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Error estimates for stochastic differential games: the adverse stopping case

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We obtain error bounds for monotone approximation schemes of a particular Isaacs equation. This is an extension of the theory for estimating errors for the Hamilton–Jacobi–Bellman equation. To obtain the upper error bound, we consider the ‘Krylov regularization’ of the Isaacs equation to build an approximate sub-solution of the scheme. To get the lower error bound, we extend the method of Barles & Jakobsen (2005, *SIAM J. Numer. Anal.*) which consists in introducing a switching system whose solutions are local super-solutions of the Isaacs equation.

Keywords: Isaacs equation; Hamilton–Jacobi–Bellman equation; stochastic differential games; finite differences; error estimates.

1. Introduction

The aim of this paper is to give error bounds for approximation schemes of a particular non-convex Isaacs equation. More precisely, we consider the following equation

$$\min\left\{\sup_{\alpha\in\mathcal{A}} L^{\alpha}(x,\mathcal{D}u(x));\, u(x)-\psi(x)\right\}=0,\quad x\in\mathbb{R}^N, \tag{1.1}$$

where

$$L^{\alpha}(x,\mathcal{D}u(x)) = L^{\alpha}(x,u(x),Du(x),D^2u(x)),$$
$$L^{\alpha}(x,t,p,X) = -\operatorname{tr}[a^{\alpha}(x)X] - b^{\alpha}(x)p + c^{\alpha}(x)t - f^{\alpha}(x).$$

Here $\mathcal{A}=\{\alpha_1,\ldots,\alpha_M\}$ denotes the set of controls, assumed to be finite; the case of a compact set will be dealt with in Section 4.3. The coefficients $(a^{\alpha},b^{\alpha},c^{\alpha},f^{\alpha})$ are, for each $\alpha\in\mathcal{A}$, bounded and Lipschitz functions $\mathbb{R}^N\to\mathcal{S}^N\times\mathbb{R}^N\times\mathbb{R}\times\mathbb{R}$, where $\mathcal{S}^N$ denotes the set of $N\times N$ symmetric matrices; ψ is a bounded Lipschitz function from $\mathbb{R}^N$ into $\mathbb{R}$. Under classical assumptions, (1.1) has a unique

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bounded viscosity solution, denoted u. The regularity of u depends on the properties of the coefficients a, b, c and f.

This problem is a particular case of stochastic differential games, called the adverse stopping case. In fact, we can note that in (1.1) we have two players, A and B. Player A has a set of controls and wants to minimize the gain. Player B can only decide to stop the game with the objective of maximizing the gain.

Then we consider monotone approximation schemes of (1.1), of the following form:

$$\min\{S(h, x, u_h(x), u_h); u_h(x) - \psi(x)\} = 0, \quad x \in \mathbb{R}^N, \tag{1.2}$$

where S is a consistent, monotonic and uniformly continuous approximation of $\sup_{\alpha\in\mathcal{A}} L^\alpha$. We will note $u_h \in C_b(\mathbb{R}^N)$ the solution of (1.2), which is the approximation of u, and h the mesh size. This abstract notation was introduced by Barles & Souganidis (1991) to display clearly the monotonicity of the scheme: $S(h, x, r, v)$ is non-decreasing in r and non-increasing in v. Typical approximation schemes that we will consider are classical finite differences (Kushner & Dupuis, 2001), generalized finite differences (Bonnans *et al.*, 2004; Bonnans & Zidani, 2003) and Markov chain approximations (Kushner & Dupuis, 2001).

Until now, results on convergence rates for monotone approximation schemes of the equation with one player have been obtained; i.e. for the following equation:

$$\sup_{\alpha\in\mathcal{A}} L^\alpha(x, \mathcal{D}u(x)) = 0, \quad x \in \mathbb{R}^N, \tag{1.3}$$

and the scheme

$$S(h, x, u_h(x), u_h) = 0, \quad x \in \mathbb{R}^N. \tag{1.4}$$

Error estimates for this equation have been obtained by Krylov (1997, 2000), and these results were extended by Barles & Jakobsen (2002, 2005). Error estimates for a stochastic impulse control problem were recently obtained by the authors (Bonnans *et al.*, 2005). During the review of this paper, we learned that also Jakobsen was working on the convergence analysis for monotone approximations of (1.1). In Jakobsen (2004b), the author obtained error estimates in the case of a finite-differences scheme with matrix a independent of x, using a penalization approach, and in Jakobsen (2004a) the author obtained error estimates in the 1D case but for general Isaacs equations.

By using the methods introduced by Barles & Jakobsen (2005), we give the convergence rate for monotonic approximation schemes of the two players equation. The upper estimate of $u - u_h$ is easier to obtain than the lower estimate. The proof involves a 'Krylov regularization' of (1.1), i.e. the perturbed equation

$$\min\left\{\sup_{\alpha,|e|\leqslant\varepsilon} L^\alpha(x+e, \mathcal{D}u^\varepsilon(x)); u^\varepsilon(x) - \psi(x)\right\} = 0, \quad x \in \mathbb{R}^N, \tag{1.5}$$

and its viscosity solution u^ε. A regularization of u^ε by convolution gives an approximate smooth sub-solution of (1.1), denoted u_ε which is also an approximate sub-solution of (1.2). So, by using the consistency property, we obtain the upper bound, after choosing an optimal parameter of regularization. Unfortunately, we cannot proceed in a similar way to build a smooth super-solution of (1.1) which would lead to the lower estimate on $u - u_h$. Instead of a smooth super-solution, we build a sequence of local smooth super-solutions. In particular, we introduce the following switching system which approximates (1.1)

$$\min\left\{\max\left\{L^{\alpha_i}(x, \mathcal{D}v_i(x)); v_i(x) - \min_{j\neq i}\{v_j(x) + k\}\right\}; v_i(x) - \psi(x)\right\} = 0, \tag{1.6}$$

for $x \in \mathbb{R}^N$ and $i \in \mathcal{I} = \{1, \dots, M\}$. For literature on switching systems, see Capuzzo-Dolcetta & Evans (1994), Evans & Friedman (1979) and Ishii & Koike (1991a,b). We consider the viscosity solution $v = (v_1, \dots, v_M)$ of this system, and give an estimate of the rate of convergence of v to u. Then we consider a perturbed system

$$\min\left\{\max\left\{\inf_{|e|\leqslant\varepsilon} L^{\alpha_i}(x+e, \mathcal{D}v_i^\varepsilon(x)); v_i^\varepsilon(x) - \min_{j\neq i}\{v_j^\varepsilon(x)+k\}\right\}; v_i^\varepsilon(x) - \psi(x)\right\} = 0, \tag{1.7}$$

for all $i \in \mathcal{I}$ and $x \in \mathbb{R}^N$, and its viscosity solution denoted $v^\varepsilon = (v_1^\varepsilon, \dots, v_M^\varepsilon)$. We regularize v^ε by convolution obtaining v_ε, and this function allows us to build a local super-solution of (1.1). Then, by applying the consistency and the monotonicity of the scheme, we obtain the desired bound. With our result, we can prove an upper bound of $h^{1/2}$ and a lower bound of $h^{1/5}$ for a classical finite-differences scheme and for a generalized finite-difference scheme.

The paper is organized as follows: in Section 2 we introduce the assumptions on (1.1) and scheme (1.2). In Section 3, we obtain the rate of convergence of the solution v of (1.6) to u. In the first part of Section 4, we obtain an upper bound on $u - u_h$, and in the second part of this section we use the rate obtained in Section 3 to derive the lower bound on $u - u_h$. In Section 5, we apply our results to the generalized finite-difference scheme taken from Bonnans & Zidani (2003), and study conditions under which a general Markov chain approximation gives better estimates than this scheme. Finally in Appendix, we give some auxiliary theorems which are used throughout the paper.

We conclude this introduction with some notations. In the sequel, C is a positive constant independent of the parameters ε and h. By $|\cdot|$ we denote the standard Euclidean norm in any $\mathbb{R}^M$-type space. In particular, if $X \in \mathcal{S}^N$, then $|X|^2 = \mathrm{tr}(XX^\top)$, where $X^\top$ is the transpose of X, i.e. $|X|$ is the Frobenius norm. If g is a bounded function from $\mathbb{R}^N$ into $\mathbb{R}$, $\mathbb{R}^M$ or the space of $N \times P$ matrices, we set

$$|g|_0 := \sup_{x\in\mathbb{R}^N} |g(x)|.$$

If g is also Lipschitz continuous, we set

$$[g]_1 := \sup_{\substack{x,y\in\mathbb{R}^N\\ x\neq y}} \frac{|g(x)-g(y)|}{|x-y|}, \quad |g|_1 := |g|_0 + [g]_1.$$

We denote by $\leqslant$ the component-wise ordering in $\mathbb{R}^N$, and by $\preceq$ the ordering in the sense of positive semi-definite matrices in $\mathcal{S}(N)$. The space $C_b(\mathbb{R}^N)$ (respectively, $C_{b,l}(\mathbb{R}^N)$) will denote the space of continuous and bounded functions (respectively, bounded and Lipschitz continuous functions) from $\mathbb{R}^N$ to $\mathbb{R}$.

2. Well-posedness of the Isaacs equation and of the scheme

Throughout this paper, we suppose that the following assumptions are satisfied:

(A1) There exist λ, K, such that, for all $x \in \mathbb{R}^N$ and $\alpha \in \mathcal{A}$, we have that $a^\alpha(x) = \frac{1}{2}\sigma^\alpha(x)(\sigma^\alpha(x))^\top$, where $\sigma^\alpha(x)$ is, for each x, an $N \times P$ matrix, and

$$c^\alpha \geqslant \lambda > 0; \quad |\sigma^\alpha|_1 + |b^\alpha|_1 + |c^\alpha|_1 + |f^\alpha|_1 \leqslant K,$$

(A2) $\lambda > \sup_\alpha\{[\sigma^\alpha]_1 + [b^\alpha]_1\}$.

DEFINITION 2.1 The function $u \in C(\mathbb{R}^N)$ is called a viscosity sub-solution (respectively, super-solution) of (1.1) if, for every $x \in \mathbb{R}^N$,

$$\min\left\{\sup_\alpha L^\alpha(x, u(x), D\varphi(x), D^2\varphi(x)); u(x) - \psi(x)\right\} \leqslant 0 \quad \text{(respectively, } \geqslant 0\text{)},$$

for each $\varphi \in C^2(\mathbb{R}^N)$ such that $u-\varphi$ has a local maximum (respectively, a local minimum) at x. Finally, we call u a viscosity solution of (1.1) if it is both a sub-solution and a super-solution.

We refer to Jakobsen (2004b, Lemma A.1) for the proof of the following result.

PROPOSITION 2.2 Assume (A1) and (A2). Then the following statements hold:

(i) If u_1 and u_2 are, respectively, a viscosity sub-solution and a viscosity super-solution of (1.1), and $u_1, u_2 \in C_b(\mathbb{R}^N)$, then $u_1 \leqslant u_2$ in $\mathbb{R}^N$.
(ii) There exists a unique viscosity solution u of (1.1), in the space $C_{b,l}(\mathbb{R}^N)$.

Consider the scheme (1.2), with $h > 0$ and $S : \mathbb{R}^+ \times \mathbb{R}^N \times \mathbb{R} \times C_b(\mathbb{R}^N) \to \mathbb{R}$. We make the following assumptions:

(S1) Monotonicity: for all $h > 0$, $r \in \mathbb{R}^N$, $m \geqslant 0$, $x \in \mathbb{R}^N$ and bounded and continuous functions u, v such that $u \leqslant v$ in $\mathbb{R}^N$,

$$S(h, x, r+m, u+m) \geqslant \lambda m + S(h, x, r, v).$$

(S2) Regularity: for all $h > 0$ and $\phi \in C_b(\mathbb{R}^N)$, $x \mapsto S(h, x, \phi(x), \phi)$ is bounded and continuous; $r \mapsto S(h, x, r, \phi)$ is uniformly continuous for bounded r, uniformly with respect to $x \in \mathbb{R}^N$.
(S3) There exist $n, k_i > 0$, $i \in J \subseteq \{1, \ldots, n\}$, and a constant $K_c > 0$ such that for all $h > 0$ and x in $\mathbb{R}^N$, and for every smooth $\phi \in C^n(\mathbb{R}^N)$ such that $|D^i\phi|_0$ is bounded, for every $i \in J$, the following holds:

$$\left|\sup_\alpha L^\alpha(x, \mathcal{D}\phi) - S(h, x, \phi(x), \phi)\right| \leqslant K_c Q(\phi),$$

where $Q(\phi) := \sum_{i\in J} |D^i\phi|_0 h^{k_i}$.

