

# RESONANCES OF AN ELASTIC PLATE IN A COMPRESSIBLE CONFINED FLUID

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## Summary

We present a theoretical study of the resonances of a fluid-structure problem, an elastic plate placed in a duct in the presence of a compressible fluid. The case of a rigid plate has been largely studied. Acoustic resonances are then associated to resonant modes trapped by the plate. Due to the elasticity of the plate we need to solve a quadratic eigenvalue problem in which the resonance frequencies  $k$  solve the equations  $\gamma(k) = k^2$  where  $\gamma$  are the eigenvalues of a self-adjoint operator of the form  $A + kB$ . First we show how to study the eigenvalues located below the essential spectrum by using the Min-Max principle. Then we study the fixed-point equations. We establish sufficient conditions on the characteristics of the plate and of the fluid that ensure the existence of resonances. Such conditions are validated numerically.

## 1. Introduction

Trapped modes for rigid obstacles in two-dimensional acoustic waveguides have been widely studied. Our aim is to extend these studies to the case of an elastic obstacle. More precisely we are interested in determining the resonance frequencies of an elastic plate in a duct in the presence of a uniform flow of a compressible fluid. A resonance phenomenon occurs when, without any acoustic source, fluid vibrations and plate deformations periodic in time and with finite energy can exist. The condition on the energy requires the fluid vibrations to be localized around the plate: thus resonant modes are also called trapped modes. Determination of resonances is of practical importance because of the following consequence: if  $\omega_c$  is a resonance frequency, the solution of the transient problem for a causal source of frequency  $\omega_c$  behaves like  $te^{-i\omega_c t}$  for large times. Since the solution is unbounded in time it can cause damage. In particular these phenomena are of great importance for the offshore industry, especially when dealing with structures such as floating bridges or floating airports which consist of a floating platform supported by cylindrical legs (**1**): the interaction of surface waves with obstacles placed vertically in a fluid layer can give rise to resonances.

Most of the studies of acoustic resonances deal with the case of a fluid at rest and a rigid obstacle, but take into account obstacles of various shapes. Trapped modes are known to occur for a large class of symmetric obstacles placed on the centreline of the guide. They have been studied numerically in the case of cylinders (**2**, **3**) and criteria on the geometry

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of the obstacle for existence were obtained thanks to variational methods (**4**, **5**, **6**). In the particular case of plates, it has been proved that resonances exist for sufficiently long plates (**7**) and that the number of trapped modes increases with the length of the plate. Precise estimations of the number of trapped modes versus the plate length were obtained for small lengths of the plate (**8**, **9**) and later for general lengths (**10**). Note that the existence of trapped mode about a plate is not systematic since if homogeneous Dirichlet conditions are considered on the channel walls, then no mode exists for a centred plate (at least below the cut-off frequency of antisymmetric modes of the duct (**9**, **11**, **12**)).

To our knowledge, no study of trapped modes in the presence of an elastic obstacle, in presence or not of a compressible flow in a duct has been made. The stability of a finite elastic plate in a waveguide has been studied but in the case of an incompressible flow (**13**). Also the influence of the elastic deformations of a plate on a flow has been considered when the fluid is not confined in a duct: when the fluid is compressible (without flow (**14**) and with flow (**15**)) or when the fluid is incompressible (plate with free ends (**16**, **17**) and plate with fixed ends (**18**)). In both cases (duct with incompressible fluid or no duct) no trapped mode can exist, but the coupling between an elastic plate and a flow can lead to an instability.

The investigation of an elastic plate in a compressible confined fluid induces new difficulties. In particular it cannot be formulated as a classical eigenvalue problem: the resonance frequencies  $k$  satisfy a quadratic problem of the general form  $A + kB = k^2I$  where  $A$  and  $B$  are self-adjoint operators and  $I$  is the identity operator;  $B$  is a coupling operator between the plate and the fluid. The general method to find the resonance frequencies consists in looking for the eigenvalues  $\gamma(k)$  of  $A + kB$  and then to solve the nonlinear eigenvalue problem  $\gamma(k) = k^2$ . In this paper, we consider the case of a fluid at rest. The influence of the flow is just briefly analyzed in the case of a rigid plate in section 4.1.3 and illustrated in section 5.1.2. The general case of an elastic plate in a flow will be treated in another article.

Concerning the characteristics of the trapping obstacle, the plate is clamped on one edge and free on the other. Variational arguments will be used to prove the existence of resonances. The required mathematical technique is based on the spectral theory for self-adjoint operators. Such techniques have been widely used to treat problems of guided waves (similar to problems of resonances) in unbounded domains (open waveguides) appearing in various areas of physics: in acoustics (**20**, **21**), in electromagnetism (**22**, **23**), in hydrodynamics (**24**, **25**) or in elasticity (**26**, **27**). All these problems have in common that the essential spectrum is bounded from below: then isolated eigenvalues below the essential spectrum can exist and be studied thanks to the Min-Max principle.

The beginning of this paper will be devoted to the description of the mathematical framework. The second part will present the nonlinear eigenvalue problem and the induced difficulties. In the third part the classical existence results for trapped modes in two simpler configurations will be recalled: a rigid plate in a fluid at rest (easily extended to the presence of a flow) and an elastic plate in vacuum. Then thanks to comparisons with these two simple configurations, lower estimates of the number of trapped modes will be deduced. Finally the accuracy of the estimates will be checked numerically.

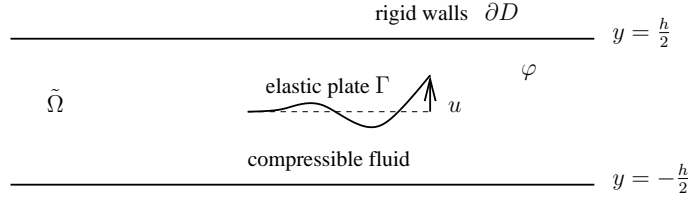


Fig. 1 Geometry of the problem

## 2. The mathematical framework

### 2.1 The equations for the resonant modes

We consider a two-dimensional infinite duct  $\tilde{D} = \{(x, y); -h/2 < y < h/2\}$  of height  $h$  and of boundary  $\partial\tilde{D}$ . The elastic plate is defined at rest by  $\Gamma = \{(x, y); y = 0 \text{ and } 0 < x < L\}$ . The propagation domain is noted  $\tilde{\Omega}$  (see Fig. 1):  $\tilde{\Omega} = \tilde{D} \setminus \Gamma$ . In the time-harmonic regime of frequency  $\omega$ , Euler's equations lead to the Helmholtz equation for the velocity potential  $\phi$  ( $\mathbf{v} = \nabla\phi$  where  $\mathbf{v}$  is the perturbation velocity and the factor  $e^{-i\omega t}$  is omitted):  $(\Delta + k^2)\phi = 0$ . The acoustic wavenumber  $k = \omega/c$  (where  $c$  is the sound velocity) will be called frequency in the sequel.

For the elastic plate, its vertical displacement  $w(x, t)$  follows the equation

$$D \frac{\partial^4 w}{\partial x^4} + \rho_p a \frac{\partial^2 w}{\partial t^2} = -[p], \quad (2.1)$$

where  $\rho_p$  is the mass density of the plate,  $a$  the thickness of the plate,  $D = Ea^3/12(1-\nu^2)$  the flexural rigidity with  $E$  the Young modulus and  $\nu$  the Poisson ratio;  $[p]$  denotes the pressure jump through the plate,  $[p] = p(x, 0^+) - p(x, 0^-)$ , where  $p(x, 0^\pm) = \lim_{\epsilon \rightarrow 0^+} p(x, \pm\epsilon)$ . From Euler's equations is deduced that the acoustic pressure  $p$  is linked to  $\phi$  through the relation  $p = i\rho\omega\phi$ . Then (2.1) is replaced by

$$\tilde{\xi} (\mu^4 u^{iv}(x) - k^2 u) = -i(k/c)[\phi],$$

with  $w(x, t) = \text{Re} [u(x)e^{-i\omega t}]$  and  $u^{iv}(x) = d^4 u/dx^4$ . We have introduced the parameters  $\mu^4 = D/\rho_p a c^2$  and  $\tilde{\xi} = a\rho_p/\rho$ .  $\tilde{\xi}$  measures the fluid-structure coupling whereas  $\mu$  measures the elasticity of the plate. Note that  $\mu$  and  $\tilde{\xi}$  are not non-dimensional parameters; they are introduced just to simplify the formulae.

Concerning the boundary conditions, on the guide walls  $\partial\tilde{D}$  the slip condition reads  $\partial\phi/\partial y = 0$ . On the plate, if we let  $\zeta$  denote the displacement in the fluid, the continuity of the displacement at the fluid/plate interface reads  $u = \zeta_y$  on both sides of  $\Gamma$ . Using the link between the displacement and the fluid velocity  $\mathbf{v} = \partial\zeta/\partial t$  we get easily that

$$\frac{\partial\phi}{\partial y} = -i\omega u \text{ on } \Gamma.$$

The plate is taken to be clamped-free:  $u(0) = 0 = u'(0)$  and  $u''(L) = 0 = u'''(L)$ . Our method is general and could be applied to the case of a clamped-clamped plate.

