Propagation of acoustic waves in junction of thin slots

Adrien SEMIN (Team AN-EDP, University Paris XI ; and Team-project POems, INRIA Rocquencourt)

Joint work with Patrick JOLY and Bertrand MAURY

CEMRACS, Wednesday, August 13^{th} 2008

Outline

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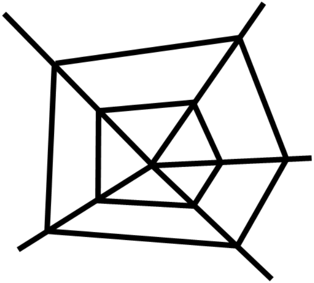
Scientific context

- I'm currently on the first PhD year, with Patrick Joly (INRIA Rocquencourt) et Bertrand Maury (University Paris XI)
- This work is a continuation of the PhD of Sébastien Tordeux (study of the Helmholtz equation between an half-space of $\mathbb{R}^n$ and a "one-dimensional" domain)

Motivations



- Goal : study the propagation of an acoustic wave in a network of thin slots

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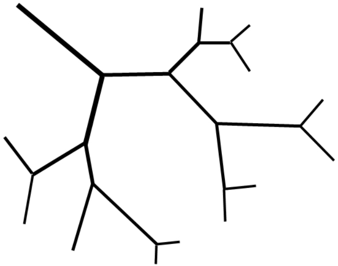

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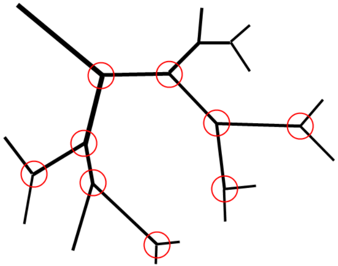

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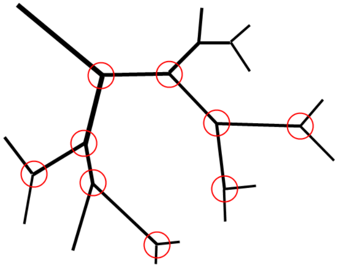
- Issue : establish the propagation in the junctions (red circles)

Motivations



- Goal : study the propagation of an acoustic wave in a network of thin slots
- Issue : establish the propagation in the junctions (red circles)
- Our geometry studied just below : two slots of same thickness and one junction

Outline

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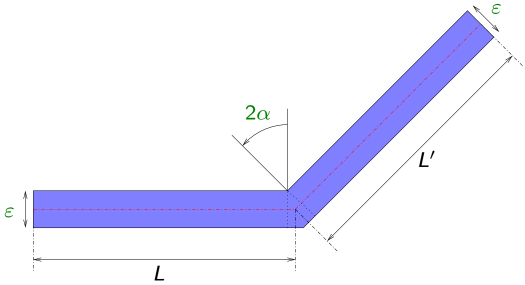
The geometry



- Ω^ε is the blue colored domain.
- When $\varepsilon \to 0$, Ω^ε tends to a 1D domain (colored in red), which can be parametrized by $s \in]-L, 0[\cup]0, L'[$.

The equation we study

Find $u^\varepsilon(t,x) \in \mathbb{R}_+ \times \Omega^\varepsilon$ such that

$$\begin{cases} \dfrac{\partial^2 u^\varepsilon}{\partial t^2}(t,x) - \Delta u^\varepsilon(t,x) & = & 0 \text{ in } \mathbb{R}_+ \times \Omega^\varepsilon \\ \dfrac{\partial u^\varepsilon}{\partial n} & = & 0 \text{ on } \mathbb{R}_+ \times \partial\Omega^\varepsilon \\ u^\varepsilon(0,.) & = & f \text{ on } \Omega^\varepsilon \\ \dfrac{\partial u^\varepsilon}{\partial t}(0,.) & = & g \text{ on } \Omega^\varepsilon \end{cases}$$

+ natural hypothesis on the Cauchy data f and g.
The associated energy is

$$\mathcal{E}^\varepsilon(t,u) = \frac{1}{\varepsilon}\int_{\Omega^\varepsilon} \left|\frac{\partial u}{\partial t}\right|^2 + |\nabla_x u|^2 dx$$

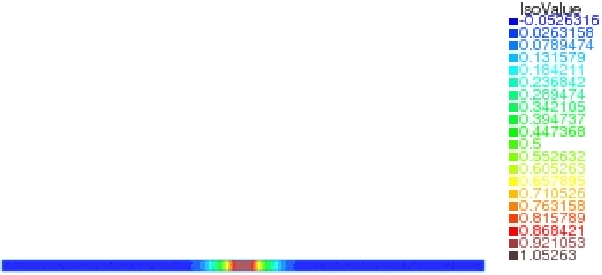
For the function u^ε, we have $\mathcal{E}^\varepsilon(t,u^\varepsilon)$ constant.

