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Laurent Bourgeois

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Abstract: This paper is devoted to a conditional stability estimate related to the ill-posed Cauchy problems for the Laplace's equation in domains with $C^{1,1}$ boundary. It is an extension of an earlier result of [19] for domains of class C^{∞}. Our estimate is established by using a global Carleman estimate near the boundary in which the exponential weight depends on the distance function to the boundary. Furthermore, we prove that this stability estimate is nearly optimal and induces a nearly optimal convergence rate for the method of quasi-reversibility introduced in [15] to solve the ill-posed Cauchy problems.

Key-words: ill-posed problem, conditional stability, Carleman estimate, quasi-reversibility, distance function

Centre de recherche INRIA Paris – Rocquencourt
Domaine de Voluceau, Rocquencourt, BP 105, 78153 Le Chesnay Cedex
Téléphone : +33 1 39 63 55 11 — Télécopie : +33 1 39 63 53 30

Stabilité conditionnelle pour les problèmes de Cauchy elliptiques mal posés : le cas d'un domaine de classe $C^{1,1}$ (partie I)

Résumé : Ce document concerne une estimation de stabilité conditionnelle relative aux problèmes de Cauchy mal posés pour l'équation de Laplace dans un domaine de classe $C^{1,1}$. Ce résultat constitue une généralisation d'un résultat antérieur [19] pour un domaine de classe C^{∞}. Notre estimation est obtenue en utilisant une inégalité de Carleman globale près du bord, dans laquelle le poids exponentiel dépend de la fonction distance au bord. De plus, nous montrons que cette estimation de stabilité est quasi-optimale et implique une vitesse de convergence quasi-optimale pour la méthode de quasi-réversibilité introduite dans [15] pour résoudre les problèmes de Cauchy mal posés.

Mots-clés : problème mal posé, stabilité conditionnelle, inégalité de Carleman, quasi-réversibilité, fonction distance

1 Introduction

The question of stability for ill-posed elliptic Cauchy problems is a central question in the fields of inverse problems and controllability. A number of authors have been providing some answers since the first contributions of F. John [13] and L.E. Payne [18]. In the meantime, the so-called Carleman estimates have become a very efficient tool to derive not only unique continuation properties (see for instance [5, 20]) but also stability estimates (see for instance [12, 16]). The obtained stability estimates take some various forms, depending on the geometry of the domain and on the regularity of the function. But a classical and general result is that the stability estimates are, following the vocabulary introduced by F. John [13], of Hölder type in a subdomain which does not include a neighborhood of the boundary where limit conditions are unknown, and of logarithmic type in the whole domain, as it will be described again in the following paper. In this sense, we can say that the physical presence of the boundary has a strong influence on the stability of ill-posed elliptic problems. A more specific question is the influence of the regularity of the boundary on this stability. The case of a Lipschitz boundary was already considered in [1] in the context of more complex inverse elliptic boundary problems with unknown boundaries (see also [21]). In particular, with the help of doubling inequalities, the authors of [1] derived some logarithmic stability estimates of the H^1 norm for functions of class $C^{1,\alpha}$ with $0 < \alpha < 1$, the exponent of the logarithm being however unspecified. We are here mostly interested in finding the optimal exponent of the logarithmic stability estimates. In the following paper, we specify such exponent in the case of a domain with $C^{1,1}$ boundary for functions in H^2. The case of a domain with Lipschitz boundary is considered in [3]. The choice of the functional space H^2 is motivated by a particular application of our stability estimate, which is the derivation of a convergence rate for the method of quasi-reversibility to regularize the ill-posed Cauchy problems for the elliptic operator P [15]. We prove that the exponent of our logarithmic stability estimate is any $\kappa < 1$ and that the value 1 cannot be improved. In this sense, our stability estimate in nearly optimal.
The starting point of our study is the article of K.-D. Phung [19], who obtained the following conditional stability estimate for the operator $P = -\Delta . - k.$, $k \in \mathbb{R}$. For a bounded and connected domain $\Omega \subset \mathbb{R}^N$ of class C^∞, if Γ_0 is an open part of $\partial\Omega$, then for all $\kappa \in]0, 1[$, there exist constants $C, \delta_0 > 0$ such that for all $\delta \in]0, \delta_0[$, for all function $u \in H^2(\Omega)$ which satisfies

$$||u||_{H^2(\Omega)} \leq M, \quad ||Pu||_{L^2(\Omega)} + ||u||_{H^1(\Gamma_0)} + ||\partial_n u||_{L^2(\Gamma_0)} \leq \delta, \tag{1}$$

where M is a constant,

$$||u||_{H^1(\Omega)} \leq C \frac{M}{(\log(M/\delta))^\kappa}. \tag{2}$$

A similar estimate holds with $||u||_{H^1(\omega)}$ replacing $||u||_{H^1(\Gamma_0)} + ||\partial_n u||_{L^2(\Gamma_0)}$ in (1) for any open domain $\omega \Subset \Omega$. The label "conditional" stems from the first inequality of (1), which is required to obtain stability. We also notice that despite $u \in H^2(\Omega)$, we only estimate $||u||_{H^1(\Omega)}$ in (2), which is due to the estimation of the function u up to the part of the boundary $\partial\Omega$ which is complementary to Γ_0 (see the proof of proposition 4). In [19], the proof of (2) for C^∞ domains is

mainly based on an interior Carleman estimate [8, 11], as well as a Carleman estimate near the boundary [17]. These Carleman estimates were obtained by using pseudo-differential operators. Precisely, the analysis of stability near the boundary follows from a Carleman estimate in the half-space after using a local mapping from the cartesian coordinates to the geodesic normal coordinates. This technique introduces some limitation concerning the regularity of the domain.
The main goal of this paper is to prove that the stability estimate (2) still holds for domains of class $C^{1,1}$ with the same assumptions. This result is obtained by another technique to derive the same estimates near the boundary. We use global Carleman estimates near the boundary directly on the initial geometry, by following the friendly method of [9], and the exponential weight is roughly speaking a function of the distance to the boundary.
Our paper is organized as follows. The second section recalls an important result concerning the local regularity of the distance function to the boundary, which is related to the regularity of the domain. Section 3 is devoted to the derivation of our stability estimate with the help of a Carleman inequality. In section 4 we prove that such stability estimate in nearly optimal. Lastly, in section 5 we derive some convergence rates for the method of quasi-reversibility to regularize the ill-posed Cauchy problems.

2 About the regularity of the distance function

We consider a bounded and connected domain $\Omega \subset \mathbb{R}^N$ of class $C^{1,1}$. For $x \in \overline{\Omega}$, we denote $d_{\partial\Omega}(x)$ the distance function to the boundary $\partial\Omega$, and we define the set

$$\pi_{\partial\Omega}(x) = \{y \in \partial\Omega, \quad |x-y| = d_{\partial\Omega}(x)\},$$

where $|.|$ denotes the euclidean norm in $\mathbb{R}^N$. At any point $y \in \partial\Omega$, the outward unit normal is denoted $n(y)$.
There are a number of contributions concerning the regularity of function $d_{\partial\Omega}$ near the boundary. Among these, the following theorem is proved in [7] (see theorem 4.3, p. 219).

Theorem 1 *: If the domain $\Omega \subset \mathbb{R}^N$ is of class $C^{1,1}$, then for all $x_0 \in \partial\Omega$, there exists a neighborhood $\mathcal{W}(x_0)$ of x_0 such that if $W(x_0) = \mathcal{W}(x_0) \cap \overline{\Omega}$,*

$$\forall x \in W(x_0), \quad \pi_{\partial\Omega}(x) = \{P_{\partial\Omega}(x)\}$$

is a singleton and the map : $W(x_0) \to \mathbb{R}^n$

$$x \mapsto P_{\partial\Omega}(x)$$

is Lipschitz continuous in $W(x_0)$. Moreover,

$$\forall x \in W(x_0), \quad \nabla d_{\partial\Omega}(x) = -n(P_{\partial\Omega}(x)).$$

As a result, $\nabla d_{\partial\Omega}$ is Lipschitz continuous in $W(x_0)$, so $d_{\partial\Omega} \in C^{1,1}(W(x_0))$, in particular the components of $\nabla^2 d_{\partial\Omega}$ belong to $L^\infty(W(x_0))$.

Remark 1 *:* As proved by a counterexample in [7], p. 222, when Ω is only of class $C^{1,\alpha}$, with $0 \leq \alpha < 1$, then $d_{\partial\Omega}$ may be not differentiable in a neighborhood of $\partial\Omega$. In particular, $\nabla d_{\partial\Omega}$ is not a C^0 function in a neighborhood of $\partial\Omega$.

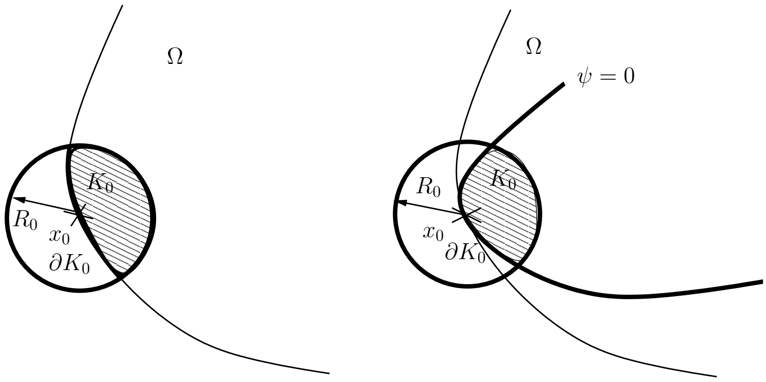

Figure 1: Two cases for definition of K_0 and ∂K_0

3 A stability estimate in domains of class $C^{1,1}$

3.1 A Carleman estimate near the boundary

We consider $x_0 \in \partial\Omega$, $R_0 > 0$, and the set $B = \Omega \cap B(x_0, R_0)$. We define $\tilde{H}_0^2(B)$ as the restrictions to B of functions in $H_0^2(B(x_0, R_0))$.
Let the function ψ satisfy $\psi \in C^1(\overline{B})$, $\nabla\psi \neq 0$ on $\overline{B}$, and $\nabla^2\psi \in (L^\infty(B))^{N\times N}$.
We define for $\varepsilon \geq 0$,

$$K_\varepsilon = \{x \in \overline{B}, \quad \psi(x) \geq \varepsilon\}.$$

In the following, ψ is chosen such that only two cases occur (see figure 1). In the first case $K_0 = \overline{B}$, the boundary of K_0 in then included in $\partial\Omega \cup \partial B(x_0, R_0)$ and we denote $\partial K_0 = \overline{B} \cap \partial\Omega$. In the second case $\{x, \psi(x) > 0\} \cap \partial\Omega = \emptyset$, the boundary of K_0 is then included in $\{x, \psi(x) = 0\} \cup \partial B(x_0, R_0)$ and we denote $\partial K_0 = \{x \in \overline{B}, \quad \psi(x) = 0\}$.
Denoting $\phi(x) = e^{\alpha\psi(x)}$ for $\alpha > 0$, we have the following lemma.