Two couplings between the plate and the fluid vibrations exist: the first one is due to the

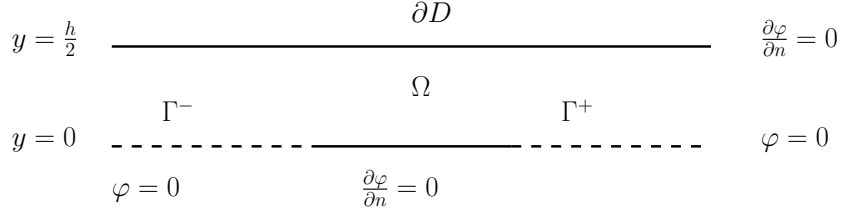


Fig. 2 Half duct

driving force  $[p]$  in the plate equation and the second is linked to the boundary condition satisfied on the plate. Introducing the new velocity potential  $\varphi = \phi/c$ , then looking for resonance frequencies is equivalent to solve the following problem: find  $k \in \mathbb{R}$  such that there exist  $(\varphi, u) \neq 0$  with finite energy (meaning that  $\int_{\tilde{\Omega}} |\varphi|^2 < \infty$  and  $\int_{\Gamma} |u|^2 < \infty$ ) satisfying

$$\begin{aligned} -\Delta\varphi &= k^2\varphi & \text{in } \tilde{\Omega}, & & \mu^4 u^{\text{iv}} + i(k/\tilde{\xi})[\varphi] &= k^2 u & \text{on } \Gamma, \\ \frac{\partial\varphi}{\partial y} &= 0 & \text{on } \partial\tilde{D}, & & \frac{\partial\varphi}{\partial y} &= -iku & \text{on } \Gamma, \\ u(0) &= 0 = u'(0), & & & u''(L) &= 0 = u'''(L). \end{aligned}$$

## 2.2 Symmetry considerations and reduction to a half-guide

We restrict ourselves to the case of a plate placed in the centre of the duct. Then every velocity potential  $\varphi$  can be decomposed in the form  $\varphi_s + \varphi_a$  where  $\varphi_s$  ( $\varphi_a$ ) is symmetric (antisymmetric) with respect to the mid-plane  $y = 0$ . Since the symmetric velocity potential satisfies  $\partial\varphi_s/\partial y = 0 = [\varphi_s]$  on the whole plane  $y = 0$ , it is a solution of a problem ‘without’ a plate since it implies  $u = 0$ . Therefore the symmetric problem cannot give rise to trapped modes and our study will concern only the antisymmetric problem.

We remark that the case of an off-centre plate is more difficult to deal with, contrary to the centred plate case where the split into symmetric and antisymmetric spaces is straightforward. However the geometry of the off-centre plate permits an ingenious similar split of spaces (for instance by restricting to fields with zero vertical mean value  $\int_{-h/2}^{h/2} \varphi(x, y) dy = 0$ ) so that the trapped mode corresponds to an eigenvalue below the cut-off of an appropriate operator (28, 29, 30). Note that this new split does not work for other off-centre convex symmetric bodies.

In the following only the upper half-guide defined by  $\Omega = \mathbb{R} \times ]0, h/2[$  will be considered and  $\varphi_a$  will be called  $\varphi$ . The upper rigid wall of  $\Omega$  is noted  $\partial D$  and  $\Gamma^+ = \{(x, 0); x > L\}$  and  $\Gamma^- = \{(x, 0); x < 0\}$  are introduced (Fig. 2). Restricting ourselves to antisymmetric resonant modes, looking for resonance frequencies is equivalent to solving the following problem:

find  $k \in \mathbb{R}$  such that there exist  $(\varphi, u) \in H^1(\Omega) \times H^2(\Gamma)$ ,  $(\varphi, u) \neq 0$  satisfying

$$\begin{aligned} -\Delta\varphi &= k^2\varphi & \text{in } \Omega, & & \mu^4 u^{iv} + i(k/\xi)\varphi &= k^2 u & \text{on } \Gamma, \\ \varphi &= 0 & \text{on } \Gamma^+ \cup \Gamma^-, & & \frac{\partial\varphi}{\partial y} &= 0 & \text{on } \partial D, & & \frac{\partial\varphi}{\partial y} &= -iku & \text{on } \Gamma, \\ u(0) &= 0 = u'(0), & & & u''(L) &= 0 = u'''(L), & & & & & \end{aligned} \quad (2.2)$$

where  $\xi = \tilde{\xi}/2$ . If  $(k, \varphi, u)$  is solution,  $(-k, \bar{\varphi}, \bar{u})$  is also solution, and thus we consider  $k \geq 0$ .

We note that  $k^2$  appears as an eigenvalue in the right hand side of the equations, but  $k$  appears also in the left hand side. We are led to solve a generalized eigenvalue problem which is a classical situation in fluid-structure problems. As we will see later, it is not the case when the plate is rigid: then we are led to a linear eigenvalue problem (even in presence of a flow thanks to a change of variable). Before presenting how we proceeded to find the trapped modes, we will first derive the mathematical framework of the eigenvalue problem.

### 2.3 Variational formulation

Let us derive the variational formulation associated to (2.2). We define the scalar product on  $L^2(\Omega) \times L^2(\Gamma)$  by

$$((\varphi, u), (\psi, v)) = \int_{\Omega} \varphi \bar{\psi} + \xi \int_{\Gamma} u \bar{v},$$

and the associated norm is denoted  $\|\cdot\|$ . Let us set

$$U = \{\varphi \in H^1(\Omega) / \varphi = 0 \text{ on } \Gamma^+ \cup \Gamma^-\} \quad \text{and} \quad V = \{u \in H^2(\Gamma) / u(0) = 0 = u'(0)\}. \quad (2.3)$$

We introduce  $W = U \times V$  with norm  $\|(\varphi, u)\|_W = \{\|\varphi\|_{H^1(\Omega)}^2 + \|u\|_{H^2(\Gamma)}^2\}^{1/2}$ . We consider the following bilinear form for  $(\varphi, u)$  and  $(\psi, v) \in W$ :

$$a_{elas}(k; (\varphi, u), (\psi, v)) = \int_{\Omega} \nabla\varphi \cdot \nabla\bar{\psi} + \xi\mu^4 \int_{\Gamma} u'' \bar{v}'' + ik \int_{\Gamma} (\varphi\bar{v} - \bar{\psi}u).$$

This bilinear form is continuous on  $W \times W$  and symmetric.

With these notations the problem to solve has the following variational formulation:

$$\left\{ \begin{array}{l} \text{Find } k \in \mathbb{R}^+ \text{ such that there exists } (\varphi, u) \in W, (\varphi, u) \neq 0, \text{ satisfying} \\ a_{elas}(k; (\varphi, u), (\psi, v)) = k^2 ((\varphi, u), (\psi, v)), \forall (\psi, v) \in W. \end{array} \right.$$

### 2.4 Operator form and self-adjointness

We denote by  $A_{elas}(k)$  the operator associated to  $a_{elas}(k)$ , defined as follows:

$$D(A_{elas}(k)) = \left\{ \begin{array}{l} (\varphi, u) \in W : \Delta\varphi \in L^2(\Omega), \quad u^{iv} \in L^2(\Gamma), \\ \partial\varphi/\partial y = 0 \text{ on } \partial D, \quad \partial\varphi/\partial y = -iku \text{ on } \Gamma, \quad u''(L) = 0 = u'''(L). \end{array} \right\}$$

and

$$\forall (\varphi, u) \in D(A_{elas}(k)), \quad A_{elas}(k)(\varphi, u) = (-\Delta\varphi, \mu^4 u^{iv} + i(k/\xi)\varphi).$$

Then for every  $(\varphi, u) \in D(A_{elas}(k))$  and for every  $(\psi, v) \in W$ ,  $(A_{elas}(k)(\varphi, u), (\psi, v)) = a_{elas}(k; (\varphi, u), (\psi, v))$ . Note that all the boundary conditions have to be understood in the weak sense. Now the problem of finding resonant modes can be written:

$$\left| \begin{array}{l} \text{Find } k \in \mathbb{R}^+ \text{ such that there exists } (\varphi, u) \in D(A_{elas}(k)), (\varphi, u) \neq (0, 0), \\ \text{satisfying: } A_{elas}(k)(\varphi, u) = k^2(\varphi, u). \end{array} \right.$$

We have to study the eigenvalues of the operator  $A_{elas}(k)$ . We prove first that it is self-adjoint. This will be achieved by proving the coercivity of  $a_{elas}(k; \cdot, \cdot) + (\cdot, \cdot)$ .

LEMMA 2.1. For  $k \in \mathbb{R}$ , the operator  $A_{elas}(k)$  is self-adjoint.

PROOF.  $A_{elas}(k)$  is clearly symmetric. To prove that it is a self-adjoint operator, we will prove there exists  $\lambda > 0$  such that  $a_{elas}(k; (\varphi, u), (\psi, v)) + \lambda((\varphi, u), (\psi, v))$  is coercive on  $W \times W$ . If it was false, there would exist a sequence  $(\varphi_n, u_n) \in W$  such that

$$\|(\varphi_n, u_n)\|_W = 1, \quad a_{elas}(k; (\varphi_n, u_n), (\varphi_n, u_n)) + n \|(\varphi_n, u_n)\|^2 \leq 1/n. \quad (2.4)$$

We cannot conclude that  $\|(\varphi_n, u_n)\| \rightarrow 0$  because  $a_{elas}(k; \cdot, \cdot)$  is not positive. However since  $(\varphi_n, u_n)$  is bounded in  $W$  and  $a_{elas}(k)$  is continuous,  $a_{elas}(k; (\varphi_n, u_n), (\varphi_n, u_n))$  is bounded. Therefore we deduce that  $n \|(\varphi_n, u_n)\|^2$  is bounded and so

$$\varphi_n \rightarrow 0 \text{ in } L^2(\Omega) \text{ strongly and } u_n \rightarrow 0 \text{ in } L^2(\Gamma) \text{ strongly.}$$

Since  $(\varphi_n, u_n)$  is bounded in  $W$ , we can extract from  $(\varphi_n, u_n)$  a subsequence, still denoted  $(\varphi_n, u_n)$ , such that

$$\varphi_n \rightharpoonup 0 \text{ in } H^1(\Omega) \text{ weakly and } u_n \rightharpoonup 0 \text{ in } H^2(\Gamma) \text{ weakly.}$$

Thanks to the trace theorem and by compact injection of  $H^{\frac{1}{2}}(\Gamma)$  in  $L^2(\Gamma)$  and of  $H^2(\Gamma)$  in  $H^1(\Gamma)$  we get

$$\varphi_n \rightarrow 0 \text{ in } L^2(\Gamma) \text{ strongly and } u_n \rightarrow 0 \text{ in } H^1(\Gamma) \text{ strongly.}$$

Thus we get

$$\int_{\Gamma} \varphi_n \bar{u}_n - \bar{\varphi}_n u_n \rightarrow 0.$$

As inequality (2.4)<sub>2</sub> reads

$$\int_{\Omega} |\nabla \varphi_n|^2 + \xi \mu^4 \int_{\Gamma} |u_n''|^2 + ik \left( \int_{\Gamma} \varphi_n \bar{u}_n - \bar{\varphi}_n u_n \right) + n \|(\varphi_n, u_n)\|^2 \leq \frac{1}{n},$$

this implies  $\|(\varphi_n, u_n)\|_W \rightarrow 0$  which is in contradiction with the hypothesis.  $\square$

### 3. Spectral study

#### 3.1 Essential spectrum

Let us denote by  $\sigma_{elas}(k)$  the spectrum of the operator  $A_{elas}(k)$  and by  $\sigma_{ess}(k)$  its essential spectrum. Note that both these spectra depend a priori on  $k$ . Our purpose is now to determine  $\sigma_{ess}(k)$ . As usual the essential spectrum is linked to the propagative modes of the duct whereas the discrete spectrum is linked to the trapped modes. Therefore to establish a lower bound of the essential spectrum we need first to calculate the duct modes. This is achieved in Appendix A.