REMARK 2.3 (S1) and (S2) imply that S is decreasing with respect to v (take $m = 0$), and increasing with respect to r (take $v = u + m$).

In the following, we say that a function $v \in C_b(\mathbb{R}^N)$ is a sub-solution (respectively, super-solution) to the scheme (1.2) if

$$\min\{S(h, x, v(x), v); v(x) - \psi(x)\} \leqslant 0 \quad \text{(respectively, } \geqslant 0) \quad \text{for all } x \in \mathbb{R}^N.$$

Under assumptions (S1) and (S2), we have the existence of a comparison principle for bounded continuous solutions of (1.2); i.e.

THEOREM 2.4 Let u_h (respectively, v_h) be a bounded, continuous sub-solution (respectively, super-solution) of (1.2). Then we have $u_h(x) \leqslant v_h(x)$, for all $x \in \mathbb{R}^N$.

Proof. The proof is an easy extension of Barles & Jakobsen (2002, Lemma 2.3). We assume that $m := \sup_x(u_h(x) - v_h(x)) > 0$ and obtain a contradiction. Let $\{x_n\}_n$ be a sequence in $\mathbb{R}^N$ such that $\delta_n := u_h(x_n) - v_h(x_n)$ converges to m as $n \to \infty$. Then $\delta_n > 0$ for large enough n. By using the notion of sub- and super-solution, and the fact that $\min(A, B) - \min(C, D) \geqslant \min(A - C, B - D)$, we get

$$0 \geqslant \min\{S(h, x_n, u_h(x_n), u_h) - S(h, x_n, v_h(x_n), v_h); u_h(x_n) - v_h(x_n)\}.$$

Since $u_h(x_n) - v_h(x_n) = \delta_n > 0$, by using (S1), we have

$$
\begin{aligned}
0 &\geqslant S(h, x_n, u_h(x_n), u_h) - S(h, x_n, v_h(x_n), v_h) \\
&\geqslant S(h, x_n, v_h(x_n) + \delta_n, v_h + m) - S(h, x_n, v_h(x_n), v_h) \\
&\geqslant S(h, x_n, v_h(x_n) + m, v_h + m) + \omega(m - \delta_n) - S(h, x_n, v_h(x_n), v_h) \\
&\geqslant \lambda m + \omega(m - \delta_n),
\end{aligned}
$$

where $\omega(t) \to 0$ when $t \to 0^+$ due to (S2). Letting $n \to \infty$ yields $m \leqslant 0$ which is a contradiction. □

In all the sequel, we will use a sequence of mollifiers $(\rho_\varepsilon)_\varepsilon$ defined as follows:

$$\rho_\varepsilon(x) = \varepsilon^{-N} \rho(x/\varepsilon), \tag{2.1}$$

where $\rho \in C^\infty(\mathbb{R}^N)$, $\int_{\mathbb{R}^N} \rho(x) dx = 1$, $\operatorname{supp}\{\rho\} \subseteq \bar{B}(0, 1)$ and $\rho \geqslant 0$. If g is a continuous function from $\mathbb{R}^N$ into $\mathbb{R}$, then we define the mollification of g as follows:

$$g_\varepsilon(x) := \int_{\mathbb{R}^N} g(x - e)\rho_\varepsilon(e)\, \mathrm{d}e. \tag{2.2}$$

Moreover, if g is a Lipschitz function, then

$$|g(x) - g_\varepsilon(x)| \leqslant \varepsilon [g]_1. \tag{2.3}$$

If $g \in C_b^n(\mathbb{R}^N)$ (respectively, $C_{b,l}^n(\mathbb{R}^N)$), then

$$|D^i g_\varepsilon(x)| \leqslant C\varepsilon^{-i}|g|_0 \quad (\text{respectively, } C\varepsilon^{1-i}|g|_0) \quad \forall\, i = 1, \ldots, n. \tag{2.4}$$

3. Switching system

Consider the following switching system approximation of (1.1):

$$\min\left\{\max\left(L^{\alpha_i}(x, \mathcal{D}v_i(x)); v_i(x) - \min_{j \neq i}\{v_j(x) + k\}\right); v_i(x) - \psi(x)\right\} = 0, \tag{3.1}$$

for $i \in \mathcal{I} = \{1, \ldots, M\}$ and $x \in \mathbb{R}^N$. In particular, we have an equation for every control. A viscosity solution theory for the switching system can be found in Ishii & Koike (1991a,b) and Yamada (1987). We recall here the definition of viscosity solution for a general switching system of the form:

$$F_i(x, v, Dv_i, D^2 v_i) = 0, \quad i = 1, \ldots, M, \tag{3.2}$$

where $F_i \colon \mathbb{R}^N \times \mathbb{R}^M \times \mathbb{R}^N \times \mathcal{S}^N \to \mathbb{R}$.

DEFINITION 3.1 The function $v = (v_1, \ldots, v_M) \in C(\mathbb{R}^N)^M$ is called a viscosity sub-solution (respectively, super-solution) of (3.2) if, for every $i \in \mathcal{I}$ and $x \in \mathbb{R}^N$,

$$F_i(x, v(x), D\varphi(x), D^2\varphi(x)) \leqslant 0 \quad (\text{respectively, } \geqslant 0),$$

for each $\varphi \in C^2(\mathbb{R}^N)$ such that $v_i - \varphi$ has a local maximum (respectively, a local minimum) at x. Finally, we call v a viscosity solution of (3.2) if it is both a sub-solution and a super-solution.

Lemma A.2 implies a comparison principle for (3.1), and then the existence of a unique viscosity solution of (3.1) in $C_{b,l}(\mathbb{R}^N)^M$, denoted $v = (v_1, \ldots, v_M)$.

We perturb the system (3.1) and build the following auxiliary system

$$\min\left\{\max\left(\sup_{|e|\leqslant\varepsilon} L^{\alpha_i}(x+e, \mathcal{D}v_i^\varepsilon(x)); v_i^\varepsilon(x) - \min_{j\neq i}\{v_j^\varepsilon(x)+k\}\right); v_i^\varepsilon(x) - \psi(x)\right\} = 0. \tag{3.3}$$

Lemma A.2 applied to (3.3), implies the existence of a unique viscosity solution of (3.3), denoted $v^\varepsilon = (v_1^\varepsilon, \ldots, v_M^\varepsilon)$, in $C_{b,l}(\mathbb{R}^N)^M$. The following lemma is a consequence of Theorem A.3.

LEMMA 3.2 Under assumptions (A1) and (A2), we have that

$$|v_i - v_i^\varepsilon| \leqslant C\varepsilon, \tag{3.4}$$

where C only depends on K, λ and $[\psi]_1$.

For every i, let $v_{i\varepsilon}$ be the mollification of v_i^ε, defined as in (2.2). Since v_i^ε is a Lipschitz function, uniformly with respect to $\varepsilon > 0$ sufficiently small, (2.3) implies

$$|v_i^\varepsilon(x) - v_{i\varepsilon}(x)| \leqslant \max_i [v_i^\varepsilon]_1 \varepsilon; \tag{3.5}$$

Lemma A.2 implies that $\max_i [v_i^\varepsilon]_1$ remains bounded when $\varepsilon \downarrow 0$ (this argument will be used several times in the paper).

LEMMA 3.3 The function $v_{i\varepsilon} - R$ is, for all i, a sub-solution of (1.1), for some

$$R := C\left(k + \varepsilon + \frac{k}{\varepsilon^2}\right), \tag{3.6}$$

where the constant C depends only on K, λ and $[\psi]_1$.

Proof. Let R satisfy (3.6). We have to prove that, for large enough C,

$$\min\left\{\sup_\alpha L^\alpha(x, \mathcal{D}(v_{i\varepsilon}(x) - R); v_{i\varepsilon}(x) - R - \psi(x)\right\} \leqslant 0 \quad \forall x \in \mathbb{R}^N, \tag{3.7}$$

for all $i \in \mathcal{I}$. Since $v_{i\varepsilon} \in C^\infty(\mathbb{R}^N)$, the definition of viscosity sub-solution is equivalent to the notion of classical sub-solution. Therefore, we have to prove that one of the following statements holds for all $x \in \mathbb{R}^N$:

$$v_{i\varepsilon}(x) - R \leqslant \psi(x) \quad \forall i \in \mathcal{I}, \tag{3.8a}$$

$$\sup_\alpha L^\alpha(x, \mathcal{D}(v_{i\varepsilon}(x) - R)) \leqslant 0 \quad \forall i \in \mathcal{I}. \tag{3.8b}$$

For every $x \in \mathbb{R}^N$, set

$$I^\varepsilon(x) := \{i \in \mathcal{I} | v_i^\varepsilon(x) = \psi(x)\}. \tag{3.9}$$

Let $\tilde{x} \in \mathbb{R}^N$. Denote by $B(\tilde{x}, 2\varepsilon)$ the ball centred on $\tilde{x}$ with radius 2ε. Then we have the following two possibilities:

Case A: There exists $y \in B(\tilde{x}, 2\varepsilon)$ such that $I^\varepsilon(y) \neq \varnothing$. We claim that (3.8a) holds. We have $v_{i_0}^\varepsilon(y) = \psi(y)$, for some $i_0 \in I^\varepsilon(y)$. Let $i \notin I^\varepsilon(y)$. The function $v_i^\varepsilon(x) - |x-y|^2/\varepsilon_1$ has, for sufficiently small

$\varepsilon_1 > 0$, a local maximum at a point x_ε such that $|x_\varepsilon - y| \leqslant \varepsilon$. Since v^ε is the viscosity solution of (3.3), one of the following statements holds:

$$v_i^\varepsilon(x_\varepsilon) \leqslant \psi(x_\varepsilon), \tag{3.10a}$$

$$\max\left\{\sup_{|e|\leqslant\varepsilon} L^{\alpha_i}\left(x_\varepsilon + e, v_i^\varepsilon(x_\varepsilon), \frac{2}{\varepsilon_1}(x_\varepsilon - y), \frac{2I}{\varepsilon_1}\right); \right.$$
$$\left. v_i^\varepsilon(x_\varepsilon) - \min_{j\neq i}\{v_j^\varepsilon(x_\varepsilon) + k\}\right\} \leqslant 0. \tag{3.10b}$$

If $v_i^\varepsilon(x_\varepsilon) \leqslant \psi(x_\varepsilon)$, since v_i^ε and ψ are Lipschitz, we obtain

$$v_i^\varepsilon(y) \leqslant \psi(y) + \left([\psi]_1 + \max_i [v_i^\varepsilon]_1\right)\varepsilon. \tag{3.11}$$

Otherwise, with (3.10b), we have

$$v_i^\varepsilon(y) \leqslant \max_i [v_i^\varepsilon]_1 \varepsilon + v_{i_0}^\varepsilon(x_\varepsilon) + k \leqslant 2\max_i [v_i^\varepsilon]_1 \varepsilon + v_{i_0}^\varepsilon(y) + k. \tag{3.12}$$

Since either (3.11) or (3.12) holds, we deduce that

$$v_i^\varepsilon(y) - C\varepsilon - k \leqslant \psi(y) \quad \forall\, i \in \mathcal{I}, \tag{3.13}$$

where C depends on $[\psi]_1$ and $\max_i [v_i^\varepsilon]_1$. Since $y \in B(\tilde{x}, 2\varepsilon)$, and ψ and v_i^ε are Lipschitz, this implies $v_i^\varepsilon(\tilde{x}) \leqslant \psi(\tilde{x}) + k + C\varepsilon$, for all $i \in \mathcal{I}$. Applying (3.5), we obtain $v_{i\varepsilon}(\tilde{x}) \leqslant \psi(\tilde{x}) + R$, for all $i \in \mathcal{I}$, which implies (3.7).