Exact 2D solution (computed with FreeFem++)

We took the lengths of the slots $L = L' = 8$, and we compute over $t \in [0, 8]$.

Here, its very difficult to see, but there exist a reflexion phenomena on the left slot.

Restriction on the left slot - $t = 0$

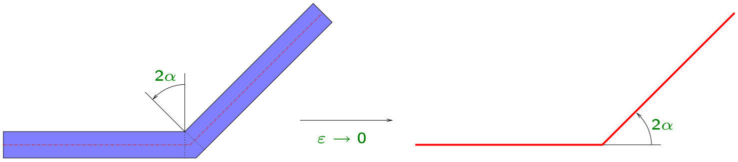


We can see the form of the initial signal f, which is a 1D Gaussian centered on $-L/2$

Restriction on the left slot - $t = 8$

We can see the form of the solution u^{ε}, which seems to be a derivate of a Gaussian. Numerical simulations show that the amplitude of this signal is like ε.

Limit problem (known since a very long time)



When ε tends to zero :

- u^{ε} tends (in a meaning not precised here) to a limit function u^0 which is 1D with respect to the space,
- u^0 satisfies the 1D time-domain equation $\frac{\partial^2 u^0}{\partial t^2} - \frac{\partial^2 u^0}{\partial s^2} = 0$ on each segment of the $1D$ domain
- $u^0(s = 0^-) = u^0(s = 0^+)$ and $\frac{\partial u^0}{\partial s}(s = 0^-) = \frac{\partial u^0}{\partial s}(s = 0^+)$ (Kirchoff laws)

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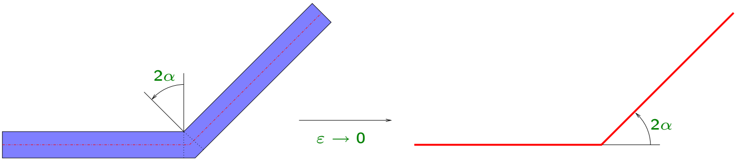

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$\Rightarrow$ u^0 does not depend on the parameter α , however u^ε does.

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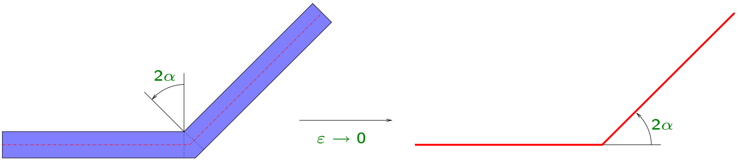

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$\Rightarrow$ u^0 does not depend on the parameter α , however u^ε does.
Our goal : study more precisely the behaviour of u^ε with respect to ε

Outline

Generalities about this method

There exists a lot of litterature about this method, and in a first approximation we can distinguish two schools (which seem to have only few cross-over references) :

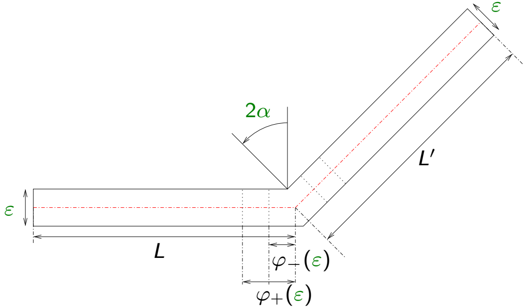
- The British school (D.G. Crighton and al.)
- The Russian school (A.M. Il'in and al.)

There exists an alternate method (the multiscale method), which leads to the same calculus and to the same conclusions.

Using the method

- Use of a overlapping domain decomposition
- Use of an ansatz on each part of the domain decomposition
- Injection of the ansatz in the equations (formally)
- Use of matching conditions (formally)
- Justify the whole development, and give error estimates *a posteriori* (not treated here)

Overlapping domain decomposition

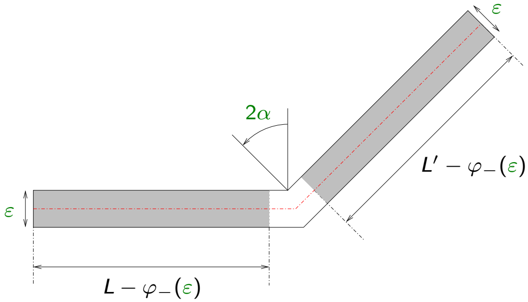


Two functions φ_-, $\varphi_+ \in C^1(\mathbb{R}, \mathbb{R})$ such that

- $0 < \varphi_-(\varepsilon) < \varphi_+(\varepsilon)$
- $\lim_{\varepsilon \to 0} \varphi_\pm(\varepsilon) = 0$ et $\lim_{\varepsilon \to 0} \varphi_\pm(\varepsilon)/\varepsilon = +\infty$