Lemma 1 *: Let define $u \in C_0^\infty(B(x_0, R_0))$ and $v = u\, e^{\lambda\phi}$ with $\lambda > 0$.*

With the following definitions

$$p_1 = 2\alpha^2\lambda \int_{K_0} \phi(\nabla\psi.\nabla v)^2\, dx,$$

$$p_2 = \alpha^4\lambda^3 \int_{K_0} \phi^3|\nabla\psi|^4 v^2\, dx, \quad p_3 = \alpha^2\lambda \int_{K_0} \phi|\nabla\psi|^2|\nabla v|^2\, dx,$$

$$d_1 = 2\alpha\lambda \int_{K_0} \phi\nabla^t v.\nabla^2\psi.\nabla v\, dx, \quad d_2 = -\alpha\lambda \int_{K_0} \phi(\Delta\psi)|\nabla v|^2\, dx,$$

$$d_3 = 2\alpha^3\lambda \int_{K_0} \phi|\nabla\psi|^2(\nabla\psi.\nabla v)v\, dx, \quad d_4 = 4\alpha^2\lambda \int_{K_0} \phi(\nabla^t\psi.\nabla^2\psi.\nabla v)v\, dx,$$

$$d_5 = 2\alpha^3\lambda^3 \int_{K_0} \phi^3(\nabla^t\psi.\nabla^2\psi.\nabla\psi)v^2\,dx, \quad d_6 = \alpha^3\lambda^3 \int_{K_0} \phi^3(\Delta\psi)|\nabla\psi|^2 v^2\,dx,$$

$$b_1 = -2\alpha\lambda \int_{\partial K_0} \phi(\nabla\psi.\nabla v)\frac{\partial v}{\partial n}\,d\Gamma, \quad b_2 = \alpha\lambda \int_{\partial K_0} \phi\frac{\partial\psi}{\partial n}|\nabla v|^2\,d\Gamma,$$

$$b_3 = -2\alpha^2\lambda \int_{\partial K_0} \phi|\nabla\psi|^2 v\frac{\partial v}{\partial n}\,d\Gamma, \quad b_4 = -\alpha^3\lambda^3 \int_{\partial K_0} \phi^3|\nabla\psi|^2\frac{\partial\psi}{\partial n}v^2\,d\Gamma,$$

$$p_0 = \int_{K_0} (k + \alpha^2\lambda\phi|\nabla\psi|^2 - \alpha\lambda\phi(\Delta\psi))^2 v^2\,dx, \quad p = \int_{K_0} (Pu)^2\,e^{2\lambda\phi}\,dx,$$

we have

$$p_1 + p_2 + p_3 + d_1 + d_2 + d_3 + d_4 + d_5 + d_6 + b_1 + b_2 + b_3 + b_4 \le p_0 + p.$$

Proof: We first find an expression of Pu as a function of v. Since $u = ve^{-\lambda\phi}$,

$$\frac{\partial u}{\partial x_j} = \left(\frac{\partial v}{\partial x_j} - \alpha\lambda\phi\frac{\partial\psi}{\partial x_j}v\right)e^{-\lambda\phi},$$

$$\begin{aligned}
\frac{\partial^2 u}{\partial {x_j}^2} &= \left(\frac{\partial^2 v}{\partial {x_j}^2} - \alpha\lambda\phi\frac{\partial^2\psi}{\partial {x_j}^2}v - \alpha^2\lambda\phi(\frac{\partial\psi}{\partial x_j})^2 v - \alpha\lambda\phi\frac{\partial\psi}{\partial x_j}\frac{\partial v}{\partial x_j}\right)e^{-\lambda\phi} \\
&- \left(\frac{\partial v}{\partial x_j} - \alpha\lambda\phi\frac{\partial\psi}{\partial x_j}v\right)\alpha\lambda\phi\frac{\partial\psi}{\partial x_j}e^{-\lambda\phi} \\
&= \left(\frac{\partial^2 v}{\partial {x_j}^2} - \alpha\lambda\phi\frac{\partial^2\psi}{\partial {x_j}^2}v - \alpha^2\lambda\phi(\frac{\partial\psi}{\partial x_j})^2 v - 2\alpha\lambda\phi\frac{\partial\psi}{\partial x_j}\frac{\partial v}{\partial x_j}\right. \\
&+ \left.\alpha^2\lambda^2\phi^2(\frac{\partial\psi}{\partial x_j})^2 v\right)e^{-\lambda\phi},
\end{aligned}$$

whence

$$\begin{aligned}
\Delta u + k\,u &= \big(\Delta v + k\,v - \alpha\lambda\phi(\Delta\psi)v - \alpha^2\lambda\phi|\nabla\psi|^2 v - 2\alpha\lambda\phi(\nabla\psi.\nabla v) \\
&+ \alpha^2\lambda^2\phi^2|\nabla\psi|^2 v\big)\,e^{-\lambda\phi}.
\end{aligned}$$

The above equation can be rewritten

$$-Pu\,e^{\lambda\phi} = M_1 v + M_2 v + M_3 v,$$

by denoting

$$\begin{aligned}
M_1 v &= \Delta v + \alpha^2\lambda^2\phi^2|\nabla\psi|^2 v \\
M_2 v &= -2\alpha\lambda\phi(\nabla\psi.\nabla v) - 2\alpha^2\lambda\phi|\nabla\psi|^2 v \\
M_3 v &= k\,v + \alpha^2\lambda\phi|\nabla\psi|^2 v - \alpha\lambda\phi(\Delta\psi)v.
\end{aligned}$$

It follows that

$$||M_1 v + M_2 v||^2_{L^2(K_0)} = ||Pu\,e^{\lambda\phi} + M_3 v||^2_{L^2(K_0)},$$

whence

$$(M_1 v, M_2 v)_{L^2(K_0)} \le ||Pu\,e^{\lambda\phi}||^2_{L^2(K_0)} + ||M_3 v||^2_{L^2(K_0)}.$$

We now develop that left-hand side term. Since $M_1 v$ and $M_2 v$ are both the sum of two terms, with obvious notations we have

$$(M_1 v, M_2 v)_{L^2(K_0)} = I_{11} + I_{12} + I_{21} + I_{22}.$$

By integration by parts in K_0, we obtain by using the Einstein notation for repeated indices,

$$\begin{aligned}
I_{11} &= -2\alpha\lambda \int_{K_0} \phi(\Delta v)(\nabla\psi.\nabla v)\, dx \\
&= 2\alpha\lambda \int_{K_0} \frac{\partial v}{\partial x_i} \frac{\partial .}{\partial x_i}(\phi \frac{\partial \psi}{\partial x_j} \frac{\partial v}{\partial x_j})\, dx - 2\alpha\lambda \int_{\partial K_0} \phi \frac{\partial v}{\partial n}(\nabla\psi.\nabla v)\, d\Gamma \\
&= 2\alpha\lambda \int_{K_0} \phi \frac{\partial v}{\partial x_i} \frac{\partial^2 \psi}{\partial x_i \partial x_j} \frac{\partial v}{\partial x_j}\, dx + 2\alpha^2\lambda \int_{K_0} \phi \frac{\partial v}{\partial x_i} \frac{\partial \psi}{\partial x_i} \frac{\partial \psi}{\partial x_j} \frac{\partial v}{\partial x_j}\, dx \\
&+ 2\alpha\lambda \int_{K_0} \phi \frac{\partial v}{\partial x_i} \frac{\partial \psi}{\partial x_j} \frac{\partial^2 v}{\partial x_i \partial x_j}\, dx - 2\alpha\lambda \int_{\partial K_0} \phi \frac{\partial v}{\partial n}(\nabla\psi.\nabla v)\, d\Gamma \\
&= 2\alpha^2\lambda \int_{K_0} \phi(\nabla\psi.\nabla v)^2\, dx + 2\alpha\lambda \int_{K_0} \phi \nabla^t v.\nabla^2\psi.\nabla v\, dx \\
&+ \alpha\lambda \int_{K_0} \phi\nabla\psi.\nabla(|\nabla v|^2)\, dx - 2\alpha\lambda \int_{\partial K_0} \phi \frac{\partial v}{\partial n}(\nabla\psi.\nabla v)\, d\Gamma.
\end{aligned}$$

The third term of the above sum can be rewritten

$$\begin{aligned}
I'_{11} &:= \alpha\lambda \int_{K_0} \phi\nabla\psi.\nabla(|\nabla v|^2)\, dx \\
&= -\alpha\lambda \int_{K_0} \frac{\partial .}{\partial x_i}(\phi \frac{\partial \psi}{\partial x_i})|\nabla v|^2\, dx + \alpha\lambda \int_{\partial K_0} \phi \frac{\partial \psi}{\partial n}|\nabla v|^2\, d\Gamma \\
&= -\alpha\lambda \int_{K_0} \phi(\Delta\psi)|\nabla v|^2\, dx - \alpha^2\lambda \int_{K_0} \phi|\nabla\psi|^2|\nabla v|^2\, dx \\
&+ \alpha\lambda \int_{\partial K_0} \phi \frac{\partial \psi}{\partial n}|\nabla v|^2\, d\Gamma.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
I_{12} &= -2\alpha^2\lambda \int_{K_0} \phi|\nabla\psi|^2 v\Delta v\, dx \\
&= 2\alpha^2\lambda \int_{K_0} \nabla(\phi|\nabla\psi|^2 v).\nabla v\, dx - 2\alpha^2\lambda \int_{\partial K_0} \phi|\nabla\psi|^2 v \frac{\partial v}{\partial n}\, d\Gamma \\
&= 2\alpha^3\lambda \int_{K_0} \phi|\nabla\psi|^2(\nabla\psi.\nabla v)v\, dx + 2\alpha^2\lambda \int_{K_0} \phi v\nabla(|\nabla\psi|^2).\nabla v\, dx \\
&+ 2\alpha^2\lambda \int_{K_0} \phi|\nabla\psi|^2|\nabla v|^2\, dx - 2\alpha^2\lambda \int_{\partial K_0} \phi|\nabla\psi|^2 v \frac{\partial v}{\partial n}\, d\Gamma.
\end{aligned}$$

$$\begin{aligned}
I_{21} &= -2\alpha^3\lambda^3 \int_{K_0} \phi^3|\nabla\psi|^2(\nabla\psi.\nabla v)v\, dx = -\alpha^3\lambda^3 \int_{K_0} \phi^3|\nabla\psi|^2 \frac{\partial \psi}{\partial x_i} \frac{\partial (v^2)}{\partial x_i}\, dx \\
&= \alpha^3\lambda^3 \int_{K_0} \mathrm{div}(\phi^3|\nabla\psi|^2\nabla\psi)v^2\, dx - \alpha^3\lambda^3 \int_{\partial K_0} \phi^3|\nabla\psi|^2 \frac{\partial \psi}{\partial n} v^2\, d\Gamma
\end{aligned}$$

$$
\begin{aligned}
&= 3\alpha^4\lambda^3\int_{K_0}\phi^3|\nabla\psi|^4v^2\,dx + \alpha^3\lambda^3\int_{K_0}\phi^3\mathrm{div}(|\nabla\psi|^2\nabla\psi)v^2\,dx\\
&- \alpha^3\lambda^3\int_{\partial K_0}\phi^3|\nabla\psi|^2\frac{\partial\psi}{\partial n}v^2\,d\Gamma.
\end{aligned}
$$

