution of the linear equation :

$$\frac{\partial y_1}{\partial t}+Ay_1=0 \quad \text{in } Q_T, \qquad \frac{\partial y_1}{\partial n_A}=0 \quad \text{on } \Sigma_T, \qquad y_1(0)=y_0 \quad \text{in } \Omega,$$

and y_2 is the weak solution of :

$$\frac{\partial y_2}{\partial t}+Ay_2=\phi \quad \text{in } Q_T, \qquad \frac{\partial y_2}{\partial n_A}=\psi \quad \text{on } \Sigma_T, \qquad y_2(0)=0 \quad \text{in } \Omega,$$

with $\phi - f(\cdot, y_v) \in L^q(Q_T)$ and $\psi - g(\cdot, y_v, v) \in L^\sigma(\Sigma_T)$. Thanks to assumption (A10), y_1 belongs to $C^{2,1}(\overline{Q}_T) \subset L^\delta(0,T;W^{1,d}(\Omega))$. On the other hand, from Proposition 3.3, we deduce that y_2 belongs to $L^\delta(0,T;W^{1,d}(\Omega))$ for every (δ, d) satisfying (23).

If $q \geq N+2$, for every (δ, d) satisfying (23), we have $\frac{N}{2d} + \frac{1}{\delta} > \frac{(N+1)}{2\sigma}$. Since $\sigma > N+1$, then $\frac{N+1}{2\sigma} < \frac{1}{2}$ and we can choose (δ, d) close enough to $\frac{N+2}{N+1}\sigma$ to have $\frac{1}{\delta} + \frac{N}{2d} < \frac{1}{2}$. We can make the same kind of remark if $q < N+2$. □

Remark 3.1. If we replace the assumption (A10) by $y_0 \in C(\overline{\Omega})$, we can prove that the solution of the state equation belongs to $L^\delta(\epsilon, T; W^{1,d}(\Omega))$ for every $\epsilon > 0$ and every (δ, d) satisfying (23). This argument can be used to prove Theorem 2.1 in the case when $y_0 \in C(\overline{\Omega})$ (see Remark 6.1).

4 Adjoint state

Consider the following terminal boundary value problem :

$$
\begin{aligned}
&-\frac{\partial p}{\partial t} + Ap + ap = \mu_{Q_T} \quad \text{in } Q_T,\\
&\frac{\partial p}{\partial n_A} + bp = \mu_{\Sigma_T} \quad \text{on } \Sigma_T, \quad p(T) = \mu_{\overline{\Omega}_T} \quad \text{on } \overline{\Omega},
\end{aligned}
\tag{24}
$$

where $\mu\mu_{Q_T} + \mu_{\Sigma_T} + \mu_{\overline{\Omega}_T}$ is a bounded Radon measure on $\overline{Q}_T \setminus \overline{\Omega}_0$, μ_{Q_T} is the restriction of μ to Q_T, μ_{Σ_T} is the restriction of μ to Σ_T and $\mu_{\overline{\Omega}_T}$ is the restriction of μ to $\overline{\Omega}_T$. Assumptions on a and b are specified below.

Theorem 4.1. *Let (a, b) be in $L^q(Q_T) \times L^\sigma(\Sigma_T)$ such that $a \geq C_0$, $b \geq C_0$, $\|a\|_{q,Q_T} \leq M$, $\|b\|_{\sigma,\Sigma_T} \leq M$, for some $M > 0$. Let μ be in $\mathcal{M}_b(\overline{Q}_T \setminus \overline{\Omega}_0)$. Equation (24) admits a unique weak solution $p \in L^1(0,T;W^{1,1}(\Omega))$. For every $\delta > 1$, $d > 1$, satisfying $\frac{N}{2d} + \frac{1}{\delta} < \frac{1}{2}$, p belongs to $L^{\delta'}(0,T;W^{1,d'}(\Omega))$ and we have:*

$$
\|p\|_{L^{\delta'}(0,T;W^{1,d'}(\Omega))} \leq C_7 \|\mu\|_{\mathcal{M}_b(\overline{Q}_T \setminus \overline{\Omega}_0)},
$$

where $C_7 = C_7(T, \Omega, N, C_0, M, d, \delta)$ is independent of a and b. Moreover, *there exists a function* $p(0)$ *belonging to* $L^1(\Omega)$ *such that :*

$$\int_{Q_T} p\big(\frac{\partial y}{\partial t} + Ay + ay\big) + \int_{\Sigma_T} p\big(\frac{\partial y}{\partial n_A} + by\big)\langle y, \mu\rangle_{b,\overline{Q}_T\setminus\overline{\Omega}_0} - \int_{\Omega} y(0)p(0) = +$$

for every $y \in Y = \{y \in W(0,T) \mid \frac{\partial y}{\partial t} + Ay \in L^q(Q_T),\ \frac{\partial y}{\partial n_A} \in L^\sigma(\Sigma_T)\}$.

Proof. The existence of p in $L^1(0,T;W^{1,1}(\Omega))$ and the Green formula stated in Theorem 4.1 are already proved in [25]. We say that a pair $(\delta, d) \in \mathbb{R}^2$ satisfies the condition $(C_{q\sigma})$ if and only if

$$(C_{q\sigma}) \quad \begin{cases} \frac{N\sigma}{\sigma-2} < d \le \frac{N\sigma}{N-1} \quad \text{and} \quad \frac{2d}{d-N} < \delta \le \sigma & \text{if} \ \ \sigma \le q, \\ \frac{Nq}{q-2} < d \le \frac{N\sigma}{N-1} \quad \text{and} \quad \frac{2d}{d-N} < \delta \le q & \text{if} \ \ N \le q < \sigma, \\ \frac{Nq}{q-2} < d \le \inf(\frac{N\sigma}{N-1}, \frac{Nq}{N-q}) \quad \text{and} \quad \frac{2d}{d-N} < \delta \le q & \text{if} \ \ q < N. \end{cases}$$

Since $q > N/2 + 1$, $\sigma > N + 1$ and since $q\sigma + q > qN + 2\sigma$, we notice that the set of pairs (δ, d) satisfying $(C_{q\sigma})$ is nonempty (see [25]). Following [25], p belongs to $L^{\delta'}(0,T;W^{1,d'}(\Omega))$ for every (δ, d) obeying $(C_{q\sigma})$. Thus $p \in L^{q'}(Q_T)$, $p|_{\Sigma_T} \in L^{\sigma'}(\Sigma_T)$, and p is also the weak solution of

$$-\frac{\partial p}{\partial t} + Ap\phi + \mu_{Q_T} = 0 \text{ in } Q_T, \quad \frac{\partial p}{\partial n_A}\psi + \mu_{\Sigma_T} = 0 \text{ on } \Sigma_T, \quad p(T) = \mu_{\overline{\Omega}_T} \text{ on } \Omega,$$

where $\phi = ap \in L^1(Q_T)$, $\psi = bp \in L^1(\Sigma_T)$. By using the method of [25] we can prove that p belongs to $L^{\delta'}(0,T;W^{1,d'}(\Omega))$ for every (δ, d) satisfying $\frac{N}{2d} + \frac{1}{\delta} < \frac{1}{2}$, and that

$$\|p\|_{L^{\delta'}(0,T;W^{1,d'})} \le K(\|\phi\|_{1,Q_T} + \|\psi\|_{\Sigma_T} + \|\mu\|_{\mathcal{M}_b(\overline{Q}_T\setminus\overline{\Omega}_0)}) \le K\|\mu\|_{\mathcal{M}_b(\overline{Q}_T\setminus\overline{\Omega}_0)}.$$

□

Theorem 4.2. *Let* $M > 0$*, and* (a, b) *be in* $L^q(Q_T) \times L^\sigma(\Sigma_T)$ *satisfying* $a \ge C_0$*,* $b \ge C_0$*,* $\|a\|_{q,Q_T} \le M$*,* $\|b\|_{\sigma,\Sigma} \le M$*. Let* ξ *be in* $L^\delta(0,T;(L^d(\Omega))^N)$ *for some* $\delta > 1$*,* $d > 1$*, obeying* $\frac{N}{2d} + \frac{1}{\delta} < \frac{1}{2}$*. Then the equation*

$$-\int_0^T \langle y, \frac{dz}{dt}\rangle_1\, dt + \sum_{i,j}\int_{Q_T} a_{ij}D_j y D_i z\, dxdt + \int_{Q_T} ayz\, dxdt + \int_{\Sigma_T} byz\, dsdt$$

$$= \sum_i \int_{Q_T} \xi_i D_i z \, dxdt - \int_\Omega y(T) z(T) \, dx \quad \textit{for all } z \in W(0,T), \tag{25}$$

admits a unique solution y in $L^2(0,T,H^1) \cap C([0,T];L^2)$ such that $y(0)=0$. This solution belongs to $C^{\alpha,\alpha/2}(\overline{Q}_T)$ for some $0<\alpha<1$, and there exists $C_8 = C_8(T,\Omega,N,C_0,q,\sigma,d,\delta,M)$, not depending on ξ, such that

$$\|y\|_{C^{\alpha,\alpha/2}(\overline{Q}_T)} \le C_8 \|\xi\|_{L^\delta(0,T;(L^d(\Omega))^N)}. \tag{26}$$