Overlapping domain decomposition

Slots zones

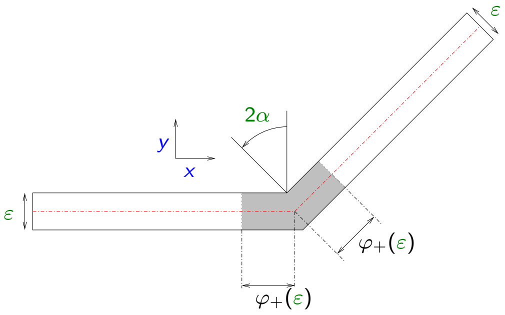


- Left slot : $\Omega_-(\varepsilon) = (s, \nu) \in]-L, -\varphi_-(\varepsilon)[\times] - \varepsilon/2, \varepsilon/2[$
- Right slot : $\Omega_+(\varepsilon) = (s, \nu) \in]\varphi_-(\varepsilon), L'[\times] - \varepsilon/2, \varepsilon/2[$
- Coordinates scaling : $(s, \nu) \mapsto (S, \mu) = (s, \nu/\varepsilon)$

We note $\widehat{\Omega}_\pm(\varepsilon)$ the set $\Omega_\pm(\varepsilon)$ in the scaled coordinates, and $\widehat{\Omega}_\pm$ its limit when ε tends to 0.

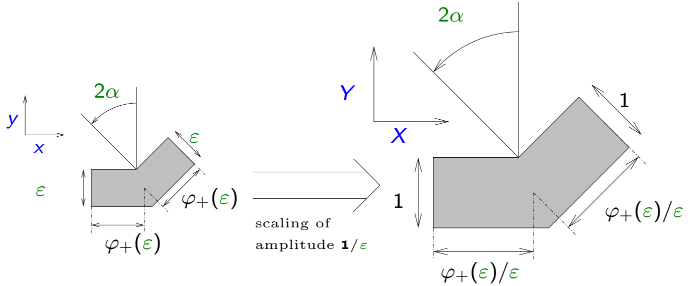
Overlapping domain decomposition

Near-field zone



- Near-field zone : $\Omega_I(\varepsilon)$
- Coordinates scaling : $(x, y) \mapsto (X, Y) = (x/\varepsilon, y/\varepsilon)$

Near-field given with the new scaled coordinates



- $\Omega_I(\varepsilon)$ converges to the point $(0,0)$
- $\widehat{\Omega}_I(\varepsilon)$ converges to one semi-infinite canonical domain $\widehat{\Omega}_I$ (thanks to the fact that $\varphi_+(\varepsilon)/\varepsilon \to +\infty$).

Ansatz (slots zones)

Ansatz

On the slots $\Omega_+(\varepsilon)$ et $\Omega_-(\varepsilon)$, we are looking for u^ε on the form

$$u^\varepsilon(t, S, \varepsilon\mu) = \sum_{k=0}^{\infty} \varepsilon^k u_{k,\pm}(t, S, \mu) + o(\varepsilon^\infty)$$

with $u_{k,\pm}$ defined on $\mathbb{R}_+ \times \widehat{\Omega}_\pm$

Equations (slots zones)

Putting this previous ansatz on the wave equation with Neumann condition on $\mu = \pm 1/2$ gives that

- $\forall k \in \mathbb{N}$, $u_{k,\pm}(t, S, \mu) = u_{k,\pm}(t, S)$
- $\forall k \in \mathbb{N}$, we have the following 1D time-domain equations

$$
\begin{aligned}
\frac{\partial^2 u_{k,-}}{\partial t^2} - \frac{\partial^2 u_{k,-}}{\partial s^2} &= 0, \ (t, s) \in \mathbb{R}_+ \times]-L, 0[\\
\frac{\partial^2 u_{k,+}}{\partial t^2} - \frac{\partial^2 u_{k,+}}{\partial s^2} &= 0, \ (t, s) \in \mathbb{R}_+ \times]0, L'[
\end{aligned}
$$

Ansatz (near-field zone)

Ansatz

On the near-field zone $\Omega_I(\varepsilon)$, we are looking for u^ε on the form

$$u^\varepsilon(t, \varepsilon X, \varepsilon Y) = \sum_{k=0}^{\infty} \varepsilon^k U_k(t, X, Y) + o(\varepsilon^\infty)$$