Case B: For all $y \in B(\tilde{x}, 2\varepsilon)$, we have that $I^\varepsilon(y) = \varnothing$. We claim that, for all $e \in B(0, \varepsilon)$, $(v_1^\varepsilon(\cdot - e), \ldots, v_M^\varepsilon(\cdot - e))$ is a viscosity sub-solution of the following system

$$\max\left\{L^{\alpha_i}(x, \mathcal{D}w_i(x)); w_i(x) - \min_{j\neq i}\{w_j(x) + k\}\right\} = 0, \quad x \in B(\tilde{x}, \varepsilon). \tag{3.14}$$

Fix $e_1 \in B(0, \varepsilon)$ and $i \in \mathcal{I}$. Let $\varphi \in C^2(\mathbb{R}^N)$ be such that $v_i^\varepsilon(\cdot - e_1) - \varphi(\cdot)$ has a local maximum x_{e_1} in the ball $B(\tilde{x}, \varepsilon)$. Then $v_i^\varepsilon(\cdot) - \varphi(\cdot + e_1)$ has a local maximum at $x_{e_1} - e_1$. Since $x_{e_1} - e_1 \in B(\tilde{x}, 2\varepsilon)$, we have that $v_i^\varepsilon(x_{e_1} - e_1) > \psi(x_{e_1} - e_1)$, and since v^ε is the viscosity solution of (3.3), we obtain

$$\max\left\{\sup_{|e|\leqslant\varepsilon} L^{\alpha_i}\left(x_{e_1} - e_1 + e, v_i^\varepsilon\left(x_{e_1} - e_1\right), D\varphi\left(x_{e_1}\right), D^2\varphi\left(x_{e_1}\right)\right); \right.$$
$$\left. v_i^\varepsilon\left(x_{e_1} - e_1\right) - \min_{j\neq i}\left\{v_j^\varepsilon\left(x_{e_1} - e_1\right) + k\right\}\right\} \leqslant 0.$$

Taking $e = e_1$, we obtain

$$\begin{cases} L^{\alpha_i}\left(x_{e_1}, v_i^\varepsilon\left(x_{e_1} - e_1\right), D\varphi\left(x_{e_1}\right), D^2\varphi\left(x_{e_1}\right)\right) \leqslant 0, \\ v_i^\varepsilon\left(x_{e_1} - e_1\right) - \min_{j\neq i}\left\{v_j^\varepsilon\left(x_{e_1} - e_1\right) + k\right\} \leqslant 0. \end{cases}$$

This being true for an arbitrary $e_1 \in B(0,\varepsilon)$ and $i \in \mathcal{I}$, we obtain that, for all $|e| \leqslant \varepsilon$, $(v_1^\varepsilon(\cdot - e), \dots, v_M^\varepsilon(\cdot - e))$ is a viscosity sub-solution of (3.14). Applying Barles & Jakobsen (2002, Lemma A.3 and Lemma 2.7), since $v_{i\varepsilon}(\cdot)$ is limit of convex combination of $v_i^\varepsilon(\cdot - e)$, for $e \in B(0,\varepsilon)$, then $(v_{1\varepsilon}(\cdot), \dots, v_{M\varepsilon}(\cdot))$ is a viscosity sub-solution of (3.14). Moreover, since it is a smooth function, it is a sub-solution of (3.14) in the classical sense, and we have

$$L^{\alpha_i}(\tilde{x}, \mathcal{D}v_{i\varepsilon}(\tilde{x})) \leqslant 0 \quad \forall\, i \in \mathcal{I}. \tag{3.15}$$

We know that $|v_i^\varepsilon(y) - v_j^\varepsilon(y)| \leqslant k$ for all $i, j \in \mathcal{I}$ and $y \in B(\tilde{x}, \varepsilon)$. Consequently,

$$D^n v_{i\varepsilon}(\tilde{x}) - D^n v_{j\varepsilon}(\tilde{x}) \leqslant \frac{Ck}{\varepsilon^n} \quad \forall\, n \geqslant 1,$$

where C depends only on ρ defined in (2.1). It follows that

$$L^{\alpha_i}(\tilde{x}, \mathcal{D}v_{j\varepsilon}(\tilde{x})) - L^{\alpha_i}(\tilde{x}, \mathcal{D}v_{i\varepsilon}(\tilde{x})) \leqslant \frac{Ck}{\varepsilon^2} \quad \forall\, i, j \in \mathcal{I}.$$

Combining this with (3.15), we get $L^{\alpha_i}(\tilde{x}, \mathcal{D}v_{j\varepsilon}(\tilde{x})) \leqslant Ck/\varepsilon^2$, for all i and j in $\mathcal{I}$, for some C depending on ρ and K, and hence, $\sup_\alpha L^\alpha(\tilde{x}, \mathcal{D}v_{i\varepsilon}(\tilde{x})) \leqslant Ck/\varepsilon^2$ for all i in $\mathcal{I}$. Using assumption (A1), we have that for all i in $\mathcal{I}$, $\sup_\alpha L^\alpha(\tilde{x}, \mathcal{D}v_{i\varepsilon}(\tilde{x}) - Ck/(\lambda\varepsilon^2)) \leqslant 0$. Therefore, (3.7) also holds in this case. □

THEOREM 3.4 For every $i \in \mathcal{I}$ and for all $x \in \mathbb{R}^N$ we have

$$0 \leqslant v_i - u \leqslant Ck^{1/3}, \tag{3.16}$$

where C depends only on λ and K from (A1).

Proof.

(a) We prove the first inequality of (3.16). Let $w = (u, \dots, u)$ be the vector whose M components are equal to u. We claim that w is a viscosity sub-solution of (3.1). Let $\varphi \in C^2(\mathbb{R}^N)$ be such that $u - \varphi$ has a local maximum at $x_0 \in \mathbb{R}^N$. Since u is a viscosity sub-solution of (1.1), either $u(x_0) \leqslant \psi(x_0)$ or $\sup_\alpha L^\alpha(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0)) \leqslant 0$. If the latter holds, then

$$L^{\alpha_i}(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0)) \leqslant 0 \quad \forall\, i \in \mathcal{I}.$$

Combining both cases, we obtain

$$\min\left\{ \max\left(L^{\alpha_i}(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0)); u(x_0) - \min_{j \neq i}\{u(x_0) + k\} \right); u(x_0) - \psi(x_0) \right\} \leqslant 0 \quad \forall\, i \in \mathcal{I}.$$

Therefore, w is a viscosity sub-solution of (3.1). By the comparison principle (Lemma A.2), the first inequality of (3.16) holds.

(b) We now prove the second inequality in (3.16). By Lemma 3.3 and Proposition 2.2, we have that $v_{i\varepsilon} - R \leqslant u$, for all $i \in \mathcal{I}$, and $x \in \mathbb{R}^N$, where R satisfies (3.6). Applying (3.5) and (3.4), we obtain

$$v_i - u \leqslant |v_i - v_i^\varepsilon| + |v_i^\varepsilon - v_{i\varepsilon}| + |v_{i\varepsilon} - u| \leqslant C\left(\frac{k}{\varepsilon^2} + \varepsilon + k\right) \quad \forall\, i \in \mathcal{I}, \ \forall\, x \in \mathbb{R}^N,$$

where C depends on K, λ, $\max_i [v_i^\varepsilon]_1$, $[\psi]_1$. Minimizing with respect to ε, we obtain the desired upper bound, with $\varepsilon = k^{1/3}$. □

REMARK 3.5 We obtain the same estimate in the case of only one player (see Barles & Jakobsen, 2005, Theorem 2.3) by extending the same methods.

4. Bounds on $u - u_h$

We state in this section our main results: the upper and lower bounds on $u - u_h$.

4.1 *Upper bound on* $u - u_h$

Perturb (1.1) so as to obtain

$$\min\left\{\sup_{\alpha,|e|\leqslant\varepsilon} L^\alpha(x+e, \mathcal{D}u^\varepsilon(x)); u^\varepsilon(x) - \psi(x)\right\} = 0, \quad x \in \mathbb{R}^N. \tag{4.1}$$

Under assumptions (A1) and (A2), by Proposition 2.2, (4.1) has a unique viscosity solution $u^\varepsilon \in C_{b,l}(\mathbb{R}^N)$. In view of Theorem A.4, we have that $|u - u^\varepsilon| \leqslant C\varepsilon$, for some C depending on λ, K and $[\psi]_1$. Define the contact domain of u^ε as

$$X(u^\varepsilon) := \{x \in \mathbb{R}^N \,|\, u^\varepsilon(x) = \psi(x)\}.$$

Let u_ε be the mollification of u^ε, defined as in (2.2). Since u^ε is a Lipschitz function, uniformly with respect to $\varepsilon > 0$ sufficiently small, (2.3) implies

$$|u^\varepsilon(x) - u_\varepsilon(x)| \leqslant [u^\varepsilon]_1\varepsilon, \tag{4.2}$$

where $[u^\varepsilon]_1$ remains bounded.

THEOREM 4.1 Assume that (A1), (A2) and (S1)–(S3) hold and let the approximation scheme (1.2) have a unique solution u_h in $C_b(\mathbb{R}^N)$. Then, for sufficiently small $h > 0$, we have

$$u - u_h \leqslant Ch^\ell \quad \forall x \in \mathbb{R}^N, \tag{4.3}$$

where $\ell := \min_{i\in J}\{k_i/i\}$, C depends only on λ, K, $[\psi]_1$ and K_c, the constants k_i and K_C being defined in (S3).

Proof. We claim that

$$\min\{S(h, x, u_\varepsilon(x) - R_1, u_\varepsilon - R_1); u_\varepsilon(x) - R_1 - \psi(x)\} \leqslant 0 \quad \forall x \in \mathbb{R}^N, \tag{4.4}$$

for some $R_1 > 0$ of the form $R_1 := \lambda^{-1}Q(u_\varepsilon) + C\varepsilon$, where $Q(\cdot)$ was defined in (S3) and C depends only on $[\psi]_1$ and $[u^\varepsilon]_1$. Indeed, we will prove a slightly stronger result: for any $x \in \mathbb{R}^N$, at least one of the following two statements holds:

$$u_\varepsilon(x) - C\varepsilon \leqslant \psi(x), \tag{4.5a}$$

$$S(h, x, u_\varepsilon(x) - K_C\lambda^{-1}Q(u_\varepsilon), u_\varepsilon - K_C\lambda^{-1}Q(u_\varepsilon)) \leqslant 0. \tag{4.5b}$$

Fix an $\tilde{x} \in \mathbb{R}^N$. We have the following alternatives:

Case A: There exists $y \in B(\tilde{x}, 2\varepsilon)$, such that $y \in X(u^\varepsilon)$, i.e. $u^\varepsilon(y) = \psi(y)$. Since u^ε and ψ are uniformly Lipschitz, for some C depending only on $[\psi]_1$ and $[u^\varepsilon]_1$, we obtain (4.5a) at point $x = \tilde{x}$.

