On hypochordal graphs

M.-C. Costa [*]   C. Picouleau [†]   H. Topart [†]

Abstract

We introduce graphs called *hypochordal*: for any path of length 2, there exists a chord or another path of length 2 between its two endpoints. We show that such graphs are 2-vertex-connected and moreover in the case of an edge or a vertex deletion, the distance between any pair of nonadjacent vertices remains unchanged.

We give properties of hypochordal graphs, then we study the class of minimum hypochordal graphs and finally we give some complexity results for classical combinatorial problems.

Keywords: graph, tree, vertex-connected, $\mathcal{NP}$-complete

1 Introduction

The purpose of this paper is to introduce a new class of graphs which are called *hypochordal graphs*. In a hypochordal graph any pair of vertices cannot have a unique common neighbour. We show that this local property is equivalent to the following global property: the distance between any pair of nonadjacent vertices remains unchanged in case of a vertex (or an edge) deletion. We are interested in several characterizations, and properties of these graphs. We also study the complexity status of classical problems.

The paper is organized as follow: In the next section we formally define the hypochordal graphs as graphs such that for any path $[u, y, v]$ there exists the chord $[u, v]$ or there exists a path $[u, z, v]$ with $z \neq y$. Some alternative characterizations of hypochordal graphs are given in the third section. Then since the definition of hypochordal graphs looks like the definition of chordal graphs, which explains the name, we emphasize the differences between these two classes of graphs. In the fifth section we are interested in *minimum hypochordal graphs*: hypochordal graphs with a minimum number of edges. In order to characterize these graphs we introduce a new type of graph partition. In the sixth section we prove the $\mathcal{NP}$-completeness of classical problems: hamiltonian cycle, vertex colouring, maximum clique, maximum stable set. Finally we conclude and we give some perspectives.

Applications of hypochordal graphs to networks design is quite natural. A lot of articles deals with the restoration of networks in case of a single component (link/edge or node/vertex) failure [9, 1]. Such a failure in two-connected networks may induce a large increasing of the distance between two nodes, even

[*]ENSTA UMA (CEDRIC), 32 boulevard Victor, 75739 Paris cedex 15 (France). Email: Marie-Christine.Costa@ensta.fr

[†]CEDRIC Laboratory, CNAM, 292 rue Saint Martin, 75003 Paris (France). Email: chp@cnam.fr, Helene.Topart@cnam.fr

if the network is also chordal. Thus it may be interesting to build networks with a hypochordal topology, and, if possible, with a minimum number of links.

2 Definition and notations

Here we only consider simple undirected graphs. Let $G = (V, E)$ be such a graph with $n(G) = n = |V|$ vertices and $m(G) = m = |E|$ edges. Except when mentioned, the graphs will be considered connected and $n \geq 3$ (the connected graphs with $n \leq 2$ vertices being trivially hypochordal). As in [2], we denote by $[x, y]$ the edge between the vertices x and y. The distance between two vertices x and y is the minimum number of edges of a path $[x, \ldots, y]$ and will be denoted by $d(x, y)$. For $x \in V$, $N_i^x = \{y \in V, d(x, y) = i\}$ is the set of vertices of G at distance i from x. The set of neighbours of x is $N(x) = N_1^x$. We say that y is a twin of x if $N(y) = N(x)$ (x and y are nonadjacent). We denote by $\delta(x)$ the degree of the vertex x and by $\delta(G)$ the minimum degree of a vertex of G. P_k (respectively C_k) is a path (respectively a cycle) of k vertices. Internally-vertex-disjoint paths are paths with common endpoints that have no other vertices in common.

Let S and S' be disjoint sets of vertices of G. We denote by $S - S'$ the relation existing between the sets S and S' if they satisfy $\forall x \in S, \forall y \in S', [x, y] \in E$. For all other definitions, refer to [2].

We now formalise the definition of hypochordal graphs.

Definition 1. *A graph $G = (V, E)$ is hypochordal if for every triple of vertices u, v, y such that $[u, y, v]$ is a P_3, we have $[u, v] \in E$ or there exists $z \neq y$ such that $[u, z, v]$ is a P_3.*

Some examples of hypochordal graphs are given in Figure 1 and 2.

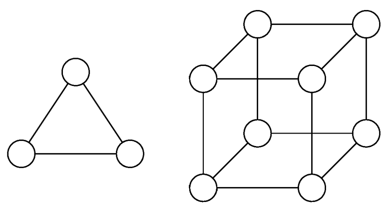


Figure 1: Examples of connected hypochordal graphs – C_3 and the cube

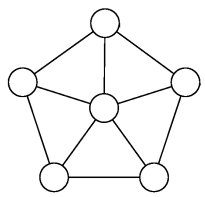


Figure 2: A no perfect hypochordal graph – the 5-wheel

3 Different characterizations of hypochordal graphs

In this section we prove the following property which gives several equivalent characterizations of hypochordal graphs.

Proposition 1. *Let $G = (V, E)$ be a connected graph. Then the following definitions are equivalent:*

1. *G is hypochordal;*

2. *every P_3 is included in a C_3 or a C_4;*

3. $\forall u, v \in V, u \neq v, |N(u) \cap N(v)| = 1 \Rightarrow [u, v] \in E$*;*

4. *the distance between any pair of nonadjacent vertices is unchanged by the deletion of any third vertex;*

5. *the distance between any pair of nonadjacent vertices is unchanged by the deletion of any edge;*

6. *for any pair of nonadjacent vertices, there exist two internally-vertex-disjoint shortest paths between them.*

While the equivalences between 1, 2 and 3 are immediate, the equivalences with $4, 5$ and 6 need a little proof:

Proof. We show that $1 \Rightarrow 4 \Rightarrow 5 \Rightarrow 6 \Rightarrow 1$.

($1 \Rightarrow 4$) Let G be a hypochordal graph, u, v a pair of nonadjacent vertices and y a third vertex, $y \neq u$, $y \neq v$. We denote by $\mu = [u = u_0, u_1, \ldots, u_k = v]$ a shortest path between u and v.
If $y \notin \mu$, then the distance between u and v is unchanged by the deletion of y.
If $y = u_j$ then $[u_{j-1}, u_j = y, u_{j+1}]$ is a P_3 and $[u_{j-1}, u_{j+1}] \notin E$ since μ is a shortest path. So there exists $z \neq y$ such that $[u_{j-1}, z, u_{j+1}]$ is a P_3, $z \neq u_i, \forall i$ otherwise μ is not a shortest path and $[u = u_0, \ldots, u'_j = z, \ldots, u_k = v]$ is another shortest path between u and v.

($4 \Rightarrow 5$) Consider the deletion of an edge $[x, y] \in E$. For any two nonadjacent vertices a, b, $\{a, b\} \neq \{x, y\}$, we may assume without loss of generality that $y \notin \{a, b\}$. From 4, deleting the vertex y will not increase the distance between a and b. Hence, deleting the edge $[x, y]$ will not increase this distance either.