Moreover, if μ belongs to $\mathcal{M}_b(\overline{Q}_T \setminus \overline{\Omega}_0)$, and if p is the weak solution of (24) associated with μ, then we have

$$\sum_i \int_{Q_T} \xi_i D_i p \, dxdt = \langle y, \mu \rangle_{b,\overline{Q}_T \setminus \overline{\Omega}_0}.$$

Proof. The existence of a unique solution y in $L^2(0,T,H^1) \cap C([0,T];L^2)$ for equation (25) is already proved in [22], Chapter 3, Theorem 5.1 when $\xi \equiv 0$ but the result can be extended to equation (25) by the same method. i) In the case when ξ belongs to $\mathcal{D}(Q_T;\mathbb{R}^N)$, y belongs to $C(\overline{Q}_T)$ and $\|y\|_{C(\bar{Q}_T)} + \|y\|_{L^2(0,T;H^1)} \le K \|\xi\|_{L^\delta(0,T;(L^d(\Omega))^N)}$ where K depends on C_0, q, σ, d, δ, M, and m_0 (see [25]). By denseness arguments we recover the same result for every $\xi \in L^\delta(0,T;(L^d(\Omega))^N)$. Now, by using the same trick as in ([28], Corollary 3.1), (26) follows from ([8], Theorem 1.3, Chapter 3).

ii) Consider a sequence $(\phi_n, \psi_n, k_n)_n$ in $C(\overline{Q}_T) \times C(\overline{\Sigma}_T) \times C(\overline{\Omega}_T)$ such that :

$$\lim_{n\to\infty} \|\phi_n\|_{1,Q_T} = \|\mu_{Q_T}\|_{\mathcal{M}_b(Q_T)}, \quad \lim_{n\to\infty} \|\psi_n\|_{1,\Sigma_T} = \|\mu_{\Sigma_T}\|_{\mathcal{M}_b(\Sigma_T)},$$

$$\lim_{n\to\infty} \|k_n\|_{1,\Omega_T} = \|\mu_{\overline{\Omega}_T}\|_{\mathcal{M}(\overline{\Omega}_T)},$$

$$\int_Q \phi_n h \, dxdt \longrightarrow \langle \mu_{Q_T}, h \rangle_{b,Q_T}, \qquad \int_\Sigma \psi_n h \, dsdt \longrightarrow \langle \mu_{\Sigma_T}, h \rangle_{b,\Sigma_T},$$

$$\int_\Omega k_n h \, dx \longrightarrow \langle \mu_{\Omega_T}, h \rangle_{\overline{\Omega}}$$

for every $h \in C(\overline{Q}_T)$. Let $p_n \in W(0,T) \cap C(\overline{Q}_T)$ be the weak solution of

$$-\frac{\partial p_n}{\partial t} + Ap_n + ap_n = \phi_n \text{ in } Q_T, \ \frac{\partial p_n}{\partial n_A} + bp_n = \psi_n \text{ on } \Sigma_T, \ p_n(T) = k_n \text{ on } \Omega.$$

We can prove that $(p_n)_n$ converges to p for the weak-star topology of $L^{\delta'}(0,T; W^{1,d'}(\Omega))$, for every (δ, d) obeying (23). Moreover, if we set zp_n in the variational formulation of y, we obtain:

$$-\int_0^T \langle y, \frac{dp_n}{dt}\rangle_1 \, dt + \sum_{i,j1}^{N} \int_{Q_T} a_{ij} D_j y D_i p_n \, dxdt + \int_{Q_T} ayp_n \, dxdt$$

$$+ \int_{\Sigma_T} byp_n \, dsdt = \sum_i \int_{Q_T} \xi_i D_i p_n \, dxdt - \int_\Omega k_n y(T) \, dx. \tag{27}$$

On the other hand, we have

$$-\int_0^T \langle y, \frac{dp_n}{dt}\rangle_1 \, dt + \sum_{i,j1}^{N} \int_{Q_T} a_{ij} D_j p_n D_i y \, dxdt + \int_{Q_T} ap_n y \, dxdt$$

$$+ \int_{\Sigma_T} bp_n y \, dsdt = \int_{Q_T} \phi_n y \, dxdt + \int_{\Sigma_T} \psi_n y \, dsdt. \tag{28}$$

By making the difference between (27) and (28), it follows that

$$\sum_i \int_{Q_T} \xi_i D_i p_n \, dxdt = \int_{Q_T} \phi_n y \, dxdt + \int_{\Sigma_T} \psi_n y \, dsdt + \int_\Omega k_n y(T) \, dx.$$

We complete the proof by passing to the limit in the above equality. □

Remark 4.1. Since we have proved that the solution of (25) belongs to $C(\overline{Q}_T)$, we can now claim that it also belongs to $W(0,T)$.

5 A problem equivalent to $(\mathcal{P})$

Introduce the following problem:

$(\widetilde{\mathcal{P}})$ $\quad \inf\{\tilde{J}(z,\varphi,w,u) \mid (z,\varphi) \in C(\overline{\Omega} \times [0,1]) \times C([0,1]),$

$\quad (w,u) \in W_{ad} \times U_{ad}, \;\; (z,\varphi,w,u)$ satisfies (29) and (30)$\}$,

where

$$\tilde{J}(z,\varphi,w,u) = \int_0^1 \int_\Omega u(\tau) F(x, \varphi(\tau), z(x,\tau)) \, dxd\tau$$

$$+\int_0^1\int_\Omega u(\tau) = G(s,\varphi(\tau),z(s,\tau),w(s,\tau))\,dsd\tau + \int_\Omega L(x,\varphi(1),z(x,1))\,dx,$$

$$W_{ad} := \{w \in L^\sigma(\Gamma\times]0,1[) \mid w(s,\tau) \in K_V(s) \ \text{ for a.e. } (s,\tau) \in \Gamma\times]0,1[\},$$

$$U_{ad} := C([0,1];\mathbb{R}_+^*)\{u \in C([0,1]) \mid u(\tau) > 0 \text{ on } [0,1]\},$$

$$\begin{cases} \dfrac{\partial z}{\partial \tau} + uAz + uf(\cdot,\varphi(\tau),z(\cdot,\tau)) = 0 \text{ in } \Omega\times]0,1[, \\ \\ \dfrac{\partial z}{\partial n_A} + g(\cdot,\varphi(\tau),z(\cdot,\tau),w(\cdot,\tau)) = 0 \text{ on } \Gamma\times]0,1[, \\ \\ \varphi'(\tau) = u(\tau) \text{ in }]0,1[, \ z(0) = y_0 \text{ in } \Omega, \quad \varphi(0) = 0, \end{cases} \tag{29}$$

and

$$\Phi(\varphi(\tau), z(\tau)) \in \mathcal{C} \qquad \text{for all } \tau \in [0,1]. \tag{30}$$

As in [33, 29], we claim that $(\mathcal{P})$ and $(\widetilde{\mathcal{P}})$ are equivalent in the following sense.

Proposition 5.1. *i) If (z,φ,w,u) is admissible for $(\widetilde{\mathcal{P}})$, then there exists a triplet (y,v,T) admissible for $(\mathcal{P})$ such that*

$$z = y(\cdot,\varphi(\cdot)), \quad w = v(\cdot,\varphi(\cdot)), \quad \varphi(1) = T, \quad \tilde{J}(z,\varphi,w,u) = J(y,v,T).$$

ii) If (y,v,T) is admissible for $(\mathcal{P})$, then for every u in U_{ad} satisfying $\int_0^1 u(\tau)\,d\tau T$, the quadruplet (z,φ,w,u) defined by

$$\varphi(\tau) = \int_0^\tau u(s)\,ds, \ \ z = y(\cdot,\varphi(\cdot)), \ \ w = u(\cdot,\varphi(\cdot)),$$

is admissible for $(\widetilde{\mathcal{P}})$, and we have $J(y,v,T) = \tilde{J}(z,\varphi,w,u)$.

5.1 Preliminary results for $(\widetilde{P})$

As in [29], due to Theorem 3.1, we can state the following proposition.

Proposition 5.2. *For every $u \in U_{ad}$ and every $w \in W_{ad}$, equation (29) admits a unique solution $(z_{wu},\varphi_u) \in (C(\overline{Q}_1) \cap W(0,1)) \times C^1([0,1])$. Moreover, for every (δ,d) satisfying (23), z_{wu} belongs to $L^\delta(0,T;W^{1,d}(\Omega))$. In* *particular, there exists* $\delta > 2,\ d > 2$, *satisfying* $\frac{N}{2d} + \frac{1}{\delta} < \frac{1}{2}$, *such that* z_{wu} *belongs to* $L^\delta(0,1;W^{1,d}(\Omega))$.