### 3.1.1 A lower bound for the essential spectrum

The essential spectrum has the following property.

LEMMA 3.1.

$$\sigma_{ess} \subset [(\pi/h)^2, +\infty[.$$

PROOF. Suppose  $\gamma \in \sigma_{ess}(k)$  and consider a singular sequence  $(\varphi_n, u_n) \in D(A_{elas}(k))$  (as defined in (31)) such that:

$$\|(\varphi_n, u_n)\| = 1, \quad (3.1)$$

$$\|(A_{elas}(k) - \gamma I)(\varphi_n, u_n)\| \rightarrow 0, \quad (3.2)$$

$$(\varphi_n, u_n) \rightharpoonup 0 \quad \text{weakly in } L^2(\Omega) \times L^2(\Gamma). \quad (3.3)$$

By (3.1) and (3.2),

$$\gamma = \lim_{n \rightarrow \infty} a_{elas}(k; (\varphi_n, u_n), (\varphi_n, u_n)). \quad (3.4)$$

Thanks to the characterization of Lemma 2.1,  $\exists \lambda > 0$  such that  $a_{elas}(k; (\varphi, u), (\psi, v)) + \lambda((\varphi, u), (\psi, v))$  is coercive on  $W \times W$ , we deduce that  $(\varphi_n, u_n)$  is bounded in  $W$ . Consequently, by (3.3),  $\varphi_n \rightharpoonup 0$  in  $U$  and  $u_n \rightharpoonup 0$  in  $V$ . Then as in Lemma 2.1 we deduce by compactness arguments that

$$\int_{\Gamma} (\varphi_n \bar{u}_n - \bar{\varphi}_n u_n) \rightarrow 0.$$

Let us define the sub-domain  $\Omega_b = \{(x, y) \in \Omega; 0 < x < L\}$ . Then from Lemma A.1 of Appendix A we get

$$a_{elas}(\varphi_n, \varphi_n) \geq \int_{\Omega \setminus \Omega_b} |\nabla \varphi_n|^2 \geq (\pi/h)^2 \|\varphi_n\|_{L^2(\Omega \setminus \Omega_b)}, \quad (3.5)$$

because the restriction of  $\varphi_n$  to  $\Omega \setminus \Omega_b$  vanishes at  $y = 0$ . Since  $\varphi_n$  is bounded in  $U$ ,  $\varphi_n \rightharpoonup 0$  in  $H^1(\Omega_b)$  and  $\varphi_n \rightarrow 0$  in  $L^2(\Omega_b)$  because  $\Omega_b$  is compact. Thus we get from (3.1)

$$\|\varphi_n\|_{L^2(\Omega \setminus \Omega_b)} = \|\varphi_n\|_{L^2(\Omega)} - \|\varphi_n\|_{L^2(\Omega_b)} \rightarrow 1. \quad (3.6)$$

Finally, we deduce from (3.4), (3.5), (3.6) and since  $\int_{\Gamma} |u_n''|^2 \geq 0$  that  $\gamma \geq (\pi/h)^2$ .  $\square$

### 3.1.2 The singular sequences

We will now construct singular sequences to prove that the previous inclusion is in fact an equality. Notice in particular that the essential spectrum does not depend on  $k$ .

THEOREM 3.2.

$$\sigma_{ess} = [(\pi/h)^2, +\infty[.$$

PROOF. Let  $\gamma \geq (\pi/h)^2$ ,  $\delta = \sqrt{\gamma - (\pi/h)^2} \in \mathbb{R}$  and consider the sequence  $(\varphi_n, 0)_{n \geq 1}$  of  $D(A_{elas}(k))$  defined by

$$\varphi_n(x, y) = \frac{1}{\sqrt{n}} \psi\left(\frac{x}{n}\right) e^{i\delta x} \theta_1(y),$$

where  $\theta_1(y) = 2 \sin(\pi y/h) / \sqrt{h}$  (see Appendix A) and  $\psi$  is a  $C^\infty$  compactly supported function such that  $\psi(x) = 0$  for  $0 < x < L$ . Thanks to  $\psi$ , the sequence  $(\varphi_n, 0)_{n \in \mathbb{N}}$  belongs to  $D(A_{elas}(k))$  since  $\partial\varphi_n/\partial y = 0$  on  $\Gamma$ . It has also the following properties:

(a)  $\|(\varphi_n, 0)\|$  is independent of  $n$ . Indeed:

$$\|\varphi_n\|_{L^2(\Omega)}^2 = \int_{\mathbb{R}} \psi^2(x) dx.$$

(b)  $\varphi_n$  converges weakly to 0 in  $L^2(\Omega)$ . Indeed by Lebesgue's theorem,  $(\varphi_n, \varphi)_{L^2(\Omega)} \rightarrow 0$  for every  $\varphi \in \mathcal{D}(\Omega)$  which remains true, by the density of  $\mathcal{D}(\Omega)$  in  $L^2(\Omega)$ , for every  $\varphi \in L^2(\Omega)$ .

(c) Finally, since  $(-\Delta + \gamma) [e^{i\delta x} \theta_1(y)] = 0$ , we have

$$(-\Delta + \gamma)\varphi_n = \left( \frac{1}{n^2} \psi''\left(\frac{x}{n}\right) + 2i\delta \frac{1}{n} \psi'\left(\frac{x}{n}\right) \right) \frac{1}{\sqrt{n}} e^{i\delta x} \theta_1(y),$$

and thus  $\|(A_{elas}(k) - \gamma I)(\varphi_n, 0)\| = \|(-\Delta + \gamma)\varphi_n\|_{L^2(\Omega)} \rightarrow 0$ . Consequently  $\gamma \in \sigma_{ess}$ .  $\square$

### 3.2 Discrete spectrum and Min-Max principle

Here we are interested in the point spectrum of the operator  $A_{elas}(k)$ . The eigenvalues  $\gamma$  such that  $\gamma < (\pi/h)^2$  form the discrete spectrum; they can be studied by means of the Min-Max principle as will be developed in the following.

The study of the so-called *embedded eigenvalues*  $\gamma$  such that  $\gamma > (\pi/h)^2$  is much more complicated. Note that the eigenvalues in the discrete spectrum associated with the antisymmetric problem become embedded when considering the general problem (problem without any restriction on the symmetry of the velocity potential). For the problem we consider now, the question of the existence of embedded eigenvalues above  $(\pi/h)^2$  is still open. When the fluid is at rest, embedded eigenvalues were found numerically for many obstacle shapes **(32)**.

Eigenvalues  $\gamma$  below the essential spectrum are related to resonance frequencies  $k$  ( $\gamma = k^2$ ). The discrete resonance frequencies  $k$  are such that  $k < k_c = \pi/h$ . The frequency  $k_c$  has a physical interpretation: it is the cut-off frequency of the half waveguide (no wave can propagate when  $k < k_c$ ). Indeed looking back to the duct modes we get easily that if  $k < k_c$ ,  $\beta_n^\pm \in i\mathbb{R}$  for all  $n$ , and all the duct modes (see Appendix A) are evanescent.

The mathematical tool we will use to study the discrete spectrum of the operator  $A_{elas}(k)$  is the Min-Max principle. Let us recall it. For every integer  $n \geq 1$  and any  $k \geq 0$  the Min-Max values are defined as:

$$\gamma_{elas}^n(k) = \inf_{F \in \mathcal{V}_n(W)} \sup_{(\varphi, u) \in F, (\varphi, u) \neq 0} \mathcal{R}_{elas}(k; \varphi, u), \quad (3.7)$$

where  $\mathcal{V}_n(W)$  denotes the set of all  $n$ -dimensional subspaces of  $W$  and  $\mathcal{R}_{elas}(k; \varphi, u)$  is the Rayleigh quotient defined as follows:

$$\mathcal{R}_{elas}(k; \varphi, u) = \frac{a_{elas}(k; (\varphi, u), (\varphi, u))}{\|(\varphi, u)\|^2}.$$



We let  $\mathcal{N}(A_{elas}(k))$  denote the number of eigenvalues below  $(\pi/h)^2$ . For the problem we are interested in, the Min-Max principle reads as follows.

**THEOREM 3.3.** For every  $k \geq 0$ , the sequence  $(\gamma_{elas}^n(k))_{n \geq 1}$  is increasing and converges to  $(\pi/h)^2$ . Moreover, the following alternative holds:

- $\gamma_{elas}^n(k) < (\pi/h)^2$ . Then  $A_{elas}(k)$  has at least  $n$  eigenvalues (counted with their multiplicity) below  $(\pi/h)^2$  (that is,  $\mathcal{N}(A_{elas}(k)) \geq n$ ) and these eigenvalues are the first  $n$  Min-Max values  $\gamma_{elas}^1(k), \gamma_{elas}^2(k), \dots, \gamma_{elas}^n(k)$ .
- $\gamma_{rigid}^n(k) = (\pi/h)^2$  and then  $A_{elas}(k)$  has at most  $n - 1$  eigenvalues strictly lower than  $(\pi/h)^2$  (that is,  $\mathcal{N}(A_{elas}(k)) \leq n - 1$ ).

See for example **(33)** for a proof. To summarize,

$$\mathcal{N}(A_{elas}(k)) = \max \{n \in \mathbb{N}; \gamma_{elas}^n(k) < (\pi/h)^2\}, \quad (3.8)$$

where we set for convenience  $\gamma_{elas}^0(k) = -\infty$ . This convention will be used throughout the paper. It can be proved by standard arguments (the so-called Neumann comparison principle **(24)**) that  $\mathcal{N}(A_{elas}(k))$  is a finite number.