($5 \Rightarrow 6$) Let u and v be nonadjacent vertices linked by a shortest path $\mu = [u = u_0, u_1, \ldots, u_k = v]$. For any $u_i \in \mu, (u_i \neq u, v)$, u_{i-1} and u_{i+1} are at distance two, otherwise μ is not minimal.
From 5, for any $1 \leq i \leq k-1$, the deletion of $[u_{i-1}, u_i]$ does not change the distance between u_{i-1} and u_{i+1}. Thus, there exists u'_i, ($u'_i \neq u_i$) such that $[u_{i-1}, u'_i]$ and $[u'_i, u_{i+1}]$ belong to G, and $u'_i \neq u_j, u'_i \neq u'_j, \forall j$ since μ is a shortest path. Then, there exist two internally-vertex-disjoint shortest path between u and v (see Figure 3): $\mu_1 = [u = u_0, u_1, u'_2, \ldots, u_{2i-1}, u'_{2i}, \ldots, u_k = v]$ and $\mu_2 = [u = u_0, u'_1, u_2, \ldots, u'_{2i-1}, u_{2i}, \ldots, u_k = v]$.

($6 \Rightarrow 1$) Let u and v be nonadjacent vertices such that $[u, y, v]$ is a P_3: then $d(u, v) = 2$. Since there exist two internally-vertex-disjoint shortest paths between u and v, there exists z such that $[u, z, v]$ is a P_3 and $z \neq y$. So G is hypochordal. □

Remark 1. *Let $[u, v] \in E$ be an edge of G. Since G is connected and $n \geq 3$, $[u, v]$ belongs to a P_3. G being hypochordal, $[u, v]$ belongs to a C_3 or a C_4. Hence, the deletion of the edge $[u, v]$ increases the distance between u and v by 1 or 2.*

One can observe that a graph such that every P_3 belongs to a C_3 is nothing more than a complete graph.

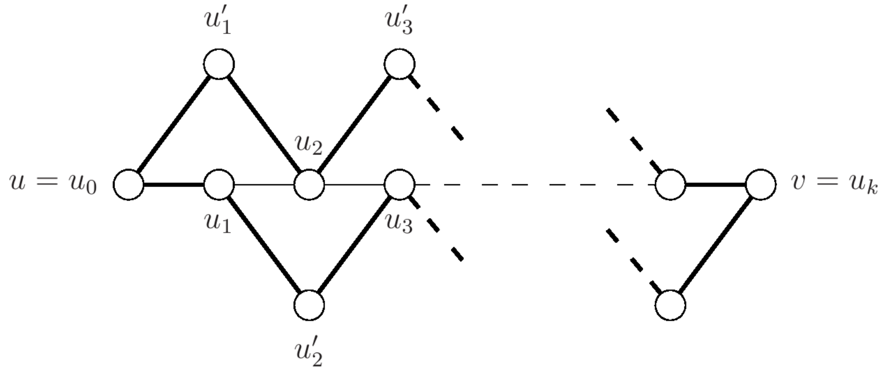


Figure 3: Two internally-vertex-disjoint paths between u and v

4 Relationship with other classes of graphs

Note first that, from Proposition 1 (.6) the class of connected hypochordal graphs (with at least three vertices) is included in the class of 2-vertex-connected graphs.

There is no inclusion relation between the classes of 2-vertex-connected chordal graphs (graphs such that every $C_k, k \geq 4$, has a chord) and hypochordal graphs. The graph of Figure 4 is a 2-vertex-connected interval graph hence it is chordal; but is not hypochordal. Complete graphs K_n are both chordal and hypochordal. Bipartite complete graphs K_{n_1,n_2} for $n_1, n_2 \geq 2$ and hypercubes are hypochordal and not chordal. Moreover, there are hypochordal graphs that are not perfect, see for example the 5-wheel of Figure 2. Thus we are interested in comparing hypochordal graphs to 2-vertex-connected graphs and to 2-vertex-connected chordal graphs:

- If G is a 2-vertex-connected graph, the deletion of a vertex (respectively an edge) can increase the distance between the vertices by up to $n - 4$ (respectively $n - 2$). Just consider C_n, $n \geq 4$.
- If G is a 2-vertex-connected chordal graph, the deletion of any edge of G increases by at most one the distance between any pair of vertices, but the deletion of a vertex of G may increase the distance between two vertices by an arbitrary long value. Consider the graph given in Figure 4, the vertices u and v and the deletion of x.

Therefore when interested in networks where a link (edge) or a node (vertex) failure induces small changes in the distance between any pair of nodes, since 2-vertex-connected chordal graphs are not good enough, one can consider hypochordal graph topology. Moreover in the next section we will show that hypochordal graphs can have a number of edges less than two times the number of vertices.

Several classes of graphs as, chordal graphs or perfect graphs for instance, have some “pleasant” properties such as heredity or monotonicity. As defined in [3], a property Π is called *hereditary* if it is closed under taking induced

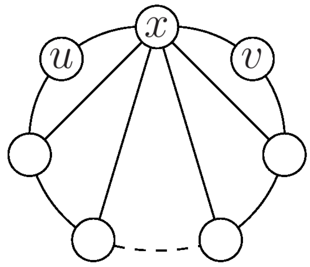


Figure 4: A 2-vertex-connected chordal graph with arbitrary big distance increase

subgraphs. In other words, a graph property Π is hereditary if it is closed under removal of vertices. It is clear that the hypochordal property, unlike the chordal property, is not hereditary. As an example, consider the 5-wheel of Figure 2, it has a C_5 as an induced subgraph, which is not hypochordal.

Observe that the class of distance hereditary graphs which in addition are 2-vertex-connected [6] is a proper subclass of the class of hypochordal graphs.

Following the definition given in [10] (page 9), a property Π is *monotone* if adding edges to a graph with property Π produces a graph satisfying Π. Contrary to the chordal property, the hypochordal property is not monotone since the cube is no longer hypochordal after adding a chord to a C_4.

5 Minimum hypochordal graph

Here we characterise the set of connected hypochordal graphs of order n with a minimum number of edges. We call them *minimum hypochordal graphs*. We first define some specific graph partitions similar to those introduced in [11]. Then we bound the number of edges of such graphs. Finally we characterise the structure of minimum hypochordal graphs.

5.1 Partitioning hypochordal graphs

Definition 2. *A graph $H = (\mathcal{B}, \mathcal{F})$ is a* C-partition *of G if*

- *each vertex B_i of H is a subset of vertices of G, called a bag, $B_i \subset V$;*
- *the bags realise a partition of V: $\bigcup_i B_i = V$, $B_i \cap B_j = \emptyset, i \neq j$;*
- *if two adjacent vertices of G are in two distinct bags A and B then $A - B$ in G, and the bags A and B are adjacent in H.*

The 'C' of C-partition stands for complete to remember that to each edge $[B_1, B_2]$ of H corresponds a complete bipartite subgraph of G induced by the vertices of $B_1 \cup B_2$.