Case B: One has $X(u^\varepsilon) \cap B(\tilde{x}, 2\varepsilon) = \emptyset$. We claim that $u^\varepsilon(\cdot - e)$ is, for all $e \in B(0, \varepsilon)$, a viscosity sub-solution of

$$\sup_\alpha L^\alpha(x, \mathcal{D}w(x)) = 0, \quad x \in B(\tilde{x}, \varepsilon). \tag{4.6}$$

Fix $e_1 \in B(\tilde{x}, \varepsilon)$, and let $\varphi \in C^2(\mathbb{R}^N)$ be such that $u^\varepsilon(\cdot - e_1) - \varphi(\cdot)$ has a local maximum at a point $x_{e_1} \in B(\tilde{x}, \varepsilon)$. Then $u^\varepsilon(\cdot) - \varphi(\cdot + e_1)$ has a local maximum at $x_{e_1} - e_1$. Since $x_{e_1} - e_1 \in B(\tilde{x}, 2\varepsilon)$, and hence, $x_{e_1} - e_1 \notin X(u^\varepsilon)$, we have, whenever $|e| \leqslant \varepsilon$,

$$\sup_{\alpha, |e| \leqslant \varepsilon} L^\alpha \left(x_{e_1} - e_1 + e, u^\varepsilon\left(x_{e_1} - e_1\right), D\varphi\left(x_{e_1}\right), D^2\varphi\left(x_{e_1}\right)\right) \leqslant 0.$$

Taking $e = e_1$, we have

$$\sup_\alpha L^\alpha \left(x_{e_1}, u^\varepsilon\left(x_{e_1} - e_1\right), D\varphi\left(x_{e_1}\right), D^2\varphi\left(x_{e_1}\right)\right) \leqslant 0.$$

This proves our claim that $u^\varepsilon(\cdot - e_1)$ is a viscosity sub-solution of (4.6). Since e_1 is an arbitrary point of $B(0, \varepsilon)$, $u^\varepsilon(\cdot - e)$ is a viscosity sub-solution of (4.6), for all $|e| \leqslant \varepsilon$. Since $u_\varepsilon(\cdot)$ is a C^∞ function, and it is the limit of convex combinations of $u^\varepsilon(\cdot - e)$ (see Barles & Jakobsen, 2002, Lemma A.3 and Lemma 2.7), hence, applying Barles & Jakobsen (2002, Lemma 2.7), we can say that $u_\varepsilon(\cdot)$ is a sub-solution of (4.6) in the classical sense. This implies

$$\sup_\alpha L^\alpha(\tilde{x}, \mathcal{D}u_\varepsilon(\tilde{x})) \leqslant 0. \tag{4.7}$$

By consistency, $S(h, \tilde{x}, u_\varepsilon(\tilde{x}), u_\varepsilon) \leqslant K_C Q(u_\varepsilon)$. Applying (S1) with $u = v = u^\varepsilon(\tilde{x}) - K_C\lambda^{-1}Q(u^\varepsilon)$ and $m = K_C\lambda^{-1}Q(u^\varepsilon)$, we obtain (4.5b) at point $x = \tilde{x}$. Combining cases A and B, we obtain (4.4). So $u_\varepsilon - R_1$ is a sub-solution of (1.2). By Theorem 2.4, $u_\varepsilon - R_1 \leqslant u_h$, i.e. $u - u_h \leqslant K_C\lambda^{-1}Q(u_\varepsilon) + C\varepsilon$, where C depends on $[u^\varepsilon]_1$ and $[\psi]_1$. Using $Q(u_\varepsilon) = \sum_{i \in J} |D^i u_\varepsilon| h^{k_i}$ and (2.4), we obtain $Q(u_\varepsilon) \leqslant C \sum_{i \in J} \varepsilon^{1-i} h^{k_i}$. The result follows by setting $\varepsilon = \max_{i \in J} h^{k_i/i}$. □

4.2 *Lower bound on $u - u_h$*

We perturb the switching system (3.1) as follows

$$\min\left\{\max\left(\inf_{|e| \leqslant \varepsilon} L^{\alpha_i}(x + e, \mathcal{D}v_i^\varepsilon(x)); v_i^\varepsilon(x) - \min_{j \neq i}\{v_j^\varepsilon(x) + k\}\right); v_i^\varepsilon(x) - \psi(x)\right\} = 0, \tag{4.8}$$

for all $i \in \mathcal{I}$ and $x \in \mathbb{R}^N$. By Lemma A.2, this system has a unique viscosity solution $v^\varepsilon = (v_1^\varepsilon, \ldots, v_M^\varepsilon)$ in $C_{b,l}(\mathbb{R}^N)^M$. Consider v_ε, the mollification of v^ε, defined as in (2.2). Since v^ε is a Lipschitz function, uniformly with respect to $\varepsilon > 0$ sufficiently small, applying Theorem A.3 and equation (2.3) we have

$$|v_i - v_i^\varepsilon| \leqslant C\varepsilon, \quad |v_i^\varepsilon - v_{i\varepsilon}| \leqslant \max_i [v_i^\varepsilon]_1 \varepsilon, \tag{4.9}$$

where C depends on λ, K and $[\psi]_1$, and $\max_i [v_i^\varepsilon]_1$ remains bounded.

LEMMA 4.2 Let $x_0 \in \mathbb{R}^N$, $i_0 \in \arg\min_{j\in\mathcal{I}} v_{j\varepsilon}(x_0)$, and assume that

$$\varepsilon \leqslant \left(12 \sup_i [v_i^\varepsilon]_1\right)^{-1} k. \tag{4.10}$$

Then the following statements hold

$$v_{i_0}^\varepsilon(y) < v_j^\varepsilon(y) + k \quad \text{for all } j \in \mathcal{I}, \text{ and } y \in B(x_0, 2\varepsilon), \tag{4.11}$$

$$\sup_\alpha L^\alpha \left(x_0, \mathcal{D}v_{i_0\varepsilon}(x_0)\right) \geqslant 0. \tag{4.12}$$

Proof. We follow the method of Barles & Jakobsen (2005, Lemma 3.4).

(a) Let us prove (4.11). Since $i_0 \in \arg\min_{j\in\mathcal{I}} v_{j\varepsilon}(x_0)$,

$$v_{i_0\varepsilon}(x_0) - \min_{j\neq i_0}\{v_{j\varepsilon}(x_0) + k\} = \max_{j\neq i_0}\{v_{i_0\varepsilon}(x_0) - v_{j\varepsilon}(x_0) - k\} \leqslant -k. \tag{4.13}$$

Since for every i, v_i^ε is Lipschitz, we apply (2.3) and we have that

$$v_{i_0}^\varepsilon(x_0) - \min_{j\neq i_0}\{v_j^\varepsilon(x_0) + k\} \leqslant -k + 2\,\varepsilon \max_i [v_i^\varepsilon]_1,$$

and, for all $y \in B(x_0, 2\varepsilon)$,

$$v_{i_0}^\varepsilon(y) - \min_{j\neq i_0}\{v_j^\varepsilon(y) + k\} \leqslant -k + 4 \max_i [v_i^\varepsilon]_1(\varepsilon + |x_0 - y|) < -k + 12\,\varepsilon \max_i [v_i^\varepsilon]_1.$$

Taking $\varepsilon \leqslant (12 \max_i [v_i^\varepsilon]_1)^{-1} k$, we obtain (4.11).

(b) We prove (4.12). We claim that $v_{i_0}^\varepsilon(\cdot - e)$ is, for all $|e| \leqslant \varepsilon$, a viscosity super-solution of

$$L^{\alpha_{i_0}}(x, Dw(x)) = 0, \quad x \in B(x_0, \varepsilon). \tag{4.14}$$

Fix $e_1 \in B(0, \varepsilon)$, and let $\varphi \in C^2(\mathbb{R}^N)$ be such that $v_{i_0}^\varepsilon(\cdot - e_1) - \varphi(\cdot)$ has a local minimum at $x_{e_1} \in B(x_0, \varepsilon)$. Then $v_{i_0}^\varepsilon(\cdot) - \varphi(\cdot + e_1)$ has a local minimum at $x_{e_1} - e_1 \in B(x_0, 2\varepsilon)$. Since $v^\varepsilon(\cdot)$ is a viscosity solution of (4.8), and $v_{i_0}^\varepsilon(x_0) - \min_{j\neq i_0}\{v_j^\varepsilon(x_0) + k\} \leqslant 0$ by (4.11), we have that

$$\inf_{|e|\leqslant\varepsilon} L^{\alpha_{i_0}} \left(x_{e_1} - e_1 + e, v_{i_0}^\varepsilon\left(x_{e_1} - e_1\right), D\varphi\left(x_{e_1}\right), D^2\varphi\left(x_{e_1}\right)\right) \geqslant 0 \quad \forall\, |e| \leqslant \varepsilon.$$

In particular, for $e = e_1$, we obtain that $v_{i_0}^\varepsilon(\cdot - e_1)$ is a viscosity super-solution of (4.14). Since e_1 is an arbitrary point in $B(0, \varepsilon)$, we obtain that $v_{i_0}^\varepsilon(\cdot - e)$ is, for all $e \in B(0, \varepsilon)$, a viscosity super-solution of (4.14). Being a limit of convex combinations of $v_{i_0}^\varepsilon(\cdot - e)$, and a smooth function, $v_{i_0\varepsilon}(\cdot)$ is a classical super-solution on (4.14), and hence $L^{\alpha_{i_0}}\left(x_0, \mathcal{D}v_{i_0\varepsilon}(x_0)\right) \geqslant 0$; relation (4.12) follows. □

Define the following two sets:

$$X := \{x \in \mathbb{R}^N | u_h(x) = \psi(x)\}; \quad Y := \{x \in \mathbb{R}^N | S(h, x, u_h, [u_h]_x) = 0\}.$$

PROPOSITION 4.3 Under assumptions (A1), (A2) and (S1)–(S3), and assuming that (1.2) has a unique solution u_h in $C_b(\mathbb{R}^N)$, we have that, if $x \in Y$, the following holds:

$$u_h(x) - u(x) \leqslant Ch^{\ell}, \tag{4.15}$$

where $\ell := \min_{i \in J}\{k_i/(3i-2)\}$ and C depends only on λ, K and K_c.

Proof. Consider the switching system (4.8), its solution $v^\varepsilon = (v_1^\varepsilon, \dots, v_M^\varepsilon)$ and mollification $v_\varepsilon = (v_{1\varepsilon}, \dots, v_{M\varepsilon})$. Let $w(y) := \min_i v_{i\varepsilon}(y)$. Define

$$m := \sup_{y \in Y}\{u_h(y) - w(y)\} = \sup_{i \in \mathcal{I}, y \in Y}\{u_h(y) - v_{i\varepsilon}(y)\}. \tag{4.16}$$

Let $\phi(y) := (1 + |y|^2)^{1/2}$. An approximation of m is, for $k > 0$, given by

$$m_k := \sup_{y \in Y}\{u_h(y) - w(y) - k\phi(y)\}. \tag{4.17}$$

Since u_h and w are bounded, ϕ is coercive and Y is a closed set, the supremum in (4.17) is attained at some point $x_0 \in Y$. By the definition of w, we also have

$$x_0 \in \arg\max_{y \in Y}\left\{u_h(y) - v_{i_0\varepsilon}(y) - k\phi(y)\right\}, \tag{4.18}$$

when $i_0 \in \arg\min_{j \in \mathcal{I}} v_{j\varepsilon}(x_0)$. In particular,

$$m_k \geqslant u_h(y) - v_{i_0\varepsilon}(y) - k\phi(y), \qquad \text{for all } y \in Y. \tag{4.19}$$

Let ε be such that (4.10) holds. Applying Lemma 4.2, and since the first- and the second-order derivatives of ϕ are bounded, we have $\sup_\alpha L^\alpha(x_0, \mathcal{D}\left(v_{i_0\varepsilon} + k\phi\right)(x_0)) \geqslant -Ck$. Combining this with (S1), (S3), (4.17) and $x_0 \in Y$, we get

$$\begin{aligned} -Ck - K_C Q\left(v_{i_0\varepsilon} + k\phi\right) &\leqslant S\left(h, x_0, \left(v_{i_0\varepsilon} + k\phi\right)(x_0), v_{i_0\varepsilon} + k\phi\right) \\ &\leqslant S(h, x_0, u_h(x_0) - m_k, u_h - m_k) \\ &\leqslant -\lambda m_k + S(h, x_0, u_h(x_0), u_h) = -\lambda m_k. \end{aligned}$$

We obtain $\lambda m_k \leqslant K_C Q\left(v_{i_0\varepsilon} + k\phi\right) + Ck$. Passing to the limit in k, we get

$$m \leqslant K_C Q\left(v_{i_0\varepsilon}\right). \tag{4.20}$$

In conclusion, we can say that for $x \in Y$ and for every $i \in \mathcal{I}$,

$$\begin{aligned} \sup_{y \in Y}\{u_h(y) - u(y)\} &\leqslant m + \sup_{y \in Y}\{w(y) - u(y)\} \\ &\leqslant m + \sup_{y \in Y}\{w(y) - v_{i\varepsilon}(y)\} + \sup_{y \in Y}\{v_{i\varepsilon}(y) - v_i^\varepsilon(y)\} \\ &\quad + \sup_{y \in Y}\{v_i^\varepsilon(y) - v_i(y)\} + \sup_{y \in Y}\{v_i(y) - u(y)\}. \end{aligned} \tag{4.21}$$

Applying (4.9) and (3.16) and the fact that $w(y) \leqslant v_{i\varepsilon}(y)$ for all $i \in \mathcal{I}$, we obtain

$$\sup_{y \in Y}\{u_h(y) - u(y)\} \leqslant m + C\varepsilon + Ck^{1/3}, \tag{4.22}$$

where C depends on $K, \lambda, [\psi]_1$ and $\max_i [v_i^\varepsilon]_1$. Using (4.20), we obtain

$$u - u_h \leqslant K_C Q\left(v_{i_0\varepsilon}\right) + C\varepsilon + Ck^{1/3} \quad \forall x \in Y.$$

The result follows by setting $\varepsilon = \max_{i \in J} h^{\frac{3k_i}{3i-2}}$ and $k = (12 \sup_i [v_i^\varepsilon]_1)\varepsilon$. □

THEOREM 4.4 Under assumptions (A1), (A2) and (S1)–(S3) and assuming that (1.2) has a unique solution u_h in $C_{b,l}(\mathbb{R}^N)$, we have that

$$u_h - u \leqslant Ch^\ell \quad \forall x \in \mathbb{R}^N, \tag{4.23}$$

where $\ell = \min_{i \in J}\{k_i/(3i-2)\}$ and C depends only on λ, K and K_c.