### 3.3 The fixed-point equations

A main difficulty of our study is that  $A_{elas}(k)$  depends on  $k$ . This problem is a ‘nonlinear’ eigenvalue problem ( $k^2$  is an eigenvalue of an operator  $A_{elas}(k)$  which depends itself of  $k$ ). The procedure to find the resonance frequencies is the following:

- use the Min-Max principle to prove the existence of isolated eigenvalues of  $A_{elas}(k)$ ,
- solve the fixed-point equations  $\gamma_{elas}^n(k) = k^2$  for  $0 \leq k < \pi/h$ ,  $n \geq 1$ . For this problem  $\mathcal{N}_{elas}$  denotes the number of resonance frequencies and  $k_{elas}^n$  is the  $n$ th resonance frequency (see Fig. 3).

Concerning the fixed-point equations, we will prove that if an eigenvalue exists for  $k = \pi/h$ , it crosses  $k^2$  once and only once. The general estimate of the number of trapped modes is given by the following lemma.

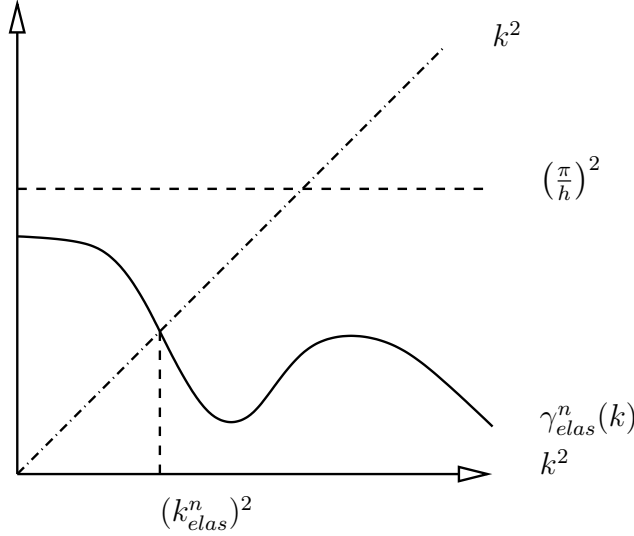
**LEMMA 3.4.** The number of resonances  $\mathcal{N}_{elas}$  is equal to  $\mathcal{N}(A_{elas}(\pi/h))$  the number of eigenvalues in  $k = \pi/h$ .

**PROOF.** This is a simple consequence of the following two properties:

- The functions  $k \rightarrow \gamma_{elas}^n(k)$  are clearly continuous and the functions  $k \rightarrow \gamma_{elas}^n(k)/k$  are decreasing. Indeed for any  $(\varphi, u) \in W$ ,

$$\mathcal{R}_{elas}(k; \varphi, u) = \left\{ \int_{\Omega} |\nabla \varphi|^2 + \xi \mu^4 \int_{\Gamma} |u''|^2 + 2k \int_{\Gamma} \text{Im}(\bar{\varphi} u) \right\} \|(\varphi, u)\|^{-2}.$$

Thus for any  $(\varphi, u) \in W$  the function  $k \rightarrow \mathcal{R}_{elas}(k; \varphi, u)/k$  is a decreasing function of  $k$  and consequently these properties are valid for the Min-Max values **(34)**. Therefore if  $\gamma_{elas}^n(k)$  crosses  $k^2$  (or equivalently  $\gamma_{elas}^n(k)/k$  crosses  $k$ ), the intersection is unique.



**Fig. 3** Definition of  $k_{elas}^n$

- For all  $n \geq 1$ ,  $\gamma_{elas}^n(0) > 0$ . Indeed:

$$\gamma_{elas}^n(0) = \inf_{F \in \mathcal{V}_n(W)} \sup_{(\varphi, u) \in F, (\varphi, u) \neq 0} \mathcal{R}_{elas}(0; \varphi, u),$$

where

$$\mathcal{R}_{elas}(0; \varphi, u) = \left\{ \int_{\Omega} |\nabla \varphi|^2 + \xi \mu^4 \int_{\Gamma} |u''|^2 \right\} \|(\varphi, u)\|^{-2} > 0.$$

To conclude, we need to determine if the function  $g^n(k) = \gamma_{elas}^n(k) - k^2$  vanishes on  $[0, \pi/h[$ . Since  $g^n(0) > 0$ , if  $\gamma_{elas}^n(\pi/h) < (\pi/h)^2$ , that is,  $n \leq \mathcal{N}(A_{elas}(\pi/h))$ , then  $g^n(\pi/h) < 0$  and therefore  $\gamma_{elas}^n(k)$  crosses  $k^2$  exactly once.

The lemma follows.  $\square$

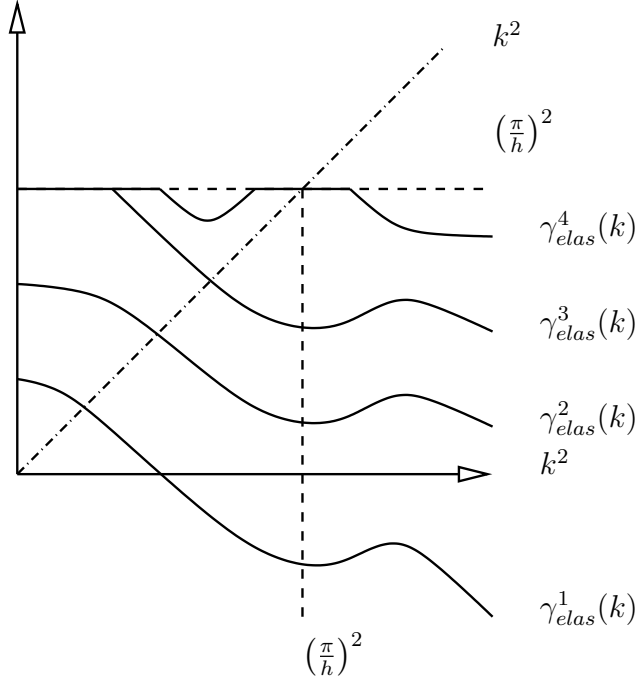
Notice that the previous results do not preclude a situation like the one illustrated in Fig. 4 where  $\mathcal{N}_{elas} = \mathcal{N}(A_{elas}(\pi/h)) = 3$  but  $\mathcal{N}(A_{elas}(0)) = 2$  and  $\mathcal{N}(A_{elas}(k)) = 4$  for some particular values of  $k$ .

Note that it is important to have a positive spectrum at  $k = 0$ . In the general case of elastic plate with flow, the spectrum is not positive for  $k = 0$ , and the previous lemma is not valid.

We can immediately deduce the following general result.

**LEMMA 3.5.** The number of resonances  $\mathcal{N}_{elas}$  is an increasing function of the length of the plate  $L$  and each resonance frequency  $k_{elas}^n$  is a decreasing function of  $L$ .

**PROOF.** Let us compare the two problems associated respectively to a plate of length  $L$  ( $\Gamma = \{(x, y); y = 0 \text{ and } 0 < x < L\}$ ) and a plate of length  $\tilde{L} = L + \delta L$  with  $\delta L > 0$  defined by  $\tilde{\Gamma} = \{(x, y); y = 0 \text{ and } -\delta L < x < L\}$ . With obvious notations, we have the following inclusion  $U \subset \tilde{U}$  (see (2.3) for definition of  $U$ ). Moreover,  $V$  can be identified as



**Fig. 4** Possible behaviours of the eigenvalues in the elastic plate case

a subspace of  $\tilde{V}$  since a function  $u \in V$  extended by 0 on  $] -\delta L, 0[$  belongs to  $\tilde{V}$ . Then one deduces from the inclusion  $W \subset \tilde{W}$  that  $\mathcal{V}_n(W) \subset \mathcal{V}_n(\tilde{W})$  and from (3.7) the inequality  $\gamma_{elas}^n(k) \leq \tilde{\gamma}_{elas}^n(k)$  for any  $k \geq 0$  and any  $n \geq 1$ . This inequality is true for  $k = \pi/h$  and we can conclude thanks to Lemma 3.4.  $\square$

#### 4. Existence results

Now we are going to establish existence results for resonances. The procedure to find the resonances of the fluid-structure coupling problem is the following. To obtain estimates on the eigenvalues of the operator  $A_{elas}(\pi/h)$ , we will compare them with the eigenvalues of two complementary simpler problems: a rigid plate in a fluid at rest and an elastic plate in vacuum. Then thanks to the Min-Max principle we will prove that the number of resonances is larger than the numbers of resonance frequencies below  $\pi/h$  of both of the two ‘sub-problems’.

##### 4.1 The sub-problems

###### 4.1.1 Elastic plate in vacuum

The problem of finding the eigenmodes of a plate in vacuum clamped at one end and free at the other end reads as follows:

$$\mu^4 u^{iv} = k^2 u \text{ on } \Gamma, \quad u(0) = 0 = u'(0), \quad u''(L) = 0 = u'''(L).$$

A simple calculation shows that there is an infinite sequence of eigenfrequencies tending to infinity. The  $n$ th frequency,  $n = 1, 2, \dots$ , is given by

$$k_{plate}^n = (\mu\alpha_n/L)^2, \quad (4.1)$$

where  $\alpha_n$  is the  $n$ th real positive root of  $\cos \alpha_n \cosh \alpha_n = -1$ . The eigenmodes are

$$u_n(x) = A_n \left[ \sinh\left(\frac{\alpha_n}{L}x\right) - \sin\left(\frac{\alpha_n}{L}x\right) \right] + B_n \left[ \cosh\left(\frac{\alpha_n}{L}x\right) - \cos\left(\frac{\alpha_n}{L}x\right) \right],$$

where

$$B_n = -A_n \frac{\sinh(\alpha_n) + \sin(\alpha_n)}{\cosh(\alpha_n) + \cos(\alpha_n)},$$

and  $A_n$  can be chosen such that  $\int_{\Gamma} |u_n|^2 = 1$ .

The following mathematical characterization of the eigenfrequencies will be used in the following.