Lastly

$$I_{22} = -2\alpha^4\lambda^3\int_{K_0}\phi^3|\nabla\psi|^4v^2\,dx.$$

If we add all terms and simplify, we finally obtain

$$(M_1v, M_2v)_{L^2(K_0)} = p_1+p_2+p_3+d_1+d_2+d_3+d_4+d_5+d_6+b_1+b_2+b_3+b_4.$$

Since

$$||M_3v||^2_{L^2(K_0)} = p_0, \quad ||Pu\,e^{\lambda\phi}||^2_{L^2(K_0)} = p,$$

this completes the proof of the lemma. ■

We obtain the following Carleman estimate in K_0.

Proposition 1 : *There exists $K, \alpha_0, \lambda_0 > 0$ such that $\forall\alpha \geq \alpha_0$, $\forall\lambda \geq \lambda_0$, $\forall u \in \tilde{H}^2_0(B)$,*

$$\alpha^4\lambda^3\int_{K_0}\phi^3|\nabla\psi|^4u^2e^{2\lambda\phi}\,dx + \alpha^2\lambda\int_{K_0}\phi|\nabla\psi|^2|\nabla u|^2e^{2\lambda\phi}\,dx$$

$$\leq K\int_{K_0}|Pu|^2e^{2\lambda\phi}\,dx + K\alpha\lambda\int_{\partial K_0}\phi|\nabla\psi||\nabla u|^2e^{2\lambda\phi}\,d\Gamma + K\alpha^3\lambda^3\int_{\partial K_0}\phi^3|\nabla\psi|^3u^2e^{2\lambda\phi}\,d\Gamma.$$

Proof : For $u \in C^\infty_0(B(x_0, R_0))$, we denote $v = u\,e^{\lambda\phi}$ and use the notations of lemma 1. Since $\nabla\psi \neq 0$ on $\overline{B}$, we have

$$p_2+d_5+d_6 \geq \alpha^3\lambda^3\int_{K_0}\phi^3|\nabla\psi|^4(\alpha + \frac{2\mu_-(\psi)+\Delta\psi}{|\nabla\psi|^2})v^2\,dx,$$

$$p_3+d_1+d_2 \geq \alpha\lambda\int_{K_0}\phi|\nabla\psi|^2(\alpha + \frac{2\mu_-(\psi)-\Delta\psi}{|\nabla\psi|^2})|\nabla v|^2\,dx,$$

where $\mu_-(\psi)$ (resp. $\mu_+(\psi)$) is the smallest (resp. largest) eigenvalue of $\nabla^2\psi$. Since $\mu_-(\psi)$ and $\Delta\psi$ belong to $L^\infty(B)$, there exists a constant c such that

$$\frac{2\mu_-(\psi)\pm\Delta\psi}{|\nabla\psi|^2} \geq c \quad \text{a.e.} \quad \text{in} \quad K_0.$$

Hence, for sufficiently large α there exists constants $K, K' > 0$ such that

$$p_2+d_5+d_6 \geq K\alpha^4\lambda^3\int_{K_0}\phi^3|\nabla\psi|^4v^2\,dx,$$

$$p_3+d_1+d_2 \geq K'\alpha^2\lambda\int_{K_0}\phi|\nabla\psi|^2|\nabla v|^2\,dx.$$

Now we look at terms d_3 and d_4.

$$|d_3| \leq 2\alpha^3\lambda\int_{K_0}\phi|\nabla\psi|^2|\nabla\psi.\nabla v||v|\,dx.$$

By using Young's formula,

$$|d_3| \leq \alpha^2\lambda \int_{K_0} \phi(\nabla\psi.\nabla v)^2\, dx + \alpha^4\lambda \int_{K_0} \phi|\nabla\psi|^4 v^2\, dx$$

$$\leq \alpha^2\lambda \int_{K_0} \phi(\nabla\psi.\nabla v)^2\, dx + \alpha^4\lambda \int_{K_0} \phi^3|\nabla\psi|^4 v^2\, dx,$$

since $\phi \geq 1$ in K_0.
Hence we have

$$p_1 + d_3 \geq \alpha^2\lambda \int_{K_0} \phi(\nabla\psi.\nabla v)^2\, dx - \alpha^4\lambda \int_{K_0} \phi^3|\nabla\psi|^4 v^2\, dx$$

$$\geq -\alpha^4\lambda \int_{K_0} \phi^3|\nabla\psi|^4 v^2\, dx.$$

$$|d_4| \leq 4\alpha^2\lambda \int_{K_0} \phi\mu(\psi)|\nabla\psi||\nabla v||v|\, dx$$

with $\mu(\psi) = \max(|\mu_-(\psi)|, |\mu_+(\psi)|)$, and by using again Young's formula,

$$|d_4| \leq 2\alpha^3\lambda^2 \int_{K_0} \phi\mu(\psi)|\nabla\psi|^2 v^2\, dx + 2\alpha \int_{K_0} \phi\mu(\psi)|\nabla v|^2\, dx.$$

Since $\mu(\psi) \in L^\infty(B)$, there exists a constant C such that

$$\frac{\mu(\psi)}{|\nabla\psi|^2} \leq C \quad \text{a.e.} \quad \text{in} \quad K_0.$$

Then, since $\phi \geq 1$ in K_0,

$$|d_4| \leq 2C\alpha^3\lambda^2 \int_{K_0} \phi^3|\nabla\psi|^4 v^2\, dx + 2C\alpha \int_{K_0} \phi|\nabla\psi|^2|\nabla v|^2\, dx.$$

We now consider the case of p_0.
We have

$$p_0 = \alpha^4\lambda^2 \int_{K_0} \phi^2|\nabla\psi|^4 (1 + \frac{k}{\alpha^2\lambda\phi|\nabla\psi|^2} - \frac{1}{\alpha}\frac{\Delta\psi}{|\nabla\psi|^2})^2 v^2\, dx.$$

For $\lambda \geq 1$ and sufficiently large α, we obtain

$$p_0 \leq 2\alpha^4\lambda^2 \int_{K_0} \phi^3|\nabla\psi|^4 v^2\, dx.$$

If we gather all the above estimates, we obtain

$$p_1 + p_2 + p_3 + d_1 + d_2 + d_3 + d_4 + d_5 + d_6 - p_0$$

$$\geq K_0\alpha^4\lambda^3 \int_{K_0} \phi^3|\nabla\psi|^4 v^2\, dx + K_1\alpha^2\lambda \int_{K_0} \phi|\nabla\psi|^2|\nabla v|^2\, dx,$$

with

$$K_0 = K - \frac{1}{\lambda^2} - \frac{2C}{\alpha\lambda} - \frac{2}{\lambda}, \quad K_1 = K' - \frac{2C}{\alpha\lambda}.$$

As a result, when α and λ are large enough, we have $K_0, K_1 > 0$.
Now let us consider $|b_i|$, $i = 1, 2, 3, 4$. We have

$$|b_1 + b_2| \leq 3\alpha\lambda \int_{\partial K_0} \phi|\nabla\psi||\nabla v|^2 \, d\Gamma,$$

$$|b_3| \leq 2\alpha^2\lambda \int_{\partial K_0} \phi|\nabla\psi|^2|\nabla v||v| \, d\Gamma,$$

$$\leq \alpha\lambda \int_{\partial K_0} \phi|\nabla\psi||\nabla v|^2 \, d\Gamma + \alpha^3\lambda \int_{\partial K_0} \phi|\nabla\psi|^3 v^2 \, d\Gamma,$$

$$|b_4| \leq \alpha^3\lambda^3 \int_{\partial K_0} \phi^3|\nabla\psi|^3 v^2 \, d\Gamma.$$

Since $\phi \geq 1$, for $\lambda \geq 1$ we have

$$|b_1 + b_2 + b_3 + b_4| \leq 4\alpha\lambda \int_{\partial K_0} \phi|\nabla\psi||\nabla v|^2 \, d\Gamma + 2\alpha^3\lambda^3 \int_{\partial K_0} \phi^3|\nabla\psi|^3 v^2 \, d\Gamma.$$

Applying lemma 1, we obtain that for sufficiently large α, λ, there exists a constant $K > 0$ such that

$$\alpha^4\lambda^3 \int_{K_0} \phi^3|\nabla\psi|^4 v^2 \, dx + \alpha^2\lambda \int_{K_0} \phi|\nabla\psi|^2|\nabla v|^2 \, dx$$

$$\leq K \int_{K_0} |Pu|^2 \, e^{2\lambda\phi} \, dx$$

$$+K\alpha\lambda \int_{\partial K_0} \phi|\nabla\psi||\nabla v|^2 \, d\Gamma + K\alpha^3\lambda^3 \int_{\partial K_0} \phi^3|\nabla\psi|^3 v^2 \, d\Gamma.$$

Now we replace v in the above estimate by its expression as a function of u.

$$v = u \, e^{\lambda\phi}, \quad \nabla v = (\nabla u)e^{\lambda\phi} + \alpha\lambda\phi u(\nabla\psi)e^{\lambda\phi}.$$

