Convergence rates for the quasi-reversibility method to solve the Cauchy problem for Laplace's equation

L Bourgeois

Laboratoire POEMS, UMR ENSTA‡/CNRS/INRIA, Ecole Nationale Supérieure des Techniques Avancées, 32 Boulevard Victor, 75739 Paris Cedex 15, France

E-mail: bourgeois@ensta.fr

Abstract. We consider the quasi-reversibility method to solve the Cauchy problem for Laplace's equation in a smooth bounded domain. We assume that the Cauchy data are contaminated by some noise of amplitude σ, so that we make a *regular* choice of ε as a function of σ, where ε is the small parameter of the quasi-reversibility method. Specifically, we present two different results concerning the convergence rate of the solution of quasi-reversibility to the exact solution when σ tends to 0. The first result is a convergence rate of type $1/(\log \frac{1}{\sigma})^\beta$ in a truncated domain, the second one holds when a *source condition* is assumed and is a convergence rate of type $\sigma^{\frac{1}{2}}$ in the whole domain.

1. Introduction

In the following article we consider the Cauchy problem for Laplace's equation in a bounded domain of $\mathbb{R}^N$. This problem is ill-posed in the sense of Hadamard (see [5] and the bibliography of [12] for a comprehensive description of the problem). Several fields of physics and engineering in which such a problem arises are described in [2], as well as some regularization techniques which can be used to solve this classical problem.

The method of quasi-reversibility was first presented in [14], in particular to solve the Cauchy problem for elliptic equations. It consists of transforming the ill-posed second-order initial problem into a family (depending on a small parameter ε) of fourth-order problems. This elegant and non-iterative method is based on a weak formulation which enables one to use a F.E.M. computation. Some computations of the method of quasi-reversibility based on finite difference schemes were actually performed in [14], [12] and [8]. A mixed formulation of quasi-reversibility was proposed in [2] to solve the problem using finite elements of class C^0, whereas finite elements of class C^1 are normally required in the case of the classical formulation.

The authors of [14] did not consider the question of the convergence rate of the quasi-reversibility solution to the exact one, in particular with data errors, say of amplitude σ. This was done in [12] by using a Carleman's estimate, albeit in a particular

‡ member of ParisTech

domain, i.e. a two-dimensional region bounded by a straight-line and a parabola. The same result is extended to more general elliptic operators in [13]. In the following paper we obtain a convergence rate estimate in any smooth domain such that the part of the boundary on which Cauchy data are given does not intersect the part on which data are unknown. This convergence rate is based on an estimate found in [21], which is itself obtained by using Carleman's estimates. We obtain a logarithmic convergence rate, i.e. of type $1/(\log \frac{1}{\sigma})^\beta$, for a simple choice of parameter $\varepsilon(\sigma)$ of type $\varepsilon(\sigma) = \sigma$, in any truncated domain which does not contain the part of the boundary on which data are unknown.

This is a natural question to wonder if it is possible to obtain a better convergence rate, in the whole domain instead of in the truncated domain mentioned before. This is done in the following paper by making a more complicated choice of parameter $\varepsilon(\sigma)$. This choice results from a methodology based on the theory of duality developed in [6], which has already been used to solve approximate controllability problems (see, e.g., [16, 18, 1, 20]). In the case of the quasi-reversibility however, there is a specific issue in applying such methodology. The formulation of quasi-reversibility can be seen as a Tikhonov regularization of a linear unbounded closed operator, while the classical results of [6] hold for bounded operators. Fortunately, these results can be extended to our case by using some proprieties of unbounded closed operators. An estimate of the convergence rate of type $\sigma^{\frac{1}{2}}$ is then obtained when a *source condition* is assumed for the exact solution (such a notion is for example described in [7, 11]). In addition, such methodology answers the question of a relevant choice of the regularization parameter ε as a function of the amplitude of the noise σ, which is a classical problem in the theory of regularization [7, 11, 4]. The effective computation of ε as a function of σ needs however to solve a minimization problem which is more complicated than the initial problem of quasi-reversibility.

This paper is organized as follows. The second section is devoted to a short presentation of the Cauchy problem that we consider. The third section presents the Tikhonov regularization for linear unbounded operators, from which we derive the method of quasi-reversibility. The fourth section gives some useful preliminary results. The fifth section is devoted to the first result which concerns the convergence rate of the solution of quasi-reversibility to the exact solution without any assumption on the exact solution, and the sixth section to the second result which concerns the convergence rate when a source condition is assumed.

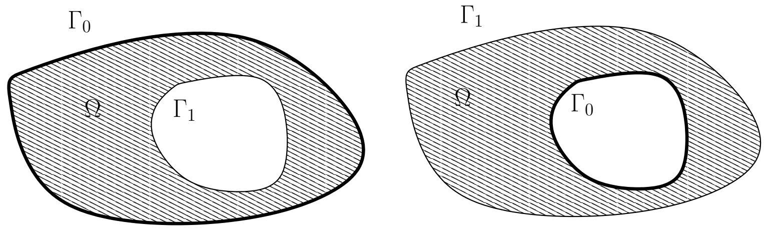
2. The Cauchy problem for Laplace's equation

Let Ω be an open set of $\mathbb{R}^N$, $N \geq 2$, which satisfies the following assumptions, referred by (A) from now on.

Assumptions (A) :

Ω is bounded, connected and of class C^2 [3]. Furthermore, Γ_0 and Γ_1 are two open subsets of $\partial\Omega$ with $\mathrm{mes}(\Gamma_0) > 0$ and $\mathrm{mes}(\Gamma_1) > 0$, $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset$ and $\overline{\Gamma_0} \cup \overline{\Gamma_1} = \partial\Omega$.

The figure below shows two geometric situations for Γ_0 and Γ_1 which are consistent with (A). However, we point out the fact that all what is presented in sections 2 and 3, lemma 1 and proposition 1 in section 4, as well as lemma 2 in section 5, still hold if $\Gamma_0 \cap \Gamma_1 = \emptyset$ but $\overline{\Gamma_0} \cap \overline{\Gamma_1} \neq \emptyset$.



The Cauchy problem consists of finding u such that

$$\Delta u = 0 \ \text{ in } \ \Omega \tag{1}$$

and

$$\left\{ \begin{array}{l} u|_{\Gamma_0} = g_0 \\ \dfrac{\partial u}{\partial n}|_{\Gamma_0} = g_1, \end{array} \right. \tag{2}$$

where n is the outward unit normal on Ω, and where g_0 and g_1 are some data.

In the following, u is *a priori* sought in $L^2(\Omega)$, while g_0 and g_1 are assumed to belong respectively to $H^{-\frac{1}{2}}(\Gamma_0)$ and $H^{-\frac{3}{2}}(\Gamma_0)$. The space $H^{-(s-\frac{1}{2})}(\Gamma_0)$ for $s = 1, 2$ is defined as the dual space of $H_{00}^{s-\frac{1}{2}}(\Gamma_0)$, which is itself defined by

$$H_{00}^{s-\frac{1}{2}}(\Gamma_0) = \{v \in L^2(\Gamma_0)\,;\, \exists w \in H^s(\Omega),\, w|_{\Gamma_0} = v,\, w|_{\Gamma_1} = 0\}.$$

$H_{00}^{s-\frac{1}{2}}(\Gamma_0)$ is hence a subspace of $H^{s-\frac{1}{2}}(\Gamma_0)$, which consists of the restrictions on Γ_0 of functions in $H^{s-\frac{1}{2}}(\partial\Omega)$. The smoothness of Ω which we assumed is consistent with a proper definition of space $H^{\frac{3}{2}}(\partial\Omega)$ and all the other spaces defined above.

As it can be seen for example in [5], problem (1) (2) is ill-posed in the sense of Hadamard because the existence of a solution u and its stability with respect to the data g_0 and g_1 don't hold even if these data are very smooth.

We introduce the space $D(\Delta, \Omega)$ defined by

$$D(\Delta, \Omega) = \{v \in L^2(\Omega)\,;\, \Delta v \in L^2(\Omega)\},$$

which is an Hilbert space when equipped with the norm $||.||_{D(\Delta,\Omega)}$ defined by

$$||v||_{D(\Delta,\Omega)} = \left(||v||^2_{L^2(\Omega)} + ||\Delta v||^2_{L^2(\Omega)}\right)^{\frac{1}{2}}.$$

To simplify the analysis, we assume that the pair (g_0, g_1) is such that there exists $p \in D(\Delta, \Omega)$ with

$$\left\{ \begin{array}{l} p|_{\Gamma_0} = g_0 \\ \dfrac{\partial p}{\partial n}|_{\Gamma_0} = g_1. \end{array} \right.$$

For example, if $g_0 \in H^{\frac{3}{2}}(\Gamma_0)$ and $g_1 \in H^{\frac{1}{2}}(\Gamma_0)$, the pair (g_0, g_1) satisfies such a property with $p \in H^2(\Omega)$. We recall that for $p \in D(\Delta, \Omega)$, the traces $p|_{\Gamma_0}$ and $(\partial p/\partial n)|_{\Gamma_0}$ have a meaning respectively in $H^{-\frac{1}{2}}(\Gamma_0)$ and $H^{-\frac{3}{2}}(\Gamma_0)$ [17].