Note that G is a C-partition of G, thus a C-partition of G always exists. Note also that there is no condition concerning the adjacency of the vertices inside a bag. Hence the graph with a single vertex is a C-partition of any graph (H has a single bag equal to V).

Definition 3.

– *a* tree-C-partition T *of* G *is a C-partition such that* T *is a tree;*

– *a* stable-C-partition H *of* G *is a C-partition such that every bag is a stable set;*

– *a* 2-stable-C-partition H *of* G *is a stable-C-partition such that every bag is of size less than or equal to* 2*:* $\forall B_i \in \mathcal{B}, |B_i| \leq 2$.

Now we begin the process of building up results towards the purpose of proving that minimum hypochordal graphs with minimum degree 2 (and $n \geq 5$) are exactly the 2-vertex-connected graphs with a 2-tree-stable-C-partition.

Proposition 2. *If* G *has a C-partition* $H = (\mathcal{B}, \mathcal{F})$ *with* $|\mathcal{B}| \geq 2$*,* H *connected and* $\forall B_i \in \mathcal{B}, |B_i| \geq 2$*, then* G *is hypochordal.*

Proof. Let u, v be two distinct vertices of G. They can either be in the same bag or not.

- Case u and v are in the same bag A: The C-partition being connected with at least two bags, there exist a bag B adjacent to A. Hence $A - B$ and $N(u) \cap N(v) \supseteq B$. Thus $|N(u) \cap N(v)| \geq |B| \geq 2$.

- Case u and v are in distinct bags A and B: Suppose that $|N(u) \cap N(v)| = 1$, $N(u) \cap N(v) = \{w\}$; w is either in the bag A (or symmetrically B) or in a third bag C.

  – Case $w \in A$: Since v and w are adjacent in G and belong to two distinct bags A and B of H, we have $A - B$. Hence $[u, v] \in E$.

  – Case $w \in C$: Using the same argument as above, we have $A - C$ and $B - C$. Thus $N(u) \cap N(v) \supseteq C$ and therefore $|N(u) \cap N(v)| \geq |C| \geq 2$, a contradiction.

□

Proposition 3. *If* G *is 2-vertex-connected and has a tree-C-partition* $T = (\mathcal{B}, \mathcal{F})$ *with* $|\mathcal{B}| \geq 3$*, then* G *is hypochordal.*

Proof. Let $A \in \mathcal{B}$ be a bag of T such that $|N(A)| \geq 2$. The subgraph of G induced by the vertices of $V \setminus A$ is disconnected. Since G is 2-vertex-connected, $|A| \geq 2$. Hence the bags of size 1 can only be leaves of T.

In the case where every bag of T but one are of size 1, we create another tree-C-partition T' of G by aggregating every bag of size 1 into a single bag of size greater than or equal to 2. Proposition 2 shows that G is hypochordal.

In any other case, we remove the bags of size 1 of T, the remaining graph is called T' and $|V(T')| \geq 2$. From Proposition 2, the subgraph G' of G induced by the vertices of the bags of T' is hypochordal. Let u, v be two vertices of G, with at least one common neighbour w:

- Case u, v are vertices of G': G' being hypochordal, $|N(u) \cap N(v)| = 1$ implies $[u, v] \in E$.

- Case u, v are both in bags of size 1: u and v are in bags that are leaves of T so w belongs to a bag A which is not a leaf of T. $N(u) = N(v) = A$, with $|A| \geq 2$.

- Case u is in a bag of size 1 and v belongs to G': w can either be in the same bag as v or in another bag of T.
  - If v and w are in the same bag, since u and w are adjacent in G, we have $[u, v] \in E$.
  - If w is in a bag A and v in a bag B of T, then $N(u) \cap N(v) = N(u) = A$ with $|A| \geq 2$.

□

Definition 4. *Let $G = (V, E)$ be a graph, the graph $2G = (U, F)$ is as follows: $U = V \times \{1, 2\}$; any vertex v of G has two corresponding vertices v_1 and v_2 in $2G$; if $[u, v] \in E$ then $[u_i, v_j] \in F, i, j \in \{1, 2\}$.*

In the literature, the graphs $2G$ are also called *duplex graphs*, see [7] for example.

See Figure 5 for an example.

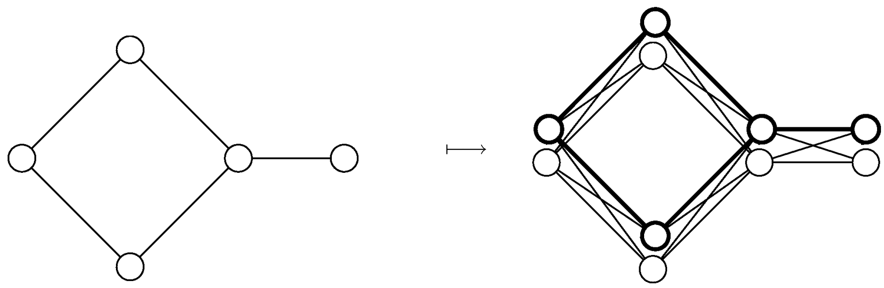


Figure 5: A graph G and the corresponding graph $2G$

Note that G is a 2-stable-C-partition of $2G$.

Proposition 4. *For any connected graph G with at least two vertices, the graph $2G$ is hypochordal.*

Proof. G is a C-partition of $2G$ with at least two bags, each bag being of size 2. We conclude using Proposition 2. □

We define a transformation $G \mapsto \widetilde{2G}$ similar to Definition 4. In fact, it is the same transformation except for pendant vertices which are not duplicated. See Figure 6 for an example.

Definition 5. *Let $G = (V, E)$ be a graph. Let $V_1 \subset V$ be the set of vertices of G with degree 1 and $V_2 = V \setminus V_1$. The graph $\widetilde{2G} = (U, F)$ is as follows: $U = V_1 \cup V_2 \times \{1, 2\}$; any vertex $u \in V_1$ corresponds to a vertex u_1 of $\widetilde{2G}$; any vertex $v \in V_2$ has two corresponding vertices v_1 and v_2 in $\widetilde{2G}$; if $[u, v] \in E$ then $[u_i, v_j] \in F, i, j \in \{1, 2\}$.*

Note again that G is a 2-stable-C-partition of $\widetilde{2G}$

Proposition 5. *For any connected graph G with at least three vertices, the graph $\widetilde{2G}$ is hypochordal.*