We obtain

$$2\alpha^4\lambda^3 \int_{K_0} \phi^3|\nabla\psi|^4 u^2 e^{2\lambda\phi} \, dx + \alpha^2\lambda \int_{K_0} \phi|\nabla\psi|^2|\nabla u|^2 e^{2\lambda\phi} \, dx$$

$$+2\alpha^3\lambda^2 \int_{K_0} \phi^2|\nabla\psi|^2 u(\nabla\psi.\nabla u)e^{2\lambda\phi} \, dx$$

$$\leq K \int_{K_0} |Pu|^2 e^{2\lambda\phi} \, dx + K\alpha\lambda \int_{\partial K_0} \phi|\nabla\psi||\nabla u|^2 e^{2\lambda\phi} \, d\Gamma$$

$$+2K\alpha^2\lambda^2 \int_{\partial K_0} \phi^2|\nabla\psi|u(\nabla\psi.\nabla u)e^{2\lambda\phi} \, d\Gamma + 2K\alpha^3\lambda^3 \int_{\partial K_0} \phi^3|\nabla\psi|^3 u^2 e^{2\lambda\phi} \, d\Gamma.$$

We now use the following Young's inequalities :

$$\left|2\alpha^3\lambda^2 \int_{K_0} \phi^2|\nabla\psi|^2 u(\nabla\psi.\nabla u)e^{2\lambda\phi} \, dx\right|$$

$$\leq \frac{1}{r}\alpha^4\lambda^3 \int_{K_0} \phi^3|\nabla\psi|^4 u^2 e^{2\lambda\phi} \, dx + \alpha^2\lambda r \int_{K_0} \phi|\nabla\psi|^2|\nabla u|^2 e^{2\lambda\phi} \, dx,$$

with $r > 0$, and

$$\left| 2\alpha^2\lambda^2 \int_{\partial K_0} \phi^2 |\nabla\psi| u (\nabla\psi.\nabla u) e^{2\lambda\phi}\, d\Gamma \right|$$

$$\leq \alpha\lambda \int_{\partial K_0} \phi |\nabla\psi||\nabla u|^2 e^{2\lambda\phi}\, d\Gamma + \alpha^3\lambda^3 \int_{\partial K_0} \phi^3 |\nabla\psi|^3 u^2 e^{2\lambda\phi}\, d\Gamma.$$

As a conclusion, if we choose $1/2 < r < 1$, we obtain $K > 0$ such that for α, λ large enough, and for all $u \in C_0^\infty(B(x_0, R_0))$,

$$\alpha^4\lambda^3 \int_{K_0} \phi^3 |\nabla\psi|^4 u^2 e^{2\lambda\phi}\, dx + \alpha^2\lambda \int_{K_0} \phi |\nabla\psi|^2 |\nabla u|^2 e^{2\lambda\phi}\, dx$$

$$\leq K \int_{K_0} |Pu|^2 e^{2\lambda\phi}\, dx + K\alpha\lambda \int_{\partial K_0} \phi |\nabla\psi||\nabla u|^2 e^{2\lambda\phi}\, d\Gamma + K\alpha^3\lambda^3 \int_{\partial K_0} \phi^3 |\nabla\psi|^3 u^2 e^{2\lambda\phi}\, d\Gamma.$$

By density, the above result remains true for $u \in \tilde{H}_0^2(B)$. ■

3.2 Two stability estimates near the boundary

We consider a bounded and connected domain $\Omega \subset \mathbb{R}^N$ with a $C^{1,1}$ boundary $\partial\Omega$, and Γ_0 an open domain of $\partial\Omega$ such that there exist $x_0 \in \Gamma_0$ and $\tau > 0$ with $\partial\Omega \cap B(x_0, \tau) \subset \Gamma_0$.
In this section we apply the Carleman estimate of proposition 1 to obtain two stability estimates near the boundary. We use approximately the same method as in [19], with however two main differences. First, we use global Carleman estimates involving weights $e^{\alpha\psi_1}, e^{\alpha\psi_2}$, where the functions ψ_1, ψ_2 are defined hereafter and depend on the distance function to the boundary, instead of a Carleman estimate in the half-space after a local change of coordinates. Second, as concerns proposition 4, we use the level curves of a well-chosen weight instead of a perturbation of the domain in order to introduce the open domain $\omega_1 \Subset \Omega$ in the right-hand side of the estimate. Before deriving these two stability estimates, we recall the following useful proposition, which is proved in [19] with the help of an interior Carleman estimate, and which is not influenced by the regularity of the domain.

Proposition 2 *: Let ω_0, ω_1 be two open domains such that $\omega_0, \omega_1 \Subset \Omega$. There exist $s, c, \varepsilon_0 > 0$ such that $\forall \varepsilon \in]0, \varepsilon_0[$, $\forall u \in H^2(\Omega)$,*

$$||u||_{H^1(\omega_1)} \leq \frac{c}{\varepsilon}\left(||Pu||_{L^2(\Omega)} + ||u||_{H^1(\omega_0)}\right) + \varepsilon^s\, ||u||_{H^1(\Omega)}.$$

For all $x_0 \in \partial\Omega$, we can choose the set $W(x_0)$ in theorem 1 as $\overline{B}$ where $B = \Omega \cap B(x_0, R_0)$, for some R_0 with $0 < R_0 < 1$. In the following, we will use the two functions ψ_1, ψ_2 defined in $\overline{\Omega}$ by :

$$\psi_1(x) = R - d_{\partial\Omega}(x) - \frac{1}{2} r(x)^2, \tag{3}$$

$$\psi_2(x) = \gamma \circ r(x) d_{\partial\Omega}(x) + (1 - \gamma \circ r(x)) \tilde{d}_{\partial\Omega}(x), \tag{4}$$

$$\tilde{d}_{\partial\Omega}(x) = d_{\partial\Omega}(x) + \frac{1}{2}(d_{\partial\Omega}(x)^2 - r(x)^2), \tag{5}$$

with $r(x) = |x - x_0|$.
Here, $R > 0$ is chosen such that $\psi_1 > 0$ on $\overline{B}$. We easily prove that for sufficiently small R_0 and $r_0 < R_0$, $\{\tilde{d}(x) > \varepsilon\} \cap B(x_0, R_0) \neq \emptyset$ for all ε with $0 \leq \varepsilon \leq r_0$. Furthermore, γ is a C^2 function on $[0, R_0]$ such that $\gamma = 1$ on the segment $[0, r_0]$, and which is non increasing on $[r_0, R_0]$ with $0 < \gamma(R_0) < 1$. Lastly we assume that $\gamma'(r) + 2\gamma(r) > 0$ on $[0, R_0]$. Such a function γ exists, take for example $\gamma(r) = \tilde{\gamma}(r - r_0)$ for $r \in [r_0, R_0]$ with $\tilde{\gamma}(r) = (2r^2 + 2r + 1)e^{-2r}$. Since $\gamma(r) \in [0, 1]$, we have $\{\psi_2(x) > \varepsilon\} \cap B(x_0, R_0) \neq \emptyset$ for all ε with $0 \leq \varepsilon \leq r_0$. We have the following result.

Lemma 2 *: The two functions ψ_1 and ψ_2 satisfy the following properties : for $i = 1, 2$, $\psi_i \in C^1(\overline{B})$, $\nabla\psi_i \neq 0$ on $\overline{B}$, and $\nabla^2\psi_i \in (L^\infty(B))^{N\times N}$.*

Proof: Theorem 1 implies that $d_{\partial\Omega}(x) \in C^1(\overline{B})$ and $\nabla^2 d_{\partial\Omega} \in (L^\infty(B))^{N\times N}$, which implies the same properties for ψ_1 and ψ_2.
We first verify that $\nabla\psi_1 \neq 0$ in $\overline{B}$. Using theorem 1, we obtain that in $\overline{B}$,

$$\nabla\psi_1(x) = n(y) - (x - x_0),$$

where $y = P_{\partial\Omega}(x)$. If for some $x \in \overline{B}$ we had $\nabla\psi_1(x) = 0$, then we would have $|x - x_0| = 1$, which is impossible since $R_0 < 1$.
We consider now $\nabla\psi_2$. A straightforward calculation leads to

$$\nabla\psi_2 = \nabla d_{\partial\Omega} - \frac{1}{2}\nabla(\gamma\circ r)(d_{\partial\Omega}^2 - r^2) + (1 - \gamma\circ r)(d_{\partial\Omega}\nabla d_{\partial\Omega} - (x - x_0)).$$

Now using the fact that $\nabla d_{\partial\Omega} = -n(y)$ and $\nabla(\gamma\circ r) = \gamma'\circ r(x)(x - x_0)/|x - x_0|$, we obtain

$$\nabla\psi_2 = -n(y) - \frac{1}{2}\gamma'(r)(d_{\partial\Omega}^2 - r^2)\frac{x - x_0}{|x - x_0|} + (1 - \gamma(r))(-d_{\partial\Omega}n(y) - (x - x_0)).$$

If $x \in \overline{b}$ with $b = \Omega \cap B(0, r_0)$, then $\nabla\psi_2 = -n(y) \neq 0$. Now assume that $\nabla\psi_2(x) = 0$ for some $x \in \overline{B} \setminus \overline{b}$. For any $\tau(y) \perp n(y)$, we have

$$\nabla\psi_2(x).\tau(y) = 0 = -(x - x_0).\tau\left(\frac{1}{2}\frac{\gamma'(r)}{|x - x_0|}(d_{\partial\Omega}^2 - r^2) + 1 - \gamma(r)\right).$$

Since $d_{\partial\Omega}(x) \leq r(x)$ on $\overline{B}$, $\gamma' \leq 0$ and $1 - \gamma > 0$ on $]r_0, R_0]$, we have necessarily $(x - x_0).\tau(y) = 0$, whence $x - x_0 = -\eta\, n(y)$ for some $\eta \in \mathbb{R}$.
Furthermore,

$$\nabla\psi_2(x).n(y) = 0 = -1 + \frac{1}{2}\gamma'(r)(d_{\partial\Omega}^2 - \eta^2)\mathrm{sgn}(\eta) - (1 - \gamma(r))(d_{\partial\Omega} - \eta),$$

that is

$$-\frac{1}{2}\gamma'(r)(\eta^2 - d_{\partial\Omega}^2)\mathrm{sgn}(\eta) + (1 - \gamma(r))(\eta - d_{\partial\Omega}) = 1.$$