Setting $f = -\Delta p \in L^2(\Omega)$, problem (1) (2) is then equivalent to the problem of finding $u \in L^2(\Omega)$ such that

$$\Delta u = f \text{ in } \Omega \tag{3}$$

and

$$\begin{cases} u|_{\Gamma_0} = 0 \\ \dfrac{\partial u}{\partial n}|_{\Gamma_0} = 0. \end{cases} \tag{4}$$

The problem (3) (4) will be the Cauchy problem we are interested in from now on.

3. The Tikhonov regularization and the method of quasi-reversibility

We denote by H a separable Hilbert space (the scalar product is $(.,.)$, the norm is $||.||$) and by $A\ : H \to H$ a linear unbounded operator of domain $D(A)$ (these notions are detailed, e.g., in [3]). We assume that A is a closed operator. Hence, $D(A)$ equipped with the graph norm $||u||_{D(A)} = (||u||^2 + ||Au||^2)^{1/2}$ is an Hilbert space. We assume that $D(A)$ is dense in H. Then we can define the adjoint operator of A, denoted by $A^*\ : H \to H$, its domain being denoted by $D(A^*)$. It can be proved that A^* is a closed operator and that $D(A^*)$ is dense in H (see [3]).

We assume that A is injective, but not necessarily onto. For a given $f \in H$, the problem of finding $u \in D(A)$ such that

$$Au = f \tag{5}$$

may have no solution and hence is an ill-posed problem.

For a given $\varepsilon > 0$, we consider the regularized problem of finding $u \in D(A)$ such that

$$(Au, Av) + \varepsilon(u, v) = (f, Av), \quad \forall v \in D(A). \tag{6}$$

The following theorem is proved in [14].

Theorem 1 :

For $f \in H$, the problem (6) admits a unique solution $u_\varepsilon \in D(A)$ and we have the estimates

$$||u_\varepsilon|| \leq \frac{1}{\sqrt{\varepsilon}}||f||, \quad ||Au_\varepsilon|| \leq ||f||. \tag{7}$$

If moreover problem (5) admits a (unique) solution u in $D(A)$, then u_ε converges when ε tends to 0 to u in $D(A)$ for the graph norm, with the following estimate

$$||A(u_\varepsilon - u)|| \leq \sqrt{2\varepsilon}||u||. \tag{8}$$

We can easily verify that the solution u_ε of (6) is the solution of the minimization problem

$$\inf_{v \in D(A)} \Big(\frac{1}{2}||Av - f||^2 + \frac{\varepsilon}{2}||v||^2\Big). \tag{9}$$

u_ε is also the solution of the system

$$\begin{cases} Av - f \in D(A^*) \\ A^*(Av - f) + \varepsilon v = 0. \end{cases}$$

In conclusion, the ill-posed problem (5) has been transformed into the well-posed problem (6). This regularization method is the so-called Tikhonov regularization, here in the case of a linear unbounded closed operator.

Now we assume that f is contaminated by noise, which corresponds to the realistic situation of inverse problems. We denote by f^σ the noisy data, with $||f^\sigma - f|| \leq \sigma$. The solution of problem (6), when f is replaced by f^σ, is denoted $u_{\varepsilon,\sigma}$. According to the definition given in [4], a *regular* choice of parameter $\varepsilon(\sigma)$ must satisfy $u_{\varepsilon(\sigma),\sigma} \to u$ when $\sigma \to 0$. Some regular choices exist, as the following theorem shows it.

Theorem 2 :

We assume that for $f \in H$, problem (5) admits a (unique) solution u in $D(A)$. For $f^\sigma \in H$ with $||f^\sigma - f|| \leq \sigma$, if the function $\varepsilon(\sigma)$ satisfies both conditions

$$\lim_{\sigma \to 0} \varepsilon(\sigma) = 0 \quad \text{and} \quad \lim_{\sigma \to 0} \frac{\sigma}{\sqrt{\varepsilon(\sigma)}} = 0, \tag{10}$$

then the solution $u_{\varepsilon(\sigma),\sigma} := u_\sigma$ of problem (6) with f^σ instead of f, converges when σ tends to 0 to u in $D(A)$ for the graph norm, with the following estimate

$$||A(u_\sigma - u)|| \leq C\,\sqrt{\varepsilon(\sigma)}, \tag{11}$$

where $C > 0$ is independent on σ.

Proof : using the estimates (7) and (8) of theorem 1, we obtain

$$||u_{\varepsilon(\sigma),\sigma} - u|| \leq ||u_{\varepsilon(\sigma),\sigma} - u_{\varepsilon(\sigma),0}|| + ||u_{\varepsilon(\sigma),0} - u|| \leq \frac{\sigma}{\sqrt{\varepsilon(\sigma)}} + ||u_{\varepsilon(\sigma),0} - u||$$

and

$$||A(u_{\varepsilon(\sigma),\sigma} - u)|| \leq ||A(u_{\varepsilon(\sigma),\sigma} - u_{\varepsilon(\sigma),0})|| + ||A(u_{\varepsilon(\sigma),0} - u)|| \leq \sigma + \sqrt{2\varepsilon(\sigma)}||u||.$$

We complete the proof by using theorem 1 and the properties (10) of function $\varepsilon(\sigma)$. ■

As already pointed out in [14], we can see the method of quasi-reversibility as a particular case of the Tikhonov regularization. The Cauchy problem for the Laplace's equation we saw in section 2, precisely (3) (4), has the form (5) when $H = L^2(\Omega)$ and A is the Laplace operator of domain $D(A) = D(\Delta, \Omega, \Gamma_0)$. Here, $D(\Delta, \Omega, \Gamma_0)$ denotes the set

$$D(\Delta, \Omega, \Gamma_0) = \{v \in D(\Delta, \Omega)\,;\, v|_{\Gamma_0} = 0,\, \frac{\partial v}{\partial n}|_{\Gamma_0} = 0\},$$

which is dense in $L^2(\Omega)$. A is a closed operator [14], so that $D(\Delta, \Omega, \Gamma_0)$ equipped with the graph norm $||.||_{D(\Delta,\Omega)}$ is an Hilbert space. The injectivity of A is a consequence of the Cauchy uniqueness (see proposition 1 in next section). However, A is not onto, due to ill-posedness of the Cauchy problem.

We assume that for $f \in L^2(\Omega)$, the (unique) solution $u \in L^2(\Omega)$ of (3) (4) exists, and that $f^\sigma \in L^2(\Omega)$ is a noisy data with $||f^\sigma - f||_{L^2(\Omega)} \leq \sigma$.

We can specify the regularized problem (6) in this particular case, which consists of finding $u \in D(\Delta, \Omega, \Gamma_0)$ such that

$$\int_\Omega \Delta u \Delta v \, dx + \varepsilon \int_\Omega uv \, dx = \int_\Omega f^\sigma \Delta v \, dx, \quad \forall v \in D(\Delta, \Omega, \Gamma_0). \tag{12}$$

Theorem 1 ($\sigma = 0$) and theorem 2 ($\sigma \neq 0$) can be directly applied to that particular case. Problem (12) is called a quasi-reversibility formulation, and the solution $u_{\varepsilon,\sigma} \in D(\Delta, \Omega, \Gamma_0)$ of (12) is called the solution of quasi-reversibility.

Remark 1 : $A \,: L^2(\Omega) \to L^2(\Omega)$ being the Laplace operator, other choices of the domain $D(A)$ are possible, which correspond to other formulations of quasi-reversibility. If for example we choose $D(A) = D^1(\Delta, \Omega, \Gamma_0)$, where

$$D^1(\Delta, \Omega, \Gamma_0) = \{v \in H^1(\Omega),\, \Delta v \in L^2(\Omega)\,;\, v|_{\Gamma_0} = 0,\, \frac{\partial v}{\partial n}|_{\Gamma_0} = 0\},$$

A is not a closed operator, but $D^1(\Delta, \Omega, \Gamma_0)$ equipped with the norm $||.||_{D^1(\Delta,\Omega)}$ defined by

$$||v||_{D^1(\Delta,\Omega)} = \left(||v||^2_{H^1(\Omega)} + ||\Delta v||^2_{L^2(\Omega)}\right)^{\frac{1}{2}}$$

is an Hilbert space. Hence, we can obtain another quasi-reversibility formulation, which consists of finding $u \in D^1(\Delta, \Omega, \Gamma_0)$ such that

$$\int_\Omega \Delta u \Delta v \, dx + \varepsilon \int_\Omega (\nabla u . \nabla v + uv) \, dx = \int_\Omega f^\sigma \Delta v \, dx, \quad \forall v \in D^1(\Delta, \Omega, \Gamma_0).$$

Theorems 1 and 2 may easily be adapted to such formulation, which is discussed in [2]. It's advantage is that it can be used when the domain Ω is not of C^2-class, but only Lipschitz continuous.

4. Some preliminary results

Let us start with the following lemma, which roughly speaking states that a function which belongs to $D(\Delta, \Omega, \Gamma_0)$ is smooth, except in a vicinity of $\overline{\Gamma_1}$.

Lemma 1 *:*

We suppose that Ω satisfies assumptions (A). If $\alpha \in C^\infty(\overline{\Omega})$ vanishes in a (volumic) vicinity of $\overline{\Gamma_1}$ and $v \in D(\Delta, \Omega, \Gamma_0)$, then $\alpha v \in H^2(\Omega)$ and there exists a constant $C(\alpha, \Omega, \Gamma_0)$ such that

$$||\alpha v||_{H^2(\Omega)} \leq C ||v||_{D(\Delta,\Omega)}, \quad \forall v \in D(\Delta, \Omega, \Gamma_0). \tag{13}$$

Proof : the proof is mainly based on the following statement that for all $s \in \mathbb{R}$, if $u \in H^s(\mathbb{R}^N)$ and $\Delta u \in H^s(\mathbb{R}^N)$, then $u \in H^{s+2}(\mathbb{R}^N)$ (see, e.g., [19]). This result will be denoted (R_s) in the following proof.