Proposition 5.3. *Let* (a,b) *be in* $L^q_{loc}([0,\infty[;L^q(\Omega)) \times L^\sigma_{loc}([0,\infty[;L^\sigma(\Gamma))$ *satisfying* $a \geq C_0$ *and* $b \geq C_0$. *Let* ϕ *be in* $L^q(Q_1)$, ψ *be in* $L^\sigma(\Sigma_1)$, ξ *be in* $L^\delta(0,1;(L^d(\Omega))^N)$ *for some* $\delta > 2$, $d > 2$ *satisfying* $N/2d + 1/\delta < 1/2$. *Let* u *be in* U_{ad}, *and set* $\varphi(\tau) \int_0^\tau u(s)\,ds$, $a^\varphi = a(\cdot,\varphi(\cdot))$, $b^\varphi = b(\cdot,\varphi(\cdot))$. *Then the function* $z \in L^2(0,1;H^1) \cap C([0,1];L^2)$ *satisfying* $z(0) = 0$ *in* Ω *and the equation :*

$$\int_0^1 \Big[-\langle z, \frac{d\chi}{d\tau}\rangle_1 + \int_\Omega u\big(\sum_{i,j} a_{ij}D_jzD_i\chi + a^\varphi z\chi\big)\,dx + \int_\Gamma ub^\varphi z\chi\,ds\Big]d\tau =$$
$$\int_0^1\int_\Omega \big(\phi\chi + \sum_i \xi_iD_i\chi\big)\,dxd\tau + \int_0^1\int_\Gamma \psi\chi\,dsd\tau - \int_\Omega z(1)\chi(1)\,dx$$

for all $\chi \in W(0,1)$, *belongs to* $C^{\alpha,\alpha/2}(\overline{\Omega}\times[0,1])$ *for some* $\alpha > 0$, *and*

$$\|z\|_{C^{\alpha,\alpha/2}(\overline{\Omega}\times[0,1])} \leq C_9(\|\phi\|_{q,\Omega\times]0,1[} + \|\psi\|_{\sigma,\Gamma\times]0,1[} + \|\xi\|_{L^\delta(0,1;(L^d(\Omega))^N)}).$$

The constant C_9 *only depends on the bounds* $\tilde{T}$, m, M, *where* $\tilde{T} \geq \varphi(1)$, $0 < m \leq \min_{\tau\in[0,1]} u(\tau)$, $\|a\|_{q,\Omega\times]0,\varphi(1)[} \leq M$ *and* $\|b\|_{\sigma,\Gamma\times]0,\varphi(1)[} \leq M$.

Proof. Let φ^{-1} be the inverse mapping of φ and let $\tilde{T} \geq \varphi(1)$. Consider the function $y \in C(\overline{Q}_{\tilde{T}})$ satisfying $y(\cdot,0)0$ on $\overline{\Omega}$ and

$$\begin{cases} \displaystyle\int_0^{\tilde{T}} \Big[-\langle y, \frac{d\chi}{dt}\rangle_1 + \int_\Omega \big(\sum_{i,j} a_{ij}D_jyD_i\chi + ay\chi\big)\,dx + \int_\Gamma by\chi\,ds\Big]dt = \\ \displaystyle\int_0^{\varphi(1)}\int_\Omega \frac{1}{u(\varphi^{-1}(t))}\Big(\phi(x,\varphi^{-1}(t)) + \sum_i \xi_i(x,\varphi^{-1}(t))D_i\chi\Big)\,dxdt \\ \displaystyle\quad + \int_0^{\varphi(1)}\int_\Gamma \frac{1}{u(\varphi^{-1}(t))}\psi(s,\varphi^{-1}(t))\chi\,dsdt - \int_\Omega y(\tilde{T})\chi(\tilde{T})\,dx \end{cases}$$

for every $\chi \in W(0,\tilde{T})$. As in Theorem 4.2 we can prove that y belongs to $C^{\alpha,\alpha/2}(\overline{Q}_{\tilde{T}})$ for some $\alpha > 0$, and that

$$\|y\|_{C^{\alpha,\alpha/2}(\overline{Q}_{\tilde{T}})} \leq C_{10}\big(\|\phi(\cdot,\varphi^{-1}(\cdot))\|_{q,\Omega\times]0,\tilde{T}[}$$

$$+\|\psi(\cdot,\varphi^{-1}(\cdot))\|_{\sigma,\Gamma\times]0,\tilde{T}[}+\|\xi(\cdot,\varphi^{-1}(\cdot))\|_{L^{\delta}(0,\tilde{T};(L^d(\Omega))^N)}\big),$$

where $C_{10}=C_{10}(\tilde{T},\Omega,N,q,\bar{\sigma},C_0,\delta,d,m,M)$. Now we verify that $y|_{\Omega\times]0,\varphi(1)[}=z(\cdot,\varphi^{-1}(\cdot))$. Then the estimate for z can be deduced from the one for y. $\square$

Remark 5.1. For the same reasons as in Remark 4.1, we can also prove that z belongs to $W(0,1)$.

Let $(\tilde{w},\tilde{u})$ be in $W_{ad}\times U_{ad}$ and let $k>0$. We set

$$W_{ad}(\tilde{w},k)=\{w\in W_{ad}\mid |w(s,\tau)-\tilde{w}(s,\tau)|\leq k \quad \text{for a.e. } (s,\tau)\in\Gamma\times]0,1[\},$$

$$U_{ad}(\tilde{u})=\{u\in U_{ad}\mid 2\|\tilde{u}\|_{C([0,1])}\geq u(\tau)\geq\frac{\min_{\tau\in[0,1]}\tilde{u}(\tau)}{2}\},$$

and we denote by d the distance defined by

$$d((w_1,u_1),(w_2,u_2))=\mathcal{L}^N(\{(s,\tau)\in\Sigma_1\ \mid w_1\neq w_2\})+\|u_1-u_2\|_{\infty,]0,1[}.$$

Proposition 5.4. *Let $(\tilde{w},\ \tilde{u})$ be in $W_{ad}\times U_{ad}$, let $k>0$. Then $(W_{ad}(\tilde{w},k)\times U_{ad}(\tilde{u}),d)$ is a complete metric space and the mapping which associates $(z_{wu},\varphi_u,\tilde{J}(z_{wu},\varphi_u,w,u))$ with (w,u) is continuous from $(W_{ad}(\tilde{w},k)\times U_{ad}(\tilde{u}),$ $d)$ into $C(\overline{\Omega}\times[0,1])\times C([0,1])\times\mathbb{R}$. Moreover, $\tilde{J}(z_{wu},\varphi_u,w,u)$ is bounded from below on $W_{ad}(\tilde{w},k)\times U_{ad}(\tilde{u})$.*

Proof. The first part of the proposition is proved in [29]. It remains to show the continuity result. Let $(w_n,u_n)_n$ be a sequence converging to (w,u) in $W_{ad}(\tilde{w},k)\times U_{ad}(\tilde{u})$. We denote by (z_n,φ_n) (respectively (z,φ)) the solution of (29) associated with (w_n,u_n) (respectively (w,u)). It is obvious that $(\varphi_n)_n$ converges to φ in $C^1([0,1])$. On the other hand, the function ξz_n-z belongs to $W(0,1)\cap C(\overline{\Omega}\times[0,1])$, and satisfies $\xi(0)=0$ and :

$$-\int_0^1\langle\xi,\frac{d\chi}{d\tau}\rangle_1\,d\tau+\sum_{i,j}\int_0^1\int_\Omega u_nD_i\xi D_j\chi\,dxd\tau+\int_0^1\int_\Omega u_na_n\xi\chi\,dxd\tau$$

$$+\int_0^1\int_\Gamma u_nb_n\chi\,dsd\tau=\int_0^1\int_\Omega[uf(x,\varphi,z)-u_nf(x,\varphi_n,z)]\chi\,dxd\tau$$

$$+\int_0^1\int_\Gamma [ug(s,\varphi,z,w)-u_n g(s,\varphi_n,z,w_n)]\chi\, dsd\tau$$

$$+\sum_{i,j}\int_0^1\int_\Omega (u-u_n)D_i z D_j\chi\, dxd\tau$$

for every $\chi \in W(0,1)$ satisfying $\chi(1)=0$, with $a_n(\cdot,\tau)=\int_0^1 f'_y(\cdot,\varphi_n,z+\theta(z_n-z))\,d\theta$, and $b_n(\cdot,\tau)=\int_0^1 g'_y(\cdot,\varphi_n,z+\theta(z_n-z),w_n)\,d\theta$. With a change of variable as in ([29], Lemma 5.1), with Proposition 3.3 and Theorem 4.2 we obtain

$$\|z_n-z\|_{C(\overline{\Omega}\times[0,1])}\le C\Bigg(\|uf(\cdot,\varphi,z)-u_n f(\cdot,\varphi_n,z)\|_{q,\Omega\times]0,1[}$$

$$+\|ug(\cdot,\varphi,z,w)-u_n g(\cdot,\varphi_n,z,w_n)\|_{\sigma,\Sigma_1}+\|u-u_n\|_{C([0,1])}\|z\|_{L^\delta(0,1;W^{1,d})}\Bigg),$$

with $\frac{N}{2d}+\frac{1}{\delta}<\frac{1}{2}$, and C is independent of n. The convergence of $(z_n)_n$ to z in $C(\overline{\Omega}\times[0,1])$ follows from this estimate, from (A3)-(A4) and from the convergence of $(w_n,u_n)_n$ and of $(\varphi_n)_n$. □

Theorem 5.1. *Let u_1 be in U_{ad}, w be in W_{ad}, denote by (z_1,φ_1) the solution of (29) corresponding to (w,u_1). Let u_0 be in $C([0,1])$, $\varphi_0(\tau)\int_0^\tau u(s)\,ds$ and let h be the weak solution in $W(0,1)\cap C(\overline{\Omega}\times[0,1])$ of*

$$\frac{d}{d\tau}\int_\Omega h(\tau)\chi\,dx+u_1\int_\Omega\Bigg(\sum_{i,j1}^N a_{ij}D_j hD_i\chi+f'_y(x,\varphi_1,z_1)h\chi\Bigg)dx$$

$$+u_1\int_\Gamma g'_y(\varphi_1,z_1,w)h\chi\,ds=u_1\varphi_0\left(\int_\Omega f'_t(\varphi_1,z_1)\chi\,dx+\int_\Gamma g'_t(\varphi_1,z_1,w)\chi\,ds\right)$$

$$+u_0\Bigg(\int_\Omega\sum_{i,j1}^N a_{ij}D_j z_1 D_i\chi\,dx+\int_\Omega f(\cdot,\varphi_1,z_1)\chi\,dx+\int_\Gamma g(\cdot,\varphi_1,z_1,w)\chi\,ds\Bigg)=0$$

for every $\chi\in H^1(\Omega)$, and $h(0)=0$ in Ω.