Proof. If $x \in X$ we have that $u_h(x) = \psi(x) \leqslant u(x)$, therefore (4.23) holds. If $x \in Y$, then by Theorem 2.4, we have that $u_h(x) - u(x) \leqslant Ch^\ell$. Since $X \cup Y = \mathbb{R}^N$, the conclusion follows. □

4.3 *Extension to the case of a compact control set*

In this section, we show that our results extend to the case of a precompact set of controls. We endow the set of controls with the distance $d(\alpha, \alpha') := |\Phi^\alpha - \Phi^{\alpha'}|_0$, where $\Phi^\alpha := (a^\alpha, b^\alpha, c^\alpha, f^\alpha)$. We suppose that $\sup_{\alpha \in \mathcal{A}} |\Phi^\alpha|_1 < +\infty$. Precompactness of $\mathcal{A}$ is equivalent to the following condition:

(A3) For every $\delta > 0$, there are $M \in \mathbb{N}$ and $\{\alpha_i\}_{i=1}^M \subset \mathcal{A}$, such that

$$\sup_{\alpha \in \mathcal{A}} \inf_{1 \leqslant i \leqslant M} (|\sigma^\alpha - \sigma^{\alpha_i}|_0 + |b^\alpha - b^{\alpha_i}|_0 + |c^\alpha - c^{\alpha_i}|_0 + |f^\alpha - f^{\alpha_i}|_0) \leqslant \delta.$$

Consider the viscosity solution u of

$$\min\left\{\sup_{\alpha \in \mathcal{A}} L^\alpha(x, \mathcal{D}u(x)); u(x) - \psi(x)\right\} = 0, \quad x \in \mathbb{R}^N.$$

Existence, uniqueness and the Lipschitz continuity of u are proved in Jakobsen (2004b, Lemma A.1). Fix δ and consider w_δ, the viscosity solution of

$$\min\left\{\sup_{i \in \mathcal{I}_M} L^{\alpha_i}(x, \mathcal{D}w_\delta(x)); w_\delta(x) - \psi(x)\right\} = 0, \quad x \in \mathbb{R}^N,$$

where $\mathcal{I}_M := \{1, \ldots, M\}$, and M is given by (A3). As in Barles & Jakobsen (2005, Lemma 3.3), we can show, by adapting their methods, that

$$|u - w_\delta|_0 \leqslant C\delta, \tag{4.24}$$

where C depends only on K and λ. If we denote by u_h the approximation of u and by $w_{h,\delta}$ the approximation of w_δ, then we have $u_h \leqslant w_{h,\delta}$, and $w_{h,\delta} - w_\delta \leqslant Ch^{\bar{\gamma}}$, where $\bar{\gamma} = \min_{i \in J}\{k_i/(3i-2)\}$, with k_i

given by (S3). From the proof of Proposition 4.3, we can see that C is independent of the size of $\mathcal{I}_M$. Then we have that

$$-Ch^{\gamma} \leqslant u_h - u \leqslant u_h - w_{h,\delta} + w_{h,\delta} - w + w - u \leqslant Ch^{\bar{\gamma}} + C_1\delta, \tag{4.25}$$

where $\gamma = \min_{i\in J}\{k_i/i\}$, $\bar{\gamma} = \min_{i\in J}\{k_i/(3i-2)\}$, k_i is given by (S3). All constants being independent of the size of $\mathcal{I}_M$, we can choose δ of the order of $h^{\bar{\gamma}}$ and we obtain the same result as in Theorem 4.4.

REMARK 4.5 It may happen that only $w_{h,\delta}$ is actually computed, and in that case it is useful to estimate $u - w_{h,\delta}$. Since $|u - w_\delta| \leqslant C\delta$, it follows from previous discussion that

$$-C(\delta + h^{\gamma}) \leqslant w_{h,\delta} - u \leqslant C(\delta + h^{\bar{\gamma}}).$$

5. Specific approximation schemes

In this section, we apply our previous results to some specific monotone discretization schemes.

5.1 *Finite differences, 1D problem*

Let x be in $\mathbb{R}$, ϕ in $C^n(\mathbb{R})$, h in $\mathbb{R}^+$ and define

$$\delta_\pm\phi(x) = \frac{\phi(x\pm h) - \phi(x)}{h}, \quad \Delta\phi(x) = \frac{\phi(x+h) - 2\phi(x) + \phi(x-h)}{h^2}.$$

In particular, by a Taylor expansion, we obtain

$$|\delta_\pm\phi(x) - D\phi(x)| \leqslant \frac{1}{2}h|D^2\phi|, \quad |\Delta_h\phi(x) - D^2\phi(x)| \leqslant \frac{1}{12}h^2|D^4\phi|.$$

Consider the finite-difference scheme in $\mathbb{R}$:

$$S(h,x,r,\phi) := \sup_{\alpha}\{-a^{\alpha}(x)\Delta\phi(x) - b_+^{\alpha}(x)\delta_+\phi(x) + b_-^{\alpha}(x)\delta_-\phi(x) + c^{\alpha}(x)r - f^{\alpha}(x)\}, \tag{5.1}$$

where $b_+^{\alpha}(x) = \max(b^{\alpha}(x), 0)$, and $b_-^{\alpha}(x) = \max(-b^{\alpha}(x), 0)$. For the consistency hypothesis (S3), we obtain, from the above Taylor expansion, $Q(\phi) = |D^2\phi|h + |D^4\phi|h^2$, i.e. $k_2 = 1$ and $k_4 = 2$. Then, by (4.3) and (4.23), we have

$$-Ch^{1/5} \leqslant u - u_h \leqslant Ch^{1/2}. \tag{5.2}$$

REMARK 5.1 Consider a general scheme S: $\mathbb{R}^+ \times \mathbb{R}^N \times \mathbb{R} \times C_b(\mathbb{R}^N) \to \mathbb{R}$, which satisfies (S1), (S2) and (S3), for some $k_i > 0$, $i \in J$. To obtain an equally good or a better estimate than (5.2), we must have

$$\min_{i\in J}\frac{k_i}{i} \geqslant \frac{1}{2}, \quad \min_{i\in J}\frac{k_i}{3i-2} \geqslant \frac{1}{5}. \tag{5.3}$$

In particular, the k_i which give an equally good or a better estimate than (5.2) are

$$\text{(i) } k_i \geqslant i/2, \quad \text{for } i \leqslant 4; \quad \text{(ii) } k_i \geqslant (3i-2)/5, \quad \text{for } i \geqslant 4. \tag{5.4}$$

Indeed, let $i \leqslant 4$. If $k_i \geqslant i/2$, then we also have $k_i \geqslant (3i-2)/5$. Moreover, if $i > 4$, we have $k_i \geqslant (3i-2)/5$ and also $k_i \geqslant i/2$. Hence, we obtain (5.3).
If the inequalities in (5.4) are strictly satisfied, then also the inequalities in (5.3) are strictly satisfied and we obtain a better estimate.

5.2 *Markov chain approximation*

The scheme (5.1) may be viewed as a particular Markov chain approximation of (1.1). We consider now a general Markov chain approximation of (1.1) on a regular grid, and we want to find conditions on the transition probabilities to obtain an estimate as in (5.2). We consider a discretization step $h \in \mathbb{R}$ and a regular grid of discretization G^h. With the coordinate $k = (k_1, \ldots, k_N)$ in $\mathbb{Z}^N$ we associate the point $x_k \in \mathbb{R}^N$ of the form $x_k := (k_1 h, \ldots, k_N h)$. Let X_q^h be the state of the Markov chain at time $q \geqslant 0$, with transition probabilities $p(x_k, y|\alpha)$, α being the control value. Let Δt^h be an interpolation interval satisfying $\Delta t^h \to 0$ as $h \to 0$, and let $\mathbb{E}_{k,q}^{h,\alpha}$ be the conditional expectation of X_{q+1}^h, given $\{X_q^h = x_k\}$ and the control value α. A possible adaptation for the cost function to this Markov chain is the following:

$$W^h(x, \alpha) = \Delta t^h \left[\sum_{q \geqslant 0} f^\alpha(X_q^h)(1 + c^\alpha(x)\Delta t^h)^{-q-1} \right].$$

Applying the dynamic programmic principle for the controlled chain $\{X_q^h,\ q \geqslant 0\}$, at state $x_k \in G_h$, we obtain the following relation:

$$u_h(x_k) = \max\left\{ \inf_\alpha \left(\frac{1}{1 + c^\alpha(x_k)\Delta t^h} \left(\sum_y p(x_k, y|\alpha) u_h(y) + f^\alpha(x_k)\Delta t^h \right) \right); \psi(x_k) \right\}. \tag{5.5}$$

Since $1 + c^\alpha(x_k)\Delta t^h \geqslant 0$ for all α, (5.5) may be written in the form (1.2), with

$$S(h, x_k, r, \phi) = \sup_\alpha \left\{ -\frac{1}{\Delta t^h} \sum_y p(x_k, y|\alpha)\phi(y) - f^\alpha(x_k) + \frac{1}{\Delta t^h} r + c^\alpha(x_k) r \right\}. \tag{5.6}$$

With the above definition for S, the assumptions (S1) and (S2) are satisfied. Suppose that (S3) is satisfied and we want to look for simple conditions on the probabilities $p(x_k, y)$ and on k_i defined in (S3), under which we obtain an equally good or a better estimate than (5.2). We note $\mathbb{P}_{x,y} = \sum_y p(x, y|\alpha)$. Using remark 5.1, we obtain the following:

THEOREM 5.2 Let S be defined as in (5.6). Suppose that (S3) is satisfied for some k_i, $i \in J$.

(i) We have an equally good or a better estimate than (5.2) if, and only if,

(a1) $\left\| \frac{1}{\Delta t^h} \mathbb{P}_{x,y}(x - y) - b^\alpha(x) \right\| = K_C h^{k_1}$,

(b1) $\left\| \frac{1}{2\Delta t^h} \mathbb{P}_{x,y}(y - x)^2 - a^\alpha(x) \right\| = K_C h^{k_2}$,

(c1) $\left\| \frac{1}{i!\Delta t^h} \mathbb{P}_{x,y}(y - x)^i \right\| = K_C h^{k_i}$, for $i = 3, 4$,

with

$$k_1 \geqslant \frac{1}{2},\ k_2 \geqslant 1,\ k_3 \geqslant \frac{3}{2} \text{ and } k_4 \geqslant 2. \tag{5.7}$$

(ii) Moreover, we have a better lower bound if, and only if, in addition, k_4 satisfies (5.7) with strict inequality.

(iii) We have a better upper bound if, and only if, all the inequalities in (5.7) are satisfied with strict inequalities.