If v belongs to $D(\Delta, \Omega, \Gamma_0)$ and α is a function of $C^\infty(\overline{\Omega})$ that vanishes near $\overline{\Gamma_1}$, let us denote by $\tilde{v}$ the trivial extension of v in $\mathbb{R}^N$ and $\tilde{\alpha}$ an extension in $C_0^\infty(\mathbb{R}^N)$ of α. We have in $\mathbb{R}^N$

$$\Delta(\tilde{\alpha}\tilde{v}) = \tilde{v}\, \Delta\tilde{\alpha} + 2 \sum_{i=1}^N \frac{\partial \tilde{\alpha}}{\partial x_i} \frac{\partial \tilde{v}}{\partial x_i} + \tilde{\alpha}\, \Delta\tilde{v}. \tag{14}$$

It is clear that $\tilde{v}\,\Delta\tilde{\alpha} \in L^2(\mathbb{R}^N)$. If $\partial\tilde{\alpha}/\partial x_i$ is denoted by $\tilde{\beta}_i$, we have

$$\tilde{\beta}_i \frac{\partial \tilde{v}}{\partial x_i} = \frac{\partial(\tilde{\beta}_i \tilde{v})}{\partial x_i} - \frac{\partial \tilde{\beta}_i}{\partial x_i} \tilde{v}, \tag{15}$$

which implies that $\tilde{\beta}_i \partial\tilde{v}/\partial x_i \in H^{-1}(\mathbb{R}^N)$. Now, let us prove that $\tilde{\alpha}\,\Delta\tilde{v} \in L^2(\mathbb{R}^N)$. For any test function $\varphi \in C_0^\infty(\mathbb{R}^N)$, we have, in the sense of distributions in $\mathbb{R}^N$,

$$\langle \tilde{\alpha}\Delta\tilde{v}, \varphi\rangle = \langle \tilde{v}, \Delta(\tilde{\alpha}\varphi)\rangle = \int_\Omega v\Delta(\alpha\varphi)\,dx.$$

Since $v \in D(\Delta, \Omega)$ and $\alpha\varphi$ is a smooth function, we can perform the following double integration by parts [17].

$$\int_\Omega v\Delta(\alpha\varphi)\,dx = \left\langle v, \frac{\partial(\alpha\varphi)}{\partial n}\right\rangle_{H^{-\frac{1}{2}}(\partial\Omega), H^{\frac{1}{2}}(\partial\Omega)} - \left\langle \frac{\partial v}{\partial n}, \alpha\varphi\right\rangle_{H^{-\frac{3}{2}}(\partial\Omega), H^{\frac{3}{2}}(\partial\Omega)} + \int_\Omega \Delta v(\alpha\varphi)\,dx.$$

Since $v|_{\Gamma_0} = 0$, $(\partial v/\partial n)|_{\Gamma_0} = 0$ and $(\alpha\varphi)$ vanishes near $\overline{\Gamma_1}$, we have

$$\langle \tilde{\alpha}\Delta\tilde{v}, \varphi\rangle = \int_\Omega (\alpha\Delta v)\varphi\,dx.$$

It follows that $\tilde{\alpha}\Delta\tilde{v} \in L^2(\mathbb{R}^N)$ and from (14) that $\Delta(\tilde{\alpha}\tilde{v}) \in H^{-1}(\mathbb{R}^N)$. From the statement (R_s) for $s = -1$, it follows that $\tilde{\alpha}\tilde{v} \in H^1(\mathbb{R}^N)$, for any $\tilde{\alpha} \in C_0^\infty(\mathbb{R}^N)$ that vanishes near $\overline{\Gamma_1}$ in Ω. This result implies that we have now $\tilde{\beta}_i\partial\tilde{v}/\partial x_i \in L^2(\mathbb{R}^N)$ from (15), and hence that $\Delta(\tilde{\alpha}\tilde{v}) \in L^2(\mathbb{R}^N)$ from (14). By using once again (R_s) for $s = 0$, it follows that $\tilde{\alpha}\tilde{v} \in H^2(\mathbb{R}^N)$, and hence $\alpha v \in H^2(\Omega)$. The estimate (13) is then a consequence of the closed graph theorem (see, e.g., [3]). ■

From lemma 1, we derive the two following propositions. We need to define truncated domains, given by

$$\Omega_\rho = \Omega/\{x \in \Omega\,;\, d(x, \overline{\Gamma_1}) \leq \rho\}. \tag{16}$$

Proposition 1 :
We suppose that Ω satisfies assumptions (A). For $f \in L^2(\Omega)$, there is at most one solution $u \in L^2(\Omega)$ which satisfies (3) (4).

Proof : this result is for example a consequence of the uniqueness theorem proved in [10] in a more general case (see the appendix B, p. 75). This theorem is obtained with the help of a Carleman's estimate and states that for a Lipschitz continuous domain Ω, if $u \in H^2(\Omega)$ satisfies (3) (4) with $f = 0$, then $u = 0$ in Ω.

We consider the set Ω_ρ defined by (16) and a cut-off function $\alpha \in C^\infty(\overline{\Omega})$ which equals 1 on $\overline{\Omega_\rho}$ and vanishes in a (volumic) vicinity of $\overline{\Gamma_1}$. It follows from lemma 1 that $\alpha u \in H^2(\Omega)$ and hence that $u \in H^2(\Omega_\rho)$, $\forall\rho > 0$. Using now the uniqueness theorem described above in the Lipschitz continuous domain Ω_ρ, we deduce that $u = 0$ in Ω_ρ, $\forall\rho > 0$. Using Lebesgue's theorem, it follows that $u = 0$ in Ω. ■

Proposition 2 :
We suppose that Ω satisfies assumptions (A). If we define

$$H^2(\Omega, \Gamma_0) = \{v \in H^2(\Omega)\,;\, v|_{\Gamma_0} = 0,\ \frac{\partial v}{\partial n}|_{\Gamma_0} = 0\},$$

then $H^2(\Omega, \Gamma_0)$ is dense in $D(\Delta, \Omega, \Gamma_0)$ for the norm $||.||_{D(\Delta,\Omega)}$.

Proof : since $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset$, there exists a function $\alpha \in C^\infty(\overline{\Omega})$ such that $\alpha = 0$ in a vicinity of $\overline{\Gamma_0}$ and $\alpha = 1$ in a vicinity of $\overline{\Gamma_1}$. Now we use the fact that $H^2(\Omega)$ is dense in $D(\Delta, \Omega)$ for the norm $||.||_{D(\Delta,\Omega)}$ [17].

Suppose $u \in D(\Delta, \Omega, \Gamma_0)$. There exists a sequence $(u_n) \in H^2(\Omega)$ which converges to u for the norm $||.||_{D(\Delta,\Omega)}$. It remains to prove that the sequence (u'_n) defined by

$$u'_n = \alpha u_n + (1 - \alpha)u$$

belongs to $H^2(\Omega, \Gamma_0)$ and converges to u for the same norm. First, since the function $(1 - \alpha)$ belongs to $C^\infty(\overline{\Omega})$ and vanishes in a vicinity of $\overline{\Gamma_1}$, it follows from lemma 1 that $(1 - \alpha)u$ belongs to $H^2(\Omega)$. Moreover, we deduce from $u|_{\Gamma_0} = 0$ and $(\partial u/\partial n)|_{\Gamma_0} = 0$ that $(1 - \alpha)u$ belongs to $H^2(\Omega, \Gamma_0)$.

Besides, the αu_n belongs to $H^2(\Omega, \Gamma_0)$. It is clear that $\alpha u_n \to \alpha u$ in $L^2(\Omega)$. In order to complete the proof, we have to verify that $\Delta(\alpha u)$ belongs to $L^2(\Omega)$ and that $\Delta(\alpha u_n) \to \Delta(\alpha u)$ in $L^2(\Omega)$. We have in Ω

$$\Delta(\alpha u) = u\,\Delta\alpha + 2\sum_{i=1}^{N} \frac{\partial\alpha}{\partial x_i}\frac{\partial u}{\partial x_i} + \alpha\,\Delta u.$$

The functions $(u\Delta\alpha)$ and $(\alpha\Delta u)$ belong to $L^2(\Omega)$. If $\partial\alpha/\partial x_i$ is denoted by β_i, we have

$$\beta_i\frac{\partial u}{\partial x_i} = \frac{\partial(\beta_i u)}{\partial x_i} - \frac{\partial\beta_i}{\partial x_i}u.$$