LEMMA 4.1. Let

$$\gamma_{plate}^n = \min_{F \in \mathcal{V}_n(V)} \max_{u \in F, u \neq 0} \mathcal{R}_{plate}(u), \quad (4.2)$$

where  $V$  is defined in (2.3),  $\mathcal{V}_n(V)$  denotes the set of all  $n$ -dimensional subspaces of  $V$  and

$$\mathcal{R}_{plate}(u) = \mu^4 \left( \int_{\Gamma} |u''|^2 \right) / \left( \int_{\Gamma} |u|^2 \right).$$

Then

$$\gamma_{plate}^n = (k_{plate}^n)^2 \quad \forall n \geq 1. \quad (4.3)$$

PROOF. Again the operator  $C_{plate}$  defined by  $C_{plate}u = \mu^4 u^{iv}$  for  $u \in D(C_{plate})$  with

$$D(C_{plate}) = \{u \in H^4(\Gamma); u(0) = u'(0) = u''(L) = u'''(L) = 0\}$$

is self-adjoint and has a compact resolvent. As a consequence, its spectrum consists of a sequence of eigenvalues which are the  $(k_{plate}^n)^2$ ,  $n \geq 0$  defined above and the eigenfunctions  $(u_n, n \geq 0)$  form an orthonormal basis of  $L^2(\Gamma)$ . The lemma results from the classical Min-Max characterization of the eigenvalues.  $\square$

#### 4.1.2 The rigid plate case

We will now consider a rigid plate in a fluid at rest and then extend the results to the presence of a flow. The case of the rigid plate in a fluid at rest has been widely studied before (5, 10). Therefore we just recall briefly the classical results in order to introduce some notations that will be useful for the study of the elastic plate case.

In this simple case the bilinear form reduces to  $a_{rigid}(\varphi, \psi) = \int_{\Omega} \nabla \varphi \cdot \nabla \bar{\psi}$  and thus the eigenvalues no longer depend on the frequency. We denote by  $A_{rigid}$  the unbounded operator of  $L^2(\Omega)$  associated to  $a_{rigid}$ . Resonance frequencies  $k$  correspond here to eigenvalues  $k^2$  of  $A_{rigid}$ :  $A_{rigid} \varphi = k^2 \varphi$ . For every integer  $n \geq 1$ , we set

$$\gamma_{rigid}^n = \inf_{F \in \mathcal{V}_n(U)} \sup_{\varphi \in F, \varphi \neq 0} \mathcal{R}_{rigid}(\varphi), \quad (4.4)$$

where  $\mathcal{V}_n(U)$  denotes the set of all  $n$ -dimensional subspaces of  $U$  defined in (2.3) and  $\mathcal{R}_{rigid}(\varphi)$  is the Rayleigh quotient defined as follows:

$$\mathcal{R}_{rigid}(\varphi) = \left( \int_{\Omega} |\nabla \varphi|^2 \right) / \left( \int_{\Omega} |\varphi|^2 \right).$$

The essential spectrum of the operator  $A_{rigid}$  is the same as in the elastic case  $\sigma_{ess} = [(\pi/h)^2, +\infty[$ . We let  $\mathcal{N}_{rigid}$  denote the number of resonances below  $\pi/h$  (clearly equal to the number of eigenvalues of  $A_{rigid}$  below  $(\pi/h)^2$ ) and by  $k_{rigid}^n$  the  $n$ th resonance frequency. Since the spectrum of the operator  $A_{rigid}$  is positive, we have clearly  $\mathcal{N}_{rigid} = \max\{n \in \mathbb{N}; \gamma_{rigid}^n < (\pi/h)^2\} = \mathcal{N}(A_{rigid})$ . We use the following lemma (proved in (10)).

LEMMA 4.2. For every  $n \geq 1$ ,

$$\min [(\pi/h)^2, ((n-1)\pi/L)^2] \leq \gamma_{rigid}^n \leq (n\pi/L)^2.$$

From this lemma, we easily deduce the following existence result for trapped modes.

THEOREM 4.3. If  $m < (L/h) \leq m+1$  then  $\mathcal{N}_{rigid} = m$  or  $m+1$  and  $(n-1)\pi/L \leq k_{rigid}^n \leq n\pi/L$  for  $n = 1$  to  $m$ .

Three remarks can be mentioned.

1. The number of resonance frequencies varies like  $L/h$ . Thus less resonance frequencies exist when the guide is wider and when the plate is smaller.
2. The number of resonance frequencies is not known exactly, but with a good accuracy of only one. Note that when  $\gamma_{rigid}^{m+1} < (\pi/h)^2$  we have the extra estimation  $m\pi/L \leq k_{rigid}^{m+1} < \pi/h$ .
3. For a short plate  $L < h$ , Theorem 4.3 indicates that  $\mathcal{N}_{rigid} = 0$  or  $1$  (at most one resonance exists). Thus the existence of a resonance is not ensured. In fact it has been proved that for  $L < h$  exactly one resonance exists (5).

#### 4.1.3 Extension to a fluid in flow

The results of the previous paragraph can be easily extended to the presence of a flow. For a uniform flow parallel to the duct boundaries of Mach number  $0 \leq M < 1$ , looking for resonance frequencies is equivalent to solving the following problem (35):

for  $M \in [0, 1[$ , find  $k \in \mathbb{R}$  such that there exist  $\varphi \in H^1(\Omega)$ ,  $\varphi \neq 0$ , satisfying

$$\begin{aligned} - (1 - M^2) \frac{\partial^2 \varphi}{\partial x^2} - \frac{\partial^2 \varphi}{\partial y^2} - 2ikM \frac{\partial \varphi}{\partial x} &= k^2 \varphi \quad \text{in } \Omega, \\ \frac{\partial \varphi}{\partial y} &= 0 \quad \text{on } \partial D \cup \Gamma, \quad \varphi = 0 \quad \text{on } \Gamma^+ \cup \Gamma^-. \end{aligned} \tag{4.5}$$

Contrary to the no-flow case, both  $k$  and  $k^2$  are involved in the convected Helmholtz equation. Thus in this form the problem of finding resonances cannot be reduced to an eigenvalue problem. However a classical eigenvalue problem can be recovered by using a change of variables and unknown (called also Prandtl–Glauert transformation (36)):

$\hat{x} = x/\sqrt{1-M^2}$ ,  $\hat{y} = y$  and  $\hat{\varphi}(\hat{x}, \hat{y}) = \varphi(x, y) \exp(i\hat{k}M\hat{x})$  where  $\hat{k} = k/\sqrt{1-M^2}$ . The problem becomes:

$$\begin{aligned} -\left(\frac{\partial^2}{\partial \hat{x}^2} + \frac{\partial^2}{\partial \hat{y}^2}\right)\hat{\varphi} &= \hat{k}^2\hat{\varphi} \quad \text{in } \hat{\Omega}, \\ \frac{\partial \hat{\varphi}}{\partial \hat{y}} &= 0 \quad \text{on } \partial \hat{D} \cup \hat{\Gamma}, \quad \hat{\varphi} = 0 \quad \text{on } \hat{\Gamma}^+ \cup \hat{\Gamma}^-, \end{aligned}$$

where the new extended domain  $\hat{\Omega}$ , still of height  $h/2$ , contains a plate of length  $\hat{L} = L/\sqrt{1-M^2}$ . As a consequence, the resonance frequencies for a rigid plate of length  $L$  in a fluid in flow  $k_{rigid}^{M,n}$  are deduced from those for a fluid at rest but with a longer plate (length  $\hat{L}$ ),  $\hat{k}_{rigid}^n$  thanks to the relation  $k_{rigid}^{M,n} = \hat{k}_{rigid}^n \sqrt{1-M^2}$ . We let  $\mathcal{N}_{rigid}^M$  denote the number of resonance frequencies and from Theorem 4.3 corresponding to the no-flow case it is straightforward to deduce the next result.

**THEOREM 4.4.** If  $m < (L/h)/\sqrt{1-M^2} \leq m+1$  then  $\mathcal{N}_{rigid}^M = m$  or  $m+1$  and

$$(n-1)(1-M^2)\pi/L \leq k_{rigid}^{M,n} \leq n(1-M^2)\pi/L \quad \text{for } n = 1 \text{ to } m.$$

Therefore the number of resonance frequencies varies like  $(L/h)/\sqrt{1-M^2}$ . Note that this tends to infinity when the flow becomes sonic ( $M \rightarrow 1$ ). Note also that the discrete resonant frequencies are such that  $k < k_c^M = \hat{k}_c \sqrt{1-M^2} = \pi\sqrt{1-M^2}/h$ . As in the no-flow case, the frequency  $k_c^M$  is the cut-off frequency of the half waveguide.

If a wake is introduced behind the plate and the Kutta condition is applied on the trailing edge of the plate to determine the amplitude  $F$  of the wake, then no real resonance exists **(37)**. Indeed a vortex uses up energy for its formation leading to a lossy system. Since our aim is to derive existence results for resonances, we preferred not to take into account a wake ( $F = 0$ ) to remain with a self-adjoint problem for which classical tools of spectral theory can be used. In spite of this approximation we think that our results for a problem without wake give relevant information about the problem with a wake: since the resonance frequencies, real for  $F = 0$ , move continuously in the complex plane when  $F \neq 0$  **(37)**, we conjecture that the frequencies without wake are close to the real parts of the frequencies with a wake. We have checked numerically this assertion for  $L = 4$  and  $L = 6$  and  $M = 0, 0.3, 0.5$  and  $0.7$  and good agreement has been found.

## 4.2 Comparison results

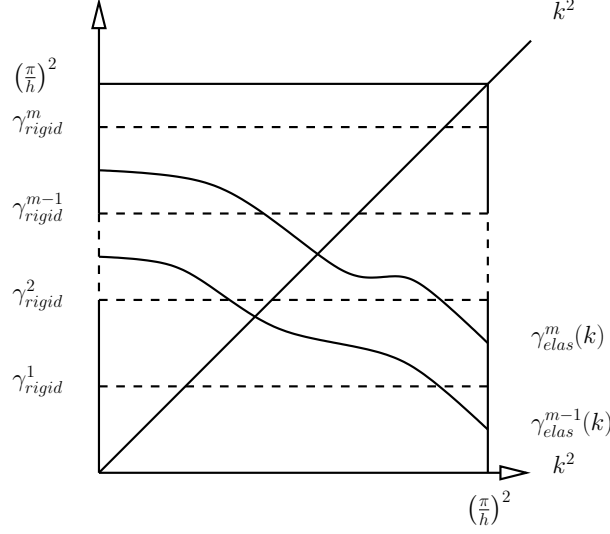
### 4.2.1 Comparison with the case of a rigid plate

First we compare an elastic plate in a fluid to a rigid plate in a fluid. We prove the following comparison result for the Min-Max functions.