Proof. We give the proof for $N = 1$; the general case follows immediately. Fix $x \in \mathbb{R}$, and let $\phi \in C^n(\mathbb{R})$, such that $D^i\phi$ is bounded for $i = 1, \ldots, n$. Set $\Delta^\phi := |\sup_\alpha L^\alpha(x, \mathcal{D}\phi(x)) - S(h, x, \phi(x), \phi)|$. An upper bound of Δ^ϕ is

$$\left|\sup_\alpha\left(-\operatorname{tr}[a^\alpha(x)D^2\phi(x)] - b^\alpha(x)D\phi(x) + \frac{1}{\Delta t^h}\sum_y p(x, y|\alpha)(\phi(y) - \phi(x))\right)\right|.$$

From the Taylor expansion of $\phi(y)$ up to order 4, we deduce that

$$\Delta^\phi \leqslant \Delta_1^\phi + \Delta_2^\phi + \Delta_3^\phi + \Delta_4^\phi, \tag{5.8}$$

where

$$\Delta_1^\phi := \sup_\alpha \left|-b^\alpha(x)D\phi(x) + \frac{1}{\Delta t^h}\mathbb{P}_{x,y}D\phi(x)(y - x)\right|,$$

$$\Delta_2^\phi := \sup_\alpha \left|-\operatorname{tr}[a^\alpha(x)D^2\phi(x)] + \frac{1}{2\Delta t^h}\mathbb{P}_{x,y}D^2\phi(x)(y - x)^2\right|,$$

$$\Delta_3^\phi := \sup_\alpha \left|\frac{1}{3!\Delta t^h}\mathbb{P}_{x,y}D^3\phi(x)(y - x)^3\right|,$$

$$\Delta_4^\phi := \sup_\alpha \left|\frac{1}{4!\Delta t^h}\mathbb{P}_{x,y}D^4\phi(c)(y - x)^3\right|,$$

where $c \in [x, y]$ if $y \geqslant x$, $c \in [y, x]$ otherwise. Suppose now that conditions (a1)–(c1) and (5.7) are satisfied. Then $J = \{1, 2, 3, 4\}$, and applying Remark 5.1, we obtain the result. Moreover, if $k_4 > 2$, then $k_i/(3i - 2) > 1/5$ for all i. Hence, we obtain a strictly better lower bound. Since $k_i/i \geqslant 1/2$ for all i in J, if all k_i satisfy (5.7) with strict inequalities, we have a better upper bound.
Suppose now that we have an equally good or better estimate than (5.2). Then we have

$$\Delta^\phi \leqslant K_C \sum_{i\in J} |D^i\phi| h^{k_i}, \tag{5.9}$$

with $\min_{i\in J} k_i/i \geqslant 1/2$ and $\min_{i\in J} k_i/(3i - 2) \geqslant 1/5$ where $J = \{1, 2, 3, 4\}$. From (5.9), (5.8) and Remark 5.3, we have that (a1)–(d1) are satisfied with k_i as in (5.7). If the lower bound is strictly greater than $1/5$, since $k_i/(3i - 2) > 1/5$ for $i = 1, 2, 3$, then we must have $k_4 > 2$. If the upper bound is strictly greater than $1/2$, since $k_i \geqslant i/2$ for all i, then we must have $k_i > i/2$ for all i. □

REMARK 5.3

(i) We have that conditions (a1) and (b1) imply the consistency in the sense of Kushner (see Kushner & Dupuis, 2001), i.e.

$$\|\mathbb{E}(y - x) - b^\alpha(x)\Delta t^h\| \leqslant \Delta t^h r_1, \quad \|\operatorname{Cov}(y) - 2a^\alpha(x)\| \leqslant \Delta t^h r_2.$$

In Kushner & Dupuis (2001) we have $r_i = o(1)$, for $i = 1, 2$. Our error estimate requires the more restrictive conditions $r_1 = h^{k_1}$ and $r_2 = h^{k_2} + \Delta t^h h^{2k_1} + \Delta t^h$.

(ii) We remark that to obtain (a1)–(d1), we use the inequality

$$\Delta_i^\phi \leqslant |D^i\phi| \|\mathbb{E}(y - x)^i - a^i\|, \tag{5.10}$$

for some a^i and for all ϕ. This inequality is sharp, since $\|\mathbb{E}(y-x)^i - a^i\|$ is the optimal constant for which we have this upper bound (for any function ϕ). Indeed, let B be an i-linear symmetric form. The optimal constant C for which

$$|D^i\phi(x)B| \leqslant C|D^i\phi| \;\forall\, \phi \in C^n(\mathbb{R}^N) \text{ such that } D^i\phi \text{ is bounded } \forall\, i,$$

is $C = |B|$. Indeed, we may identify a^i and $\mathbb{E}(y-x)^i$ with i-linear symmetric forms, and the displayed inequality reduces to the Cauchy–Schwarz inequality for i-linear symmetric forms.

(iii) Let $\Delta^\phi := |\sum_{i=1}^4 D^i\phi(x)\mathbb{E}(y-x)^i - a^i D^i\phi(x)|$, for some a^i. We have that the optimal constants C_i such that

$$\Delta \leqslant \sum_{i=1}^4 C_i|D^i\phi(\mathbb{E}(y-x)^i - a^i)| \;\forall\, \phi \in C^n(\mathbb{R}^N) \text{ such that } D^i\phi \text{ is bounded } \forall\, i,$$

are $C_i = 1$, for all i.

5.3 *A counter-example*

We give here an example of a finite-difference scheme for which the k_i do not satisfy the conditions given in Remark 5.1, and we will show that we obtain an estimate worse than (5.2). Consider the following equation

$$\sup_\alpha\{-\operatorname{tr}[a^\alpha(x)D^2u(x)] + c^\alpha(x)u(x) - f^\alpha(x)\} = 0, \quad x \in \mathbb{R}^2, \tag{5.11}$$

with

$$b^\alpha(x) = 0, \quad a^\alpha(x) = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \quad \forall\, x, \alpha.$$

Let h be the discretization step and Δt^h be the interpolation interval. We consider the following transition probabilities:

$$p(x, x - he_2|\alpha) = \frac{1}{2}; \quad p(x, x \pm he_1 + he_2|\alpha) = \frac{1}{4}.$$

In particular, if we choose $\Delta t^h = \frac{1}{4}h^2$, we have that these probabilities satisfy

$$\mathbb{E}(y-x) = \frac{1}{2}\begin{pmatrix} 0 \\ -h \end{pmatrix} + \frac{1}{4}\begin{pmatrix} h \\ h \end{pmatrix} + \frac{1}{4}\begin{pmatrix} -h \\ h \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

and

$$\mathbb{E}(y-x)^2 = \frac{1}{2}\begin{pmatrix} 0 & 0 \\ 0 & h^2 \end{pmatrix} + \frac{1}{4}\begin{pmatrix} h^2 & h^2 \\ h^2 & h^2 \end{pmatrix} + \frac{1}{4}\begin{pmatrix} h^2 & -h^2 \\ -h^2 & h^2 \end{pmatrix}$$

$$= \Delta t^h \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}.$$

We have that

$$
\begin{aligned}
&\sum_y \frac{p(x,y)}{h^2} D^3\phi(x)(y-x)^3 \\
&\quad = \sum_y \frac{p(x,y)}{h^2}\left[\sum_{i=0}^{3} \frac{\partial^3\phi}{\partial x_1^i \partial x_2^{3-i}}(x)(y_1-x_1)^i(y_2-x_2)^{3-i}\right] \\
&\quad = -\frac{1}{2}\frac{\partial^3\phi}{\partial x_2^3}(x)h + \frac{1}{4}\left(\frac{\partial^3\phi}{\partial x_1^3}(x)h + \frac{\partial^3\phi}{\partial x_1^2 x_2}(x)h + \frac{\partial^3\phi}{\partial x_1 x_2^2}(x)h + \frac{\partial^3\phi}{\partial x_2^3}(x)h\right) \\
&\qquad + \frac{1}{4}\left(-\frac{\partial^3\phi}{\partial x_1^3}(x)h + \frac{\partial^3\phi}{\partial x_1^2 x_2}(x)h - \frac{\partial^3\phi}{\partial x_1 x_2^2}(x)h + \frac{\partial^3\phi}{\partial x_2^3}(x)h\right) \\
&\quad = \frac{1}{2}\frac{\partial^3\phi}{\partial x_1^2 x_2}(x)h.
\end{aligned}
$$

Hence, we can write (S3) in the following way:

$$
\begin{aligned}
&\left|\sup_\alpha L^\alpha(x,\phi(x),D\phi(x),D^2\phi(x)) - S(x,h,\phi(x),\phi)\right| \\
&\quad \leqslant \frac{1}{2}\left|\frac{\partial^3\phi}{\partial x_1^2\partial x_2}(x)\right| h + \left(\frac{1}{2}\left|\frac{\partial^4\phi}{\partial x_1^4}(x)\right| + \frac{1}{2}\left|\frac{\partial^4\phi}{\partial x_1^2\partial x_2^2}\right|(x) + \left|\frac{\partial^4\phi}{\partial x_2^4}\right|(x)\right)h^2.
\end{aligned}
$$

So, we have $k_3 = 1$ and $k_4 = 2$, and by applying equation (2.3) and Theorems 4.1 and 4.4 we obtain

$$
-Ch^{1/7} \leqslant u - u_h \leqslant Ch^{1/3}. \tag{5.12}
$$

5.4 *The generalized finite-difference scheme*

We consider the generalized finite-difference scheme defined in Bonnans & Zidani (2003). Let $\varphi = \{\varphi_k\}$ be a real-valued function over $\mathbb{Z}^N$. Let $\xi \in \mathbb{Z}^N$ and consider the finite-difference operator

$$
\Delta_\xi \varphi_k := \varphi_{k+\xi} + \varphi_{k-\xi} - 2\varphi_k.
$$

If ϕ belongs to $C^2(\mathbb{R}^N)$, and $\varphi_k := \phi(x_k)$ for all k, then we have

$$
\Delta_\xi \varphi_k := \phi(x_{k+\xi}) + \phi(x_{k-\xi}) - 2\phi(x_k).
$$

We consider

$$
(D_k u_h(x_k))_i = \begin{cases} \dfrac{u_h(x_{k+e_i}) - u_h(x_k)}{h}, & \text{if } b_i^\alpha(x_k) \geqslant 0, \\ \dfrac{u_h(x_k) - u_h(x_{k-e_i})}{h}, & \text{if } b_i^\alpha(x_k) \leqslant 0. \end{cases}
$$

Let $\mathcal{S}$ be a finite set of $\mathbb{Z}^N \setminus \{0\}$ containing $\{e_1, \ldots, e_N\}$. We consider the following transition probabilities:

$$p^\alpha(x_k, x_k|\alpha) = 1 - \Delta t^h \sum_{i=1}^{N} \left(\frac{|b_i^\alpha(x_k)|}{h} + 2\sum_{\xi\in\mathcal{S}} \alpha_{k,\xi} \right),$$

$$p^\alpha(x_k, x_k \pm e_i h|\alpha) = \Delta t^h \left(\frac{b_i^{\alpha\pm}(x_k)}{h} + \alpha_{k,e_i} \right),$$

$$p^\alpha(x_k, x_k + \xi h|\alpha) = \Delta t^h \alpha_{k,\xi}, \quad \text{for } \xi \in \mathcal{S},\ \ \xi \neq e_i,$$

$$p^\alpha(x_k, y|\alpha) = 0, \quad \text{for } y \notin x_{k+\mathcal{S}}.$$

Then we can write (5.6) in the following way:

$$S(h, x_k, r, \phi) = \sup_\alpha \left\{ -\sum_{\xi\in\mathcal{S}} \alpha_{k,\xi} \Delta_\xi \phi(x_k) - b^\alpha(x_k) D_k \phi(x_k) + c^\alpha(x_k) r - f^\alpha(x_k) \right\}. \tag{5.13}$$

We make the strong consistency hypothesis on the matrix

$$a^\alpha(x) = \sum_{i,j} h^2 \xi_i \xi_j \alpha_{k,\xi} e_i e_j^\top \quad \forall\, k \in \mathbb{Z}^N.$$

The scheme defined in (5.13) satisfies (S1) and (S2). We consider a function $\phi \in C^2(\mathbb{R}^N)$. By applying a Taylor expansion, we obtain

$$\text{(S3)}\quad \left| \sup_{\alpha\in\mathcal{A}} L^\alpha(x, \phi, D\phi, D^2\phi) - S(x, h, \phi(x), \phi) \right| \leqslant \sup_{\alpha\in\mathcal{A}} |b^\alpha|_0 |D^2\phi|_0 h + \sup_{\alpha\in\mathcal{A}} |\sigma^\alpha|_0^2 |D^4\phi|_0 h^2.$$

So we can say that $k_2 = 1$ and $k_4 = 2$. Applying equation (2.3) and Theorems 4.1 and 4.4, we obtain the same estimate as in the case of one player (see Barles & Jakobsen, 2005).