But, since $\gamma' \leq 0$, $1 - \gamma > 0$ and $d_{\partial\Omega} \leq |\eta| \leq R_0 < 1$,

$$-\frac{1}{2}\gamma'(r)(\eta^2 - d_{\partial\Omega}^2)\mathrm{sgn}(\eta) + (1 - \gamma(r))(\eta - d_{\partial\Omega}) \leq -\frac{1}{2}\gamma'(r) + 1 - \gamma(r),$$

and $-\gamma'/2 + 1 - \gamma < 1$ since $\gamma' + 2\gamma > 0$, which is a contradiction. ■

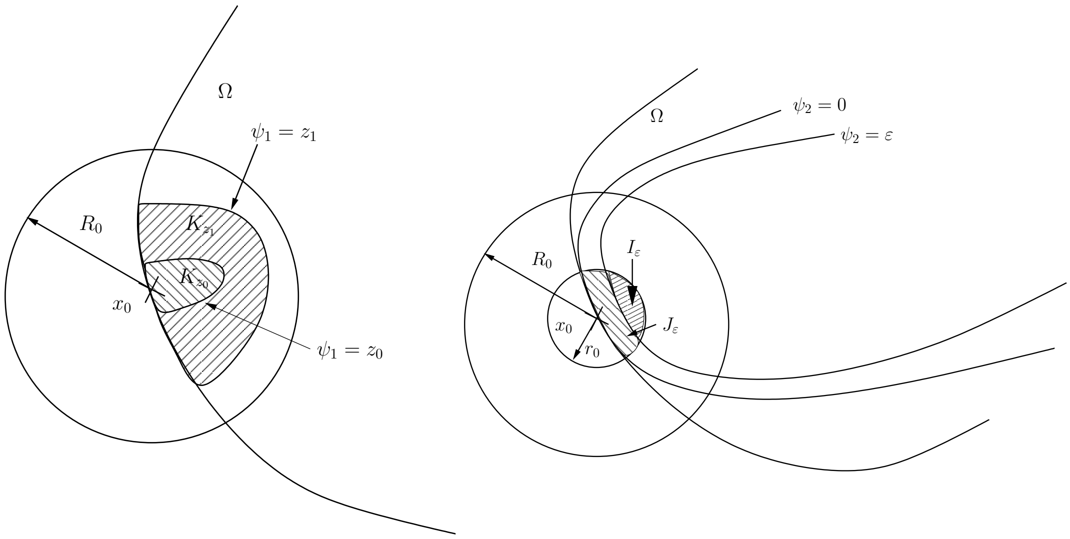

Figure 2: Left : proof of proposition 3. Right : proof of proposition 4

Now we prove the two following estimates.

Proposition 3 *: Let $x_0 \in \Gamma_0$ and $\tau > 0$ such that $\partial\Omega \cap B(x_0, \tau) \subset \Gamma_0$. There exists a neighborhood ω_0 of x_0, there exist $s, c, \varepsilon_0 > 0$ such that $\forall \varepsilon \in]0, \varepsilon_0[$, $\forall u \in H^2(\Omega)$,*

$$||u||_{H^1(\Omega\cap\omega_0)} \leq \frac{c}{\varepsilon}\left(||Pu||_{L^2(\Omega)} + ||u||_{H^1(\Gamma_0)} + ||\partial_n u||_{L^2(\Gamma_0)}\right) + \varepsilon^s\, ||u||_{H^1(\Omega)}.$$

Proposition 4 *: Let $x_0 \in \partial\Omega$. There exist a neighborhood ω of x_0 and an open domain $\omega_1 \Subset \Omega$ such that for all $\kappa \in]0, 1[$, there exist $c, \varepsilon_0 > 0$ such that $\forall \varepsilon \in]0, \varepsilon_0[$, $\forall u \in H^2(\Omega)$,*

$$||u||_{H^1(\Omega\cap\omega)} \leq e^{c/\varepsilon}\left(||Pu||_{L^2(\Omega)} + ||u||_{H^1(\omega_1)}\right) + \varepsilon^\kappa\, ||u||_{H^2(\Omega)}.$$

Proof of proposition 3 : We apply proposition 1 with function $\psi = \psi_1$ defined by (3). Here $K_0 = \overline{B}$ since $\psi_1 > 0$ on $\overline{B}$ and $\partial K_0 = \overline{B} \cap \partial\Omega$ (see the definition at the beginning of section 3.1 and the left figure of 1). We assume that $R_0 < \tau$ so that $\partial K_0 \subset \Gamma_0$. We consider z_0 and z_1 such that $0 < z_1 < z_0 < R$, with $\sqrt{2(R - z_1)} < R_0$. This last condition implies that $\{x \in \overline{\Omega},\quad \psi_1(x) \geq z_1\} \subset B(x_0, R_0)$. Next, we define $v = \chi u$, where χ is a function in $C_0^\infty(B(x_0, R_0))$ such that $\chi = 1$ on K_{z_1}.
Thus we have $v \in \tilde{H}_0^2(B)$, and there exists $K, \lambda_0 > 0$ such that for fixed (sufficiently large) α and for all $\lambda \geq \lambda_0$,

$$\int_{K_0}(v^2 + |\nabla v|^2)e^{2\lambda\phi}\, dx \leq K\int_{K_0}|Pv|^2 e^{2\lambda\phi}\, dx + K\lambda^2\int_{\partial K_0}(v^2 + |\nabla v|^2)e^{2\lambda\phi}\, d\Gamma.$$

We hence obtain

$$\int_{K_{z_0}}(u^2 + |\nabla u|^2)e^{2\lambda\phi}\, dx \leq K'\int_{K_0}|Pu|^2 e^{2\lambda\phi}\, dx$$

$$+K' \int_{K_0 \setminus K_{z_1}} (u^2 + |\nabla u|^2) e^{2\lambda\phi}\, dx + K'\lambda^2 \int_{\partial K_0} (u^2 + |\nabla u|^2) e^{2\lambda\phi}\, d\Gamma.$$

By denoting $h(z) = e^{\alpha z}$, and since $\psi_1 \geq z_0$ in K_{z_0}, $\psi_1 \leq R$ in K_0 and $\psi_1 < z_1$ in $K_0 \setminus K_{z_1}$ (see the left figure of 2), it follows that

$$e^{2\lambda h(z_0)} ||u||^2_{H^1(K_{z_0})} \leq K' e^{2\lambda h(R)} ||Pu||^2_{L^2(K_0)} + K' e^{2\lambda h(z_1)} ||u||^2_{H^1(K_0)}$$

$$+K'\lambda^2 e^{2\lambda h(R)} \left(||u||^2_{H^1(\partial K_0)} + ||\partial_n u||^2_{L^2(\partial K_0)} \right),$$

and thus for sufficiently large λ,

$$||u||_{H^1(K_{z_0})} \leq K''\lambda e^{\lambda(h(R)-h(z_0))} \left(||Pu||_{L^2(K_0)} + ||u||_{H^1(\partial K_0)} + ||\partial_n u||_{L^2(\partial K_0)} \right)$$

$$+K'' e^{-\lambda(h(z_0)-h(z_1))} ||u||_{H^1(K_0)}.$$

Taking into account the fact that $h(R) - h(z_0) > 0$ and $h(z_0) - h(z_1) > 0$, by changing variable $\lambda \to \varepsilon$ we obtain that there exist $s, c, \varepsilon_0 > 0$ such that for all ε, $0 < \varepsilon < \varepsilon_0$, for all $u \in H^2(\Omega)$,

$$||u||_{H^1(K_{z_0})} \leq \frac{c}{\varepsilon} \left(||Pu||_{L^2(K_0)} + ||u||_{H^1(\partial K_0)} + ||\partial_n u||_{L^2(\partial K_0)} \right) + \varepsilon^s \, ||u||_{H^1(K_0)}.$$

This ends the proof since $K_0 \subset \overline{\Omega}$, $\partial K_0 \subset \Gamma_0$ and $K_{z_0} = \{x \in \overline{\Omega},\, d_{\partial\Omega}(x) + r^2(x)/2 \leq R - z_0\}$ can be written $\overline{\Omega \cap \omega_0}$, where ω_0 is a neighborhood of x_0. ■

In order to prove proposition 4, we need the two following lemmas.

Lemma 3 *: Let s, β, A and B denote four non negative reals such that $\beta \leq B$. If $\exists\, c, \varepsilon_0 > 0$ such that $\forall \varepsilon$, $0 < \varepsilon < \varepsilon_0$,*

$$\beta \leq \frac{c}{\varepsilon} A + \varepsilon^s \, B,$$

then

$$\beta \leq C \, A^{\frac{s}{s+1}} B^{\frac{1}{s+1}},$$

where $C(s) = \max(D(s), \tilde{D}(s))$,

$$D(s) = c^{\frac{s}{s+1}} (s^{\frac{1}{s+1}} + s^{-\frac{s}{s+1}}), \quad \tilde{D}(s) = \left(c/s\varepsilon_0^{(s+1)} \right)^{\frac{s}{s+1}}.$$

$C(s)$ is a bounded function on each interval $[0, s_0]$.

Proof : We denote ε_{min} and f_{min} the minimizer and the minimum of

$$f(\varepsilon) = \frac{c}{\varepsilon} A + \varepsilon^s \, B$$

respectively, that is

$$\varepsilon_{min} = (\frac{cA}{sB})^{\frac{1}{s+1}}, \quad f_{min} = D(s) A^{\frac{s}{s+1}} B^{\frac{1}{s+1}},$$

with

$$D(s) = c^{\frac{s}{s+1}} (s^{\frac{1}{s+1}} + s^{-\frac{s}{s+1}}).$$

One should distinguish two cases. First, if $\varepsilon_0 > \varepsilon_{min}$, the result follows with $C = D(s)$.
If $\varepsilon_0 \leq \varepsilon_{min}$, one has

$$\varepsilon_0 \leq (\frac{cA}{sB})^{\frac{1}{s+1}},$$

and hence

$$B \leq A\left(c/s\varepsilon_0^{(s+1)}\right).$$

Using assumption $\beta \leq B$, we obtain

$$\beta \leq B^{\frac{s}{s+1}} B^{\frac{1}{s+1}} \leq \tilde{D}(s) A^{\frac{s}{s+1}} B^{\frac{1}{s+1}},$$

with

$$\tilde{D}(s) = \left(c/s\varepsilon_0^{(s+1)}\right)^{\frac{s}{s+1}},$$

and the result follows with $C = \tilde{D}(s)$. To prove that $C(s)$ is a bounded function of $s \in [0, s_0]$ for fixed ε_0, we just have to verify that $D(s)$ and $\tilde{D}(s)$ are continuous on $[0, s_0]$, in particular at 0. ■

Lemma 4 *: If $\Omega \subset \mathbb{R}^N$ is a bounded, connected and Lipschitz continuous domain, and if $d_{\partial\Omega}(x)$ denotes the distance of x to $\partial\Omega$, then $\forall r \in]0, 1/2[$, $\forall u \in H^r(\Omega)$,*

$$||\frac{u}{d^r_{\partial\Omega}}||_{L^2(\Omega)} \leq C\,||u||_{H^r(\Omega)},$$

with $C > 0$ depending only on r and on Ω.