**LEMMA 4.5.** For every integer  $n \geq 1$  and every real  $k \geq 0$ ,

$$\gamma_{elas}^n(k) \leq \gamma_{rigid}^n,$$

where  $\gamma_{elas}^n(k)$  is defined by (3.7) and  $\gamma_{rigid}^n$  is defined by (4.4).



**Fig. 5** Existence of at least  $m$  resonances

PROOF. Taking  $u = 0$ , we get the following identity:

$$\mathcal{R}_{rigid}(\varphi) = \mathcal{R}_{elas}(k; \varphi, 0) \quad \forall \varphi \in U.$$

The lemma follows from the inclusion  $\{(\varphi, 0); \varphi \in U\} \subset W$ .  $\square$

We deduce from this lemma the next theorem.

**THEOREM 4.6.**  $\mathcal{N}_{elas} \geq \mathcal{N}_{rigid}$  and  $k_{elas}^n \leq k_{rigid}^n$  for  $n = 1$  to  $\mathcal{N}_{rigid}$ .

PROOF. By definition of  $\mathcal{N}_{rigid}$ ,  $\gamma_{rigid}^n < (\pi/h)^2$  for  $n = 1$  to  $\mathcal{N}_{rigid}$ . Then, by the previous lemma,  $\gamma_{elas}^n(k) < (\pi/h)^2$  for  $n = 1$  to  $\mathcal{N}_{rigid}$  and for every  $k$ . By the Min-Max principle, this inequality proves that the operator  $A_{elas}(k)$  has at least  $\mathcal{N}_{rigid}$  eigenvalues below the essential spectrum, equal to  $\gamma_{elas}^1(k), \gamma_{elas}^2(k), \dots, \gamma_{elas}^{\mathcal{N}_{rigid}}(k)$ . This is in particular true for  $k = \pi/h$  and it is easy to conclude thanks to Lemma 3.4.  $\square$

This result is illustrated in Fig. 5 in the case  $\gamma_{rigid}^m < (\pi/h)^2 \leq \gamma_{rigid}^{m+1}$  (that is,  $m < L/h \leq m + 1$ ): then  $\mathcal{N}_{rigid} = m$ .

#### 4.2.2 Comparison with the case of a plate in vacuum

Now we compare the resonances of an elastic plate in a fluid at rest to the resonances of the same plate in vacuum.

**LEMMA 4.7.** For every integer  $n \geq 1$  and every real  $k \geq 0$ ,

$$\gamma_{elas}^n(k) \leq \gamma_{plate}^n,$$

where  $\gamma_{elas}^n(k)$  is defined by (3.7) and  $\gamma_{plate}^n$  is defined by (4.2).

PROOF. Taking  $\varphi = 0$ , we get the following identity:  $\mathcal{R}_{plate}(u) = \mathcal{R}_{elas}(k; 0, u) \quad \forall u \in V$ . The lemma follows from the inclusion  $\{(0, u); u \in V\} \subset W$ .  $\square$

Proceeding as above, we deduce the next theorem.

THEOREM 4.8.  $\mathcal{N}_{elas} \geq \mathcal{N}_{plate}$  where

$$\mathcal{N}_{plate} = \max \{n \in \mathbb{N} : \gamma_{plate}^n = (\mu\alpha_n/L)^4 < (\pi/h)^2\}, \quad (4.6)$$

and  $k_{elas}^n \leq k_{plate}^n$  for  $n = 1$  to  $\mathcal{N}_{plate}$  where  $k_{plate}^n$  and  $\gamma_{plate}^n$  are defined by (4.1) and (4.3), respectively.

Note that  $\mathcal{N}_{plate}$  is the number of eigenfrequencies  $k_{plate}^n$  below the cut-off frequency of the duct  $\pi/h$ . These are the resonance frequencies of the plate which cannot radiate away in the fluid. Note also that when the plate is very rigid ( $\mu \rightarrow \infty$ ) then all the eigenfrequencies of the plate tend to infinity and  $\mathcal{N}_{plate} \rightarrow 0$  (in fact  $\mathcal{N}_{plate} = 0$  as soon as  $\mu > (L/\alpha_1)\sqrt{\pi/h}$ ).

We remark that it is easy to prove that  $(n-1)\pi < \alpha_n < n\pi$  for all  $n \in \mathbb{N}^*$ , and also that  $\alpha_n \sim (n-1/2)\pi$  when  $n \rightarrow \infty$ . Therefore we get that if  $m < (L/\mu)(\pi h)^{-1/2} \leq m+1$  then  $\mathcal{N}_{plate} = m$  or  $m+1$  ( $\mathcal{N}_{plate}$  behaves like  $L/\sqrt{h}$ ).

### 4.3 Conclusion and comments

The final result is as follows.

THEOREM 4.9.

$$\mathcal{N}_{elas} \geq \max(\mathcal{N}_{rigid}, \mathcal{N}_{plate}) \text{ and } \begin{cases} k_{elas}^n \leq k_{rigid}^n & \text{for } n = 1 \text{ to } \mathcal{N}_{rigid}, \\ k_{elas}^n \leq k_{plate}^n & \text{for } n = 1 \text{ to } \mathcal{N}_{plate}. \end{cases}$$

In particular the conditions on  $k_{elas}^n$  imply that  $k_{elas}^n \leq \min(k_{rigid}^n, k_{plate}^n)$  for  $n = 1$  to  $\min(\mathcal{N}_{rigid}, \mathcal{N}_{plate})$ . This theorem means that the number of trapped modes when the plate is elastic and the fluid is at rest is larger than two bounds:

- the number of trapped modes in the fluid when the plate is rigid,
- the number of resonances of the plate below the cut-off frequency.

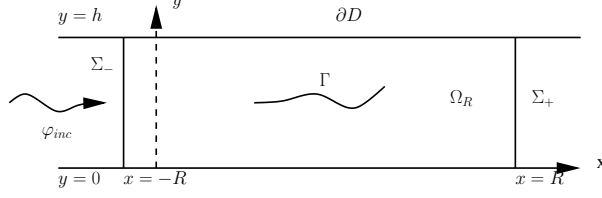
Several comments can be made.

- when the plate is sufficiently rigid ( $\mu > \mu_1 = (L/\alpha_1)\sqrt{\pi/h}$ ) then  $\mathcal{N}_{plate} = 0$  and we get  $\max(\mathcal{N}_{rigid}, \mathcal{N}_{plate}) = \mathcal{N}_{rigid}$ .
- $\mathcal{N}_{plate}$  and  $\mathcal{N}_{rigid}$  behave respectively like  $L/\sqrt{h}$  and  $L/h$ . Therefore the behaviour versus  $L$  is the same whereas it is different versus  $h$ . We can define a particular duct height for which  $\mathcal{N}_{plate} \simeq \mathcal{N}_{rigid}$ :  $h_c \simeq \pi\mu^2$ .

## 5. Numerical study

In order to illustrate our theoretical results, we have determined numerically the resonances of an elastic plate in presence of a flow. The numerical method consists in solving the problem with a source: the plate is submitted to an incident wave of variable frequency  $k$ . A bounded domain  $\Omega_R$  is defined around the plate (see Fig. 6), and for a fixed energy of the incident wave  $\int_{\Omega_R} |\varphi_{inc}|^2 = 1$  the quantity  $E_{fluid}(k) = \int_{\Omega_R} |\varphi|^2$ , proportional to the energy





**Fig. 6** Bounded domain introduced for the numerical resolution

diffracted in the fluid, is calculated. Each peak of the energy corresponds to a resonance frequency.

Transparent boundary conditions on the vertical boundaries  $\Sigma_{\pm}$  are obtained thanks to the modal decomposition of the velocity potential outside the bounded domain. These boundary conditions read  $\partial\varphi_d/\partial n = -T^{\pm}\varphi_d$  on  $\Sigma_{\pm}$  where the diffracted field is defined as  $\varphi_d = \varphi - \varphi_{inc}$  and where the Dirichlet-to-Neumann operators  $T^{\pm}$  satisfy:

$$\text{for all } w(y) \in H^{1/2}(\Sigma_{\pm}), \quad T^{\pm}w = \mp \sum_{n=0}^{\infty} i\beta_n^{\pm} \int_{\Sigma_{\pm}} w \theta_n dy \quad \text{with} \quad \theta_n = \frac{2}{\sqrt{h}} \sin \zeta_n y$$

(see (38) for more details), where  $\beta_n^{\pm}$  and  $\zeta_n$  are defined in Appendix A.

Resolution is achieved thanks to the finite element code MELINA (39). The numerical method couples finite elements introduced in the bounded domain to two families of modal representations: in addition to the spectral representation of the velocity potential outside the bounded domain, the vertical displacement of the plate is expanded on the eigenmodes of the plate in vacuum. All the numerical studies presented hereafter have been performed in a duct of height  $h = 4$ .

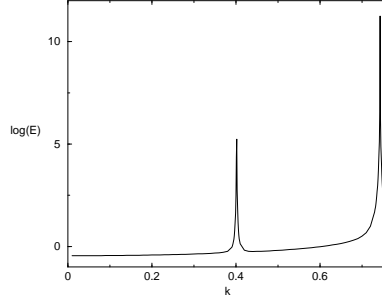
### 5.1 Case of a rigid plate

In the case of a rigid plate, we have proved that the number of resonances increases with  $L$  or  $M$ . More precisely the resonances existence criterium is: if  $m < L/h\sqrt{1-M^2} \leq m+1$ , then  $m$  or  $m+1$  resonance frequencies exist and satisfy  $(n-1)(1-M^2)\pi/L \leq k_{rigid}^{M,n} \leq n(1-M^2)\pi/L$  for  $n = 1$  to  $m$ . In the following we will check numerically the validity of these theoretical predictions.