Since β_i is a function of $C^\infty(\overline{\Omega})$ which vanishes in a vicinity of $\overline{\Gamma_1}$, it follows from lemma 1 that $\beta_i u$ belongs to $H^2(\Omega)$ and hence that $\beta_i\partial u/\partial x_i$ belongs to $H^1(\Omega)$. So $\Delta(\alpha u)$ belongs to $L^2(\Omega)$. Writing

$$\Delta(\alpha u_n) = u_n\,\Delta\alpha + 2\sum_{i=1}^{N} \frac{\partial\alpha}{\partial x_i}\frac{\partial u_n}{\partial x_i} + \alpha\,\Delta u_n,$$

we observe that to prove convergence of $\Delta(\alpha u_n)$ to $\Delta(\alpha u)$ in $L^2(\Omega)$, we have to prove convergence of $\beta_i\partial u_n/\partial x_i$ to $\beta_i\partial u/\partial x_i$ for all i. Writing again

$$\beta_i\frac{\partial u_n}{\partial x_i} = \frac{\partial(\beta_i u_n)}{\partial x_i} - \frac{\partial\beta_i}{\partial x_i}u_n,$$

this convergence is a consequence of the estimate (13) of lemma 1. ■

5. Convergence rate in the general case

In section 3, precisely in theorems 1 and 2, we proved convergence of the solution of quasi-reversibility to the exact solution u for the norm $||.||_{D(\Delta,\Omega)}$ when $\varepsilon \to 0$ (without noise) or $\sigma \to 0$ (in presence of noise for a particular choice of function $\varepsilon(\sigma)$ which satisfies condition (10)). In the following section, we are interested in the convergence rate. For that purpose, we follow the method used in [12]. The authors, when the domain Ω is a two-dimensional region bounded by a straight-line and a parabola, used a Carleman's estimate and found an estimate, for the $H^2(\Omega_\rho)$ norm, of the difference between the solution of quasi-reversibility and the exact solution. Here Ω_ρ is a truncated domain in the sense of (16).

In the following section, Ω is a C^∞-class domain satisfying (A) (see the definition of assumption (A) at the beginning of section 2). We obtain a similar estimate as the one obtained in [12] by using a powerful result proved in [21]. This result was established by employing Carleman's estimates and some techniques developed in [15]. It consists of the following lemma.

Lemma 2 *:*
If Ω is a C^∞-class domain which satisfies assumptions (A), then

$\forall \beta \in]0,1[,\ \exists\, c>0,\ \forall \lambda>0,\ \forall v \in H^2(\Omega),$

$$||v||_{H^1(\Omega)} \le e^{c\lambda}\Big(||\Delta v||_{L^2(\Omega)} + ||v||_{L^2(\Gamma_0)} + ||\frac{\partial v}{\partial n}||_{L^2(\Gamma_0)}\Big) + \frac{1}{\lambda^\beta}||v||_{H^2(\Omega)}. \tag{17}$$

From lemma 2, we derive the following theorem.

Theorem 3 *:*
Let Ω be a C^∞-class domain which satisfies assumptions (A) and Ω_ρ be a truncated domain in the sense of (16). We assume that for $f \in L^2(\Omega)$, problem (3) (4) admits a (unique) solution u in $L^2(\Omega)$. If u_ε is the solution of problem (12) with $\sigma = 0$, then we have

$$\forall \rho>0,\ \forall \beta \in]0,1[,\ \exists\, c>0,\ \forall \varepsilon>0, \quad ||u_\varepsilon - u||_{H^2(\Omega_\rho)} \le c\,\frac{1}{\left[\log(\frac{1}{\varepsilon})\right]^\beta}.$$

For $f^\sigma \in L^2(\Omega)$ with $||f^\sigma - f||_{L^2(\Omega)} \le \sigma$, if $u_\sigma = u_{\varepsilon(\sigma),\sigma}$ is the solution of problem (12) with $\sigma > 0$, and if $\varepsilon(\sigma)$ is a function which satisfies (10), then we have

$$\forall \rho>0,\ \forall \beta \in]0,1[,\ \exists\, c>0,\ \forall \sigma>0, \quad ||u_\sigma - u||_{H^2(\Omega_\rho)} \le c\,\frac{1}{\left[\log(\frac{1}{\varepsilon(\sigma)})\right]^\beta}.$$

Proof : we use the estimate (17) of lemma 2 for $v_\varepsilon = u_\varepsilon - u$ in a truncated domain Ω_ρ, which is a C^∞-class domain because Ω is a C^∞-class domain and $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset$. Since $v_\varepsilon \in D(\Delta,\Omega,\Gamma_0)$, it follows from lemma 1 that $v_\varepsilon \in H^2(\Omega_\rho)$ and we have

$$\forall \beta \in]0,1[,\ \exists\, c>0,\ \forall \lambda>0, \quad ||v_\varepsilon||_{H^1(\Omega_\rho)} \le e^{c\lambda}\,||\Delta v_\varepsilon||_{L^2(\Omega_\rho)} + \frac{1}{\lambda^\beta}||v_\varepsilon||_{H^2(\Omega_\rho)}. \tag{18}$$

By using the estimate (13) of lemma 1 for $v_\varepsilon \in D(\Delta,\Omega,\Gamma_0)$, we have

$$||v_\varepsilon||_{H^2(\Omega_\rho)} \le C\,(||v_\varepsilon||_{L^2(\Omega)} + ||\Delta v_\varepsilon||_{L^2(\Omega)}).$$

Using now theorem 1, in particular the estimate (8) and the fact that u_ε is bounded in $L^2(\Omega)$, there exists constants $c_1 > 0$ and $c_2 > 0$ such that

$$||v_\varepsilon||_{L^2(\Omega)} \le c_1, \quad ||\Delta v_\varepsilon||_{L^2(\Omega)} \le c_2\sqrt{\varepsilon},$$

and then constants $c_3 > 0$ and $c_4 > 0$ such that

$$||v_\varepsilon||_{H^1(\Omega_\rho)} \le c_3\sqrt{\varepsilon}e^{c\lambda} + \frac{c_4}{\lambda^\beta}.$$

By using again the estimate (13) of lemma 1 for v_ε in the domain Ω_ρ, we deduce that for $\rho' > \rho > 0$, we have

$$||v_\varepsilon||_{H^2(\Omega_{\rho'})} \le C\,(||v_\varepsilon||_{L^2(\Omega_\rho)} + ||\Delta v_\varepsilon||_{L^2(\Omega_\rho)}).$$

Hence, there exists constants $c_3' > 0$ and $c_4' > 0$ such that

$$||v_\varepsilon||_{H^2(\Omega_{\rho'})} \le c_3'\sqrt{\varepsilon}e^{c\lambda} + \frac{c_4'}{\lambda^\beta}.$$

Since the function $g(\lambda) = (c_4'/c_3')^2(e^{-2c\lambda}/\lambda^{2\beta})$ is a strictly decreasing function of $\lambda > 0$ which ranges from $+\infty$ for $\lambda = 0^+$ to 0 for $\lambda \to +\infty$, there exists a unique $\lambda(\varepsilon) > 0$ such that $\varepsilon = g(\lambda)$, which is equivalent to

$$c_3'\sqrt{\varepsilon}e^{c\lambda} = \frac{c_4'}{\lambda^\beta}.$$

For that choice of $\lambda(\varepsilon)$, we have $||v_\varepsilon||_{H^2(\Omega_{\rho'})} \leq 2c_4'/\lambda^\beta$, and since $\lambda(\varepsilon) \sim \log(1/\varepsilon)/2c$ when $\varepsilon \to 0$, it follows that there exists $c > 0$ such that

$$||v_\varepsilon||_{H^2(\Omega_{\rho'})} \leq c\,\frac{1}{\left[\log(\frac{1}{\varepsilon})\right]^\beta},$$

which completes the proof of the first estimate of the theorem.

The second estimate of the theorem is obtained the same way by using theorem 2 instead of theorem 1, in particular the estimate (11) and the fact that u_σ is bounded in $L^2(\Omega)$. ■

The function $\varepsilon(\sigma) = \sigma$, for example, satisfies condition (10). For that choice, the second estimate of theorem 3 is simply

$$||u_\sigma - u||_{H^2(\Omega_\rho)} \leq c\,\frac{1}{\left[\log(\frac{1}{\sigma})\right]^\beta}.$$

6. Convergence rate with a source condition

In the previous section, we established a logarithmic convergence rate of the solution of quasi-reversibility $u_{\varepsilon(\sigma),\sigma}$ to the exact solution u when $\sigma \to 0$ in the truncated domains Ω_ρ, with the assumption that $\varepsilon(\sigma)$ is a function which satisfies (10). In the following section, we are interested in improving this convergence rate for a better choice of $\varepsilon(\sigma)$, and possibly in the whole domain Ω. For that purpose, we will come to assume that the exact solution u satisfies a so-called source condition, precisely $u \in \text{Range}(A^*)$. Some source conditions on u exist in a more generalized form (see for example [7]). In the context of Tikhonov regularization, they enable one to obtain information on the convergence rate of the regularized solution to the exact one. Throughout this section, we suppose that Ω is a C^2-class domain which satisfies assumptions (A) (see the definition of (A) at the beginning of section 2).

The following developments are divided into four steps. First, we transform the formulation of quasi-reversibility into a constrained minimization problem (P). Secondly, we form the dual problem (P^*) associated to problem (P) with the help of the theory developed in [6]. Thirdly, we establish a relationship between the solutions of (P) and (P^*). Finally, we find a relationship between these two solutions and the solution of quasi-reversibility, and then find a way to compute $\varepsilon(\sigma)$ in order to obtain a better convergence rate for $u_{\varepsilon(\sigma),\sigma} - u$.