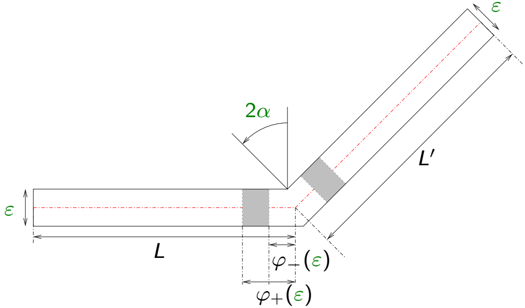
with U_k defined on $\mathbb{R}_+ \times \widehat{\Omega}_I$

Equations (near-field zone)

Putting this previous ansatz on the wave equation with Neumann condition on $\partial\widehat{\Omega}_I$ gives that

$$
\begin{aligned}
\Delta_{X,Y} U_0 &= 0 \text{ in } \mathbb{R}_+ \times \widehat{\Omega}_I \\
\Delta_{X,Y} U_1 &= 0 \text{ in } \mathbb{R}_+ \times \widehat{\Omega}_I \\
\forall k \in \mathbb{N},\ \Delta_{X,Y} U_{k+2} &= \frac{\partial^2 U_k}{\partial t^2} \text{ in } \mathbb{R}_+ \times \widehat{\Omega}_I \\
\forall k \in \mathbb{N},\ \nabla_{X,Y} U_k . \vec{n} &= 0 \text{ on } \mathbb{R}_+ \times \partial\widehat{\Omega}_I
\end{aligned}
$$

Use of matching conditions



On the gray domains, we have, with evident notations :

$$\begin{aligned} u^{\varepsilon}(t,s,\nu) &= \sum_{k=0}^{\infty} \varepsilon^k u_{k,\pm}(t,s) + o(\varepsilon^{\infty}) \\ &= \sum_{k=0}^{\infty} \varepsilon^k U_k\left(t, \frac{s}{\varepsilon}, \frac{\nu}{\varepsilon}\right) \end{aligned}$$

for $\pm s \in]\varphi_-(\varepsilon), \varphi_+(\varepsilon)[$ and $\nu \in]-\varepsilon/2, \varepsilon/2[$

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for $\pm s \in]\varphi_-(\varepsilon), \varphi_+(\varepsilon)[$ and $\nu \in]-\varepsilon/2, \varepsilon/2[$

This relation links the behaviour of $u_{k,\pm}(t,s)$ at $s = 0$ to the behaviour of $U_k(t,S,\mu)$ at $S = \pm\infty$

Main results

Theorem

There exist three unique families $(u_{k,-})_{k\in\mathbb{N}}$, $(u_{k,+})_{k\in\mathbb{N}}$ *and* $(U_k)_{k\in\mathbb{N}}$ *satisfying the coupled problem described just above.*

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Theorem

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Theorem

There exists two cut-off functions χ_-^ε *and* χ_+^ε *such that*

- $\chi_\pm^\varepsilon = 1$ *on* $\Omega_\pm(\varepsilon) \setminus \Omega_I(\varepsilon)$ *and* $\chi_\pm^\varepsilon = 0$ *on* $\Omega^\varepsilon \setminus \Omega_\pm(\varepsilon)$
- *For any* $n \in \mathbb{N}$, *if we call*

$$\widetilde{u}_n^\varepsilon = \sum_{k=0}^{n} \varepsilon^k \left(\chi_-^\varepsilon u_{k,-} + \chi_+^\varepsilon u_{k,+} + (1 - \chi_-^\varepsilon \chi_+^\varepsilon U_k\right)$$

then there exists a constant $C(t)$ *that only depends on the time,* k *and the Cauchy data such that, for* ε *small enough,*

$$\mathcal{E}^\varepsilon(t, u^\varepsilon - \widetilde{u}_n^\varepsilon) \leqslant C(t)\varepsilon^{n+1}$$

Outline

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General idea

- Let u_n^ε the $1D$ function (with respect to the space) defined by

$$u_n^\varepsilon(t,s) = \begin{cases} \sum_{k=0}^{n} \varepsilon^k u_{k,+}(t,s), & s > 0 \\ \sum_{k=0}^{n} \varepsilon^k u_{k,-}(t,s), & s < 0 \end{cases}$$

- Then u_n^ε satisfies the following problem

$$\begin{cases} \dfrac{\partial^2 u_n^\varepsilon}{\partial t^2} - \dfrac{\partial^2 u_n^\varepsilon}{\partial s^2} = 0, & t \in \mathbb{R}_+, \ s \in]-L, 0[\cup]0, L'[\\ \dfrac{\partial u_n^\varepsilon}{\partial s}(t,s) = 0, & t \in \mathbb{R}_+, \ s \in \{-L, L'\} \\ u_n^\varepsilon(0,s) = f, & s \in]-L, 0[\cup]0, L'[\\ \dfrac{\partial u_n^\varepsilon}{\partial t}(0,s) = g, & s \in]-L, 0[\cup]0, L'[\end{cases}$$

Jump conditions

The only missing information to fully characterize is the following gaps

- $[u_n^\varepsilon(t)] := u_n^\varepsilon(t, 0^+) - u_n^\varepsilon(t, 0^-)$
- $\left[\frac{\partial u_n^\varepsilon}{\partial s}(t)\right] := \frac{\partial u_n^\varepsilon}{\partial s}(t, 0^+) - \frac{\partial u_n^\varepsilon}{\partial s}(t, 0^-)$

Thanks to the matching conditions, we are able to give these information.