Let μ *be in* $\mathcal{M}(\overline{\Omega} \times [0,1])$ *and* ψ *be the weak solution in* $L^1(0,1;W^{1,1}(\Omega))$ *of*

$$\begin{cases} -\dfrac{\partial \psi}{\partial \tau} + u_1 A\psi + u_1 f'_y(x,\varphi_1,z_1)\psi = \mu_{\Omega\times]0,1[} & \text{in } Q_1, \\ u_1\left(\dfrac{\partial \psi}{\partial n_A} + g'_y(s,\varphi_1,z_1,w)\psi\right) = \mu_{\Gamma\times]0,1[} & \text{on } \Sigma_1, \\ \psi(1) = \mu_{\overline{\Omega}\times\{1\}} & \text{on } \overline{\Omega}. \end{cases}$$

($\mu_{\Omega\times]0,1[}$ *is the restriction of* μ *to* Q_1*,* $\mu_{\Gamma\times]0,1[}$ *is the restriction of* μ *to* Σ_1*, and* $\mu_{\overline{\Omega}\times\{1\}}$ *is the restriction of* μ *to* $\overline{\Omega}\times\{1\}$*.) Then the following Green formula holds :*

$$\langle \mu, h\rangle_{\overline{Q}_1} = \langle \mu, h\rangle_{b,\overline{Q}_1\setminus\overline{\Omega}_0}$$

$$\int_0^1 \left(\int_\Omega \psi f'_t(\cdot,\varphi_1,z_1)\,dx + \int_\Gamma \psi g'_t(\cdot,\varphi_1,z_1,w_1)\,ds\right) u_1\varphi_0\,d\tau$$

$$= \int_0^1 u_0\bigg(\int_\Omega [\psi f(\varphi_1,z_1) + \sum_{i,j1}^N a_{ij} D_j z_1 D_i\psi]\,dx + \int_\Gamma \psi g(\varphi_1,z_1,w_1)\,ds\bigg)d\tau.$$

Proof. This theorem can be deduced from Theorems 4.1 and 4.2 with an appropriate change of variable in time. □

5.2 Derivative of the state variable with respect to u

Theorem 5.2. *Let* $\tilde{u}$ *be in* U_{ad}*,* u_1, u_2 *be in* $U_{ad}(\tilde{u})$*,* w *be in* W_{ad} *and* ρ *satisfying* $0<\rho<1$*. Denote by* $(z_\rho,\varphi_\rho),(z_1,\varphi_1)$ *the solutions of (29) corresponding respectively to* (w,u_ρ) *and to* (w,u_1)*, where* $u_\rho = u_1+\rho(u_2-u_1)$*. Then the following holds*

$$z_\rho = z_1 + \rho h + r_\rho, \quad \text{with} \quad \lim_{\rho\to 0}\frac{1}{\rho}\|r_\rho\|_{C(\overline{\Omega}\times[0,1])} = 0, \tag{31}$$

$$\begin{aligned} \tilde{J}(z_\rho,\varphi_\rho,w,u_\rho) &= \rho[\tilde{J}'_z(z_1,\varphi,w,u_1)h + \tilde{J}'_\varphi(z_1,\varphi_1,w,u_1)(\varphi_2-\varphi_1)] \\ &+ \tilde{J}(z_1,\varphi_1,w,u_1) + \tilde{J}(z_1,\varphi_1,w,u_2) - \tilde{J}(z_1,\varphi_1,w,u_1) + o(\rho), \end{aligned} \tag{32}$$

where $\varphi_2(\tau) = \int_0^\tau u_2(s)\,ds$*, and* h *is the weak solution in* $W(0,1) \cap C(\overline{\Omega} \times [0,1])$ *of*

$$\begin{aligned}
&\frac{d}{d\tau}\int_\Omega h(\tau)\chi\,dx + u_1(\tau)\int_\Omega \Big(\sum_{i,j1}^N a_{ij} D_j h D_i\chi + a_1 h\chi\Big)\,dx \\
&+u_1\int_\Gamma b_1 h\chi\,ds = u_1(\varphi_1-\varphi_2)(\int_\Omega f'_{1t}\chi\,dx + \int_\Gamma g'_{1t}\chi\,ds) \\
&+(u_1-u_2)\Big(\int_\Omega \sum_{i,j1}^N a_{ij} D_j z_1 D_i \chi\,dx + \int_\Omega f_1\chi\,dx + \int_\Gamma g_1\chi\,ds\Big)
\end{aligned} \tag{33}$$

$$\text{for every } \chi \in H^1(\Omega), \qquad \text{and} \quad h(0)0 \ \text{ in } \Omega,$$

with $a_1 = f'_y(x,\varphi_1,z_1)$*,* $b_1 = g'_y(s,\varphi_1,z_1,w)$*,* $f'_{1t} = f'_t(x,\varphi_1,z_1)$*,* $g'_{1t} = g'_t(s,\varphi_1,z_1,w)$*,* $f_1 = f(x,\varphi_1,z_1)$*, and* $g_1 = g(s,\varphi_1,z_1,w)$*.*

Proof. Due to Proposition 5.2, z_1 belongs to $L^\delta(0,1;W^{1,d})$ for some $\delta > 2$, $d > 2$ satisfying $\frac{N}{2d} + \frac{1}{\delta} < \frac{1}{2}$. From Proposition 5.3 we deduce that h belongs to $C^{\alpha,\alpha/2}(\overline{\Omega}\times[0,T])$ for some $\alpha > 0$. The function $r = \frac{1}{\rho}r_\rho\frac{1}{\rho}(z_\rho - z_1) - h$ belongs to $W(0,1)$, satisfies $r(0) = 0$ and

$$\frac{d}{d\tau}\int_\Omega r(\tau)\chi\,dx + u_\rho\int_\Omega\Big(\sum_{i,j1}^N a_{ij}D_j r D_i\chi + a_\rho r\chi\Big)\,dx + u_\rho\int_\Gamma b_\rho r\chi\,ds$$

$$= \int_\Omega f_\rho\chi\,dx + \int_\Gamma g_\rho\chi\,ds - \rho(u_2-u_1)\int_\Omega\sum_{i,j1}^N a_{ij}D_j h D_i\chi\,dx \quad \text{for all } \chi\in H^1(\Omega),$$

where

$$a_\rho(x,\tau) = \int_0^1 f'_y(x,\varphi_\rho(\tau),(z_1+\theta(z_\rho-z_1))(x,\tau))\,d\theta,$$

$$b_\rho(s,\tau) = \int_0^1 g'_y(s,\varphi_\rho(\tau),(z_1+\theta(z_\rho-z_1))(s,\tau),w(s,\tau))\,d\theta,$$

$$f_\rho = (u_1a_1 - u_\rho a_\rho)h + (\varphi_1-\varphi_2)\Big(u_\rho\int_0^1 f'_t(x,\varphi_1+\theta(\varphi_\rho-\varphi_1),z_1)\,d\theta - u_1 f'_{1t}\Big),$$

$$g_\rho = (u_1 b_1 - u_\rho b_\rho)h + (\varphi_1 - \varphi_2)\left(u_\rho \int_0^1 g'_t(s, \varphi_1 + \theta(\varphi_\rho - \varphi_1), z_1, w)\, d\theta - u_1 g'_{1t}\right).$$