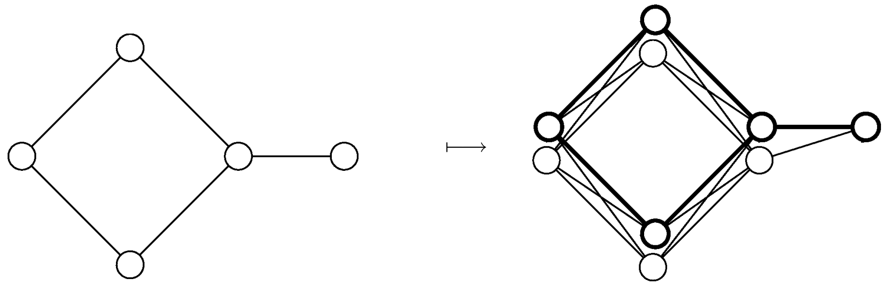


Figure 6: A graph G and the corresponding graph $\widetilde{2G}$

Proof. Let G' be the subgraph of G induced by the vertices with degree greater than or equal to 2. Since $n(G) \geq 3$ and G is connected, $n(G') \geq 1$.
If $n(G') = 1$, then G is the star $K_{1,n(G)-1}$. Thus $\widetilde{2G}$ is the bipartite complete graph $K_{2,n(G)-1}$ which is hypochordal since $n(G) - 1 \geq 2$.
If $n(G') \geq 2$, then, from Proposition 4, the induced subgraph $2G'$ of $\widetilde{2G}$ is hypochordal. Using the same argument as in the proof of Proposition 3, we show that $\widetilde{2G}$ is hypochordal. □

Proposition 6. *A connected graph G with $n \geq 5$ is a $\widetilde{2H}$ if and only if $\delta(G) \geq 2$ and G has a connected 2-stable-C-partition with at least one bag of size 2 and every vertex with degree greater than or equal to 3 is in a bag of size 2.*

Proof.
($\Rightarrow$) If $G = \widetilde{2H}$ and $n \geq 5$, then H is a connected 2-stable-C-partition of G with at least 3 bags. Hence there is at least one bag of size 2. The vertices of G in bags of size 1 have degree 2 in G.
($\Leftarrow$) Let K be the 2-stable-C-partition of G satisfying the conditions of Proposition 6. Thus every pair of vertices in bags of size 2 have the same degree.

Since vertices with degree greater than or equal to 3 are in a bag of size 2 and $\delta(G) \geq 2$, bags of size 1 can only contains vertices of G with degree 2. The neighbours of any vertex x with degree 2 in G are the two vertices of a bag of size 2: by contradiction, suppose that x has its two neighbours in two different bags, necessarily these bags are of size 1 and the two neighbours of x have degree 2. This argument implies that G has a connected component which is a cycle. As G is connected, G is a cycle where every vertex is a bag of K which is impossible since there is at least one bag of size 2 and $n \geq 5$.
So any bag of size 1 in K has a single adjacent bag, which is necessarily of size 2. We construct a new 2-stable-C-partition of G, K' such that if a vertex x of G with degree 2 is in a bag A of size 2 of K, then in K', A is replaced with two bags of size 1, each one containing a single vertex with degree 2. This way, every vertex with degree 2 is in a bag of size 1 of K', every vertex with degree greater than or equal to 3 is in a stable bag of size 2 of K'. Since K' is a 2-stable-C-partition, the two vertices of a same bag of size 2 have exactly the same neighbours. Hence $G = \widetilde{2K'}$. □

Lemma 1. *Let H be a connected graph with $n(H) \geq 3$, $\widetilde{2H}$ satisfies $m(\widetilde{2H}) = 2 \times n(\widetilde{2H}) - 4$ if and only if H is a tree.*

Proof. Let n_1 be the number of pendant vertices of H and n_2 be the number of vertices with degree greater than or equal to 2. $\widetilde{2H}$ has $n(\widetilde{2H}) = n_1 + 2n_2$ vertices. Now let us count the number of edges of $\widetilde{2H}$: any edge in H with an endpoint with degree 1 corresponds to two edges of $\widetilde{2H}$, the other edges of H correspond to four edges of $\widetilde{2H}$ (see Figure 6), hence $m(\widetilde{2H}) = 2n_1 + 4(m(H) - n_1) = 4m(H) - 2n_1$ edges.

We have the following: $m(\widetilde{2H}) = 2n(\widetilde{2H}) - 4$ if and only if $m(H) = n(H) - 1$ and the results follows. □

5.2 How many edges?

For $n = 2$ (respectively $n = 3$), there is a unique minimum connected hypochordal graph which is K_2 (respectively K_3).
Now we consider $n \geq 4$.

Lemma 2. *Let G be a connected hypochordal graph and x be a vertex of G, then $\forall i \geq 2$ and $\forall v \in N_i^x$, $|N(v) \cap N_{i-1}^x| \geq 2$.*

Proof. Let $\mu = [x = v_0, \ldots, v_{i-1}, v_i = v]$ be a shortest path from x to v. G being hypochordal, $d(x, v) = i$ in the subgraph $G \setminus \{v_{i-1}\}$. So in G we have $|N(v) \cap N_{i-1}^x| \geq 2$. □

Lemma 3. *Let G be a connected hypochordal graph with $n \geq 4$, we have $m \geq 2n - 4$. Moreover if $m = 2n - 4$ then $\delta(G) \leq 3$.*

Proof. Since G is hypochordal and $n \geq 4$, we have $\delta(G) \geq 2$.

- Case $\delta(G) = 2$: let x be a vertex with minimum degree. From Lemma 2, for each vertex $v \in N_i^x, i \geq 2$ there are at least two edges $[v, w_1]$ and $[v, w_2]$ with $w_1, w_2 \in N_{i-1}^x$. Moreover x has two neighbours in N_1^x so we have $m \geq 2 \times |\cup_{i\geq 2} N_i^x| + |N_1^x| = 2(n-3) + 2 = 2n - 4$.
- Case $\delta(G) = 3$: when $n = 4$, $G = K_4$ for which $m = 6 > 2n - 4$. We consider $n \geq 5$. Let x be a vertex with minimum degree and y a vertex at maximum distance k from x. We use the same argument as above plus the fact that y has degree at least 3. So we have $m \geq 2 \times (|\cup_{i\geq 2} N_i^x| - 1) + |N_1^x| + \delta(G) = 2(n-5) + 3 + 3 = 2n - 4$.
- Case $\delta(G) \geq 4$: we have $m = \frac{1}{2}\sum_{v\in V} \delta(v) \geq \frac{1}{2}\sum_{v\in V} \delta(G) \geq 2n > 2n - 4$.