Lemma 4 is known as Hardy's inequality and is proved for example in [10], p. 6.

Proof of proposition 4 : The first step consists in finding an estimate far away from x_0, by applying proposition 1 with function $\psi = \psi_2$ defined by (4) (5). Here $K_0 = \{x \in \overline{B},\, \psi_2(x) \geq 0\}$ and $\partial K_0 = \{x \in \overline{B},\, \psi_2(x) = 0\}$ (see the definition at the beginning of section 3.1 and the right figure of 1). We consider the domains $K_{z,z'} = \{x \in \overline{B},\, z \leq \psi_2(x) \leq z'\}$, with $0 \leq z < z' \leq r_0$. For $v \in \tilde{H}^2_0(B)$, there exists $K, \lambda_0 > 0$ such that for fixed (sufficiently large) $\alpha \geq 1$ and for all $\lambda \geq \lambda_0$,

$$\int_{K_0} (v^2 + |\nabla v|^2) e^{2\lambda\phi}\, dx \leq K \int_{K_0} |Pv|^2 e^{2\lambda\phi}\, dx + K\lambda^2 \int_{\partial K_0} (v^2 + |\nabla v|^2) e^{2\lambda\phi}\, d\Gamma.$$

Let ε be such that $0 < \varepsilon < r_0$. Denoting again $h(z) = e^{\alpha z}$, since $\psi_2 \geq \varepsilon$ in K_{ε,r_0}, $\psi_2 \leq R_0$ in K_0 and $\psi_2 = 0$ on ∂K_0, we obtain

$$e^{2\lambda h(\varepsilon)} ||v||^2_{H^1(K_{\varepsilon,r_0})} \leq K e^{2\lambda h(R_0)} ||Pv||^2_{L^2(K_0)} + K\lambda^2 e^{2\lambda h(0)} \left(||v||^2_{H^1(\partial K_0)} + ||\partial_n v||^2_{L^2(\partial K_0)}\right),$$

and hence, by using a classical trace theorem,

$$||v||_{H^1(K_{\varepsilon,r_0})} \leq K' e^{\lambda(h(R_0) - h(\varepsilon))} ||Pv||_{L^2(K_0)} + K'\lambda e^{-\lambda(h(\varepsilon) - h(0))} ||v||_{H^2(K_0)}.$$

We notice that $h(\varepsilon) - h(0) \geq \alpha\varepsilon \geq \varepsilon$ and $\lambda \leq (2/\varepsilon) e^{\varepsilon\lambda/2}$, whence there exists $d, L > 0$ such that

$$||v||_{H^1(K_{\varepsilon,r_0})} \leq L e^{d\lambda} ||Pv||_{L^2(K_0)} + L \frac{1}{\varepsilon} e^{-\varepsilon\lambda} ||v||_{H^2(K_0)}.$$

Next, $s > 0$ and $\mu > 0$ are uniquely defined by $e^{d\lambda} = 1/\mu$ and $e^{-\varepsilon\lambda} = \mu^s$. It follows in particular that $s = \varepsilon/d$, and for $0 < \mu \leq \mu_0 = e^{-d\lambda_0}$, $\forall v \in \tilde{H}_0^2(B)$,

$$\varepsilon||v||_{H^1(K_{\varepsilon,r_0})} \leq \frac{1}{\mu} L\varepsilon||Pv||_{L^2(K_0)} + \mu^s\, L||v||_{H^2(K_0)}.$$

We apply lemma 3 with $s = \varepsilon/d$, $\beta = \varepsilon||v||_{H^1(K_{\varepsilon,r_0})}$, $A = L\varepsilon||Pv||_{L^2(K_0)}$ and $B = L||v||_{H^2(K_0)}$. There exists C (independent of ε) such that for ε with $0 < \varepsilon < r_0$, for $v \in \tilde{H}_0^2(B)$,

$$||v||_{H^1(K_{\varepsilon,r_0})} \leq C \left(||Pv||_{L^2(K_0)}\right)^{\frac{\varepsilon}{\varepsilon+d}} \left(\frac{1}{\varepsilon}||v||_{H^2(K_0)}\right)^{\frac{d}{\varepsilon+d}}.$$

At this step we reproduce exactly the same calculations as in [19]. We introduce now $s > 0$, such that

$$\left(\varepsilon^{-\frac{d}{\varepsilon}(s+1)}\right)^{\frac{\varepsilon}{\varepsilon+d}} (\varepsilon^s)^{\frac{d}{\varepsilon+d}} = \left(\frac{1}{\varepsilon}\right)^{\frac{d}{\varepsilon+d}},$$

it follows that

$$||v||_{H^1(K_{\varepsilon,r_0})} \leq C \left(\varepsilon^{-\frac{d}{\varepsilon}(s+1)}||Pv||_{L^2(K_0)}\right)^{\frac{\varepsilon}{\varepsilon+d}} \left(\varepsilon^s||v||_{H^2(K_0)}\right)^{\frac{d}{\varepsilon+d}}.$$

Moreover,

$$\varepsilon^{-\frac{d}{\varepsilon}(s+1)} = e^{\frac{d}{\varepsilon}(s+1)\log\frac{1}{\varepsilon}},$$

and for small ε, if we introduce $\mu > 1$,

$$\frac{1}{\varepsilon}\log\frac{1}{\varepsilon} \leq \frac{1}{\mu-1}\frac{1}{\varepsilon^\mu}$$

(which is obtained by remarking that $\log 1/\varepsilon^{\mu-1} \leq 1/\varepsilon^{\mu-1}$ for small ε). This leads to

$$\varepsilon^{-\frac{d}{\varepsilon}(s+1)} \leq e^{\frac{d(s+1)}{(\mu-1)\varepsilon^\mu}},$$

and finally, $\forall s > 0$, $\forall \mu > 1$, $\exists\, c > 0$ such that for sufficiently small ε, $\forall v \in \tilde{H}_0^2(B)$,

$$||v||_{H^1(K_{\varepsilon,r_0})} \leq C \left(e^{c/\varepsilon^\mu}||Pv||_{L^2(K_0)}\right)^{\frac{\varepsilon}{\varepsilon+d}} \left(\varepsilon^s||v||_{H^2(K_0)}\right)^{\frac{d}{\varepsilon+d}}.$$

By using the fact that $\forall a, b \geq 0$, $\forall \rho \in [0,1]$, $a^\rho b^{1-\rho} \leq a + b$, we obtain

$$||v||_{H^1(K_{\varepsilon,r_0})} \leq C \left(e^{c/\varepsilon^\mu}||Pv||_{L^2(K_0)} + \varepsilon^s||v||_{H^2(K_0)}\right).$$

We denote $I_\varepsilon = K_\varepsilon \cap \overline{B(x_0, r_0)}$, and J_ε the complementary part of I_ε in b with $b = \Omega \cap B(x_0, r_0)$ (see the right figure of 2). Since for $x \in B(x_0, r_0)$ we have $\psi_2 = d_{\partial\Omega}$, it is easy to verify that $I_\varepsilon \subset K_{\varepsilon,r_0}$. We finally have

$$||v||_{H^1(I_\varepsilon)} \leq C \left(e^{c/\varepsilon^\mu}||Pv||_{L^2(K_0)} + \varepsilon^s||v||_{H^2(K_0)}\right). \qquad (6)$$

The second step consists in finding an estimate of $||v||_{H^1(J_\varepsilon)}$ uniformly in ε, with the help of lemma 4 in the domain b for $v \in \tilde{H}^2_0(B)$. It follows that for all $r \in]0, 1/2[$,

$$||\frac{v}{d^r_{\partial b}}||_{L^2(b)} \leq C\,||v||_{H^r(b)},$$

and since $d_{\partial b} \leq d_{\partial\Omega} = \psi_2 < \varepsilon$ in J_ε,

$$||v||_{L^2(J_\varepsilon)} \leq C\,\varepsilon^r||v||_{H^r(b)} \leq C\,\varepsilon^r||v||_{H^{1/2}(b)}.$$

By using a classical interpolation inequality and a Young's inequality, it follows that $\forall \eta > 0$,

$$||v||_{L^2(J_\varepsilon)} \leq C'\,\varepsilon^r||v||^{1/2}_{H^1(b)}||v||^{1/2}_{L^2(b)} \leq C'\left(\frac{\varepsilon^{2r}}{\eta}||v||_{H^1(b)} + \eta||v||_{L^2(b)}\right).$$

Since the above inequality is also true for the first derivatives of v, it follows that $\forall r \in]0, 1/2[$, $\exists\, C' > 0$ such that $\forall \eta > 0$,

$$||v||_{H^1(J_\varepsilon)} \leq C'\left(\frac{\varepsilon^{2r}}{\eta}||v||_{H^2(b)} + \eta||v||_{H^1(b)}\right). \tag{7}$$

Using $||v||_{H^1(b)} \leq ||v||_{H^1(I_\varepsilon)} + ||v||_{H^1(J_\varepsilon)}$, and gathering (6) and (7), we obtain

$$||v||_{H^1(b)} \leq C\left(e^{c/\varepsilon^\mu}||Pv||_{L^2(K_0)} + \varepsilon^s||v||_{H^2(B)}\right)$$
$$+C'\left(\frac{\varepsilon^{2r}}{\eta}||v||_{H^2(B)} + \eta||v||_{H^1(b)}\right).$$

Choosing $s = 2r$ and η such that $C'\eta = 1/2$, we obtain $\forall r \in]0, 1/2[$, $\forall \mu > 1$, $\exists\, c > 0$ such that for sufficiently small ε, $\forall v \in \tilde{H}^2_0(B)$,

$$||v||_{H^1(b)} \leq C\left(e^{c/\varepsilon^\mu}||Pv||_{L^2(K_0)} + \varepsilon^{2r}||v||_{H^2(B)}\right),$$

where C is a new constant. We obtain that $\forall \kappa \in]0, 1[$, $\exists\, c > 0$ such that for sufficiently small ε, $\forall v \in \tilde{H}^2_0(B)$,

$$||v||_{H^1(b)} \leq e^{c/\varepsilon}||Pv||_{L^2(K_0)} + \varepsilon^\kappa||v||_{H^2(B)}.$$