First step : a constrained minimization problem (P)

We saw in section 3 that the solution of quasi-reversibility $u_{\varepsilon,\sigma}$ is the solution of the unconstrained minimization problem

$$\inf_{v\in D(A)} \Big(\frac{1}{2}||Av - f^\sigma||^2_{L^2(\Omega)} + \frac{\varepsilon}{2}||v||^2_{L^2(\Omega)}\Big).$$

Here, $A\ : L^2(\Omega) \to L^2(\Omega)$ denotes the Laplace operator of domain $D(A) = D(\Delta, \Omega, \Gamma_0)$. Let us now focus on the following constrained minimization problem

$$(P) \quad \inf_{v\in K} \frac{1}{2}||v||^2_{L^2(\Omega)} \tag{19}$$

with

$$K = \{v \in D(A)\,;\, ||Av - f^\sigma||_{L^2(\Omega)} \leq \alpha\}. \tag{20}$$

Here $\alpha > 0$ is a real constant.

Such kind of problem has already been considered by several authors (see, e.g., [16, 18, 1, 20]) in the framework of approximate controllability problems, in which the operator A was a linear bounded operator. What is unusual here in problem (P) is that A is a linear unbounded operator, of domain $D(A)$.

We have the following proposition.

Proposition 3 *:*
For $f^\sigma \in L^2(\Omega)$ and $\alpha > 0$, problem (P) has a unique solution $\overline{u}_{\alpha,\sigma}$.

Proof : K is a convex, closed and non-empty subset of the Hilbert space $D(A)$. The non-emptiness of K is a consequence of the fact that $A(D(A))$ is dense in $L^2(\Omega)$. The proof of that denseness result can be found in [14] and relies on proposition 1. The cost function in (19) is convex, continuous and coercive on K. Hence, (P) has at least one solution according to a classical result of the theory of convex minimization (see, e.g., [3]). Since the cost function in (19) is strictly convex, there is at most one solution. ■

It is easy to check that problem (P) can be reformulated as

$$(P) \quad \inf_{v\in D(A)} J(v, Av), \tag{21}$$

where $J\ : L^2(\Omega) \times L^2(\Omega) \to \overline{\mathbb{R}}$ is defined by

$$J(v, p) = \frac{1}{2}||v||^2_{L^2(\Omega)} + I_{B(0,\alpha)}(||p - f^\sigma||_{L^2(\Omega)}). \tag{22}$$

Here, $I_{B(0,\alpha)}$ is defined by

$$\begin{cases} I_{B(0,\alpha)}(\lambda) = 0 & \text{if} \quad |\lambda| \leq \alpha \\ I_{B(0,\alpha)}(\lambda) = +\infty & \text{if} \quad |\lambda| > \alpha. \end{cases}$$

Second step : the dual problem (P^*) of problem (P)

We start by defining the dual problem (P^*) of problem (P), i.e.

$$(P^*) \qquad \sup_{p^* \in D(A^*)} -J^*(A^* p^*, -p^*), \tag{23}$$

where $J^* : L^2(\Omega) \times L^2(\Omega) \to \overline{\mathbb{R}}$ is the Fenchel conjugate function of J. We recall that J^* is defined, if we identify the dual space of $L^2(\Omega)$, i.e. $L^2(\Omega)^*$, with $L^2(\Omega)$ itself, by

$$J^*(v^*, p^*) = \sup_{v \in L^2(\Omega),\, p \in L^2(\Omega)} \Big((v, v^*)_{L^2(\Omega)} + (p, p^*)_{L^2(\Omega)} - J(v, p) \Big).$$

Our objective is to find a relationship between (P) and (P^*).

First, we have the following lemma.

Lemma 3 *:*
For $f^\sigma \in L^2(\Omega)$ and $\alpha > 0$, problems (P) and (P^*) satisfy

$$\inf (P) \geq \sup (P^*). \tag{24}$$

The main idea for proving lemma 3 is to use the well-known theorem of Fenchel-Rockafellar, which can be applied to problems of type (21) for a bounded operator A, i.e. with $D(A) = L^2(\Omega)$. The issue here is that A is an unbounded operator and that the minimization problem (21) is set in $D(A)$ instead of $L^2(\Omega)$. For sake of self-containment, we recall what the theorem of Fenchel-Rockafellar is (see [6] for the proof).

Theorem (Fenchel-Rockafellar) *:*
Let us denote V and Y two Hilbert spaces, V^* and Y^* the corresponding dual spaces. Let us denote $\Lambda : V \to Y$ a bounded operator, and $\Lambda^* : Y^* \to V^*$ its adjoint operator. Finally, let us denote $H : V \times Y \to \overline{\mathbb{R}}$ a convex, lower semi-continuous and proper function, and $H^* : V^* \times Y^* \to \overline{\mathbb{R}}$ the Fenchel conjugate function of H.

We consider the primal minimization problem

$$(P) \qquad \inf_{v \in V} H(v, \Lambda v)$$

and the dual maximization problem

$$(P^*) \qquad \sup_{y^* \in Y^*} -H^*(\Lambda^* y^*, -y^*).$$

If we have $\inf (P) < +\infty$ and if there exists $v_0 \in V$ such that $H(v_0, \Lambda v_0) < +\infty$ and $y \to H(v_0, y)$ is continuous at Λv_0, then

$$\inf (P) = \sup (P^*)$$

and problem (P^*) has at least one solution.

Proof of lemma 3 : in order to use the theorem of Fenchel-Rockafellar, we define $\tilde{A} : D(A) \to L^2(\Omega)$ as the restriction of A to $D(A)$, so that $\tilde{A}$ is now a bounded operator. We also define $\tilde{J} : D(A) \times L^2(\Omega) \to \overline{\mathbb{R}}$ as the restriction of J to $D(A) \times L^2(\Omega)$. Problem (P) is then equivalent to

$$(\tilde{P}) \qquad \inf_{v \in D(A)} \tilde{J}(v, \tilde{A} v). \tag{25}$$

We have now to check that $\tilde{J}$ satisfies the assumptions of the theorem of Fenchel-Rockafellar. It is clear that $\tilde{J}$ is a convex, lower semi-continuous and proper function. Furthermore, since $A(D(A))$ is dense in $L^2(\Omega)$, there exists $v_0 \in D(A)$ such that $||Av_0 - f^\sigma||_{L^2(\Omega)} \leq \alpha/2$. We have hence $\tilde{J}(v_0, \tilde{A}v_0) = J(v_0, Av_0) < +\infty$, and if $||p - Av_0||_{L^2(\Omega)} \leq \alpha/2$, we have $||p - f^\sigma||_{L^2(\Omega)} \leq \alpha$, and then $\tilde{J}(v_0, p) = ||v_0||^2_{L^2(\Omega)}/2$. This implies that there exists $v_0 \in D(A)$ such that $\tilde{J}(v_0, \tilde{A}v_0) < +\infty$ and $p \to \tilde{J}(v_0, p)$ is a continuous function at $\tilde{A}v_0$.

Finally, since $\inf(\tilde{P}) < +\infty$, we can apply the theorem of Fenchel-Rockafellar to problem $(\tilde{P})$, which implies that the dual problem

$$(\tilde{P}^*) \qquad \sup_{p^* \in L^2(\Omega)} -\tilde{J}^*(\tilde{A}^* p^*, -p^*) \tag{26}$$

has solutions and, furthermore,

$$\inf(\tilde{P}) = \sup(\tilde{P}^*). \tag{27}$$

Here, $\tilde{A}^* : L^2(\Omega) \to D(A)^*$ is the adjoint operator of $\tilde{A}$ and $\tilde{J}^* : D(A)^* \times L^2(\Omega) \to \overline{\mathbb{R}}$ is the Fenchel conjugate function of $\tilde{J}$. $\tilde{J}$ is defined, for $(v^*, p^*) \in D(A)^* \times L^2(\Omega)$, by

$$\tilde{J}^*(v^*, p^*) = \sup_{v \in D(A),\, p \in L^2(\Omega)} \Big(\langle v, v^* \rangle_{D(A), D(A)^*} + (p, p^*)_{L^2(\Omega)} - \tilde{J}(v, p) \Big). \tag{28}$$

Note that the space $D(A)^*$ is different from the space $D(A^*)$. The first one is the dual space of the domain $D(A)$ of unbounded operator A, while the second one is the domain of the adjoint operator A^* of A. In particular, we have

$$D(A^*) \subset L^2(\Omega) \subset D(A)^*,$$

where every inclusion is strict.

Now, instead of considering problem $(\tilde{P}^*)$ defined by (26), we consider the following problem, which is obtained by restricting the space of solutions from $L^2(\Omega)$ to $D(A^*)$

$$(P_r^*) \qquad \sup_{p^* \in D(A^*)} -\tilde{J}^*(\tilde{A}^* p^*, -p^*). \tag{29}$$

Let us show that in the above problem (P_r^*), $\tilde{A}^*$ and $\tilde{J}^*$ may be replaced by A^* and J^*. By definition of $D(A^*)$ and A^*, we have

$$(v, A^* v^*)_{L^2(\Omega)} = (Av, v^*)_{L^2(\Omega)}, \quad \forall v \in D(A), \forall v^* \in D(A^*). \tag{30}$$

By definition of $\tilde{A}^*$, we have

$$\left\langle v, \tilde{A}^* v^* \right\rangle_{D(A), D(A)^*} = (\tilde{A}v, v^*)_{L^2(\Omega)}, \quad \forall v \in D(A), \forall v^* \in L^2(\Omega). \tag{31}$$

Using the fact that $\tilde{A}$ is the restriction of A to $D(A)$, and combining (30) and (31), we obtain

$$\left\langle v, \tilde{A}^* v^* \right\rangle_{D(A), D(A)^*} = (v, A^* v^*)_{L^2(\Omega)}, \quad \forall v \in D(A), \forall v^* \in D(A^*).$$

This implies that the linear form $\tilde{A}^* v^*$ on the space $D(A)$ may be identified with the function $A^* v^* \in L^2(\Omega)$. Since $D(A)$ is dense in $L^2(\Omega)$, for $v^* \in D(A^*)$, $\tilde{A}^* v^*$ has a unique extension to $L^2(\Omega)$ which coincide with $A^* v^*$. This implies that A^* may be identified with the restriction of $\tilde{A}^*$ to $D(A^*)$.