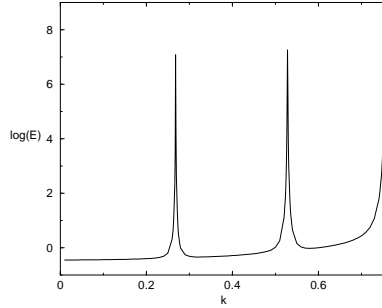
#### 5.1.1 No-flow case: influence of $L$

For a plate of length  $L = 6$ , we found numerically that  $\mathcal{N}_{rigid} = 2$ . On Fig. 7 is represented the logarithm of the diffracted energy in the fluid versus the frequency  $k$  below the critical frequency  $k_c = \pi/h = 0.79$ . To a plate of longer length  $L = 10$  corresponds  $\mathcal{N}_{rigid} = 3$  (see Fig. 8). The two first resonance frequencies are lower for  $L = 10$  than for  $L = 6$ . We studied several lengths of the plate and we found that the number of resonances is always  $\mathcal{N}_{rigid} = m + 1$ . This result has been observed before (7, 9).

The real parts of the velocity potential of the resonant modes for  $L = 10$  are drawn on Fig. 9. It appears that the test field  $\varphi_1(x, y) = \sin(\pi x/L)$  for  $0 < x < L$  and  $y > h/2$ ,  $\varphi_1$  antisymmetric with respect to the plate and  $\varphi_1 = 0$  elsewhere is very close to the



**Fig. 7** Logarithm of the energy in the fluid versus the frequency for a rigid plate,  $M = 0$  and  $L = 6$ .



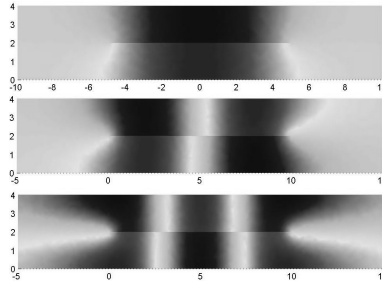
**Fig. 8** Logarithm of the energy in the fluid versus the frequency for a rigid plate,  $M = 0$  and  $L = 10$ .

first resonant mode associated to  $k_{rigid}^1 = 0.268 < \pi/L = 0.314$  ( $\varphi_1$  is the field such that  $\mathcal{R}_{rigid}(k; \varphi_1) = \int_{\Omega} |\nabla \varphi_1|^2 / \int_{\Omega} |\varphi_1|^2 = \pi^2/L^2$ ). The second resonant mode corresponds to  $k_{rigid}^2 = 0.528 < 2\pi/L = 0.528$  and is very similar to  $\varphi_2(x, y) = \sin(2\pi x/L)$ . The frequency of the third resonant mode is  $k_{rigid}^3 = 0.754 < \pi/h = 0.785$ , and we can notice that this mode is less trapped by the plate than the two other modes since  $k_{rigid}^3$  is close to  $k_c$ .

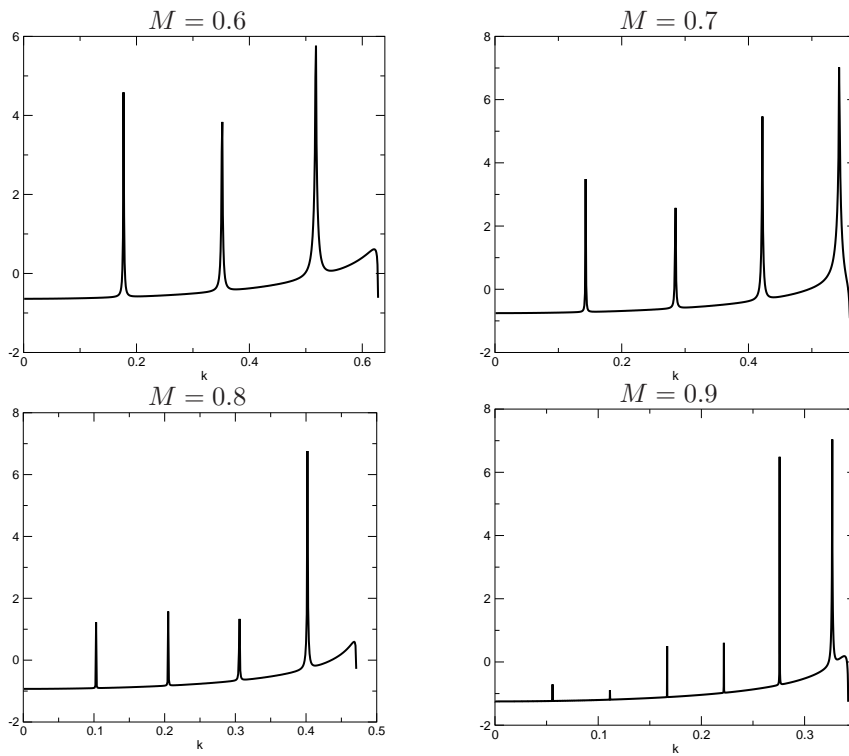
### 5.1.2 Flow case: influence of $M$

The results are presented in the case  $L = 10$  on Fig. 10:  $r = L/h\sqrt{1-M^2} = 3.1$  for  $M = 0.6$ ,  $r = 3.5$  for  $M = 0.7$ ,  $r = 4.2$  for  $M = 0.8$  and  $r = 5.7$  for  $M = 0.9$ . We have found numerically that  $\mathcal{N}_{rigid}^M = m$  if  $m < L/h\sqrt{1-M^2} < m + 1/2$  and that  $\mathcal{N}_{rigid}^M = m + 1$  if  $m + 1/2 < L/h\sqrt{1-M^2} < m + 1$  (remember that it is always  $m + 1$  when  $M = 0$ ). The frequencies are decreasing when the Mach number increases. The resonant modes for the case  $M = 0.6$  are drawn on Fig. 11. They correspond to the frequencies  $k_{rigid}^{M,1} = 0.177 < \pi(1-M^2)/L = 0.201$ ,  $k_{rigid}^{M,2} = 0.352 < 2\pi(1-M^2)/L = 0.402$  and  $k_{rigid}^{M,3} = 0.518 < 3\pi(1-M^2)/L = 0.603 < k_c^M = 0.628$ .

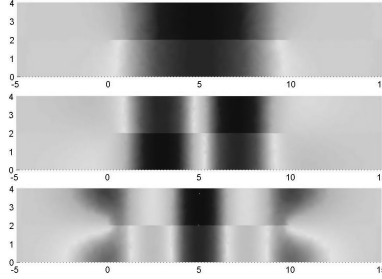
Note that these modes are very similar to the resonant modes in the no-flow case. More



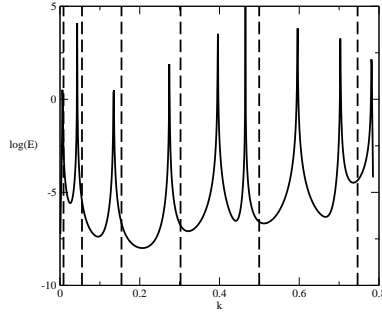
**Fig. 9** Real part of the velocity potential for the trapped modes when  $L = 10$ .



**Fig. 10** Logarithm of the energy in the fluid versus the frequency for a rigid plate for  $L = 10$  and various Mach numbers.



**Fig. 11** Real part of the velocity potential for the trapped modes when  $L = 10$ ,  $M = 0.6$ .



**Fig. 12** Logarithm of the energy in the plate versus the frequency for an elastic plate  $L = 10$  in a fluid at rest.

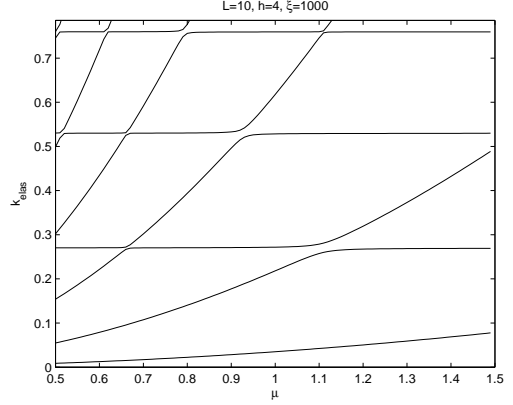
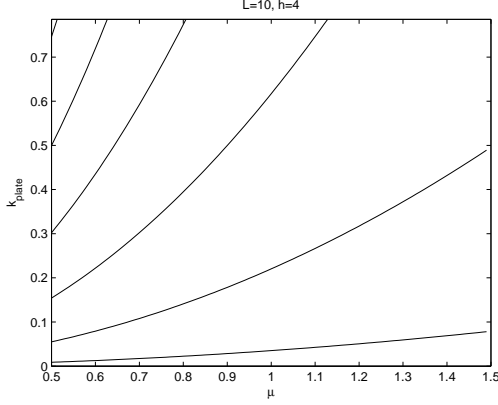
precisely the first resonance in a fluid in flow is close to the velocity potential  $\varphi_1^M = \sin(\pi x/L) \exp\{-ikMx/(1-M^2)\}$ .  $\varphi_1^M$  is built starting from the first eigenmode in a fluid at rest (see section 5.1.1):  $\hat{\varphi}_1 = \sin(\pi \hat{x}/\hat{L})$ .  $\varphi_1^M$  is deduced thanks to the transformation between uniform flow and fluid at rest defined in section 4.1.3.

## 5.2 Case of an elastic plate

### 5.2.1 Validation of theoretical estimates

We have proved theoretically that  $\mathcal{N}_{elas} \geq \max(\mathcal{N}_{rigid}, \mathcal{N}_{plate})$  ( $\mathcal{N}_{rigid}$  and  $\mathcal{N}_{plate}$  are defined respectively by (3.8) and (4.6)). Fig. 12 illustrates a typical result for an elastic plate. The eight first eigenmodes of the plate have been used in the modal expansion of the plate deformation. In solid line is represented the energy in the plate versus the frequency  $k$ . In dotted lines are drawn the  $\mathcal{N}_{plate} = 6$  eigenfrequencies of the plate (in vacuum) for  $\mu = 0.5$  and  $\xi = 5$ . As already said in the previous paragraph  $\mathcal{N}_{rigid} = 3$  for  $L = 10$  and we obtain numerically  $\mathcal{N}_{elas} = 9 \geq \max(\mathcal{N}_{rigid}, \mathcal{N}_{plate}) = 6$  resonances. Concerning the location of the resonance frequencies, we get  $k_{elas}^n \leq k_{plate}^n$  for  $n = 1$  to  $\mathcal{N}_{plate} = 6$ .