Outline

Order 0 jump conditions

We get that

- $[(u^{\varepsilon})_0(t)] = 0$
- $\left[\dfrac{\partial (u^{\varepsilon})_0}{\partial s}(t)\right] = 0$

This means that the function $(u^{\varepsilon})_0$ satisfying our problem with order 0 jump conditions is the same as the function u^0 limit of the solution u^{ε} of the 2D exact problem.

Order 1 jump conditions

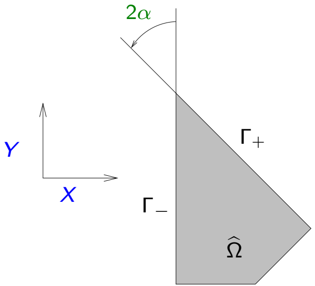
We get that

- $\left[\dfrac{\partial (u^{\varepsilon})_1}{\partial s}(t)\right] = 0$
- $[(u^{\varepsilon})_1(t)] = \varepsilon\,(K(\alpha) - \tan\alpha)\left\langle \dfrac{\partial (u^{\varepsilon})_0}{\partial s}(t)\right\rangle$, where $\langle v(t)\rangle = \frac{1}{2}\left(v(t,0^+) + v(t,0^-)\right)$

$\Rightarrow$ Approximated Dirichlet condition :

- $[(u^{\varepsilon})_1(t)] = \varepsilon\,(K(\alpha) - \tan\alpha)\left\langle \dfrac{\partial (u^{\varepsilon})_1}{\partial s}(t)\right\rangle + O(\varepsilon^2)$

Computation of $K(\alpha)$



We have to solve :

$$\begin{cases} \Delta W & = & 0 \text{ in } \widehat{\Omega} \\ \dfrac{\partial W}{\partial n} + T_+ W & = & 1 \text{ on } \Gamma_+ \\ \dfrac{\partial W}{\partial n} + T_- W & = & -1 \text{ on } \Gamma_+ \\ \dfrac{\partial W}{\partial n} & = & 0 \text{ on } \partial\widehat{\Omega} \setminus \Gamma_\pm \\ \displaystyle\int_{\widehat{\Omega}} W & = & 0 \end{cases}$$

when $T_\pm$ are positive DtN operators.

Computation of $K(\alpha)$

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when $T_\pm$ are positive DtN operators.
Then :

- This problem admits a unique solution in $H^1(\widehat{\Omega})$
- The constant $K(\alpha)$ is computed by

$$K(\alpha) = \int_{\Gamma_+} W - \int_{\Gamma_-} W > 0$$

Outline

Natural 1D problem

Let us write $\widetilde{u}^\varepsilon = (u^\varepsilon)_1$, then

$$\left\{\begin{array}{rcl}
\dfrac{\partial^2 \widetilde{u}^\varepsilon}{\partial t^2} - \dfrac{\partial^2 \widetilde{u}^\varepsilon}{\partial s^2} & = & 0,\ \ t \in \mathbb{R}_+,\ \ s \in]-L, 0[\, \cup\,]0, L'[\\
\dfrac{\partial \widetilde{u}^\varepsilon}{\partial s}(t,s) & = & 0,\ \ t \in \mathbb{R}_+,\ \ s \in \{-L, L'\} \\
\widetilde{u}^\varepsilon(0, \bullet) & = & f \\
\dfrac{\partial \widetilde{u}^\varepsilon}{\partial t}(0, \bullet) & = & g \\
\left[\dfrac{\partial \widetilde{u}^\varepsilon}{\partial s}(t)\right] & = & 0 \\
[\widetilde{u}^\varepsilon(t)] & = & \varepsilon\,(K(\alpha) - \tan(\alpha)) \left\langle \dfrac{\partial \widetilde{u}^\varepsilon}{\partial s}(t) \right\rangle
\end{array}\right.$$