Since $h \in L^2(0,1;H^1(\Omega))$, we have the estimate :

$$\frac{1}{\rho}\|r_\rho\|_{L^2(H^1)} \le K(\|f_\rho\|_{2,\Omega\times]0,1[} + \|g_\rho\|_{2,\Gamma\times]0,1[} + \rho\|u_2 - u_1\|_{C([0,1])}\|h\|_{L^2(H^1)})$$

where $K > 0$ is independent of ρ. The sequence $(z_\rho)_\rho$ converges to z_1 uniformly on $\overline{\Omega} \times [0,1]$, and $(\varphi_\rho)_\rho$ converges to φ_1 uniformly on $[0,1]$. With (A3)-(A4), we can prove that $(a_\rho)_\rho$ converges to a_1 in $L^q(\Omega\times]0,1[)$, $(b_\rho)_\rho$ converges to b_1 in $L^\sigma(\Gamma\times]0,1[)$, $(\int_0^1 f'_t(x, \varphi_1 + \theta(\varphi_\rho - \varphi_1), z_1)\, d\theta)_\rho$ converges to f'_{1t}, $(\int_0^1 g'_t(s, \varphi_1 + \theta(\varphi_\rho - \varphi_1), z_1, w)\, d\theta)_\rho$ converges to g'_{1t}. Therefore $(f_\rho)_\rho$ and $(g_\rho)_\rho$ converge to zero respectively in $L^q(\Omega\times]0,1[)$ and in $L^\sigma(\Gamma\times]0,1[)$. Thus, we first conclude that $\lim_{\rho\to 0} \frac{1}{\rho}\|r_\rho\|_{L^2(0,1;H^1(\Omega))} 0$. On the other hand, the function $\zeta_\rho(z_\rho - z_1)/\rho$ belongs to $W(0,1)$, satisfies $\zeta_\rho(0) 0$ and

$$\frac{d}{d\tau}\int_\Omega \zeta_\rho(\tau)\chi\, dx + u_\rho \int_\Omega \left(\sum_{i,j 1}^N a_{ij} D_j \zeta_\rho D_i \chi + a_\rho \zeta_\rho \chi\right) dx + u_\rho \int_\Gamma b_\rho \zeta_\rho \chi\, ds$$

$$= (u_1 - u_2)\int_\Omega \left(\sum_{i,j 1}^N a_{ij} D_j z_1 D_i \chi + f_1 \chi\right) dx + (u_1 - u_2)\int_\Gamma g_1 \chi\, ds$$

$$+ u_\rho(\varphi_1 - \varphi_2)\int_\Omega \int_0^1 f'_t(\cdot, \varphi_1 + \theta(\varphi_\rho - \varphi_1), z_1)\, d\theta\, \chi\, dx$$

$$+ u_\rho(\varphi_1 - \varphi_2)\int_\Gamma \int_0^1 g'_t(\cdot, \varphi_1 + \theta(\varphi_\rho - \varphi_1), z_1, w)\, d\theta\, \chi\, ds,$$

for every $\chi \in H^1(\Omega)$. From Proposition 5.3 we deduce that $(\zeta_\rho)_\rho$ is bounded in $C^{\alpha,\alpha/2}(\overline{\Omega} \times [0,1])$. Therefore we have:

$$\|\frac{1}{\rho} r_\rho\|_{C^{\alpha,\alpha/2}(\overline{\Omega}\times[0,1])} \le \|\zeta_\rho\|_{C^{\alpha,\alpha/2}(\overline{\Omega}\times[0,1])} + \|h\|_{C^{\alpha,\alpha/2}(\overline{\Omega}\times[0,1])}.$$

Since $(\frac{1}{\rho} r_\rho)_\rho$ tends to zero in $L^2(0,1;H^1(\Omega))$ and since the sequence $(\frac{1}{\rho} r_\rho)_\rho$ is bounded in $C^{\alpha,\alpha/2}(\overline{\Omega} \times [0,1])$, from Arzela-Ascoli Theorem, we conclude that $\lim_{\rho\to 0} \frac{1}{\rho}\|r_\rho\|_{C(\overline{\Omega}\times[0,1])} 0$.

ii) Taking (A5)-(A7) into account, (32) can be deduced from (31). □

6 Proof of Theorem 2.1

Let $(\bar{y}, \bar{v}, \bar{T})$ be an optimal solution for $(\mathcal{P})$. Then there exists an optimal solution $(\bar{z}, \bar{\varphi}, \bar{w}, \bar{u})$ for

$(\widetilde{\mathcal{P}})$ such that $\bar{z}\bar{y}(\cdot, \bar{\varphi}(\cdot))$, $\bar{w}\bar{v}(\cdot, \bar{\varphi}(\cdot))$, $\bar{u}(\tau)\bar{T}$ and $\bar{\varphi}(\tau)\tau\bar{T}$ on $[0,1]$. Optimality conditions for $(\bar{y}, \bar{u}, \bar{T})$ can be deduced from the ones for $(\bar{z}, \bar{\varphi}, \bar{w}, \bar{u})$ (see Section 6.3).

6.1 Penalized Problem

We define a function Ψ from $C^1([0,1]) \times C(\overline{\Omega} \times [0,1])$ to $C(\overline{\Omega} \times [0,1])$ by setting $\Psi(\varphi, z)(x, \tau)\Phi(\varphi(\tau), z(\tau))(x)$ for every (φ, z) in $C^1([0,1]) \times C(\overline{\Omega} \times [0,1])$ and $(x, \tau) \in \overline{\Omega} \times [0,1]$. Now we notice that the state constraint (30) is equivalent to

$$\Psi(\varphi, z) \in \widetilde{\mathcal{C}}, \tag{34}$$

where $\widetilde{\mathcal{C}} = \{f \in C(\overline{\Omega} \times [0,1]) \mid f(\tau) \in \mathcal{C}\}$. The problem $(\widetilde{\mathcal{P}})$ can be written as follows

$$\inf\{\tilde{J}(z, \varphi, w, u) \mid (w, u) \in W_{ad} \times U_{ad},\ (z, \varphi, w, u) \text{ satisfies (29), (34)}\}.$$

Notice that the interior of $\widetilde{\mathcal{C}}$ is nonempty in $C(\overline{\Omega} \times [0,1])$ because the interior of $\mathcal{C}$ is nonempty in $C(\overline{\Omega})$. Since $C(\overline{\Omega} \times [0,1])$ is separable, there exists a norm $|\cdot|_{C(\overline{\Omega}\times[0,1])}$ on $C(\overline{\Omega} \times [0,1])$, which is equivalent to the usual one,

such that $(C(\overline{\Omega} \times [0,1]), |\cdot|_{C(\overline{\Omega}\times[0,1])})$ be strictly convex and $\mathcal{M}(\overline{\Omega} \times [0,1])$, endowed with the dual norm of $|\cdot|_{C(\overline{\Omega}\times[0,1])}$ (denoted by $|\cdot|_{\mathcal{M}(\overline{\Omega}\times[0,1])}$), be also strictly convex (see [7], [24], [28]). Define the distance function to $\widetilde{\mathcal{C}}$ by $d_{\widetilde{\mathcal{C}}}(\zeta) = \inf_{z \in \widetilde{\mathcal{C}}} |\zeta - z|_{C(\overline{\Omega}\times[0,1])}$. Since $\widetilde{\mathcal{C}}$ is convex, then $d_{\widetilde{\mathcal{C}}}$ is convex and Lipschitz of rank 1, and we have:

$$\limsup_{\rho \searrow 0,\ \zeta' \to \zeta} \frac{d_{\widetilde{\mathcal{C}}}(\zeta' + \rho z) - d_{\widetilde{\mathcal{C}}}(\zeta')}{\rho} = \max\{\langle \xi, z \rangle_{\overline{\Omega}\times[0,1]} \mid \xi \in \partial d_{\widetilde{\mathcal{C}}}(\zeta)\} \tag{35}$$

for every $\zeta, z \in C(\overline{\Omega} \times [0,1])$, where $\partial d_{\widetilde{\mathcal{C}}}$ is the subdifferential in the sense of convex analysis (see [6]). Therefore, for a given $\zeta \in C(\overline{\Omega} \times [0,1])$ we have :

$$\langle \xi, z-\zeta \rangle_{\overline{\Omega}\times[0,1]} + d_{\widetilde{\mathcal{C}}}(\zeta) \leq d_{\widetilde{\mathcal{C}}}(z) \quad \text{and} \quad |\xi|_{\mathcal{M}(\overline{\Omega}\times[0,1])} \leq 1 \tag{36}$$

for all $\xi \in \partial d_{\widetilde{\mathcal{C}}}(\zeta)$ and all $z \in C(\overline{\Omega} \times [0,1])$. Moreover since $\widetilde{\mathcal{C}}$ is a closed convex subset of $C(\overline{\Omega} \times [0,1])$, for every $\zeta \notin \widetilde{\mathcal{C}}$, and every $\xi \in \partial d_{\widetilde{\mathcal{C}}}(\zeta)$, then $|\xi|_{\mathcal{M}(\overline{\Omega}\times[0,1])} 1$ (see [24], Lemma 3.4). If $\zeta \notin \widetilde{\mathcal{C}}$, since $\partial d_{\widetilde{\mathcal{C}}}(\zeta)$ is convex in $\mathcal{M}(\overline{\Omega}\times[0,1])$ and $(\mathcal{M}(\overline{\Omega}\times[0,1]), |\cdot|_{\mathcal{M}(\overline{\Omega}\times[0,1])})$ is strictly convex, then $\partial d_{\widetilde{\mathcal{C}}}(\zeta)$ is a singleton (denoted by $\{\nabla d_{\widetilde{\mathcal{C}}}(\zeta)\}$) and $d_{\widetilde{\mathcal{C}}}$ is Gâteaux-differentiable at ζ. We consider the penalized functional defined on $C(\overline{\Omega} \times [0,1]) \times C([0,1]) \times W_{ad}(\bar{w}, n) \times U_{ad}(\bar{u})$ by

$$\tilde{J}_n(z,\varphi,w,u) = \Big([(\tilde{J}(z,\varphi,w,u) - \tilde{J}(\bar{z},\bar{\varphi},\bar{w},\bar{u}) + \epsilon_n^2)^+]^2 + d_{\widetilde{\mathcal{C}}}(\Psi(\varphi,z)^2 \Big)^{\frac{1}{2}},$$

where $\epsilon_n \leq 1/n^{2\sigma}$. It is clear that $(\bar{z},\bar{\varphi},\bar{w},\bar{u})$ is an ϵ_n^2-solution of the penalized problem

$$(\widetilde{\mathcal{P}}_n) \qquad \inf\{\tilde{J}_n(z,\varphi,w,u) \mid (z,\varphi) \in C(\overline{\Omega}\times[0,1]) \times C([0,1]),$$

$$(w,u) \in W_{ad}(\bar{w},n) \times U_{ad}(\bar{u}),\ (z,\varphi,w,u) \text{ satisfies (29)}\}.$$