The third step consists in coming back to a function $u \in H^2(\Omega)$. To this end we consider a function $\chi \in C^\infty_0(B(x_0, R_0))$ such that $\chi = 1$ in $\overline{B(x_0, r_1)}$ with $0 < r_0 < r_1 < R_0$, and $v = \chi u \in \tilde{H}^2_0(B)$. Applying the previous estimate to v, and denoting $D_{z,z'} = B(x_0, z') \setminus \overline{B(x_0, z)}$ for $z < z'$, one obtain there exists a new constant C such that

$$||u||_{H^1(b)} \leq Ce^{c/\varepsilon}\left(||Pu||_{L^2(K_0)} + ||u||_{H^1(K_0\cap D_{r_1,R_0})}\right) + C\varepsilon^\kappa||u||_{H^2(B)}.$$

Given the particular definition of ψ_2, we have $\overline{K_0 \cap D_{r_1,R_0}} \subset \Omega$. Indeed, assume that $x \in \overline{K_0 \cap D_{r_1,R_0}}$ and $d_{\partial\Omega}(x) = 0$, then

$$\psi_2(x) = -\frac{1}{2}\left(1 - \gamma \circ r(x)\right)|x - x_0|^2 \leq -\frac{1}{2}(1 - \gamma(r_1))r_1^2 < 0,$$

which is not possible. We conclude that there exists a neighborhood ω of x_0 and an open domain $\omega_1 \Subset \Omega$ such that $\forall \kappa \in]0,1[$, there exist $c, \varepsilon_0 > 0$, $\forall \varepsilon \in]0, \varepsilon_0[$, $\forall u \in H^2(\Omega)$,

$$||u||_{H^1(\Omega\cap\omega)} \leq e^{c/\varepsilon} \left(||Pu||_{L^2(\Omega)} + ||u||_{H^1(\omega_1)}\right) + \varepsilon^\kappa ||u||_{H^2(\Omega)},$$

which completes the proof. ■

Remark 2 *:* As can be seen in the proof of our Carleman estimate in proposition 1, the choice of ψ_1 and ψ_2 as set in (3), (4), (5) is not possible when Ω is not $C^{1,1}$ any longer, because in such situation (see remark 1) the components of $\nabla^2 d_{\partial\Omega}$ and hence of $\nabla^2\psi_i$ $(i = 1, 2)$ may be not functions any more in the classical sense. This is the reason why for Lipschitz domains, in particular, another technique has to be used (see [3]).

3.3 Derivation of the final estimate

Our final estimate for $C^{1,1}$ domains results from propositions 2, 3 and 4. Precisely, proposition 3 enables us to "propagate" Cauchy data on Γ_0 to a neighborhood of any smooth point x_0 of Γ_0, in particular to an open domain $\omega_0 \Subset \Omega$. Proposition 2 enables us to "propagate" data from this open domain ω_0 to any other open domain $\omega_1 \Subset \Omega$. Lastly, proposition 4 enables us to propagate data on an open domain $\omega_1 \Subset \Omega$ up to a neighborhood of any point $x \in \partial\Omega$.

Theorem 2 *: Let Ω be a bounded and connected domain $\Omega \subset \mathbb{R}^N$ with a $C^{1,1}$ boundary $\partial\Omega$. If Γ_0 is an open domain of $\partial\Omega$ such that there exist $x_0 \in \Gamma_0$ and $\tau > 0$ with $\partial\Omega \cap B(x_0, \tau) \subset \Gamma_0$, then*
$\forall \kappa \in]0, 1[$, $\exists c, \varepsilon_0 > 0$, $\forall \varepsilon \in]0, \varepsilon_0[$, $\forall u \in H^2(\Omega)$,

$$||u||_{H^1(\Omega)} \leq e^{c/\varepsilon} \left(||Pu||_{L^2(\Omega)} + ||u||_{H^1(\Gamma_0)} + ||\partial_n u||_{L^2(\Gamma_0)}\right) + \varepsilon^\kappa\, ||u||_{H^2(\Omega)}. \quad (8)$$

From theorem 2 we obtain the following corollary.

Corollary 1 *: With the assumptions of theorem 2, $\forall \kappa \in]0, 1[$, $\exists C, \delta_0 > 0$ such that $\forall \delta \in]0, \delta_0[$, $\forall u \in H^2(\Omega)$ with*

$$||u||_{H^2(\Omega)} \leq M, \quad ||Pu||_{L^2(\Omega)} + ||u||_{H^1(\Gamma_0)} + ||\partial_n u||_{L^2(\Gamma_0)} \leq \delta,$$

where M is a constant,

$$||u||_{H^1(\Omega)} \leq C \frac{M}{(\log(M/\delta))^\kappa}.$$

Proof : We deduce from theorem 2 that for $\varepsilon \leq \varepsilon_0$,

$$||u||_{H^1(\Omega)} \leq e^{c/\varepsilon}\delta + M\varepsilon^\kappa. \quad (9)$$

Denoting $f(\varepsilon) = e^{c/\varepsilon}\delta + M\varepsilon^\kappa$ for $\varepsilon > 0$, the minimizer ε_{min} of f solves

$$g(\varepsilon_{min}) = \frac{M}{\delta}, \quad g(\varepsilon) := \frac{c}{\kappa}\frac{e^{c/\varepsilon}}{\varepsilon^{\kappa+1}}.$$

The function g is non increasing with $g(0+) = +\infty$ and $g(+\infty) = 0$, so that the above equation has a unique solution ε_{min} for each $\delta > 0$.
If $\varepsilon_0 > \varepsilon_{min}$, then by choosing $\varepsilon = \varepsilon_{min}$ in (9) we obtain that

$$||u||_{H^1(\Omega)} \leq (\frac{\kappa}{c}\varepsilon_0 + 1)M\varepsilon_{min}^{\kappa} = CM\varepsilon_{min}^{\kappa}. \quad (10)$$

For sufficiently small δ, ε_{min} is sufficiently small to have for some $c' > c$,

$$\frac{M}{\delta} = g(\varepsilon_{min}) \leq e^{c'/\varepsilon_{min}}.$$

It follows that $\varepsilon_{min} \leq c'/\log(M/\delta)$, and we obtain the required result by plugging this estimate in (10). If $\varepsilon_0 \leq \varepsilon_{min}$, we obtain $g(\varepsilon_0) \geq M/\delta$, and thus

$$||u||_{H^1(\Omega)} \leq M \leq g(\varepsilon_0)\delta = C\frac{M}{M/\delta}.$$

The result follows from the fact that for small δ, $M/\delta \geq (\log(M/\delta))^{\kappa}$. In our proof, C is independent of u, M, δ. ■

Remark 3 : Let Γ_1 denote the complementary part of Γ_0 in $\partial\Omega$. It follows from corollary 1 that that for all $\kappa \in]0,1[$,

$$||u||_{H^{1/2}(\Gamma_1)} + ||\partial_n u||_{H^{-1/2}(\Gamma_1)} \leq C(\kappa)\frac{M}{(\log(M/\delta))^{\kappa}},$$

for all $u \in H^2(\Omega)$ such that $Pu = 0$, $||u||_{H^2(\Omega)} \leq M$ for some constant $M > 0$ and $||u||_{H^1(\Gamma_0)} + ||\partial_n u||_{L^2(\Gamma_0)} \leq \delta$ for sufficiently small δ. This estimate should be compared to the one proved in [6] for 2D functions in $C^2(\overline{\Omega})$ with the help of a Carleman estimate obtained in [4].

It is useful to complete theorem 2 with the following one in a truncated domain, which is more classical (see for example [12]). It results from propositions 2 and 3.

Theorem 3 : *We consider a bounded and connected domain $\Omega \subset \mathbb{R}^N$ of class $C^{1,1}$. If Γ_0 is an open domain of $\partial\Omega$ such that there exist $x_0 \in \Gamma_0$ and $\tau > 0$ with $\partial\Omega \cap B(x_0, \tau) \subset \Gamma_0$, then $\exists\, s, c, \varepsilon_0 > 0$ such that $\forall \varepsilon \in]0, \varepsilon_0[$, $\forall u \in H^2(\Omega)$,*

$$||u||_{H^1(\Omega_\rho)} \leq \frac{c}{\varepsilon}\left(||Pu||_{L^2(\Omega)} + ||u||_{H^1(\Gamma_0)} + ||\partial_n u||_{L^2(\Gamma_0)}\right) + \varepsilon^s\, ||u||_{H^1(\Omega)}, \quad (11)$$

$$||u||_{H^2(\Omega_\rho)} \leq \frac{c}{\varepsilon}\left(||Pu||_{L^2(\Omega)} + ||u||_{H^{3/2}(\Gamma_0)} + ||\partial_n u||_{H^{1/2}(\Gamma_0)}\right) + \varepsilon^s\, ||u||_{H^1(\Omega)}, \quad (12)$$

where Ω_ρ is defined, for small $\rho > 0$, by $\Omega_\rho = \{x \in \Omega, \quad d(x, \overline{\Gamma_1}) > \rho\}$, and Γ_1 is the open domain of $\partial\Omega$ such that $\Gamma_0 \cap \Gamma_1 = \emptyset$ and $\partial\Omega = \overline{\Gamma_0} \cup \overline{\Gamma_1}$.