THEOREM 5.4 Assume (A1)–(A4) and (S1)–(S3). If u and u_h are solutions of (1.1) and (1.2), with S defined as in (5.13), then for h sufficiently small we obtain

$$-Ch^{1/5} \leqslant u - u_h \leqslant Ch^{1/2}.$$

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Appendix

Well-posedness of the switching system

In this Appendix, we prove the well-posedness of the switching system (3.1), for $k \geqslant 0$, under assumptions (A1) and (A2) on the coefficients (stated in Section 2). Well-posedness of the original equation (1.1) is given in Jakobsen (2004b, Lemma A.1). Let us start by stating a technical lemma which is an easy extension of Barles & Jakobsen (2005, Lemma A.2).

LEMMA A.1 Let v be a bounded and continuous sub-solution of (3.1) and $\bar{v}$ be a bounded and continuous super-solution of another equation (3.1), where L^α is replaced by $\bar{L}^\alpha$, satisfying the same assumptions with coefficients $(\bar{\sigma}^\alpha, \bar{b}^\alpha, \bar{c}^\alpha, \bar{f}^\alpha)$. Let $g \in C^2(\mathbb{R}^N \times \mathbb{R}^N)$. Consider

$$m := \sup_{i,x,y} \{v_i(x) - \bar{v}_i(y) - g(x,y)\},$$

and suppose that the 'sup' is attained at some point (i_0, x_0, y_0). Set

$$A := \{i \in \mathcal{I} | (i, x_0, y_0) \text{ realizes the sup}\}; \quad I(x_0) := \{i \in \mathcal{I} | v_i(x_0) \leqslant \psi(x_0)\}.$$

If $A \cap I(x_0) = \emptyset$, then there exists $i \in A$ such that

$$\bar{v}_i(y_0) < \min_{j \neq i}\{\bar{v}_j(y_0) + k\}. \tag{A.1}$$

Proof. We proceed by contradiction. Let j in A. If (A.1) does not hold, there exists $\ell \in \mathcal{I}$ such that

$$\bar{v}_j(y_0) \geqslant \bar{v}_l(y_0) + k. \tag{A.2}$$

Since $A \cap I(x_0) = \emptyset$, then for all $i \in A$,

$$\max\left\{L^{\alpha_i}(x_0, v_i(x_0), D_x g(x_0, y_0), D^2_{xx} g(x_0, y_0)); v_i(x_0) - \min_{j \neq i}\{v_j(x_0) + k\}\right\} \leqslant 0.$$

Hence, we obtain $v_j(x_0) \leqslant v_l(x_0) + k$, and then with (A.2),

$$v_j(x_0) - \bar{v}_j(y_0) \leqslant v_l(x_0) - \bar{v}_l(y_0). \tag{A.3}$$

Therefore, $l \in A$, and equality holds in (A.3). Then $\bar{v}_j(y_0) = \bar{v}_l(y_0) + k$. Since A is a finite set, this proves that there exists a finite sequence $j_1, \ldots, j_K \in A$ such that $\bar{v}_{j_i}(y_0) = \bar{v}_{j_{i+1}}(y_0) + k$ for $i = 1, \ldots, K-1$, and $j_1 = j_K$. So we obtain

$$0 = \sum_{i=1}^{K-1} (\bar{v}_{j_i}(y_0) - \bar{v}_{j_{i+1}}(y_0)) = (K-1)k > 0,$$

and we have a contradiction. Therefore, (A.1) holds. □

Now we can state the following lemma about comparison, existence, uniqueness and the bounds on the solution $v = (v_1, \ldots, v_M)$ of (3.1). This is an easy extension of Barles & Jakobsen (2005, Theorem A.1).

LEMMA A.2 Under assumptions (A1) and (A2), the following statements hold:

(a) If v and w are, respectively, a sub-solution and a super-solution of (3.1), such that $v_i, w_i \in C_b(\mathbb{R}^N)$ for all $i \in \mathcal{I}$, then $v \leqslant w$ in $\mathbb{R}^N$.
(b) There exists a unique viscosity solution v of (3.1), such that $v_i \in C_{b,l}(\mathbb{R}^N)$ for all $i \in \mathcal{I}$. This solution satisfies

$$\max_i |v_i|_0 \leqslant \max\left\{\lambda^{-1} \sup_\alpha |f^\alpha|_0; |\psi|_0\right\}, \tag{A.4}$$

$$\max_i [v_i]_1 \leqslant \max\left\{\sup_{i,\alpha} \frac{[c^\alpha]_1 |v_i|_0 + [f^\alpha]_1}{\lambda - [\sigma^\alpha]_1^2 - [b^\alpha]_1}; [\psi]_1\right\}. \tag{A.5}$$

Proof.

(a) This is a consequence of the comparison principle (Ishii & Koike, 1991b, Theorem 3.1). Indeed, in Ishii & Koike (1991b), the comparison principle is proved for the Dirichlet problem on a bounded domain. To extend the result to an unbounded domain, we only have to modify the test functions of Ishii & Koike (1991b) in the standard way.

(b) Existence and uniqueness follow from the comparison principle (a). Let $M := \max\{\sup_\alpha \lambda^{-1} |f^\alpha|_0; |\psi|_0\}$. It is easy to see that M and $-M$ are, respectively, super- and sub-solutions of (3.1). Hence, by the comparison principle (a) we obtain the bound on $\max_i |v_i|_0$.

To obtain the bound on $\max_i [v_i]_1$, we set

$$m := \sup_{i,x,y} \phi_i(x, y) := \sup_{i,x,y} \{v_i(x) - v_i(y) - L|x - y| - \epsilon(|x|^2 + |y|^2)\},$$

where $L > 0$. If, by setting

$$L := \max\left\{\sup_{i,\alpha}\left\{\frac{[c^\alpha]_1|v_i|_0 + [f^\alpha]_1}{\lambda - [\sigma^\alpha]_1^2 - [b^\alpha]_1}\right\}; [\psi]_1\right\},$$

we can conclude that $m \leqslant 0$, then, sending ϵ to 0, we have done so. Assume that the supremum is attained at a point (i_0, x_0, y_0). If $x_0 = y_0$, then $m \leqslant 0$, and sending ϵ to 0 we have the result. If not, since $L|x - y|$ is smooth at x_0, y_0, we can apply a doubling-of-variables argument. Set

$$P_0 := \left[I - \frac{(x_0 - y_0)}{|x_0 - y_0|}\left(\frac{x_0 - y_0}{|x_0 - y_0|}\right)^\top\right], \quad \Lambda := \begin{pmatrix} P_0 & -P_0 \\ -P_0 & P_0 \end{pmatrix}.$$

Define the following sets:

$$A := \{i \in \mathcal{I} | (i, x_0, y_0) \text{ realizes the sup}\}, \quad I(x_0) := \{i \in \mathcal{I} | v_i(x_0) \leqslant \psi(x_0)\}.$$

The maximum principle for semi-continuous functions (see Crandall *et al.*, 1992) and the definition of viscosity solutions imply that, for $i \in A$, there exist $X, Y \in \mathcal{S}^N$ such that

$$\min\left\{\max(L^{\alpha_i}(x_0, v_i(x_0), p_x, X); v_i(x_0) - \min_{j\neq i}\{v_j(x_0) + k\}); v_i(x_0) - \psi(x_0)\right\} \leqslant 0,$$

$$\min\left\{\max(L^{\alpha_i}(y_0, v_i(y_0), p_y, Y); v_i(y_0) - \min_{j\neq i}\{v_j(y_0) + k\}); v_i(y_0) - \psi(y_0)\right\} \geqslant 0,$$

where $p_x = L\frac{(x_0-y_0)}{|x_0-y_0|} + 2\varepsilon x_0$, $p_y = L\frac{(x_0-y_0)}{|x_0-y_0|} - 2\varepsilon y_0$, and

$$\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leqslant \frac{L}{|x_0 - y_0|}\Lambda + 2\varepsilon\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}.$$

Case 1: There exists $i \in A \cap I(x_0)$, i.e. $v_i(x_0) \leqslant \psi(x_0)$. Since $\bar{v}_i(y_0) \geqslant \psi(y_0)$, for all $i \in A$, we have

$$v_i(x_0) - \bar{v}_i(y_0) \leqslant \psi(x_0) - \psi(y_0) \leqslant [\psi]_1|x_0 - y_0|.$$

Case 2: The set $A \cap I(x_0)$ is empty. Then

$$\max\{L^{\alpha_i}(x_0, v_i(x_0), p_x, X); v_i(x_0) - \min_{j\neq i}\{v_j(x_0) + k\}\} \leqslant 0 \quad \forall i \in A. \tag{A.6}$$

Since $\max\left\{L^{\alpha_{i_0}}\left(y_0, v_{i_0}(y_0), p_y, Y\right); v_{i_0}(y_0) - \min_{j\neq i_0}\{v_j(y_0)+k\}\right\} \geqslant 0$, applying Lemma A.1, we obtain

$$L^{\alpha_{i_0}}\left(x_0, v_{i_0}(x_0), p_x, X\right) \leqslant 0 \leqslant L^{\alpha_{i_0}}\left(y_0, v_{i_0}(y_0), p_y, Y\right).$$

Since P_0 is a projection and, hence, is non-expansive, we have that

$$\begin{aligned}\begin{pmatrix}\sigma^\alpha(x_0)\\ \sigma^\alpha(y_0)\end{pmatrix}^\top \Lambda \begin{pmatrix}\sigma^\alpha(x_0)\\ \sigma^\alpha(y_0)\end{pmatrix} &= (\sigma^\alpha(x_0)-\sigma^\alpha(y_0))P_0(\sigma^\alpha(x_0)-\sigma^\alpha(y_0))\\ &\leqslant |\sigma^\alpha(x_0)-\sigma^\alpha(y_0)|^2.\end{aligned}$$

Now we can proceed as in the standard situation (see Barles & Jakobsen, 2005, Theorem A.1). Combining Cases 1 and 2 we obtain the result. □

By using Barles & Jakobsen (2002, Theorem A.1), we prove the following theorem.

THEOREM A.3 Let v and $\bar{v}$ be solutions of (3.1) with coefficients σ, b, c, f and $\bar{\sigma}, \bar{b}, \bar{c}, \bar{f}$, respectively. Suppose that assumptions (A1) and (A2) are satisfied for both sets of coefficients with the same λ, and $\max_i |v_i|_1 + \max_i |\bar{v}_i|_1 \leqslant M < \infty$. Then

$$\lambda \max_i |v_i - \bar{v}_i|_0 \leqslant M\left(\sup_\alpha\{|\sigma^\alpha - \bar{\sigma}^\alpha|_0 + |b^\alpha - \bar{b}^\alpha|_0 + |c^\alpha - \bar{c}^\alpha|_0 + |f^\alpha - \bar{f}^\alpha|_0\}\right),$$

where M depends on K, $\sup_i |v_i|_0$ and $\sup_i |\bar{v}_i|_0$.

Proof. We set

$$m := \sup_{i,x,y}\phi_i(x,y) := \sup_{i,x,y}\{v_i(x) - \bar{v}_i(y) - \delta|x-y|^2 - \varepsilon(|x|^2+|y|^2)\},$$

where $\delta > 0$ and $\varepsilon > 0$. The sup is attained at a point (i, x_0, y_0), so

$$m = v_i(x_0) - \bar{v}_i(y_0) - \delta|x_0-y_0|^2 - \varepsilon(|x_0|^2+|y_0|^2).$$

Let

$$A := \{i \in \mathcal{I} | (i, x_0, y_0) \text{ realize the } \sup\}, \quad I(x_0) := \{i \in \mathcal{I} | v_i(x_0) \leqslant \psi(x_0)\}.$$

The maximum principle for semi-continuous functions (see Crandall *et al.*, 1992) and the definition of viscosity solutions imply that, for $i \in A$, there exist $X, Y \in \mathcal{S}^N$ such that

$$\min\left\{\max(L^{\alpha_i}(x_0, v_i(x_0), p_x, X); v_i(x_0) - \min_{j\neq i}\{v_j(x_0)+k\}); v_i(x_0) - \psi(x_0)\right\} \leqslant 0,$$

$$\min\left\{\max(L^{\alpha_i}(y_0, \bar{v}_i(y_0), p_y, Y); \bar{v}_i(y_0) - \min_{j\neq i}\{\bar{v}_j(y_0)+k\}); \bar{v}_i(y_0) - \psi(y_0)\right\} \geqslant 0,$$

where $p_x = 2\delta(x_0-y_0) + 2\varepsilon x_0$ and $p_y = 2\delta(x_0-y_0) - 2\varepsilon y_0$, and there exists $\ell > 0$ such that

$$\begin{pmatrix}X & 0\\ 0 & -Y\end{pmatrix} \leqslant \ell\delta\begin{pmatrix}I & -I\\ -I & I\end{pmatrix} + \ell\varepsilon\begin{pmatrix}I & 0\\ 0 & I\end{pmatrix} + \mathcal{O}(k).$$

We have to study two different cases.