From Proposition 5.4 we deduce that $(W_{ad}(\bar{w},n) \times U_{ad}(\bar{u}), d)$ is a complete metric space and that the functional $(w,u) \to \tilde{J}_n(z_{wu},\varphi_u,w,u)$ is bounded from below and continuous on this metric space (here (z_{wu},φ_u) is the solution of (29) corresponding to (w,u)).

6.2 Optimality condition for $(\widetilde{P})$

Due to Ekeland's principle, for every $n \geq 1$, there exists $(w_n,u_n) \in W_{ad}(\bar{w},n) \times U_{ad}(\bar{u})$ such that

$$d((w_n,u_n),(\bar{w},\bar{u})) \leq \frac{1}{n^{2\sigma}}, \tag{37}$$

$$\tilde{J}_n(z_n,\varphi_n,w_n,u_n) \leq \tilde{J}_n(z_{wu},\varphi_u,w,u) + \frac{1}{n^{2\sigma}} d((w_n,u_n),(w,u)), \tag{38}$$

for every $(w, u) \in W_{ad}(\bar{w}, n) \times U_{ad}(\bar{u})$ ((z_n, φ_n) and (z_{wu}, φ_u) are the solutions of (29) corresponding respectively to (w_n, u_n) and to (w, u)). As in [28], [29], inequalities (37), (38) can be used to obtain optimality conditions for $(\bar{z}, \bar{\varphi}, \bar{w}, \bar{u})$. with respect same as the

Step 1. Approximate Pontryagin's principle for u_n. From now on we only consider the case when $n^{2\sigma} \geq \frac{4}{\bar{T}}$. Let u_0 be in $C([0,1])$ satisfying $\min_{\tau\in[0,1]} u_0(\tau) \geq -\frac{\bar{T}}{4}$. We set $\tilde{u}_n = u_0 + u_n$, $u_\rho^n = u_n + \rho(\tilde{u}_n - u_n)$, $w_\rho^n = w_n$. Because of (37) and because $n^{2\sigma} \geq \frac{4}{\bar{T}}$, we have $\min_{\tau\in[0,1]} u_n(\tau) \geq \frac{3\bar{T}}{4}$ and $\min_{\tau\in[0,1]} \tilde{u}_n(\tau) \geq \frac{\bar{T}}{2}$ (indeed $\bar{u}(\tau) \equiv \bar{T}$ on $[0,1]$). We denote by $(z_\rho^n, \varphi_\rho^n)$ (respectively (z_n, φ_n)) the solution of (29) corresponding to (w_ρ^n, u_ρ^n) (respectively (w_n, u_n)). There exists $\rho_0 \in]0,1[$ such that $\{(w_\rho^n, u_\rho^n) \mid 0 < \rho \leq \rho_0\}$ is a subset of $W_{ad}(\bar{w}, n) \times U_{ad}(\bar{u})$. On the other hand, with Theorem 5.2, we have

$$z_\rho^n = z_n + \rho h_n + \frac{1}{\rho} r_\rho, \quad \lim_{\rho\to 0} \frac{1}{\rho}\|r_\rho\|_{C(\overline{\Omega}\times[0,1])} = 0, \tag{39}$$

$$\tilde{J}(z_\rho^n, \varphi_\rho^n, w_\rho^n, u_\rho^n) = \tilde{J}(z_n, \varphi_n, w_n, u_n) + \rho\Delta\tilde{J}_n + o(\rho), \tag{40}$$

with

$$\Delta\tilde{J}_n = \int_0^1 \int_\Omega [u_n F_y'(x, \varphi_n, z_n) h_n + u_n F_t'(x, \varphi_n, z_n)(\tilde{\varphi}_n - \varphi_n)]\, dx d\tau$$

$$+ \int_0^1 \int_\Gamma [u_n G_y'(s, \varphi_n, z_n, w_n) h_n + u_n G_t'(s, \varphi_n, z_n, w_n)(\tilde{\varphi}_n - \varphi_n)]\, ds d\tau$$

$$+ \int_\Omega [L_y'(x, \varphi_n(1), z_n(1)) h_n(1) + L_t'(x, \varphi_n(1), z_n(1))(\tilde{\varphi}_n - \varphi_n)(1)]\, dx$$

$$+ \int_0^1 \int_\Omega F(x, \varphi_n, z_n)(\tilde{u}_n - u_n)\, dx d\tau + \int_0^1 \int_\Gamma G(s, \varphi_n, z_n, w_n)(\tilde{u}_n - u_n)\, ds d\tau,$$

where $\tilde{\varphi}_n(\tau) = \int_0^\tau \tilde{u}_n(s)\,ds$, and h_n is the weak solution in $W(0,1) \cap C(\overline{Q}_1)$ of

$$\begin{cases} \dfrac{d}{d\tau}\displaystyle\int_\Omega h_n \chi\,dx + u_n\Big(\int_\Omega [\sum_{i,j1}^N a_{ij} D_j h_n D_i \chi + f'_{ny} h_n \chi]\,dx + \int_\Gamma g'_{ny} h_n \chi\,ds\Big) = \\ -u_n\Big(\displaystyle\int_\Omega f'_t(x,\varphi_n,z_n)\chi\,dx + \int_\Gamma g'_t(s,\varphi_n,z_n,w_n)\chi\,ds\Big)\varphi_0 \\ -u_0\Big(\displaystyle\int_\Omega [\sum_{i,j1}^N a_{ij} D_j z_n D_i \chi + f(\cdot,\varphi_n,z_n)\chi]\,dx + \int_\Gamma g(\cdot,\varphi_n,z_n,w_n)\chi\,ds\Big), \\ \text{for every } \chi \in H^1(\Omega), \qquad \text{and } h_n(0) = 0 \text{ in } \Omega, \end{cases}$$

with $\varphi_0(\tau) = \int_0^\tau u_0(s)\,ds$, $f'_{ny} = f'_y(\cdot,\varphi_n,z_n)$, and $g'_{ny} = g'_y(\varphi_n,z_n,w_n)$. Taking (38), (39), (40) and the definition of $d_{\widetilde{\mathcal{C}}}$ into account, we obtain :

$$\begin{aligned} &-\nu_n \Delta\tilde{J}_n - \langle \mu_n, \Psi'_z(\varphi_n,z_n)h_n + \Psi'_\varphi(\varphi_n,z_n)\varphi_0\rangle_{\overline{\Omega}\times[0,1]} \\ &= \lim_{\rho\to 0}\frac{\tilde{J}_n(z_n,\varphi_n,w_n,u_n) - \tilde{J}_n(z^n_\rho,\varphi^n_\rho,w^n_\rho,u^n_\rho)}{\rho} \\ &\le \frac{1}{n}\|\tilde{u}_n - u_n\|_{\infty,]0,1[} \le \frac{1}{n}\|u_0\|_{\infty,]0,1[}, \end{aligned} \tag{41}$$

where ν_n and μ_n are defined by

$$\nu_n = \frac{(\tilde{J}(z_n,\varphi_n,w_n,u_n) - \tilde{J}(\bar{z},\bar{\varphi},\bar{w},\bar{u}) + \frac{1}{n^2})^+}{\tilde{J}_n(z_n,\varphi_n,w_n,u_n)},$$

$$\mu_n = \begin{cases} \frac{d_{\widetilde{\mathcal{C}}}(\Psi(\varphi_n,z_n))\nabla d_{\widetilde{\mathcal{C}}}(\Psi(\varphi_n,z_n))}{\tilde{J}_n(z_n,\varphi_n,w_n,u_n)} & \text{if } d_{\widetilde{\mathcal{C}}}(\Psi(\varphi_n,z_n)) > 0, \\ 0 & \text{if not.} \end{cases}$$