It follows immediately that if $m = 2n - 4$, we have $2 \leq \delta(G) \leq 3$. □

Lemma 4. *Let G be a connected hypochordal graph with $m = 2n - 4$ and x be a vertex of G with minimum degree $\delta(x) = 2$: $\forall i \geq 2$, $\forall u \in N_i^x, |N(u) \cap N_{i-1}^x| = 2$ and $\forall i \geq 0, N_i^x$ is a stable set.*

Proof. Let us consider the proof of Lemma 3. Since $m = 2n - 4$, every vertex in $N_i^x, i \geq 2$ has exactly two neighbours in N_{i-1}^x and there is no edge $[u, v]$ with $u, v \in N_i^x$. □

Lemma 5. *Let G be a connected hypochordal graph with $m = 2n - 4$ and x be a vertex of G with minimum degree $\delta(x) = 3$, let k be the maximum distance from a vertex to x: $\forall\, 2 \leq i \leq k-1$, $\forall u \in N_i^x, |N(u) \cap N_{i-1}^x| = 2$ and $\forall\, 0 \leq i \leq k-1, N_i^x$ is a stable set. Moreover $|N_k^x| \leq 2$. If $|N_k^x| = 1$, then $\delta(y) = 3$, where $y \in N_k^x$ and if $|N_k^x| = 2$ then $\forall y \in N_k^x, |N(y) \cap N_{k-1}^x| = 2$.*

Proof. Let us consider the proof of Lemma 3. K_4 does not satisfies $m = 2n-4$, so $n \geq 5$. For each vertex $v \in N_i^x, 2 \leq i < k$, there are exactly two edges $[v, w_1]$ and $[v, w_2]$ with $w_1, w_2 \in N_{i-1}^x$ and there is no edge $[u, v]$ with $u, v \in N_i^x$.
If $|N_k^x| \geq 3$, using the counting argument of Lemma 3, then $m \geq |N_1^x| + 2 \times |\cup_{i\geq 2} N_i^x| + \frac{1}{2}|N_k^x| = 3 + 2(n-4) + \frac{1}{2}|N_k^x| > 2n-4$. Hence $m = 2n-4$ implies that $|N_k^x| \leq 2$.

- If $|N_k^x| = 1$, let $y \in N_k^x$. $m = |N_1^x| + 2 \times |\cup_{2\leq i<k} N_i^x| + \delta(y) = 3 + 2(n-5) + \delta(y)$ and G is a minimum connected hypochordal graph, so $m = 2n-4$ which implies that $\delta(y) = 3$.
- If $|N_k^x| = 2$, $N_k^x = \{y_1, y_2\}$. Suppose that $|N(y_1) \cap N_{k-1}^x| \neq 2$. From Lemma 2, $|N(y_1) \cap N_{k-1}^x| \geq 3$. Hence, $m = |N_1^x| + 2 \times |\cup_{2\leq i<k} N_i^x| + |N(y_1) \cap N_{k-1}^x| + \delta(y_2) \geq 3 + 2(n-6) + 3 + 3 = 2n-3$. Hence $\forall y \in N_k^x, |N(y) \cap N_{k-1}^x| = 2$.

□

We use these lemmata to prove the following.

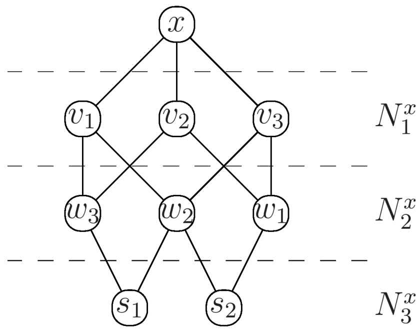
Theorem 1. *A minimum connected hypochordal graph G with $n \geq 4$ is such that $m = 2n-4$, G is bipartite and $\delta(G) = 2$ or 3. Moreover there exists a unique minimum hypochordal graph with $\delta(G) = 3$ which is the cube. Furthermore there exists an infinite family of minimum hypochordal graphs with $\delta(G) = 2$.*

Proof.

- Case $\delta(G) = 2$: if $G = K_{2,n-2}$ we have $m = 2n-4$ and G is hypochordal. For $m = 2n-4$, any hypochordal graph is necessarily bipartite because we have seen in the proof of Lemma 4 that every edge $[u, v]$ of G is such that u (or v) $\in N_i^x$ and v (or u) $\in N_{i-1}^x, i \geq 1$.
- Case $\delta(G) = 3$: the cube satisfies $m = 2n-4$ and is hypochordal.
  Let x be a vertex with minimum degree and k be the maximum distance from a vertex to x.

  First we show that $k \geq 3$.
  $2m = \sum_{v\in V} \delta(v) \geq 3n$ since G has the minimum degree 3. And $m = 2n-4$, so $2(2n-4) \geq 3n$, $n \geq 8$. If $k < 3$, $V = \{x\} \cup N_1^x \cup N_2^x$ thereby $n = 1 + 3 + |N_2^x| \leq 6$, from Lemma 5.

  Hence $k \geq 3$. Let $v_1, v_2, v_3 \in N_1^x$ be the three neighbours of x. Since G is hypochordal and N_1^x is a stable set, v_2 and v_3 must have a common neighbour $w_1 \in N_2^x$. From Lemma 5, $N(w_1) \cap N_1^x = \{v_2, v_3\}$. By symmetry, there exists a vertex $w_2 \neq w_1$ such that $N(w_2) \cap N_1^x = \{v_1, v_3\}$; and there exists another vertex $w_3 \neq w_1, w_3 \neq w_2$ such that $N(w_3) \cap N_1^x = \{v_1, v_2\}$. Hence $N_2^x \supset \{w_1, w_2, w_3\}$.



Suppose that $|N_3^x| \geq 2$: from Lemma 5, $\forall s \in N_3^x, |N(s) \cap N_2^x| = 2$. Consider the vertices w_2 and w_3 which are not adjacent (N_2^x is a stable set) and have a common neighbour. Hence, G being hypochordal, w_2 and w_3 must have at least a second common neighbour $s_1 \in N_3^x$ and $N(s_1) \cap N_2^x = \{w_2, w_3\}$. Vertices s_1 and v_3 have a unique common neighbour w_2, which is a contradiction since G is hypochordal.

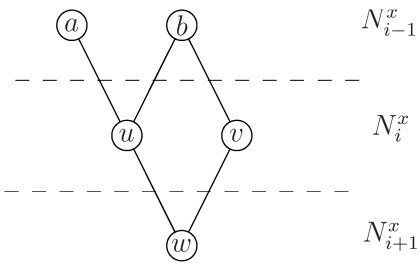
Hence $|N_3^x| = 1$. Since hypochordal graphs have no separating vertex, $k = 3$. Let $\{y\} = N_3^x$, using the same argument as above, we show that $N(y) \supset \{w_1, w_2, w_3\}$ and since $\delta(y) = 3$ by Lemma 5, $N(y) = \{w_1, w_2, w_3\}$.
To show that G is the cube, we now need to show that $N_2^x = \{w_1, w_2, w_3\}$. Assume that there exists $w_4 \in N_2^x \setminus \{w_1, w_2, w_3\}$. From Lemma 5, w_4 has two neighbours in N_1^x, none in N_2^x and none in N_3^x ($N(y) = \{w_1, w_2, w_3\}$) which contradicts $\delta(G) = 3$. □

5.3 Shape of minimum hypochordal graphs

Theorem 1 states that the sole minimum hypochordal graph with $\delta(G) = 3$ is the cube and there are no minimum hypochordal graphs for $\delta(G) \geq 4$. Hence in this section, we consider minimum hypochordal graphs with $\delta(G) = 2$ (there exists an infinite number of such graphs). We recall that these graphs satisfy $m = 2n - 4$, they are bipartite, the sets N_i^x are stable and any vertex v at distance i from x has exactly two neighbours at distance $(i-1)$ from x.