Case 1: If there exists $i \in A \cap I(x_0)$, then $v_i(x_0) \leqslant \psi(x_0)$. Since $\bar{v}_i(y_0) \geqslant \psi(y_0)$, then we have

$$v_i(x_0) - \bar{v}_i(y_0) \leqslant \psi(x_0) - \psi(y_0) \leqslant [\psi]_1 |x_0 - y_0|.$$

Case 2: If $A \cap I(x_0) = \varnothing$, then

$$\max\left\{L^{\alpha_i}(x_0, v_i(x_0), p_x, X); v_i(x_0) - \min_{j \neq i}\{v_j(x_0) + k\}\right\} \leqslant 0 \quad \forall i \in A. \tag{A.7}$$

Since, $\max\left\{L^{\alpha_{i_0}}(y_0, \bar{v}_{i_0}(y_0), p_y, Y); \bar{v}_{i_0}(y_0) - \min_{j \neq i_0}\{\bar{v}_j(y_0) + k\}\right\} \geqslant 0$, applying Lemma A.1, we obtain

$$L^{\alpha_{i_0}}\left(x_0, v_{i_0}(x_0), p_x, X\right) \leqslant 0 \leqslant \bar{L}^{\alpha_{i_0}}\left(y_0, \bar{v}_{i_0}(y_0), p_y, Y\right),$$

and then

$$\begin{aligned} 0 &\leqslant -\operatorname{tr}[\bar{a}^{\alpha_{i_0}}(y_0)Y] + \operatorname{tr}[a^{\alpha_{i_0}}(x_0)X] - \bar{b}^{\alpha_{i_0}}(y_0)p_y + b^{\alpha_{i_0}}(x_0)p_x \\ &\quad + \bar{c}^{\alpha_{i_0}}(y_0)\bar{v}_{i_0}(y_0) - c^{\alpha_{i_0}}(x_0)v_{i_0}(x_0) - \bar{f}^{\alpha_{i_0}}(y_0) + f^{\alpha_{i_0}}(x_0) \\ &=: (\mathrm{I}) + (\mathrm{II}) + (\mathrm{III}) + (\mathrm{IV}). \end{aligned}$$

As in Barles & Jakobsen (2002), we analyse each term separately:

$$\begin{aligned} (\mathrm{I}) &= \operatorname{tr}[a^{\alpha_{i_0}}(x_0)X] - \operatorname{tr}[\bar{a}^{\alpha_{i_0}}(y_0)Y] \\ &\leqslant 2\ell\delta\{|\sigma^{\alpha_{i_0}}(x_0) - \bar{\sigma}^{\alpha_{i_0}}(x_0)|^2 + |\bar{\sigma}^{\alpha_{i_0}}(x_0) - \bar{\sigma}^{\alpha_{i_0}}(y_0)|^2\} \\ &\quad + \ell\varepsilon\{|\sigma^{\alpha_{i_0}}(x_0)|^2 + |\bar{\sigma}^{\alpha_{i_0}}(y_0)|^2\}, \\ (\mathrm{II}) &= (b^{\alpha_{i_0}}(x_0) - \bar{b}^{\alpha_{i_0}}(y_0))(x_0 - y_0) \\ &\leqslant 2|b^{\alpha_{i_0}}(x_0) - \bar{b}^{\alpha_{i_0}}(x_0)|^2 + 2|x_0 - y_0|^2 \\ &\quad + |\bar{b}^{\alpha_{i_0}}(x_0) - \bar{b}^{\alpha_{i_0}}(y_0)||x_0 - y_0|, \\ (\mathrm{III}) &= \bar{c}^{\alpha_{i_0}}(y_0)\bar{u}(y_0) - c^{\alpha_{i_0}}(x_0)u(x_0) \\ &\leqslant |u(x_0)||c^{\alpha_{i_0}}(x_0) - \bar{c}^{\alpha_{i_0}}(x_0)| + |\bar{u}(y_0)||\bar{c}^{\alpha_{i_0}}(x_0) - \bar{c}^{\alpha_{i_0}}(y_0)| \\ &\quad - \lambda m, \\ (\mathrm{IV}) &= f^{\alpha_{i_0}}(x_0) - \bar{f}^{\alpha_{i_0}}(y_0) \\ &\leqslant |f^{\alpha_{i_0}}(x_0) - \bar{f}^{\alpha_{i_0}}(x_0)| + |\bar{f}^{\alpha_{i_0}}(x_0) - \bar{f}^{\alpha_{i_0}}(y_0)||x_0 - y_0|. \end{aligned}$$

Hence, we obtain

$$\begin{aligned} \lambda m &\leqslant 2\ell\delta\{|\sigma^{\alpha_{i_0}} - \bar{\sigma}^{\alpha_{i_0}}|_0^2 + |b^{\alpha_{i_0}} - \bar{b}^{\alpha_{i_0}}|_0^2\} \\ &\quad + \left\{\left|v_{i_0}\right|_0 |\bar{c}^{\alpha_{i_0}} - c^{\alpha_{i_0}}|_0 + |f^{\alpha_{i_0}} - \bar{f}^{\alpha_{i_0}}|_0\right\} \\ &\quad + 2\delta\{\ell[\bar{\sigma}^{\alpha_{i_0}}]_1^2 + [\bar{b}^{\alpha_{i_0}}]_1 + 2\}|x_0 - y_0|^2 \\ &\left\{\left|\bar{v}_{i_0}\right|_0 [\bar{c}^{\alpha_{i_0}}]_1 + [\bar{f}^{\alpha_{i_0}}]_1\right\} |x_0 - y_0| + C\varepsilon(1 + |x_0| + |y_0|). \end{aligned}$$

We sum the bounds obtained in the two cases to have a general bound of m. So we obtain

$$\begin{aligned}\lambda m \leqslant\ & 2\ell\delta\{|\sigma^{\alpha_{i_0}} - \bar{\sigma}^{\alpha_{i_0}}|_0^2 + |b^{\alpha_{i_0}} - \bar{b}^{\alpha_{i_0}}|_0^2\} + \left\{\left|v_{i_0}\right|_0 |\bar{c}^{\alpha_{i_0}} - c^{\alpha_{i_0}}|_0 + |f^{\alpha_{i_0}} - \bar{f}^{\alpha_{i_0}}|_0\right\} \\ & + 2\delta\{\ell[\bar{\sigma}^{\alpha_{i_0}}]_1^2 + [\bar{b}^{\alpha_{i_0}}]_1 + 2\}|x_0 - y_0|^2 \\ & + \left\{\left|\bar{v}_{i_0}\right|_0 [\bar{c}^{\alpha_{i_0}}]_1 + [\bar{f}^{\alpha_{i_0}}]_1 + \lambda[\psi]_1\right\}|x_0 - y_0| \\ & + C\varepsilon(1 + |x_0|^2 + |y_0|^2).\end{aligned}$$

We set $k_1 := \{2\ell[\bar{\sigma}^{\alpha_{i_0}}]_1^2 + 2[\bar{b}^{\alpha_{i_0}}]_1 + 4\}$, $k_2 := \left\{\left|\bar{v}_{i_0}\right|_0 [\bar{c}^{\alpha_{i_0}}]_1 + [\bar{f}^{\alpha_{i_0}}]_1 + \lambda[\psi]_1\right\}$. From now on we proceed as in Barles & Jakobsen (2002, Theorem A.1). Noting that $2\phi(x_0, y_0) \geqslant \phi(x_0, x_0) + \phi(y_0, y_0)$, we have

$$|x_0 - y_0| \leqslant C\delta^{-1}, \tag{A.8}$$

where C depends K. The inequality (A.8) implies that

$$|x_0 - y_0|^2 \leqslant C\delta^{-2}, \tag{A.9}$$

where C depends on K. So we obtain

$$\begin{aligned}\lambda m \leqslant\ & \left\{\left|v_{i_0}\right|_0 |\bar{c}^{\alpha_{i_0}} - c^{\alpha_{i_0}}|_0 + |f^{\alpha_{i_0}} - \bar{f}^{\alpha_{i_0}}|_0\right\} + 2\ell\delta\{|\sigma^{\alpha_{i_0}} - \bar{\sigma}^{\alpha_{i_0}}|_0^2 + |b^{\alpha_{i_0}} - \bar{b}^{\alpha_{i_0}}|_0^2\} \\ & + C(k_1 + k_2)\delta^{-1} + C\varepsilon(1 + |x_0|^2 + |y_0|^2).\end{aligned}$$

We know that $v_{i_0}(x) - \bar{v}_{i_0}(x) - 2\varepsilon|x|^2 \leqslant m$, and so

$$\begin{aligned}v_{i_0}(x) - \bar{v}_{i_0}(x) \leqslant\ & \left\{\left|v_{i_0}\right|_0 |\bar{c}^{\alpha_{i_0}} - c^{\alpha_{i_0}}|_0 + |f^{\alpha_{i_0}} - \bar{f}^{\alpha_{i_0}}|_0\right\} + 2\ell\delta\{|\sigma^{\alpha_{i_0}} - \bar{\sigma}^{\alpha_{i_0}}|_0^2 + |b^{\alpha_{i_0}} - \bar{b}^{\alpha_{i_0}}|_0^2\} \\ & + C(k_1 + k_2)\delta^{-1} + 2\varepsilon|x|^2 + C\varepsilon(1 + |x_0|^2 + |y_0|^2).\end{aligned}$$

This inequality holds for all δ, and hence we minimize with respect to δ, by noting that for $l > 0$,

$$\min_{\delta>0}(l\delta + C\delta^{-1}) = Cl^{1/2}.$$

Hence, by sending ε to zero, we obtain

$$\begin{aligned}v_{i_0}(x) - \bar{v}_{i_0}(x) \leqslant\ & \left\{\left|v_{i_0}\right|_0 |\bar{c}^{\alpha_{i_0}} - c^{\alpha_{i_0}}|_0 + |f^{\alpha_{i_0}} - \bar{f}^{\alpha_{i_0}}|_0\right\} \\ & + C\{|\sigma^{\alpha_{i_0}} - \bar{\sigma}^{\alpha_{i_0}}|_0^2 + |b^{\alpha_{i_0}} - \bar{b}^{\alpha_{i_0}}|_0^2\}^{1/2},\end{aligned}$$

where C depends on K, $\left|v_{i_0}\right|_0$, $\left|\bar{v}_{i_0}\right|_0$ and $[\psi]_1$. Since $(s^2 + t^2)^{1/2} \leqslant |s| + |t|$, we can conclude. □

Similarly, we have the following result.

THEOREM A.4 Let u and $\bar{u}$ be solutions of (1.1) with coefficients σ, b, c, f and $\bar{\sigma}, \bar{b}, \bar{c}, \bar{f}$, respectively. Suppose that assumptions (A1) and (A2) are satisfied for both sets of coefficients with the same λ, and $\max|u|_1 + \max|\bar{u}|_1 \leqslant M < \infty$. Then

$$\lambda \max|u - \bar{u}|_0 \leqslant M\left(\sup_\alpha\{|\sigma^\alpha - \bar{\sigma}^\alpha|_0 + |b^\alpha - \bar{b}^\alpha|_0 + |c^\alpha - \bar{c}^\alpha|_0 + |f^\alpha - \bar{f}^\alpha|_0\}\right),$$

where M depends on K, $\sup|u|_0$ and $\sup|\bar{u}|_0$.