Let ψ_n be the weak solution in $L^1(0,1;W^{1,1}(\Omega))$ of

$$\begin{cases} -\dfrac{\partial\psi_n}{\partial\tau} + u_n A\psi_n + u_n f'_y(\varphi_n,z_n)\psi_n \nu_n u_n F'_y(\varphi_n,z_n) + [\Psi'_z(\varphi_n,z_n)^*\mu_n]_{Q_1}, \\ u_n\left(\dfrac{\partial\psi_n}{\partial n_A} + g'_y(\varphi_n,z_n,w_n)\psi_n\right)\nu_n u_n G'_y(\varphi_n,z_n,w_n) + [\Psi'_z(\varphi_n,z_n)^*\mu_n]_{\Sigma_1}, \\ \psi_n(1)\nu_n L'_y(x,\varphi_n(1),z_n(1)) + [\Psi'_z(\varphi_n,z_n)^*\mu_n]_{\overline{\Omega}\times\{1\}}, \end{cases}$$

where $[\Psi'_z(\varphi_n, z_n)^*\mu_n]$ is the Radon measure on $\overline{Q}_1$ defined by $z \mapsto \langle\mu_n, \Psi'_z(\varphi_n, z_n)z\rangle_{\overline{Q1}_1}$ for $z \in C(\overline{Q}_1)$. From Proposition 4.1 it follows that z_n belongs to $L^\delta(0,1; W^{1,d}(\Omega))$ for some $\delta > 1$, $d > 1$ satisfying $\frac{N}{2d} + \frac{1}{\delta} < \frac{1}{2}$. Therefore, with the Green formula of Theorem 5.1, we obtain:

$$\nu_n \int_0^1 \int_\Omega u_n F'_y(x, \varphi_n, z_n) h_n \, dx d\tau + \nu_n \int_\Omega L'_y(x, \varphi_n(1), z_n(1)) h_n(1) \, dx$$

$$+\nu_n \int_0^1 \int_\Gamma u_n G'_y(s, \varphi_n, z_n, w_n) h_n \, ds d\tau + \langle \mu_n, \Psi'_z(\varphi_n, z_n) h_n \rangle_{\overline{\Omega}\times[0,1]} =$$

$$-\int_0^1 \left(\int_\Omega \psi_n u_n f'_t(x, \varphi_n, z_n) \, dx + \int_\Gamma \psi_n u_n g'_t(s, \varphi_n, z_n, w_n) \, ds \right) \varphi_0(\tau) \, d\tau$$

$$-\int_0^1 u_0 \bigg(\int_\Omega [\psi_n f_n + \sum_{i,j1}^N a_{ij} D_j z_n D_i \psi_n] \, dx + \int_\Gamma \psi_n g_n \, ds \bigg) d\tau. \tag{42}$$

with $f_n = f(\cdot, \varphi_n, z_n)$, and $g_n = g(\cdot, \varphi_n, z_n, w_n)$. We introduce $\chi_n \in BV([0,1])$ the solution of the differential equation with right hand side measure:

$$\begin{cases} d\chi_n = [\Psi'_\varphi(\varphi_n, z_n)^*\mu_n] + u_n \bigg(\displaystyle\int_\Omega \big(\psi_n f'_t(x, \varphi_n, z_n) - \nu_n F'_t(x, \varphi_n, z_n) \big) \, dx \\ \qquad + \displaystyle\int_\Gamma [\psi_n g'_t(s, \varphi_n, z_n, w_n) - \nu_n G'_t(s, \varphi_n, z_n, w_n)] \, ds \bigg) d\tau, \\ \\ \chi_n(1) = \nu_n \displaystyle\int_\Omega L'_t(x, \varphi_n(1), z_n(1)) \, dx. \end{cases}$$

With an integration by parts (see [26], p. 110), we obtain

$$\nu_n \int_0^1 u_n \bigg(\int_\Omega F'_t(x, \varphi_n, z_n) \varphi_0 \, dx + \int_\Gamma G'_t(s, \varphi_n, z_n, w_n) \varphi_0 \, ds \bigg) d\tau$$

$$+\langle \mu_n, \Psi'_\varphi(\varphi_n, z_n) \varphi_0 \rangle_{\overline{\Omega}\times[0,1]} =$$

$$\langle -d\chi_n, \varphi_0 \rangle_{[0,1]} + \int_0^1 u_n \big(\int_\Omega \psi_n f'_t(\varphi_n, z_n) \, dx + \int_\Gamma \psi_n g'_t(\varphi_n, z_n, w_n) \, ds \big) \varphi_0 \, d\tau$$

$$-\nu_n \int_\Omega L'_t(x, \varphi_n(1), z_n(1)) \varphi_0(1) \, dx + \int_0^1 \chi_n u_0 \, d\tau$$

$$+\int_0^1 u_n\Big(\int_\Omega \psi_n f'_t(\varphi_n, z_n)\,dx + \int_\Gamma \psi_n g'_t(\varphi_n, z_n, w_n)\,ds\Big)\varphi_0\,d\tau. \qquad (43)$$

From (41), (42), (43), and the definitions of $\Delta\tilde{J}_n$ and $\tilde{u}_n$, we deduce

$$-\int_0^1 u_0\mathcal{H}(\varphi_n, z_n, w_n, \psi_n, \nu_n)d\tau - \int_0^1 \chi_n u_0\,d\tau \le \frac{1}{n^{2\sigma}}\|u_0\|_{C([0,1])} \qquad (44)$$

for every $u_0 \in C([0,1])$ satisfying $\min_{\tau\in[0,1]} u_0(\tau) \ge \frac{-\bar{T}}{4}$ and for all $n^{2\sigma} \ge \frac{4}{\bar{T}}$.

Step 2. Approximate Pontryagin's principle for w_n. Let w_0 be in W_{ad}, and denote by w_{0n} $(n > 0)$ the function in $W_{ad}(\bar{w}, n)$ defined by

$$w_{0n}(x,\tau)\begin{cases} w_0(x,\tau) & \text{if } |w_0(x,\tau) - \bar{w}(x,\tau)| \le n, \\ \bar{w}(x,\tau) & \text{if not.}\end{cases}$$

As in [33, 28, 29], for every $n > 0$, we can prove that

$$\int_0^1\int_\Gamma u_n H(s, \varphi_n, z_n, w_n, \psi_n, \nu_n)\,dsd\tau$$

$$\le \int_0^1\int_\Gamma u_n H(s, \varphi_n, z_n, w_{0n}, \psi_n, \nu_n)\,dsd\tau + \frac{1}{n^{2\sigma}}\mathcal{L}^N(\Gamma\times]0,1[). \qquad (45)$$

Step 3. Observe that $\nu_n^2 + |\mu_n|^2_{\mathcal{M}(\overline{\Omega}\times[0,1])}1$. Hence there exists $\bar{\nu} \in \mathbb{R}^+$ and $\bar{\mu} \in \mathcal{M}(\overline{\Omega}\times[0,1])$ such that $\nu_n \longrightarrow \bar{\nu}$ in $\mathbb{R}$ and $\mu_n \longrightarrow \bar{\mu}$ weakly star in $\mathcal{M}(\overline{\Omega}\times[0,1])$. With the same arguments as in [28, 29], we can prove that $(\psi_n)_n$ converges to $\bar{\psi}$ for the weak-star topology of $L^{\delta'}(0,1;W^{1,d'}(\Omega))$, for every $\delta > 1$, $d > 1$ obeying $\frac{N}{2d} + \frac{1}{\delta} < \frac{1}{2}$, where $\bar{\psi}$ is the weak solution of

$$\begin{cases} -\dfrac{\partial\bar{\psi}}{\partial\tau} + \bar{u}A\bar{\psi} + \bar{u}f'_y(\bar{\varphi},\bar{z})\bar{\psi} = \bar{\nu}\bar{u}F'_y(\bar{\varphi},\bar{z}) + [\Psi'_z(\bar{\varphi},\bar{z})^*\bar{\mu}]_{\Omega\times]0,1[}, \\ \bar{u}\left(\dfrac{\partial\bar{\psi}}{\partial n_A} + g'_y(\bar{\varphi},\bar{z},\bar{w})\bar{\psi}\right) = \bar{\nu}\bar{u}G'_y(\bar{\varphi},\bar{z},\bar{w}) + [\Psi'_z(\bar{\varphi},\bar{z})^*\bar{\mu}]_{\Gamma\times]0,1[}, \\ \bar{\psi}(1) = \bar{\nu}L'_y(x,\bar{\varphi}(1),\bar{z}(1)) + [\Psi'_z(\bar{\varphi},\bar{z})^*\bar{\mu}]_{\overline{\Omega}\times\{1\}}. \end{cases} \qquad (46)$$