Lemma 6. *Let G be a minimum connected hypochordal graph with $n \geq 4$ and $\delta(G) = 2$, and x be a vertex with minimum degree. Let u and v be two vertices in N_i^x, $i \geq 2$. If $N(u) \cap N(v) \cap N_{i+1}^x \neq \emptyset$ then $|N(u) \cap N(v) \cap N_{i-1}^x| = 2$.*

Proof. By Lemma 4, we know that both u and v have two neighbours in N_{i-1}^x. Let $w \in N_{i+1}^x$ be a common neighbour of u and v. Suppose that u has a neighbour a in N_{i-1}^x which is not a neighbour of v.



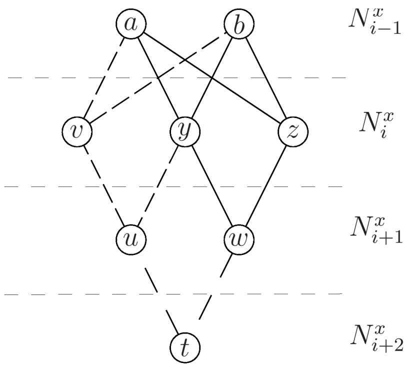
w has exactly two neighbours in N_i^x which are u and v. $[a, u, w]$ is a P_3 but neither $[a, w]$ nor $[a, v]$ are edges of G. This contradicts the fact that G is hypochordal. $\square$

Lemma 7. *Let G be a minimum connected hypochordal graph with $\delta(G) = 2$ and $n \geq 5$. For every vertex y such that $\delta(y) \geq 3$, y has a twin vertex z. Furthermore z is unique.*

Proof. Let x and y be two vertices such that $\delta(x) = 2$ and $\delta(y) \geq 3$. Thus $y \in N_i^x, i \geq 1$.

If $y \in N_1^x$, let $N_1^x = \{y, z\}$. We show that z is the twin of y. From Lemma 4, $[y, z] \notin E$ and every vertex $w \in N_2^x$ has two neighbours in N_1^x which are necessarily y and z. So $N(y) = N(z) = \{x\} \cup N_2^x$ and z is the unique twin of y.

We consider now $y \in N_i^x, i \geq 2$. From Lemma 4, $|N(y) \cap N_{i-1}^x| = 2$; let a and b be the two neighbours of y in N_{i-1}^x. Since $\delta(y) \geq 3$, y has a neighbour $w \in N_{i+1}^x$. Now $|N(w) \cap N_i^x| = 2$; let z be such that in $N(w) \cap N_i^x = \{y, z\}$. We show that $N(z) = N(y)$. G being minimum hypochordal, we know that $N(y) \subset N_{i-1}^x \cap N_{i+1}^x$. Since $N(y) \cap N(z) \cap N_{i+1}^x \neq \emptyset$, Lemma 4 and Lemma 6 ensure that $N(y) \cap N_{i-1}^x = N(z) \cap N_{i-1}^x = \{a, b\}$. Suppose there exists $u \in N_{i+1}^x \cap N(y) \setminus N(z)$. Let $v \neq z$ be such that $N(u) \cap N_i^x = \{y, v\}$. If $\delta(u) = 2$ or $\delta(w) = 2$, $N(u) \cap N(w) = \{y\}$; since $[u, w] \notin E$ and $[v, w] \notin E$, this is impossible. Hence $\delta(u) \geq 3$ and $\delta(w) \geq 3$.



Since $[u, w] \notin E$ and $N(u) \cap N(w) \neq \emptyset$, we have $|N(u) \cap N(w)| \geq 2$; so there exists $t \in N(u) \cap N(w) \cap N_{i+2}^x$; from Lemma 4 we have $N(t) \cap N_{i+1}^x = \{u, w\}$, thus $N(z) \cap N(t) = \{w\}$, a contradiction since $[z, t] \notin E$.

Now we show that y has exactly one twin z. By contradiction, suppose y has two twins z and z'. So $N(y) = N(z) = N(z')$ and $\delta(y) = \delta(z) = \delta(z') \geq 3$. Then there exists $t \in N(y) \cap N_{i+1}^x$ but $|N(t) \cap N_i^x| \geq 3$, which contradicts Lemma 4. $\square$

Theorem 2. *G is a minimum connected hypochordal graph with $n \geq 4$ and $\delta(G) = 2$ if and only if $G = \widetilde{2T}$ for an appropriate tree T with $n(T) \geq 3$.*

Proof. $(\Rightarrow)$

- Case $n = 4$: The only minimum hypochordal graph G is C_4 and $G = \widetilde{2P_3}$;

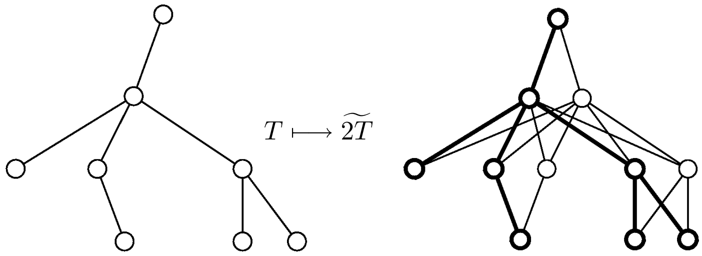


Figure 7: A tree T and the corresponding graph $\widetilde{2T}$

- Case $n \geq 5$: Due to Lemma 7, every vertex y with $\delta(y) \geq 3$ has a unique twin vertex z. We make H a 2-stable-C-partition of G as follows: each vertex w, $\delta(w) = 2$ forms a bag; each vertex y, $\delta(y) \geq 3$ with its twin z form a bag. Given the characterization of Proposition 6, $G = \widetilde{2H}$. Moreover G satisfies $m = 2n - 4$ which means from Lemma 1 that H is a tree.

($\Leftarrow$) From Proposition 5, $\widetilde{2T}$ is hypochordal and from Lemma 1, $m(\widetilde{2T}) = 2n(\widetilde{2T}) - 4$. Hence $\widetilde{2T}$ is a minimum hypochordal graph. □

Theorem 3. *A graph G with $n \geq 5$ and $\delta(G) = 2$ is a minimum hypochordal graph if and only if G is 2-vertex-connected and there exists a 2-tree-stable-C-partition T of G.*

Proof.
($\Rightarrow$) If G is a minimum hypochordal graph, G is 2-vertex-connected and from Theorem 2, $G = \widetilde{2T}$ where T is a tree. Hence, T is a 2-tree-stable-C-partition of G.
($\Leftarrow$) Let G be a graph having T as a 2-tree-stable-C-partition. Since $n(G) \geq 5$, a 2-tree-stable-C-partition of G has at least three bags; then due to Proposition 3, G is hypochordal. Hence $m(G) \geq 2n(G) - 4$.