Proof : The estimate (11) is an obvious consequence of propositions 2 and 3. The proof of (12) requires the following regularity estimate, which is easy to derive. For $\rho' > \rho$, there exists $C > 0$ such that for all $v \in H^2(\Omega)$ with $v|_{\Gamma_0} = 0$ and $(\partial_n v)|_{\Gamma_0} = 0$,

$$||v||_{H^2(\Omega_{\rho'})} \leq C(||v||_{H^1(\Omega_\rho)} + ||Pv||_{L^2(\Omega_\rho)}). \quad (13)$$

We can define $(u|_{\Gamma_0}, \partial_n u|_{\Gamma_0}) \in H^{3/2}(\Gamma_0) \times H^{1/2}(\Gamma_0)$ for $u \in H^2(\Omega)$, and a continuous extension $E : (g_0, g_1) \in H^{3/2}(\Gamma_0) \times H^{1/2}(\Gamma_0) \to \tilde{u} \in H^2(\Omega)$ such that $(\tilde{u}|_{\Gamma_0}, \partial_n \tilde{u}|_{\Gamma_0}) = (g_0, g_1)$ (see [10], p. 37).
Let us suppose that $\tilde{u} = E((u|_{\Gamma_0}, \partial_n u|_{\Gamma_0}))$. Since $v := u - \tilde{u}$ satisfies (11) with $v|_{\Gamma_0} = 0$ and $\partial_n v|_{\Gamma_0} = 0$, and since v satisfies (13) as well, we obtain that for small $\rho > 0$,

$$||v||_{H^2(\Omega_\rho)} \leq \frac{c}{\varepsilon}||Pv||_{L^2(\Omega)} + \varepsilon^s \, ||v||_{H^1(\Omega)}.$$

We obtain the estimate (12) by coming back to the function u and using the continuity of E. ■

4 About the sharpness of the stability estimate

In this section, we prove that the estimate (8) is nearly sharp in a sense we define later on. In this view, we take $P = -\Delta$, Ω is the 2D rectangle $]0, X[\times]0, Y[$ and Γ_0 is the segment $]0, Y[$ on the y axis. Ω is not a domain of class $C^{1,1}$. Nevertheless, (8) holds in Ω for functions u defined in $]0, X[\times\mathbb{R}$ such that $u \in H^2(\Omega)$ and $u(x, y+Y) = u(x, y)$, for all $(x, y) \in]0, X[\times\mathbb{R}$. We prove this simply by using propositions 2, 3, 4 and the Y-periodicity of function u along the y axis.
The estimate (8) is nearly sharp in the following sense : there does not exist a function $\varepsilon \to g(\varepsilon)$ with $\lim_{\varepsilon\to 0} g(\varepsilon)/\varepsilon = 0$, such that for some $c, \varepsilon_0 > 0$, for all $\varepsilon \in]0, \varepsilon_0[$, for all u such as described above,

$$||u||_{H^1(\Omega)} \leq e^{c/\varepsilon}\left(||\Delta u||_{L^2(\Omega)} + ||u||_{H^1(\Gamma_0)} + ||\partial_n u||_{L^2(\Gamma_0)}\right) + g(\varepsilon)\, ||u||_{H^2(\Omega)}.$$

In other words, g cannot decrease faster than ε when ε tends to 0. Since in (8) $g(\varepsilon) = \varepsilon^\kappa$ for all $\kappa < 1$, this proves that (8) is nearly sharp.
We prove this by contradiction. Assume $\lim_{\varepsilon\to 0} g(\varepsilon)/\varepsilon = 0$. We define, for $X > 0$ and $Y = 2\pi$, the following sequence of functions, which is inspired from the famous example of Hadamard.

$$u_m(x, y) = \phi(x) e_m(x, y), \quad e_m(x, y) = e^{mx} e^{imy},$$

with $m \in \mathbb{N}$ and ϕ is a C^2 function defined in $\mathbb{R}$ by

$$\begin{cases} \phi = 0 & x \leq 0 \\ \phi \geq 0 & 0 \leq x \leq A \\ \phi = 1 & x \geq A, \end{cases}$$

with $X > A > 0$.
We have of course $u_m \in H^2(\Omega)$, $u_m(x, y+Y) = u_m(x, y)$ for all $(x, y) \in]0, X[\times\mathbb{R}$, and the definition of ϕ leads to $u_m|_{\Gamma_0} = 0$ and $(\partial_x u_m)|_{\Gamma_0} = 0$. From the stability estimate, we obtain that for all $m \in \mathbb{N}$ and for all $\varepsilon < \varepsilon_0$,

$$||u_m||_{H^1(\Omega)} \leq e^{c/\varepsilon}||\Delta u_m||_{L^2(\Omega)} + g(\varepsilon)||u_m||_{H^2(\Omega)}. \tag{14}$$

After some simple calculations, we have

$$\frac{\partial u_m}{\partial x} = (m\phi + \phi')e_m(x, y), \quad \frac{\partial u_m}{\partial y} = (im\phi)e_m(x, y),$$

$$\frac{\partial^2 u_m}{\partial x^2} = (m^2\phi + 2m\phi' + \phi'')e_m(x,y), \quad \frac{\partial^2 u_m}{\partial x \partial y} = im(m\phi + \phi')e_m(x,y)$$

$$\frac{\partial^2 u_m}{\partial y^2} = -(m^2\phi)e_m(x,y), \quad \Delta u_m = (2m\phi' + \phi'')e_m(x,y).$$

Now let us consider the estimate (14). Concerning the left-hand side, we obtain after some simple calculations and by using the fact that $\phi(x) = 1$ when $x \in [A, X]$ that

$$||u_m||_{H^1(\Omega)} \geq C_1\sqrt{m}\sqrt{e^{2mX} - e^{2mA}}, \tag{15}$$

for some constant $C_1 > 0$. Concerning the right-hand side, by using the fact that $\sup_{i=0,1,2}\sup_{x\in\mathbb{R}} |\phi^{(i)}(x)| < +\infty$ and $\phi'(x) = 0$ when $x \in [A, X]$,

$$||\Delta u_m||_{L^2(\Omega)} \leq C_2\sqrt{m}\sqrt{e^{2mA} - 1}, \quad ||u_m||_{H^2(\Omega)} \leq C_3 m^{3/2}\sqrt{e^{2mX} - 1}, \tag{16}$$

for some constants $C_2, C_3 > 0$. Combining the estimates (14), (15) and (16), we obtain that for all m and all $\varepsilon < \varepsilon_0$,

$$\sqrt{e^{2mX} - e^{2mA}} \leq Ce^{c/\varepsilon}\sqrt{e^{2mA} - 1} + Cg(\varepsilon)m\sqrt{e^{2mX} - 1},$$

for some constant $C > 0$. Dividing the above equation by $\sqrt{e^{2mX} - 1}$, we obtain

$$\frac{\sqrt{1 - e^{-2m(X-A)}}}{\sqrt{1 - e^{-2mX}}} \leq Ce^{c/\varepsilon}e^{-m(X-A)}\frac{\sqrt{1 - e^{-2mA}}}{\sqrt{1 - e^{-2mX}}} + Cg(\varepsilon)m. \tag{17}$$

It remains to select η such that $0 < \eta < X - A$ and define the sequence $(\varepsilon_m)_m$ such that $\varepsilon_m = 1/(km)$ with $k = (X - A - \eta)/c > 0$. Hence we have $e^{c/\varepsilon_m - m(X-A)} = e^{-\eta m}$. The left-hand side of (17) converges to 1 when $m \to +\infty$, while the first term of the right-hand side tends to 0 when ε is replaced by ε_m, as well as the second term since $g(1/m)m \to 0$ when $m \to +\infty$. Thus, we have found a contradiction.

Remark 4 *:* To the author's knowledge, the validity of (8) for $\kappa = 1$ is an open problem, even for domains of class C^∞.

5 Application to the method of quasi-reversibility

In this section, we use the stability estimates obtained before in order to derive some convergence rates for the quasi-reversibility method, and therefore to complete the results already obtained in [14] in truncated domains. The method of quasi-reversibility, first introduced in [15], enables one to regularize the ill-posed elliptic Cauchy problems. Specifically, we consider a domain Ω as described in the statement of theorem 2, and a truncated domain Ω_ρ as defined in the statement of theorem 3.

Now we assume that $u \in H^2(\Omega)$ solves the ill-posed Cauchy problem with $(g_0, g_1) \in H^{3/2}(\Gamma_0) \times H^{1/2}(\Gamma_0)$:

$$\begin{cases} Pu = 0 \text{ in } \Omega \\ u|_{\Gamma_0} = g_0 \\ \partial_n u|_{\Gamma_0} = g_1. \end{cases} \tag{18}$$

Given some noisy data $(g_0^\sigma, g_0^\sigma) \in H^{3/2}(\Gamma_0) \times H^{1/2}(\Gamma_0)$ with

$$||g_0^\sigma - g_0||_{H^{3/2}(\Gamma_0)} + ||g_1^\sigma - g_1||_{H^{1/2}(\Gamma_0)} \leq \sigma,$$

we consider the formulation of quasi-reversibility for $\alpha > 0$: find $u_\alpha^\sigma \in H^2(\Omega)$, such that $\forall v \in H^2(\Omega)$, $v|_{\Gamma_0} = \partial_n v|_{\Gamma_0} = 0$,

$$\left\{ \begin{array}{c} (Pu_\alpha^\sigma, Pv)_{L^2(\Omega)} + \alpha(u_\alpha^\sigma, v)_{H^2(\Omega)} = 0 \\ u_\alpha^\sigma|_{\Gamma_0} = g_0^\sigma \\ \partial_n u_\alpha^\sigma|_{\Gamma_0} = g_1^\sigma. \end{array} \right. \tag{19}$$

Using Lax-Milgram theorem, we easily prove that formulation (19) is well-posed. If we denote $u_\alpha = u_\alpha^0$, which is the solution of quasi-reversibility without noise, we obtain for some constant $C_0 > 0$,

$$||u_\alpha^\sigma - u_\alpha||_{H^2(\Omega)} \leq C_0 \frac{\sigma}{\sqrt{\alpha}}. \tag{20}$$

On the other hand, we easily prove by using (18) and (19) that there exist constants $C_1, C_2 > 0$ such that

$$||u_\alpha - u||_{H^2(\Omega)} \leq C_1, \quad ||P(u_\alpha - u)||_{L^2(\Omega)} \leq C_2\sqrt{\alpha}. \tag{21}$$

Using (21) and then corollary 1, theorem 3 (combined with lemma 3) for function $u_\alpha - u \in H^2(\Omega)$, we obtain there exist $\gamma \in]0, 1/2[$, $C(\kappa) > 0$ for all $\kappa \in]0, 1[$, such that for sufficiently small $\alpha > 0$,

$$||u_\alpha - u||_{H^2(\Omega_\rho)} \leq C\, \alpha^\gamma, \tag{22}$$

$$||u_\alpha - u||_{H^1(\Omega)} \leq C(\kappa) \frac{1}{(\log(1/\alpha))^\kappa}. \tag{23}$$

Choosing $\alpha = \sigma$ in (20), we obtain exactly the same estimates for $u_\alpha^\sigma - u$ as in (22) and (23) simply by replacing the regularization parameter α by the amplitude of noise σ in the right-hand side.

Remark 5 : In [2], theorem 3 is not optimal in the sense that we can obtain the Hölder convergence rate (22) and not only a logarithmic convergence rate as stated in the theorem.

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