We also can prove that $(\chi_n)_n$ converges to the solution $\bar{\chi}$ of

$$\begin{cases} d\bar{\chi} = [\Psi'_\varphi(\bar{\varphi},\bar{z})^*\bar{\mu}] + \bar{u}(\tau)\Big(\int_\Omega [\bar{\psi}(\tau)f'_t(x,\bar{\varphi},\bar{z}) - \bar{\nu}F'_t(x,\bar{\varphi},\bar{z})]\,dx \\ \qquad + \int_\Gamma [\bar{\psi}(\tau)g'_t(s,\bar{\varphi},\bar{z},\bar{w}) - \bar{\nu}G'_t(s,\bar{\varphi},\bar{z},\bar{w})]\,ds\Big)d\tau, \\ \\ \bar{\chi}(1) = \bar{\nu}\int_\Omega L'_t(x,\bar{\varphi}(1),\bar{z}(1))\,dx, \end{cases} \tag{47}$$

for the weak-star topology of $BV([0,1])$ (see [26], p. 110 for the definition of weak-star topology of $BV([0,1])$). On the other hand, $(w_n)_n$ converges to $\bar{w}$ in $L^\sigma(\Gamma\times]0,1[)$, $(\varphi_n)_n$ converges to $\bar{\varphi}$ in $C([0,1])$, and $(z_n)_n$ converges to $\bar{z}$ in $W(0,1)\cap C(\overline{\Omega}\times[0,1])$ and in $L^\delta(0,1;W^{1,d}(\Omega))$ for some $\delta>1$, $d>1$ satisfying $\frac{N}{2d}+\frac{1}{\delta}<\frac{1}{2}$. Due to (A3)-(A7), by passing to the limit in (44) and in (45), we obtain:

$$-\int_0^1 u_0(\tau)\mathcal{H}(\bar{\varphi}(\tau),\bar{z}(\tau),\bar{w}(\tau),\bar{\psi}(\tau),\bar{\nu})d\tau - \int_0^1 \bar{\chi}(\tau)u_0(\tau)\,d\tau \le 0,$$

for every $u_0\in C([0,1])$ satisfying $\min_{\tau\in[0,1]} u_0(\tau)\ge \frac{-\bar{T}}{4}$, and

$$\int_0^1\int_\Gamma \bar{u}H(s,\bar{\varphi},\bar{z},\bar{w},\bar{\psi},\bar{\nu})\,dsd\tau \le \int_0^1\int_\Gamma \bar{u}H(s,\bar{\varphi},\bar{z},w_0,\bar{\psi},\bar{\nu})\,dsd\tau$$

for every $w_0\in W_{ad}$. By using the same arguments as in [28], from the previous integral Pontryagin principle, we can deduce the following pointwise Pontryagin principle

$$H(\cdot,\bar{\varphi}(\cdot),\bar{z}(\cdot),\bar{w}(\cdot),\bar{\psi}(\cdot),\bar{\nu})(s,\tau) = \min_{w\in K_V(s)} H(\cdot,\bar{\varphi}(\cdot),\bar{z}(\cdot),w,\bar{\psi}(\cdot),\bar{\nu})(s,\tau) \tag{48}$$

for a.e. $(s,\tau)\in\Sigma_1$. Let u be in $C([0,1])$ non identically zero. We set $u_0^\pm(\tau)\pm\frac{u(\tau)}{\|u\|_{C([0,1])}}\times\frac{\bar{T}}{4}$. It is clear that u_0^+ and u_0^- belong to $C([0,1])$, and that $\min_{[0,1]} u_0^+(\tau)\ge\frac{-\bar{T}}{4}$, $\min_{[0,1]} u_0^-(\tau)\ge\frac{-\bar{T}}{4}$. Thus:

$$\pm\frac{\bar{T}}{4\|u\|_{C([0,1])}}\left(\int_0^1 u\mathcal{H}(\bar{\varphi},\bar{z},\bar{w},\bar{\psi},\bar{\nu})d\tau + \int_0^1 u\bar{\chi}\,d\tau\right)\le 0.$$

Then we deduce that $\int_0^1 u\mathcal{H}(\bar\varphi,\bar z,\bar w,\bar\psi,\bar\nu)d\tau + \int_0^1 u\bar\chi\, d\tau = 0$, for every $u \in C([0,1])$. Therefore,

$$\mathcal{H}(\bar\varphi(\tau),\bar z(\tau),\bar w(\tau),\bar\psi(\tau),\bar\nu) = -\bar\chi(\tau) \quad \text{for a.e. } \tau \in]0,1[. \tag{49}$$

Step 4. Nontriviality condition. From the definition of μ_n and from (36), we deduce

$$\langle \mu_n, \xi - \Psi(\varphi_n, z_n)\rangle_{\overline{\Omega}\times[0,1]} \le 0 \qquad \text{for all } \xi \in \widetilde{\mathcal{C}}. \tag{50}$$

By passing to the limit in this expression, we obtain

$$\langle \bar\mu, \xi - \Psi(\bar\varphi, \bar z)\rangle_{\overline{\Omega}\times[0,1]} \le 0 \qquad \text{for all } \xi \in \widetilde{\mathcal{C}}. \tag{51}$$

To prove that $(\bar\nu,\bar\mu)$ is nonzero, we use the same arguments as in [28]. First recall that $\nu_n^2 + |\mu_n|^2_{\mathcal{M}(\overline{\Omega}\times[0,1])} = 1$. If $\lim_n \nu_n > 0$, then the proof is complete. If not, we have $\lim_n |\mu_n|_{\mathcal{M}(\overline{\Omega}\times[0,1])} = 1$. Since the interior of $\widetilde{\mathcal{C}}$ is nonempty in $C(\overline{\Omega}\times[0,1])$, there exists $\xi \in \widetilde{\mathcal{C}}$ and $\rho > 0$ such that the ball $B(\xi;\rho)$ in $C(\overline{\Omega}\times[0,1])$, centred at ξ and with radius ρ, is included in $C(\overline{\Omega}\times[0,1])$. Let ξ_n be in $B(0;\rho)$ such that $\langle \mu_n, \xi_n\rangle_{\overline{\Omega}\times[0,1]} = \frac{\rho}{2}|\mu_n|_{\mathcal{M}(\overline{\Omega}\times[0,1])}$.

Since $\xi + \xi_n \in \widetilde{\mathcal{C}}$, we have

$$\langle \mu_n, \xi + \xi_n - \Psi(\bar\varphi, \bar z)\rangle_{\overline{\Omega}\times[0,1]} \le 0.$$

By passing to the limit, we obtain

$$\frac{\rho}{2} + \langle \bar\mu, \xi - \Psi(\bar\varphi, \bar z)\rangle_{\overline{\Omega}\times[0,1]} \le 0.$$

Thus $\bar\mu \ne 0$ and the proof is complete.

6.3 Proof of Theorem 2.1

Recall that $(\bar y, \bar v, \bar T)$ and $(\bar z, \bar\varphi, \bar w, \bar u)$ are linked by the equalities $\bar z(x,\tau) = \bar y(x, \bar\varphi(\tau))$, $\bar w(x,\tau) = \bar v(x,\bar\varphi(\tau))$, $\bar u(\tau) = \bar T$ and $\bar\varphi(\tau) = \tau\bar T$ on $[0,1]$. Let $\bar\lambda$ be the Radon measure on $\overline{Q}_{\bar T}$ defined by

$$\langle \bar\lambda, \Phi'_y(\cdot, \bar y(\cdot))h\rangle_{\overline{Q}_{\bar T}} = \langle \bar\mu, \Psi'_z(\bar\varphi,\bar z)h(\cdot,\bar\varphi(\cdot))\rangle_{\overline{\Omega}\times[0,1]}$$

for every $h \in C(\overline{Q}_{\bar{T}})$. Consider $\bar{p}$ and $\bar{q}$ the solutions of (5)-(7) associated with $(\bar{y}, \bar{v}, \bar{\lambda}, \bar{\nu})$. It is obvious that $\bar{\psi}(x,\tau) = \bar{p}(x, \bar{\varphi}(\tau))$,

$\bar{\chi}(\tau) = \bar{q}(\bar{\varphi}(\tau))$. Now, taking the definitions of $\bar{z}$, $\bar{w}$ and the optimality conditions for $(\bar{z}, \bar{\varphi}, \bar{w}, \bar{u})$ into account, the proof of Theorem 2.1 is complete. $\square$

Remark 6.1. If (A10) is not fulfilled, then we have to slightly modify the proof of optimality conditions for $(\widetilde{\mathcal{P}})$ (given in §6.2), by using some arguments developed in [29]. Instead of considering the penalized problems $(\widetilde{\mathcal{P}}_n)$, we have to consider $(\widetilde{\mathcal{P}}_n^\epsilon)$ defined by

$$\inf\{\tilde{J}_n(z,\varphi,w,u) \mid (z,\varphi) \in C(\overline{\Omega} \times [0,1]) \times C([0,1]),$$

$$(w,u) \in W_{ad}(\bar{w}, n) \times U_{ad}(\bar{u}, \epsilon),\ (z,\varphi,w,u) \text{ satisfies } (29)\}$$

where $0 < \epsilon < 1$ and $U_{ad}(\bar{u},\epsilon)\{u \in U_{ad}(\bar{u}) \mid u \equiv \bar{u} \text{ in } [0,\epsilon]\}$. With such a choice, the calculations of Section 6.2 are still valid because the function z_n defined in Section 6.2 belongs to $L^\delta(\epsilon, T; W^{1,d}(\Omega))$ for every $\epsilon > 0$ (see [29] for more details). The arguments in ([29]) together with that of §6.2 lead to:

$$\mathcal{H}(\bar{\varphi}(\tau), \bar{z}(\tau), \bar{w}(\tau), \bar{\psi}(\tau), \bar{\nu}) = -\bar{\chi}(\tau) \quad \text{for a.e. } \tau \in]0,1[.$$

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