Hence, for $p^* \in D(A^*)$, $\tilde{A}^* p^* = A^* p^* \in L^2(\Omega)$, and we are now interested in the restriction of $\tilde{J}^*$ to $L^2(\Omega) \times L^2(\Omega)$. From (28), we deduce that for $(v^*, p^*) \in L^2(\Omega) \times L^2(\Omega)$

$$\tilde{J}^*(v^*, p^*) = \sup_{v \in D(A),\, p \in L^2(\Omega)} \Big((v, v^*)_{L^2(\Omega)} + (p, p^*)_{L^2(\Omega)} - J(v, p) \Big).$$

The definition of J given by (22) implies that for $(v^*, p^*) \in L^2(\Omega) \times L^2(\Omega)$, we have

$$\tilde{J}^*(v^*, p^*) = \sup_{v \in D(A)} G_{v^*}(v) + \sup_{p \in L^2(\Omega)} H_{p^*}(p),$$

where $G_{v^*} \ : \ L^2(\Omega) \to \overline{\mathbb{R}}$ and $H_{p^*} \ : \ L^2(\Omega) \to \overline{\mathbb{R}}$ are defined by

$$G_{v^*}(v) = (v, v^*)_{L^2(\Omega)} - \frac{1}{2}||v||^2_{L^2(\Omega)},$$

$$H_{p^*}(p) = (p, p^*)_{L^2(\Omega)} - I_{B(0,\alpha)}(||p - f^\sigma||_{L^2(\Omega)}).$$

Since $D(A)$ is dense in $L^2(\Omega)$ and the function G_{v^*} is continuous, we obtain

$$\tilde{J}^*(v^*, p^*) = \sup_{v \in L^2(\Omega)} G_{v^*}(v) + \sup_{p \in L^2(\Omega)} H_{p^*}(p).$$

For $(v^*, p^*) \in L^2(\Omega) \times L^2(\Omega)$, we thus obtain

$$\tilde{J}^*(v^*, p^*) = J^*(v^*, p^*).$$

Finally, problem (29) simply becomes

$$(P_r^*) \ \sup_{p^* \in D(A^*)} -J^*(A^* p^*, -p^*),$$

and then (P_r^*) coincides with (P^*).

Since problem $(\tilde{P})$ and problem (P) are equivalent, and given (26), (27) and (29), we have

$$+\infty > \inf{(P)} = \inf{(\tilde{P})} = \sup{(\tilde{P}^*)} \geq \sup{(P^*)},$$

which completes the proof. ■

A duality gap between problems (P) and (P^*) is likely to occur. We will come to prove shortly that actually, this is not the case, i.e. $\inf{(P)} = \sup{(P^*)}$. Now, let us give an explicit form for problem (P^*) given by (23). It amounts to specify $D(A^*)$, A^* and J^*.

We have the following proposition.

Proposition 4 *:*

The adjoint operator of operator A is the Laplace operator $A^* \ : \ L^2(\Omega) \to L^2(\Omega)$ of domain $D(A^*) = D(\Delta, \Omega, \Gamma_1)$, where

$$D(\Delta, \Omega, \Gamma_1) = \{v \in D(\Delta, \Omega)\,;\, v|_{\Gamma_1} = 0,\, \frac{\partial v}{\partial n}|_{\Gamma_1} = 0\}.$$

Proof : we recall that $D(A^*)$ and A^* are defined as follows.

$$D(A^*) = \{v^* \in L^2(\Omega)\,;\, \exists w \in L^2(\Omega)\,;\, (Av, v^*)_{L^2(\Omega)} = (v, w)_{L^2(\Omega)},\, \forall v \in D(A)\}.$$

Since $D(A)$ is dense in $L^2(\Omega)$, w defined as such is unique and we set $A^* v^* = w$ for all $v^* \in D(A^*)$.

First, we easily check that $D(A^*) \subset D(\Delta, \Omega, \Gamma_1)$. To do so, let us suppose $v^* \in D(A^*)$. First, for all $\phi \in C_0^\infty(\Omega) \subset D(\Delta, \Omega, \Gamma_0)$, we obtain

$$(\Delta \phi, v^*)_{L^2(\Omega)} = (\phi, w)_{L^2(\Omega)},$$

which implies $\Delta v^* = w \in L^2(\Omega)$. Therefore, $v^* \in D(\Delta,\Omega)$ and $A^* v^* = \Delta v^*$. Now let us choose $v \in H^2(\Omega,\Gamma_0) \subset D(\Delta,\Omega,\Gamma_0)$. Using a double integration by parts, it follows that

$$(\Delta v, v^*)_{L^2(\Omega)} = \left\langle \frac{\partial v}{\partial n}, v^* \right\rangle_{H^{\frac{1}{2}}(\partial\Omega), H^{-\frac{1}{2}}(\partial\Omega)} - \left\langle v, \frac{\partial v^*}{\partial n} \right\rangle_{H^{\frac{3}{2}}(\partial\Omega), H^{-\frac{3}{2}}(\partial\Omega)} + (v, \Delta v^*)_{L^2(\Omega)}.$$

$v^* \in D(A^*)$ implies $(\Delta v, v^*)_{L^2(\Omega)} = (v, \Delta v^*)_{L^2(\Omega)}$, and $v \in H^2(\Omega,\Gamma_0)$ implies $v|_{\Gamma_0} = 0$ and $(\partial v/\partial n)|_{\Gamma_0} = 0$. It follows that $v^*|_{\Gamma_1} = 0$ and $(\partial v^*/\partial n)|_{\Gamma_1} = 0$. We conclude that $D(A^*) \subset D(\Delta,\Omega,\Gamma_1)$.

By using again a double integration by parts, this time for $v \in D(\Delta,\Omega,\Gamma_0)$ and $v^* \in H^2(\Omega,\Gamma_1)$, we obtain

$$(\Delta v, v^*)_{L^2(\Omega)} = (v, \Delta v^*)_{L^2(\Omega)}.$$

We complete the proof by using proposition 2, given the assumption $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset$. This proposition implies that $H^2(\Omega,\Gamma_1)$ is dense in $D(\Delta,\Omega,\Gamma_1)$ for the norm $||.||_{D(\Delta,\Omega)}$, which means that the above relationship still holds for $v^* \in D(\Delta,\Omega,\Gamma_1)$. Thus, $D(A^*) = D(\Delta,\Omega,\Gamma_1)$. ■

We have identified $D(A^*)$ and A^*. As concerns J^*, a simple computation (see, e.g., [1]) provides

$$J^*(u^*, p^*) = \frac{1}{2}||u^*||^2_{L^2(\Omega)} + \alpha||p^*||_{L^2(\Omega)} + (f^\sigma, p^*)_{L^2(\Omega)}.$$

As a consequence, the problem (P^*) has the explicit form

$$(P^*) \qquad \sup_{p^* \in D(A^*)} -\frac{1}{2}||A^* p^*||^2_{L^2(\Omega)} - \alpha||p^*||_{L^2(\Omega)} + (f^\sigma, p^*)_{L^2(\Omega)}, \tag{32}$$

or

$$(P^*) \qquad \inf_{p^* \in D(A^*)} F(p^*), \tag{33}$$

where the function $F : D(A^*) \to \mathbb{R}$ is defined by

$$F(p^*) = \frac{1}{2}||A^* p^*||^2_{L^2(\Omega)} + \alpha||p^*||_{L^2(\Omega)} - (f^\sigma, p^*)_{L^2(\Omega)}. \tag{34}$$

Remark 2 : problem (P^*), given by (33), contrary to problem (P), given by (19), consists therefore in a minimization problem without any constraint.

We have the following proposition.

Proposition 5 *:*

For $f^\sigma \in L^2(\Omega)$ and $\alpha > 0$, problem (P^*) has a unique solution $p_{\alpha,\sigma}$.

Proof: the set $D(A^*)$ is an Hilbert space when equipped with the graph norm $||.||_{D(\Delta,\Omega)}$. The function F is convex and continuous. Existence of a solution of problem (P^*) relies therefore on the coercivity of F. Let us assume that F is not coercive. Then one could find a sequence of $p_n \in D(A^*)$ with $n \in \mathbb{N}$ and a constant C such that $||p_n||_{D(\Delta,\Omega)} \to +\infty$ when $n \to +\infty$ while $F(p_n) \leq C$ for all n. As a consequence, $||p_n||_{L^2(\Omega)} \to +\infty$. If it were not the case, one could find a subsequence of (p_n), still denoted (p_n), such that both $||p_n||_{L^2(\Omega)}$ and $||\Delta p_n||_{L^2(\Omega)}$ be bounded (because of the expression of $F(p_n)$), which would contradict $||p_n||_{D(\Delta,\Omega)} \to +\infty$.