Natural associated "energy" :

$$
\begin{aligned}
\mathcal{E}_{1D}(t,\widetilde{u}^\varepsilon) &= \int_{s=-L}^{0} \left|\frac{\partial \widetilde{u}^\varepsilon}{\partial s}(t,\sigma)\right|^2 + \left|\frac{\partial \widetilde{u}^\varepsilon}{\partial t}(t,\sigma)\right|^2 d\sigma \\
&+ \int_{s=0}^{L'} \left|\frac{\partial \widetilde{u}^\varepsilon}{\partial s}(t,\sigma)\right|^2 + \left|\frac{\partial \widetilde{u}^\varepsilon}{\partial t}(t,\sigma)\right|^2 d\sigma \\
&+ \frac{1}{\varepsilon\,(K(\alpha)-\tan(\alpha))}\,|[\widetilde{u}^\varepsilon(t)]|^2
\end{aligned}
$$

- The good point : $\mathcal{E}_{1D}(t,\widetilde{u}^\varepsilon) = \mathcal{E}_{1D}(0,\widetilde{u}^\varepsilon)$

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- The good point : $\mathcal{E}_{1D}(t,\widetilde{u}^{\varepsilon}) = \mathcal{E}_{1D}(0,\widetilde{u}^{\varepsilon})$
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$\Rightarrow$ This quantity is not an energy, and in fact numerical simulations explode in finite time.

Answer : we write the jump conditions on points which depends on ε

New jump conditions

Let us write

- $[[v(t)]]_{\varepsilon} = v(t, \varepsilon \tan(\alpha)/2) - v(t, -\varepsilon \tan(\alpha)/2)$
- $\langle\langle v(t) \rangle\rangle_{\varepsilon} = \frac{1}{2}(v(t, \varepsilon \tan(\alpha)/2) + v(t, -\varepsilon \tan(\alpha)/2))$

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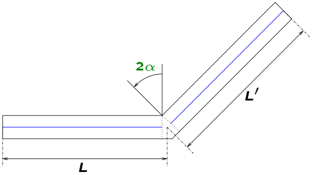
- $\langle\langle v(t) \rangle\rangle_\varepsilon = \frac{1}{2}(v(t, \varepsilon \tan(\alpha)/2) + v(t, -\varepsilon \tan(\alpha)/2))$

Then new jump conditions can be written as

- $\left[\left[\frac{\partial \widetilde{u}^\varepsilon}{\partial s}(t)\right]\right]_\varepsilon = \varepsilon \tan(\alpha) \left\langle\left\langle \frac{\partial^2 \widetilde{u}^\varepsilon}{\partial t^2}(t) \right\rangle\right\rangle_\varepsilon \quad (+O(\varepsilon^2))$
- $[[\widetilde{u}^\varepsilon(t)]]_\varepsilon = \varepsilon K(\alpha) \left\langle\left\langle \frac{\partial \widetilde{u}^\varepsilon}{\partial s}(t) \right\rangle\right\rangle_\varepsilon \quad (+O(\varepsilon^2))$

Modified 1D problem



$$\begin{cases} \dfrac{\partial^2 \widetilde{u}^\varepsilon}{\partial t^2} - \dfrac{\partial^2 \widetilde{u}^\varepsilon}{\partial s^2} & = & 0, \ t \in \mathbb{R}_+, \ s \in \left]-L, -\frac{\varepsilon \tan(\alpha)}{2}\right[\ \cup \ \left]\frac{\varepsilon \tan(\alpha)}{2}, L'\right[\\ \dfrac{\partial \widetilde{u}^\varepsilon}{\partial s}(t,s) & = & 0, \ t \in \mathbb{R}_+, \ s \in \{-L, L'\} \\ \widetilde{u}^\varepsilon(0,\bullet) & = & f \\ \dfrac{\partial \widetilde{u}^\varepsilon}{\partial t}(0,\bullet) & = & g \\ \left[\left[\dfrac{\partial \widetilde{u}^\varepsilon}{\partial s}(t)\right]\right]_\varepsilon & = & \varepsilon \tan(\alpha) \left\langle\left\langle \dfrac{\partial^2 \widetilde{u}^\varepsilon}{\partial t^2}(t) \right\rangle\right\rangle_\varepsilon \\ [[\widetilde{u}^\varepsilon(t)]]_\varepsilon & = & \varepsilon K(\alpha) \left\langle\left\langle \dfrac{\partial \widetilde{u}^\varepsilon}{\partial s}(t) \right\rangle\right\rangle_\varepsilon \end{cases}$$