We still need to show that because G has a 2-tree-stable-C-partition T, $m(G) \leq 2n(G) - 4$:

$n_1(T)$ will denote the number of bags of size 1. T being a tree, its number of edges is $m(T) = n(T) - 1$; let us denote by $m_1(T)$ the number of edges of T which are incident to bags of size 1. We observe that every internal bag of T has size 2 since G is 2-vertex-connected, so $m_1(T) = n_1(T)$.
We have $n(G) = n_1(T) + 2 \times (n(T) - n_1(T)) = 2n(T) - n_1(T)$ and $m(G) = 2m_1(T) + 4[(n(T) - 1) - m_1(T)] = 2(2n(T) - m_1(T)) - 4$. Since $m_1(T) = n_1(T)$, $m(G) = 2n(G) - 4$. □

This characterization of minimum hypochordal graphs implies that many classical combinatorial problems are polynomial in this class: minimum hypochordal graphs are bipartite, hence they are 2-colorable, their maximum clique has size two and the problem of maximum stable set is polynomial; they do not have a hamiltonian cycle, except the cube and $\widetilde{2P_k}, \forall k \geq 3$. In the next section, we consider the same combinatorial problems in the case of (general) hypochordal graphs.

6 Classical combinatorial problems

Here we study the complexity of classical combinatorial problems in the class of hypochordal graphs. For the definition of the problems considered in this section, refer to [5].

6.1 Hamiltonian cycle

We are interested in deciding if a given hypochordal graph is hamiltonian or not.

Theorem 4. *The problem* HAMILTONIAN CYCLE *is $\mathcal{NP}$-complete in hypochordal graphs.*

Proof. We know that deciding if a 3-regular graph is hamiltonian is $\mathcal{NP}$-complete. We are going to show that this problem reduces to the problem of the existence of a hamiltonian cycle in a hypochordal graph.

The problem of deciding if a hypochordal graph is hamiltonian is in $\mathcal{NP}$.

Let $G = (V, E)$ be a 3-regular graph. We build a hypochordal graph $H = (U, F)$ by the following polynomial transformation. Each vertex a of G becomes a subgraph H_a of H isomorphic to K_6. Each edge $[a, b]$ of G is substituted by a gadget (Figure 8) connected to H_a and H_b. The vertices a_1, a_2 are belonging to H_a and b_1, b_2 to H_b. G being a 3-regular graph, any vertex a has three incident

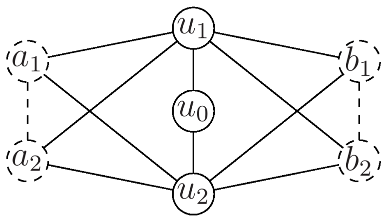


Figure 8: The gadget corresponding to the edge $[a, b]$ of G

edges $[a, b], [a, c]$ and $[a, d]$. You can see the corresponding subgraph of H on Figure 9.

We should first make sure that the graph H we have build is hypochordal. We consider the common neighbours of vertices at distance two (see Figure 9). Given the symmetry of H, we need to consider the vertices at distance two from a_1 plus the pair $\{u_1, u_2\}$. The vertices at distance two from a_1 are b_1 (and b_2), u_0 and v_1 (and v_2).

- Vertices a_1 and b_1 have two common neighbours u_1 and u_2;
- vertices a_1 and u_0 have two common neighbours u_1 and u_2;
- vertices a_1 and v_1 have two common neighbours a_5 and a_6;
- vertices u_1 and u_2 have five common neighbours a_1, a_2, b_1, b_2 and u_0.

We show that if G has a hamiltonian cycle then there is a hamiltonian cycle in H. Let $\mathcal{C}_G$ be a hamiltonian cycle of G: from any vertex $v \in V$, $\mathcal{C}_G$ induces a total order $<$ on V.

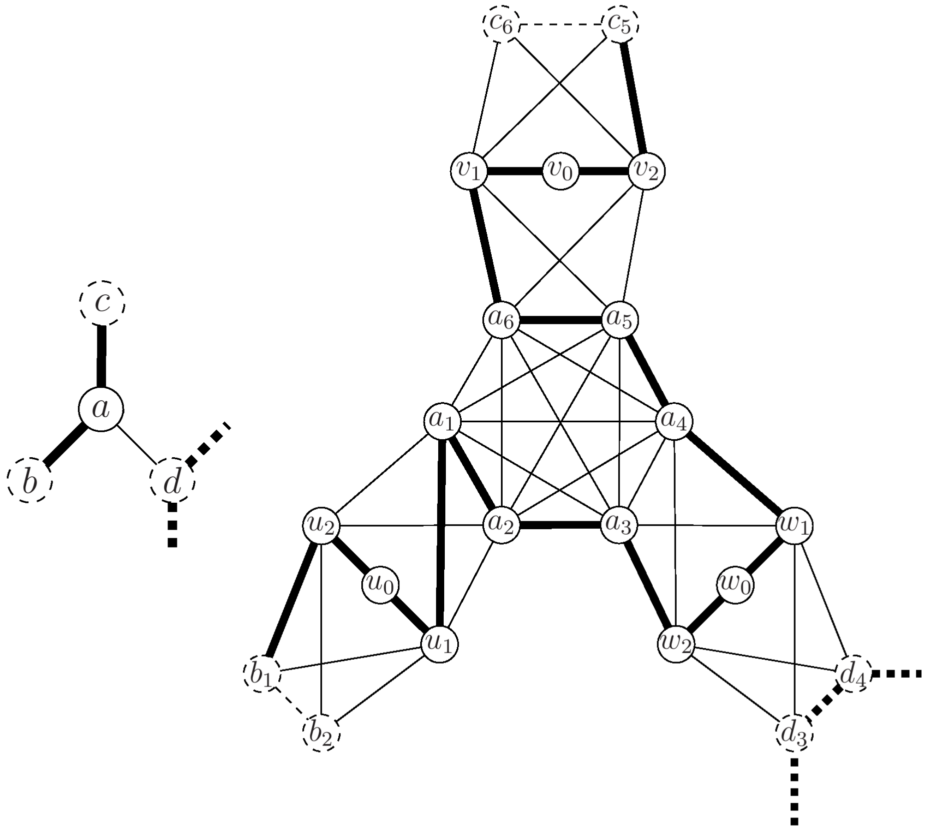


Figure 9: The neighbourhood of a vertex a of G
and the corresponding subgraph of H

For an edge $[x, y]$ of G, $[x, y] \in \mathcal{C}_G$, we associate a path in the gadget between H_x and H_y as shown in Figure 10.

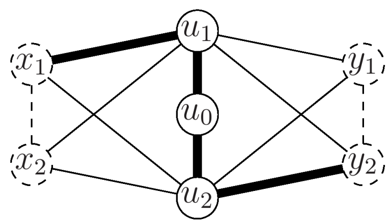


Figure 10: Path in H corresponding to an edge of the hamiltonian cycle of G

For an edge $[x, y]$ of G with $[x, y] \notin \mathcal{C}_G$ and $x < y$, we associate a path of the gadget which does not bridge over H_x and H_y, as shown in Figure 11.