Let us denote $\beta_n = ||p_n||_{L^2(\Omega)} \to +\infty$, and $\tilde{p}_n = p_n/\beta_n$. We have $||\tilde{p}_n||_{L^2(\Omega)} = 1$ and

$$\frac{F(p_n)}{\beta_n^2} = \frac{1}{2}||\Delta\tilde{p}_n||^2_{L^2(\Omega)} + \frac{1}{\beta_n}\Big(\alpha - (f^\sigma, \tilde{p}_n)_{L^2(\Omega)}\Big).$$

Using $F(p_n) \leq C$, we obtain

$$||\Delta\tilde{p}_n||^2_{L^2(\Omega)} \leq \frac{2C}{\beta_n^2} + \frac{2}{\beta_n}(||f^\sigma||_{L^2(\Omega)} - \alpha),$$

which implies $||\Delta\tilde{p}_n||_{L^2(\Omega)} \to 0$ when $n \to +\infty$.

Since the sequence $(\tilde{p}_n)$ is bounded in $L^2(\Omega)$, there exists a subsequence, still denoted $(\tilde{p}_n)$, such that $\tilde{p}_n \rightharpoonup \tilde{p}$ in $L^2(\Omega)$ weakly. Since $\Delta\tilde{p}_n \to 0$ in $L^2(\Omega)$, we have $\Delta\tilde{p} = 0$, and $\tilde{p}_n \rightharpoonup \tilde{p}$ in $D(\Delta, \Omega)$. Since $D(A^*)$ is a closed subspace of $D(\Delta, \Omega)$, it is weakly closed (see, e.g., [3]). We have therefore $\tilde{p} \in D(A^*)$ and $\Delta\tilde{p} = 0$, and the Cauchy uniqueness (i.e. proposition 1) leads to $\tilde{p} = 0$.

Finally,

$$F(p_n) \geq \beta_n\Big(\alpha - (f^\sigma, \tilde{p}_n)_{L^2(\Omega)}\Big).$$

Since $(f^\sigma, \tilde{p}_n)_{L^2(\Omega)} \to 0$, we obtain $F(p_n) \to +\infty$, which contradicts $F(p_n) \leq C$. The function F is therefore coercive.

To complete the proof, it remains to check that F is strictly convex, which ensures uniqueness of the solution of (P^*). F is the sum of the function $L : D(A^*) \to \mathbb{R}$ defined by $L(p^*) \to ||A^*p^*||^2_{L^2(\Omega)}/2$ and of a convex function. Due to Cauchy uniqueness (i.e. proposition 1), L is strictly convex, and then F is strictly convex. ■

Third step : the relationship between solutions of (P) *and* (P^*)

We have proved that problem (P) and problem (P^*) have unique solutions, denoted respectively $\overline{u}_{\alpha,\sigma}$ and $p_{\alpha,\sigma}$. We now establish a relationship between these two solutions, using the fact that the function F defined by (34) is differentiable in the sense of Gâteaux for $p^* \neq 0$.

Following the same kind of developments as in [1], we obtain the following proposition.

Proposition 6 *:*

For $f^\sigma \in L^2(\Omega)$ *and* $\alpha > 0$*, the solutions of* (P) *and* (P^*)*, i.e. respectively* $\overline{u}_{\alpha,\sigma}$ *and* $p_{\alpha,\sigma}$*, satisfy the relationship* $\overline{u}_{\alpha,\sigma} = A^*p_{\alpha,\sigma}$*, and*

$$\inf(P) = \sup(P^*).$$

Proof : we have to distinguish the case $||f^\sigma||_{L^2(\Omega)} \leq \alpha$ from the case $||f^\sigma||_{L^2(\Omega)} > \alpha$. In the first case, it is easy to check that both $\overline{u}_{\alpha,\sigma} = 0$ and $p_{\alpha,\sigma} = 0$, which implies proposition 6. In the second case, we have $\overline{u}_{\alpha,\sigma} \neq 0$ and $p_{\alpha,\sigma} \neq 0$. The first statement is obvious. To prove the second one, we use a sequence (f_n) which belongs to $D(A^*)$ such that $f_n \to f^\sigma$ in $L^2(\Omega)$ when $n \to +\infty$. Such a sequence exists, because $D(A^*)$ is dense in $L^2(\Omega)$. Denoting $p_n = f_n/||f_n||_{L^2(\Omega)}$, we have for $\varepsilon > 0$

$$F(\varepsilon p_n) = \frac{\varepsilon^2}{2}||A^*p_n||^2_{L^2(\Omega)} + \varepsilon\Big(\alpha - \frac{(f^\sigma, f_n)_{L^2(\Omega)}}{||f_n||_{L^2(\Omega)}}\Big).$$

Since $(f^\sigma, f_n)_{L^2(\Omega)}/||f_n||_{L^2(\Omega)}$ converges when $n \to +\infty$ to $||f^\sigma||_{L^2(\Omega)} > \alpha$, we have $\alpha - (f^\sigma, f_n)_{L^2(\Omega)}/||f_n||_{L^2(\Omega)} < 0$ for a sufficiently large $n = n_0$. For sufficiently small ε, $F(\varepsilon p_{n_0}) < 0 = F(0)$, and hence $p_{\alpha,\sigma} \neq 0$.

In the case $||f^\sigma||_{L^2(\Omega)} > \alpha$, by differentiation of (34), we obtain that the solution $p_{\alpha,\sigma}$ of problem (P^*) is the unique solution in $D(A^*)$ of the weak formulation

$$(A^* p_{\alpha,\sigma}, A^* q)_{L^2(\Omega)} + \alpha(\frac{p_{\alpha,\sigma}}{||p_{\alpha,\sigma}||_{L^2(\Omega)}}, q)_{L^2(\Omega)} - (f^\sigma, q)_{L^2(\Omega)} = 0, \quad \forall q \in D(A^*). \tag{35}$$

Choosing $q = p_{\alpha,\sigma}$ in (35), it follows that

$$||A^* p_{\alpha,\sigma}||^2_{L^2(\Omega)} + \alpha||p_{\alpha,\sigma}||_{L^2(\Omega)} - (f^\sigma, p_{\alpha,\sigma})_{L^2(\Omega)} = 0.$$

Using now lemma 3, we obtain

$$\frac{1}{2}||\overline{u}_{\alpha,\sigma}||^2_{L^2(\Omega)} \geq -\frac{1}{2}||A^* p_{\alpha,\sigma}||^2_{L^2(\Omega)} - \alpha||p_{\alpha,\sigma}||_{L^2(\Omega)} + (f^\sigma, p_{\alpha,\sigma})_{L^2(\Omega)},$$

which together with the previous equation leads to

$$\frac{1}{2}||\overline{u}_{\alpha,\sigma}||^2_{L^2(\Omega)} \geq \frac{1}{2}||A^* p_{\alpha,\sigma}||^2_{L^2(\Omega)}. \tag{36}$$

From (35) and since $A^{**} = A$, we also obtain that $A^* p_{\alpha,\sigma} \in D(A)$ and

$$A(A^* p_{\alpha,\sigma}) + \alpha\frac{p_{\alpha,\sigma}}{||p_{\alpha,\sigma}||_{L^2(\Omega)}} - f^\sigma = 0, \tag{37}$$

which implies in particular that

$$||A(A^* p_{\alpha,\sigma}) - f^\sigma||_{L^2(\Omega)} = \alpha, \tag{38}$$

and hence $A^* p_{\alpha,\sigma} \in K$, K being defined by (20). This result, together with (36), implies that $A^* p_{\alpha,\sigma} = \overline{u}_{\alpha,\sigma}$ from uniqueness in problem (P). Hence $\inf(P) = \sup(P^*)$. ■

Remark 3 : as a consequence of proposition 6, $\overline{u}_{\alpha,\sigma} \in \text{Range}(A^*)$.

Fourth step : back to the solution of quasi-reversibility and conclusion

We now establish, in the case $||f^\sigma||_{L^2(\Omega)} > \alpha$, a relationship between $\overline{u}_{\alpha,\sigma}$, $p_{\alpha,\sigma}$, and the solution of quasi-reversibility $u_{\varepsilon,\sigma}$, i.e. the solution of problem (12).

We have the following proposition.

Proposition 7 *:*

For $f^\sigma \in L^2(\Omega)$ and $\alpha > 0$ such that $||f^\sigma||_{L^2(\Omega)} > \alpha$, if $\overline{u}_{\alpha,\sigma}$ and $p_{\alpha,\sigma}$ are the solutions of (P) and (P^*) respectively, and if $u_{\varepsilon,\sigma}$ is the solution of (12) for $\varepsilon = \alpha/||p_{\alpha,\sigma}||_{L^2(\Omega)}$, then $\overline{u}_{\alpha,\sigma} = u_{\varepsilon,\sigma}$.