More generally, considering a large range of values for the fluid/plate coupling  $\xi$  and for the rigidity of the plate  $\mu$ , we have found numerically that  $\mathcal{N}_{elas} = \mathcal{N}_{rigid} + \mathcal{N}_{plate}$  for any value of  $\xi$  and  $\mu$ . We did not succeed in proving this result in the general case, except for the case  $\xi \rightarrow \infty$  (limit of a light fluid): then the behaviour of the fluid and of the plate



**Fig. 13**  $k_{plate}^n$ ,  $n = 1, \dots, \mathcal{N}_{plate}$  for  $\mu \in [0.5; 1.5]$

**Fig. 14**  $k_{elas}^n$ ,  $n = 1, \dots, \mathcal{N}_{elas}$  for  $\mu \in [0.5; 1.5]$ .

are uncoupled. Indeed it is easy to find that for  $(\varphi, u)$  and  $(\psi, v) \in W$  and thanks to the change of unknowns  $(\tilde{u}, \tilde{v}) = \sqrt{\xi}(u, v)$ :

$$\mathcal{R}_{elas}(k; \varphi, \tilde{u}) = \frac{\int_{\Omega} |\nabla \varphi|^2 + \mu^4 \int_{\Gamma} |\tilde{u}''|^2 - \frac{2k}{\sqrt{\xi}} \int_{\Gamma} \varphi \tilde{u}}{\int_{\Omega} |\varphi|^2 + \int_{\Gamma} |\tilde{u}|^2}.$$

We deduce that

$$\lim_{\xi \rightarrow \infty} \mathcal{R}_{elas}(k; \varphi, \tilde{u}) = \frac{\int_{\Omega} |\nabla \varphi|^2 + \mu^4 \int_{\Gamma} |\tilde{u}''|^2}{\int_{\Omega} |\varphi|^2 + \int_{\Gamma} |\tilde{u}|^2},$$

which corresponds to a fluid and an elastic plate that are not coupled.

### 5.2.2 Illustration of the coupling effect

Before studying the coupling effect, let us specify some parameters. For a plate of length  $L = 10$  and a guide of height  $h = 4$  we have determined in section 5.1 that  $\mathcal{N}_{rigid} = 3$  with  $k_{rigid}^1 = 0.268$ ,  $k_{rigid}^2 = 0.528$  and  $k_{rigid}^3 = 0.785$ . Also the eigenfrequencies for the plate in vacuum  $k_{plate}^n = (\mu \alpha_n / L)^2$  below the cut-off frequency  $\pi/h = 0.79$  are drawn on Fig. 13 as functions of the rigidity  $\mu$  of the plate ( $0.5 \leq \mu \leq 1.5$ ). We see that  $2 \leq \mathcal{N}_{plate} \leq 6$ .

In the case of a light fluid ( $\xi = 1000$ ) we have drawn on Fig. 14 the resonance frequencies  $k_{elas}^n$  as functions of the rigidity of the plate. Because the fluid/plate coupling is chosen weak ( $\xi$  large),  $k_{elas}$  is equal to  $k_{rigid}$  (horizontal lines) or to  $k_{plate}$  nearly everywhere (in accordance with the previous paragraph). Moreover we see that, even for this weak coupled system, two frequencies never coalesce (curve veering).

As expected, when increasing the fluid/plate coupling (smaller  $\xi$  values:  $\xi = 100$  and  $\xi = 10$  represented on Fig. 15), the curves of the resonance frequencies twist continuously and the frequencies  $k_{elas}$  become different from  $k_{rigid}$  and  $k_{plate}$ .

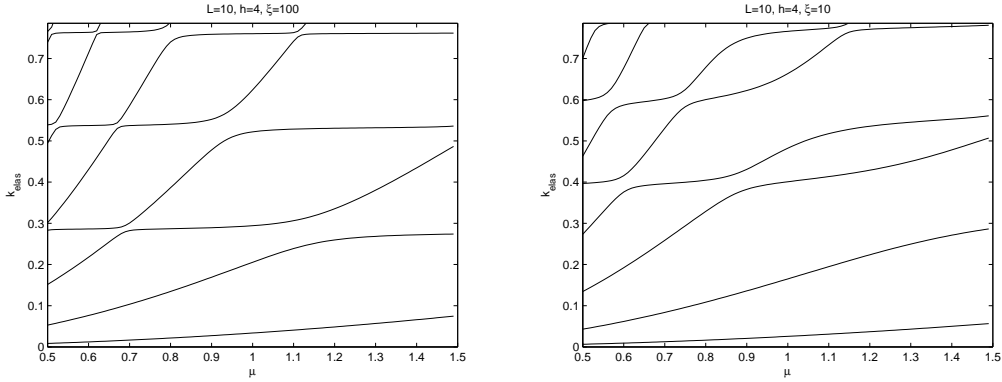


Fig. 15  $k_{elas}^n$ ,  $n = 1, \dots, \mathcal{N}_{elas}$  for  $\mu \in [0.5; 1.5]$ .

## 6. Conclusion

We have determined existence results for trapped modes about an elastic plate in presence of a compressible fluid confined in a duct. The number of trapped modes satisfies  $\mathcal{N}_{elas} \geq \max(\mathcal{N}_{rigid}, \mathcal{N}_{plate})$  where  $\mathcal{N}_{rigid}$  varies like  $L/h$  and  $\mathcal{N}_{plate}$  behaves like  $L/\sqrt{h}$ . As shown in the numerical results, we conjecture that  $\mathcal{N}_{elas} = \mathcal{N}_{rigid} + \mathcal{N}_{plate}$ . We are unable to prove this because of the nonlinearity of the eigenvalue problem: indeed even if at  $k = 0$  we know that  $\mathcal{N}_{rigid} + \mathcal{N}_{plate}$  eigenvalues  $\gamma_{elas}^n(0)$  exist and are positive, we cannot prove that they are all crossing the line  $k^2$  when  $k$  increases. The situation is even worse when the fluid is in motion since we do not know in this case how many eigenvalues are positive when  $k = 0$ . We are currently investigating this problem which will be presented in a forthcoming paper.

Our results can be easily extended to the case of a clamped-clamped plate by changing the space  $V$  for the plate deformations. In particular thanks to the Min-Max principle we easily get that  $\mathcal{N}_{plate}$  is smaller in the case of a clamped-clamped plate.

Throughout all this paper we looked for real resonances. We did not look for trapped modes associated to  $k \in \mathbb{C} \setminus \mathbb{R}$ . Such a mode, still in  $L^2(\Omega)$ , would correspond to an exponential growing in time (instability because  $e^{-i\omega t} \sim e^{\text{Im}(k)ct}$ ). It is easy to prove that in all the situations we have considered no instability exists. It is obvious when the plate is rigid (with or without flow) since the operator  $A_{rigid}$  is self-adjoint. When the plate is elastic, we have only proved that  $A_{elas}(k)$  is self-adjoint for real  $k$ . Let us prove that  $k$  cannot have an imaginary part. Suppose a trapped mode  $(\varphi, u) \neq 0$  exists, satisfying  $a_{elas}(k; (\varphi, u), (\psi, v)) = k^2 ((\varphi, u), (\psi, v)), \forall (\psi, v) \in W$ , with  $k \in \mathbb{C}$ . Taking  $(\psi, v) = (\varphi, u)$ , multiplying by  $-\bar{k}$  and taking the imaginary part leads to

$$\text{Im}(k) \left[ \int_{\Omega} |\nabla \varphi|^2 + |k\varphi|^2 + \xi \left( \int_{\Gamma} \mu^4 |u''|^2 + |ku|^2 \right) \right] = 0.$$

Therefore  $\text{Im}(k) \neq 0$  implies  $(\varphi, u) = 0$ . Of course resonant states as defined in (40) associated to a complex  $k$  exist but they do not correspond to trapped modes (since  $\varphi \notin L^2(\Omega)$ ). The key point of the above proof is that the coupling term  $2k \int_{\Gamma} \text{Im}(\bar{\varphi}u)$  disappears in this process. It is no longer the case when the elastic plate is in presence of a flow, and then instabilities can exist.

## APPENDIX A

*Modes of the duct*

The acoustic propagation in the duct, in presence of a uniform flow, can be completely described by computing the *modes* of the duct. These modes are the solutions of (4.5) with separated variables, belonging to  $T = \{\varphi \in H^1(\Omega) / \varphi = 0 \text{ on } y = 0\}$ . Let us point out that every fluid vibration can be decomposed on these modes. The modes read  $\varphi_n^\pm = \exp(i\beta_n^\pm x) \sin \zeta_n y$ ,  $n \in \mathbb{N}$ , where  $\zeta_n = \pi(2n+1)/h$ ,  $n \geq 0$ . A finite number of propagative modes exists, corresponding to  $2n+1 < kh/\pi$ , associated to the wave numbers  $\beta_n^\pm = \pm\sqrt{k^2 - \zeta_n^2}$ . The other modes for  $2n+1 > kh/\pi$  are evanescent, associated to the wave numbers  $\beta_n^\pm = \pm i\sqrt{\zeta_n^2 - k^2}$ .

The following mathematical result, related to the  $(\zeta_n)_{n \geq 0}$ , will be used.

LEMMA A.1.

$$\int_0^{h/2} |\varphi'|^2 \geq \left(\frac{\pi}{h}\right)^2 \int_0^{h/2} |\varphi|^2,$$

for every  $\varphi \in X = \{\varphi \in H^1(0, h/2); \varphi(0) = 0\}$ .

PROOF. Let us consider the operator  $C_{fluid}$  defined as follows:

$$D(C_{fluid}) = \{\varphi \in H^2(0, h/2); \varphi(0) = 0 \text{ and } \varphi'(h/2) = 0\}$$

and  $C_{fluid} \varphi = -\varphi''$  for  $\varphi \in D(C_{fluid})$ . The operator  $C_{fluid}$  is self-adjoint and has a compact resolvent. As a consequence, its spectrum consists in a sequence of eigenvalues which are the  $(\zeta_n)_{n \geq 0}$  defined above, the eigenfunctions  $\theta_n(y) = 2 \sin(\zeta_n y) / \sqrt{h}$  form an orthonormal basis of  $L^2(0, h/2)$  and one has

$$\zeta_0 = \min_{\varphi \in X, \varphi \neq 0} \frac{\int_0^{h/2} |\varphi'|^2}{\int_0^{h/2} |\varphi|^2}.$$

This proves the lemma since  $\zeta_0 = (\pi/h)^2$ . □

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