Modified associated energy

$$\begin{aligned}
\mathcal{E}'_{1D}(t,\widetilde{u}^{\varepsilon}) &= \int_{s=-L}^{-\frac{\varepsilon \tan(\alpha)}{2}} \left|\frac{\partial \widetilde{u}^{\varepsilon}}{\partial s}(t,\sigma)\right|^2 + \left|\frac{\partial \widetilde{u}^{\varepsilon}}{\partial t}(t,\sigma)\right|^2 d\sigma \\
&+ \int_{s=\frac{\varepsilon \tan(\alpha)}{2}}^{L'} \left|\frac{\partial \widetilde{u}^{\varepsilon}}{\partial s}(t,\sigma)\right|^2 + \left|\frac{\partial \widetilde{u}^{\varepsilon}}{\partial t}(t,\sigma)\right|^2 d\sigma \\
&+ \frac{1}{\varepsilon K(\alpha)} \left|[[\widetilde{u}^{\varepsilon}(t)]]_{\varepsilon}\right|^2 + \varepsilon \tan(\alpha) \left|\left\langle\left\langle \frac{\partial \widetilde{u}^{\varepsilon}}{\partial t}(t) \right\rangle\right\rangle_{\varepsilon}\right|^2
\end{aligned}$$

- The first good point : $\mathcal{E}'_{1D}(t,\widetilde{u}^{\varepsilon}) = \mathcal{E}'_{1D}(0,\widetilde{u}^{\varepsilon})$
- The second good points : all the terms of this quantity are positive

Main results

Theorem

The modified 1D problem which defines $\widetilde{u}^\varepsilon$ is well posed and admits a unique solution.

Theorem

For the function u^ε solution of the 2D exact problem, we can define $\widehat{u}^\varepsilon$ by

$$\widehat{u}^\varepsilon(t,s) = \frac{1}{\varepsilon}\int_{-\varepsilon/2}^{\varepsilon/2} u^\varepsilon(t,s,\nu)d\nu, \quad \forall |s| > \frac{\varepsilon \tan(\alpha)}{2}$$

Then there exists a constant $C(t)$ that only depends on time and Cauchy data such that

$$\mathcal{E}'_{1D}(t, \widetilde{u}^\varepsilon - \widehat{u}^\varepsilon) \leqslant C(t)\varepsilon^2$$

Corollary

For any $\delta > 0$, there exists a positive constant $C'_\delta(t)$ such that, for $\varepsilon < 2\delta/\tan(\alpha)$,

$$\|(\widetilde{u}^\varepsilon - \widehat{u}^\varepsilon)(t,\bullet)\|_{H^1(|s|>\delta)} \leqslant C'_\delta(t)\varepsilon^2$$

Outline

Natural 1D problem (“bad” problem)

Modified 1D problem (“good” problem)

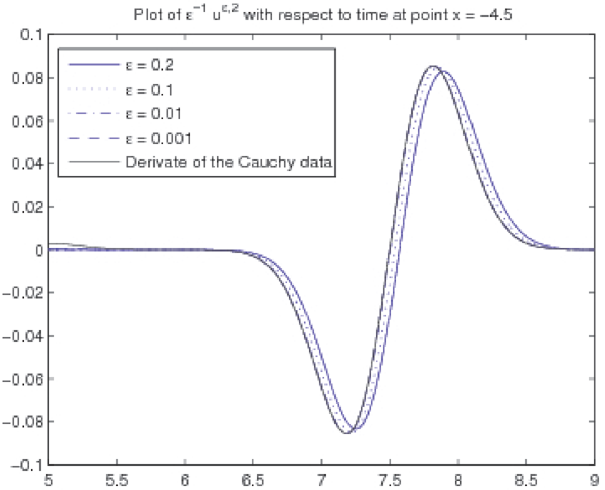
On this last simulation, we can see that the reflexion signal is like the derivate of the initial signal.
In fact, one can show that the first order of the reflexion coefficient (with respect to ε) is

$$R = -\frac{\varepsilon}{2}(\tan(\alpha) - K(\alpha))\frac{\partial}{\partial t}$$

To see this phenomena, we compute $\varepsilon^{-1}\widetilde{u}^{\varepsilon}(t, s)$ as a function of t on the point $s = -3L/4$, and we compared with the derivate of the Cauchy data (translated and multiplied by $-\frac{1}{2}(\tan(\alpha) - K(\alpha))$)

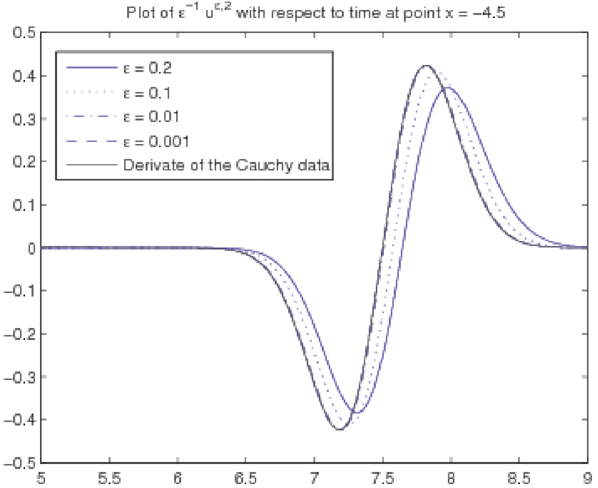
Plot of $\varepsilon^{-1}\widetilde{u}^{\varepsilon}(t, -\frac{3L}{4})$ with respect to t