Proof : the equation (37) implies

$$(A(A^* p_{\alpha,\sigma}), Av)_{L^2(\Omega)} + \alpha(\frac{p_{\alpha,\sigma}}{||p_{\alpha,\sigma}||_{L^2(\Omega)}}, Av)_{L^2(\Omega)} = (f^\sigma, Av), \quad \forall v \in D(A).$$

Since $p_{\alpha,\sigma} \in D(A^*)$ and $\overline{u}_{\alpha,\sigma} = A^* p_{\alpha,\sigma}$, we obtain

$$(A\overline{u}_{\alpha,\sigma}, Av)_{L^2(\Omega)} + \frac{\alpha}{||p_{\alpha,\sigma}||_{L^2(\Omega)}}(\overline{u}_{\alpha,\sigma}, v)_{L^2(\Omega)} = (f^\sigma, Av), \quad \forall v \in D(A).$$

It follows that $\overline{u}_{\alpha,\sigma} = u_{\varepsilon,\sigma}$ on condition that $\varepsilon = \alpha/||p_{\alpha,\sigma}||_{L^2(\Omega)}$. ■

From now on, we assume that $\alpha = \sigma$, with $||f^\sigma - f||_{L^2(\Omega)} \leq \sigma < ||f^\sigma||_{L^2(\Omega)}$, and that

$$\varepsilon(\sigma) = \frac{\sigma}{||p_{\sigma,\sigma}||_{L^2(\Omega)}}. \tag{39}$$

Making these choices, we have $\overline{u}_{\sigma,\sigma} = u_{\varepsilon(\sigma),\sigma}$, and from (38) and $\overline{u}_{\sigma,\sigma} = A^* p_{\sigma,\sigma}$, we obtain

$$||Au_{\varepsilon(\sigma),\sigma} - f^\sigma||_{L^2(\Omega)} = \sigma. \tag{40}$$

What (40) means precisely is that the choice of the regularization parameter $\varepsilon(\sigma)$ in the formulation of quasi-reversibility, defined by (39), satisfies the so-called Morozov *discrepancy principle*. This principle, which is well-known in the theory of regularization, is for example described in [11] in the framework of the Tikhonov regularization of bounded operators. The discrepancy principle is here extended to the regularization of an unbounded closed operator. It consists of adjusting the regularization parameter ε to the amplitude of noise σ, such a way that the discrepancy between $Au_{\varepsilon(\sigma),\sigma}$ and f^σ be of the same order than the discrepancy between f^σ and f.

Eventually, we find a convergence rate for $u_{\varepsilon(\sigma),\sigma} - u$, $\varepsilon(\sigma)$ being defined by (39), on the additional condition that $u \in \text{Range}(A^*)$.

We have the following theorem.

Theorem 4 :
We assume that for $f \in L^2(\Omega)$, problem (3) (4) admits a (unique) solution u in $L^2(\Omega)$, with u satisfying the source condition $u \in \text{Range}(A^*)$. For $f^\sigma \in L^2(\Omega)$ with $||f^\sigma - f||_{L^2(\Omega)} \leq \sigma < ||f^\sigma||_{L^2(\Omega)}$, if $u_{\varepsilon(\sigma),\sigma} = u_\sigma$ is the solution of problem (12) with $\varepsilon(\sigma)$ defined by (39), then we have

$$||u_\sigma - u||_{L^2(\Omega)} \leq C\sqrt{\sigma}, \tag{41}$$

$$||\Delta(u_\sigma - u)||_{L^2(\Omega)} \leq 2\sigma, \tag{42}$$

where $C > 0$ is independent on σ.

Proof : the proof is strongly inspired from the one proposed in [11], which we here extend to the case of the unbounded operator A without any difficulty. Since the solution of quasi-reversibility $u_{\varepsilon(\sigma),\sigma}$ is the solution of the minimization problem (9), we have

$$||\Delta u_\sigma - f^\sigma||^2_{L^2(\Omega)} + \varepsilon(\sigma)\,||u_\sigma||^2_{L^2(\Omega)} \leq ||\Delta u - f^\sigma||^2_{L^2(\Omega)} + \varepsilon(\sigma)\,||u||^2_{L^2(\Omega)}.$$

Since $||\Delta u_\sigma - f^\sigma||_{L^2(\Omega)} = \sigma$, $\Delta u = f$ and $||f^\sigma - f||_{L^2(\Omega)} \leq \sigma$, we obtain

$$\sigma^2 + \varepsilon(\sigma)\,||u_\sigma||^2_{L^2(\Omega)} \leq \sigma^2 + \varepsilon(\sigma)\,||u||^2_{L^2(\Omega)},$$

which implies

$$||u_\sigma||_{L^2(\Omega)} \leq ||u||_{L^2(\Omega)}. \tag{43}$$

Since

$$||u_\sigma - u||^2_{L^2(\Omega)} = ||u_\sigma||^2_{L^2(\Omega)} - 2(u_\sigma, u)_{L^2(\Omega)} + ||u||^2_{L^2(\Omega)},$$

it follows from (43) that

$$||u_\sigma - u||^2_{L^2(\Omega)} \leq 2(u - u_\sigma, u)_{L^2(\Omega)}.$$

Using the fact that $u \in \text{Range}(A^*)$, which means there exist $p \in D(A^*)$ such that $u = \Delta p$ (p is unique), and the fact that $u - u_\sigma \in D(A)$, we have

$$(u - u_\sigma, u)_{L^2(\Omega)} = (u - u_\sigma, \Delta p)_{L^2(\Omega)} = (\Delta(u - u_\sigma), p)_{L^2(\Omega)} = (f - \Delta u_\sigma, p)_{L^2(\Omega)}$$

$$(u - u_\sigma, u)_{L^2(\Omega)} = (f - f^\sigma, p)_{L^2(\Omega)} + (f^\sigma - \Delta u_\sigma, p)_{L^2(\Omega)}$$

$$\leq \Big(||f - f^\sigma||_{L^2(\Omega)} + ||f^\sigma - \Delta u_\sigma||_{L^2(\Omega)}\Big)||p||_{L^2(\Omega)} \leq 2\sigma\,||p||_{L^2(\Omega)}.$$

We finally have

$$||u_\sigma - u||^2_{L^2(\Omega)} \leq 4\sigma\,||p||_{L^2(\Omega)},$$

which implies (41). The estimate (42) is an immediate consequence of

$$||\Delta(u_\sigma - u)||_{L^2(\Omega)} \leq ||\Delta u_\sigma - f^\sigma||_{L^2(\Omega)} + ||f^\sigma - f||_{L^2(\Omega)}. \blacksquare$$

Remark 4 : from lemma 1, we have the following estimate on the truncated domains Ω_ρ defined by (16)

$$||u_\sigma - u||_{H^2(\Omega_\rho)} \leq C\sqrt{\sigma}.$$

Remark 5 (numerical point of view) : finding a function u_σ which satisfies the estimates of theorem 4 requires to solve the problem (33) instead of the initial problem (12). From a numerical point of view, problem (33) is more difficult to solve than problem (12). Problem (12) is a weak formulation which would provide a numerical solution using a single F.E.M. computation. Problem (33) is a real minimization problem which needs to iterate F.E.M. computations by using a gradient method in order to approximate the solution. Furthermore, we have to point out the fact that the cost function of (33) is not differentiable, but the important point is that it is differentiable in a vicinity of the solution.

Besides, the F.E.M. computation would bring about two additional difficulties in solving problems (12) and (33). First, concerning the choice of the finite element itself, we observe that these two problems involve functions $u \in L^2(\Omega)$ with $\Delta u \in L^2(\Omega)$. Hence, F.E.M. computation necessarily requires to handle unusual finite elements, like the Argyris' triangle (conforming method) or the Morley's triangle (nonconforming method), which have to be specifically implemented. Secondly, the F.E.M. computation would introduce a new small parameter h (i.e. the diameter of the elements), in addition to the parameter ε (12) or α (33) of each problem. Hence, this raises the difficult question of a relevant choice of h depending on the amplitude of the noise σ.

7. Conclusion

In this paper, we obtained two results concerning the convergence rate of the method of quasi-reversibility in presence of noisy data. The amplitude of that noise being σ, the first result (section 5) is a convergence rate of type $1/(\log\frac{1}{\sigma})^\beta$ in the general case, the second one (section 6) is a convergence rate of type $\sigma^{\frac{1}{2}}$ when a source condition is assumed.

These two results were obtained for a smooth domain Ω and in the restrictive case $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset$, which may be considered as strong limitations for real applications. Therefore, it would be interesting to investigate how to weaken those requirements. For example, the second result holds for a C^2-class domain due to the fact that in the quasi-reversibility formulation we have chosen, we handle functions in $L^2(\Omega)$. It would be possible to consider Lipschitz continuous domains by using a quasi-reversibility formulation involving functions in $H^1(\Omega)$ (like in remark 1), but the author failed in deriving the same kind of results as proposed in section 6 for such formulation. Besides, if we omit the assumption $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset$ in section 6, we meet with a problem which seems to be strongly related to the ill-posedness of mixed boundary value problems in $L^2(\Omega)$, such as discussed in [9] (see section 2.3).

The result obtained in section 6 appears to the author as a nice illustration of a possible extension of the theory of duality presented in [6] for an unbounded closed operator instead of a bounded operator. A minimization problem involving operator A of domain $D(A)$ is thus transformed into an equivalent minimization problem involving its adjoint operator A^* of domain $D(A^*)$. This result also answers the question how to make a relevant choice of the parameter ε of the quasi-reversibility method as a function of σ. Precisely, we proved the feasibility of applying the discrepancy principle to the case of our unbounded closed operator. In [22], another approach to the discrepancy principle for general unbounded closed operators has been recently proposed. It is based on the spectral theory of selfadjoint unbounded operators.

To complete these concluding remarks, we mention that it would be interesting to test the obtained convergence rates numerically, by using F.E.M. computations. Such task is in progress.

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