For any vertex a of G, two of its three incident edges are in $\mathcal{C}$. Assume that b, c, d are the three neighbours of a and $[a, b]$ and $[a, c]$ belong to $\mathcal{C}_G$. We have to connect the corresponding paths through H_a. There are two different cases:

- $a < d$: there are three paths to connect, one between H_a and H_b (with a_1

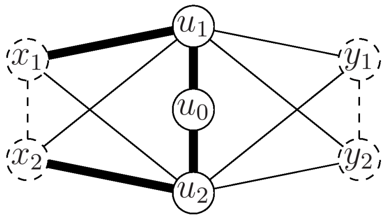


Figure 11: Path corresponding to an edge not in the hamiltonian cycle of G

as end point), one between H_a and H_c (with a_6 as endpoint) and one in the gadget corresponding to the edge $[a, d]$ (with a_3, a_4 as endpoints). We join these three paths with the vertices of H_a that are not in a path yet i.e. a_2 and a_5, see Figure 9.

- $d < a$: there are two paths to connect, one between H_a and H_b (with a_1 as endpoint) and one between H_a and H_c (with a_6 as endpoint). We join these two path with the path $[a_1, a_2, a_3, a_4, a_5, a_6]$.

Let $\mathcal{C}_H$ be the cycle of H obtained this way. For any x of H, x is either a vertex of a gadget or a vertex of a subgraph isomorphic to a K_6. In both cases, it belongs to $\mathcal{C}_H$, thus $\mathcal{C}_H$ is hamiltonian.

We will now show that if H has a hamiltonian cycle then there is a hamiltonian cycle in G. For this, let us look to the different ways to have a hamiltonian path through a gadget. Due to vertex u_0, a hamiltonian cycle can either get in and out the gadget in the same K_6 (as in Figure 11), or get in on one side and out on the other (as in Figure 10). In the second case, for any gadget, we keep the corresponding edge of G. This way, we have a hamiltonian cycle of G. □

6.2 Vertex colouring

Theorem 5. *The problem* VERTEX COLOURING *is $\mathcal{NP}$-complete in hypochordal graphs.*

Proof. We recall that the mapping $G \mapsto 2G$ constructs a hypochordal graph. Suppose having a minimum colouring of G, we obtain a colouring of $2G$ by affecting to v_1 and v_2 the same colour as the corresponding vertex v in the colouring of G. Since G is an induced subgraph of $2G$, this colouring of $2G$ is minimum.

Now consider a minimum colouring of $2G$. Since the twin vertices v_1 and v_2 are nonadjacent and have the same neighbours, we can affect them the same colour. Given such a colouring of the vertices of $2G$, we deduce a colouring of G by affecting to v the colour of its corresponding vertices in $2G$. This colouring of G is minimum, otherwise we would obtain a better colouring of $2G$ by affecting to v_1 and v_2 the same colour as the corresponding vertex v in a minimum colouring of G. □

6.3 Maximum clique

Theorem 6. *The problem* MAXIMUM CLIQUE *is $\mathcal{NP}$-complete in hypochordal graphs.*

Proof. A clique in $2G$ can contain only one of the twin vertices v_1 and v_2. Therefore given a set K of vertices of G and K' a set of vertices of $2G$ where every vertex $v \in K$ corresponds to either v_1 or v_2 in K' ($|K| = |K'|$), K is a maximum clique of G if and only if K' is a maximum clique of $2G$. □

6.4 Maximum stable set

Theorem 7. *The problem* MAXIMUM STABLE SET *is $\mathcal{NP}$-complete in hypochordal graphs.*

Proof. A stable set of $2G$ can contain the twin vertices v_1 and v_2 since they are nonadjacent. Let S be a set of vertices of G and $S' = 2S$ the set of corresponding vertices of $2G$, $|S'| = 2 \times |S|$. S is a maximum stable set of G if and only if S' is a maximum stable set of $2G$. □

7 Conclusion

In this paper, we have introduced the class of hypochordal graphs. Several equivalent characterizations are given, and we have completely characterized the class of minimum hypochordal graphs (in terms of edges). Complexity results are also proven.

We have not tackled the hypochordal recognition issue. This problem is obviously solved by a matrix multiplication and runs at worst in $\mathcal{O}(n^{2.376})$ due to the result of D. Coppersmith and S. Winograd in [4]. A challenge is to find a $\mathcal{O}(n^2)$ or $\mathcal{O}(m)$ algorithm for the recognition of hypochordal graphs. Moreover a research in progress consists in the minimum edge-completion and edge-deletion problems [8] i.e. making hypochordal an existing graph; some additional constraints like diameter or maximum degree should be taken into account.

Hypochordal graphs turn out to be interesting for shortest paths routing in networks since the distances between any pair of non adjacent vertices remain unchanged in case of one vertex or one edge deletion. The minimum hypochordal graphs can be attractive when creating a network since we have shown that their number of edges is two times minus four their number of vertices.

Acknowledgements

The authors express their gratitude to an anonymous referee, whose comments and suggestions have contributed to improving the presentation of the paper.

References

[1] Dimitris Alevras, Martin Grötschel and Roland Wessäly, *Cost-efficient network synthesis from leased lines*, Annals of Operations Research, Vol. 76 (1998) 1–20.

[2] Claude Berge, *Graphes*, Gauthier-Villars (Paris, 1983).

[3] B. Bollobàs and A. G. Thomason, *Hereditary and monotone properties of graphs*, R. L. Graham and J. Nešetřil, editors. The mathematics of Paul Erdös II, Algorithms and combinatorics, Vol. 14, SpringerVerlag (New York, 1997) 70–78.

[4] D. Coppersmith, S. Winograd, *Matrix Multiplication via Arithmetic Progressions*, Proceedings of the nineteenth annual ACM symposium on Theory of computing (New York, 1987) 1–6.

[5] M. R. Garey, D. S. Johnson, *Computers and intractability. A guide to the theory of NP-completeness*, W. H. Freeman (San Francisco, 1979).

[6] E. Howorka, *A characterization of distance-hereditary graphs*, The Quarterly Journal of Mathematics, Vol. 28, Issue 4 (1977) 417.

[7] Dieter Kratsch, Lorna Stewart , *Total domination and transformation*, Information Processing Letters, Vol. 63, (1997) 167–170.

[8] A. Natanzon and R. Shamir and R. Sharan, *Complexity classification of some edge modification problems*, Discrete Applied Mathematics, Vol. 113, Issue 1 (2001) 109–128.

[9] Sebastian Orlowski and Roland Wessäly, *Comparing Restoration Concepts Using Optimal Network Configurations with Integrated Hardware and Routing Decisions*, Journal of Network and Systems Management, Vol. 13, Issue 1 (2005) 99–118.

[10] Jeremy P. Spinrad, *Efficient graph representations*, AMS Bookstore (2003).

[11] David R. Wood, *On tree-partition-width*, European Journal of Combinatorics, Vol. 30, Issue 5 (2009) 1245–1253.