Parameters taken : $L = 6$ and $\alpha = \frac{\pi}{8}$



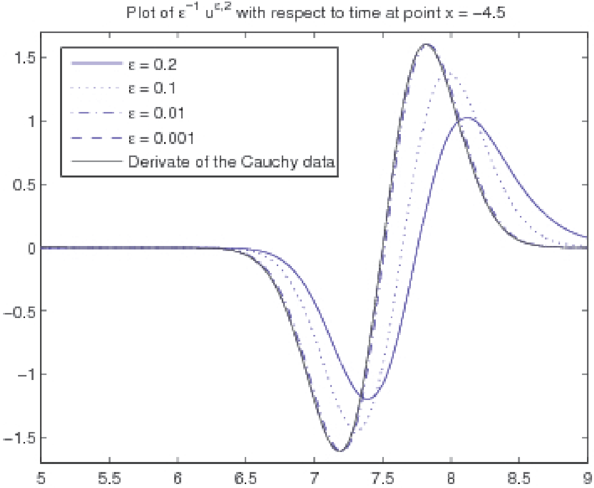
Plot of $\varepsilon^{-1}\widetilde{u}^{\varepsilon}(t, -\frac{3L}{4})$ with respect to t

Parameters taken : $L = 6$ and $\alpha = \frac{\pi}{4}$



Plot of $\varepsilon^{-1}\widetilde{u}^{\varepsilon}(t, -\frac{3L}{4})$ with respect to t

Parameters taken : $L = 6$ and $\alpha = \frac{3\pi}{8}$



Error between the exact solution and the 1D approximated solution

Theory claims that

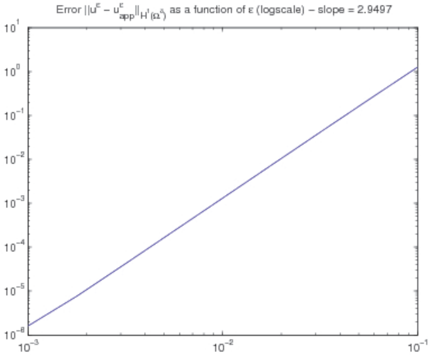
$$\|(\widetilde{u}^{\varepsilon} - \widehat{u}^{\varepsilon})(t, \bullet)\|_{\mathrm{H}^1(|s|>\delta)} \leqslant C(t)\varepsilon^2$$

We then compute $\|(\widetilde{u}^{\varepsilon} - \widehat{u}^{\varepsilon})(t, \bullet)\|_{\mathrm{H}^1(|s|>\delta)}$ with a time t large enough to see the effects of the junction, and we plot this norm with respect to ε.

Error between the exact solution and the 1D approximated solution

Theory claims that

$$\|(\widetilde{u}^{\varepsilon} - \widehat{u}^{\varepsilon})(t, \bullet)\|_{\mathrm{H}^1(|s|>\delta)} \leqslant C(t)\varepsilon^2$$



Error between the exact solution and the 1D approximated solution

Theory claims that

$$\|(\widetilde{u}^{\varepsilon} - \widehat{u}^{\varepsilon})(t, \bullet)\|_{\mathrm{H}^1(|s|>\delta)} \leqslant C(t)\varepsilon^2$$

Numerically, we get that

$$\|(\widetilde{u}^{\varepsilon} - \widehat{u}^{\varepsilon})(t, \bullet)\|_{\mathrm{H}^1(|s|>\delta)} \sim C\varepsilon^3$$

We can see that there is a superconvergence (we get an additionnal order for the error estimate), which is only due to the fact that we have two slots of the same width and the junction admits an axis of symmetry.

Outline

Conclusions

- Writing of a 1D space problem which is of order 1 with respect to ε, which differs from the exact problem with a ε^2 error,
- Results can be written with the time-domain and time-harmonic wave equation,
- The method can be extented for writing a 1D space problem at any order with respect to ε
- The method can be extented for writing a 1D space problem for junction of an arbitrary number of slots of differents widths (but those width are proportional to ε - not presented here)

Perspectives

- Writing a C++ code which can treat a network of junctions (the theory exists, and the code exists already for Matlab, but it is quite slow)
- Inverse problem (we look for the function $\widetilde{u}^{\varepsilon}(t, x)$ with respect to t, may we have some information about the geometry ?)
- Extend the results to the Maxwell equations (more difficult since we look for a vectorial function)