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**GENERALIZED COVARIATION AND EXTENDED FUKUSHIMA
DECOMPOSITION FOR BANACH SPACE VALUED PROCESSES.
APPLICATIONS TO WINDOWS OF DIRICHLET PROCESSES.**

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This paper is concerned with the notion of covariation for Banach space valued processes. In particular, we introduce a notion of quadratic variation, which is a generalization of the classical restrictive formulation of Métivier and Pellaumail. Our approach is based on the notion of χ -covariation for processes with values in two Banach spaces B_1 and B_2 , where χ is a suitable subspace of the dual of the projective tensor product of B_1 and B_2 . We investigate some C^1 type transformations for various classes of stochastic processes admitting a χ -quadratic variation and related properties. If \mathbb{X}^1 and \mathbb{X}^2 admit a χ -covariation, $F^i : B_i \rightarrow \mathbb{R}$, $i = 1, 2$ are of class C^1 with some supplementary assumptions, then the covariation of the real processes $F^1(\mathbb{X}^1)$ and $F^2(\mathbb{X}^2)$ exist.

A detailed analysis is provided on the so-called window processes. Let X be a real continuous process; the $C([-\tau, 0])$ -valued process $X(\cdot)$ defined by $X_t(y) = X_{t+y}$, where $y \in [-\tau, 0]$, is called *window* process. Special attention is given to transformations of window processes associated with Dirichlet and weak Dirichlet processes. Those will constitute a significant Fukushima decomposition for functionals of windows of (weak) Dirichlet processes. As application, we provide a new technique for representing a path-dependent random variable as its expectation plus a stochastic integral with respect to the underlying process.

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1. Introduction

Historically, covariation is defined for two real valued (\mathcal{F}_t) -semimartingales X and Y and it is denoted by $[X, Y]$. However, covariation has been extended to general processes by means of discretization techniques in ¹⁵, or via regularization, see for instance ^{22,24}. In this paper, we will adopt the regularization tools and for simplicity, we will assume that either X or Y is continuous. Now, our approach is slightly different than ²². As for notation, we assume that $T > 0$ is a fixed future time, and every process X will be indexed by $[0, T]$, but can be extended to the real line for convenience by setting $X_t = X_0$ if $t < 0$ and $X_t = X_T$ if $t \geq T$.

Definition 1.1. Let X and Y be two real processes such that X is continuous. For $\epsilon > 0$, we denote

$$[X, Y]_t^\epsilon = \int_0^t \frac{(X_{s+\epsilon} - X_s)(Y_{s+\epsilon} - Y_s)}{\epsilon} ds, \quad t \in [0, T].$$

We say that X and Y admit a **covariation** if

- i) $\lim_{\epsilon \rightarrow 0} [X, Y]_t^\epsilon$ exists in probability for every $t \in [0, T]$ and
- ii) the limiting process in i) admits a continuous modification that will be denoted by $[X, Y]$.

If $[X, X]$ exists, we say that X is a **finite quadratic variation process** (or that it has finite quadratic variation). The **quadratic variation process** $[X, X]$ is also denoted by $[X]$. If $[X] = 0$, X is called **zero quadratic variation process**.

We say that (X, Y) admits its (or X and Y admit their) **mutual covariations** if $[X]$, $[Y]$ and $[X, Y]$ exist.

Remark 1.2.

- 1) Lemma 1.3 below allows to show that, whenever $[X, X]$ exists, then $[X, X]^\epsilon$ also converges in the uniform convergence in probability (ucp) sense, see ^{22,24}. The basic results established there are still valid here, see the following items.
- 2) If X and Y are (\mathcal{F}_t) -local semimartingales, then $[X, Y]$ coincides with the classical covariation, see Corollaries 2 and 3 in ²⁴.
- 3) If X (resp. A) is a finite (resp. zero) quadratic variation process, then $[A, X] = 0$.

We recall two useful tools related to the notion of covariation for real valued processes, see Lemma 3.1 from ²³ and Proposition 1.2 in ²².

Lemma 1.3. *Let $(Z^\epsilon)_{\epsilon>0}$ be a family of continuous processes indexed by $[0, T]$. We suppose the following.*

- 1) $\forall \epsilon > 0, t \rightarrow Z_t^\epsilon$ is increasing.
- 2) There is a continuous process $(Z_t)_{t \in [0, T]}$ such that $Z_t^\epsilon \rightarrow Z_t$ in probability for any $t \in [0, T]$ when ϵ goes to zero.

Then Z^ϵ converges to Z ucp.

Proposition 1.4. *Let X and Y be two continuous processes admitting their mutual covariations. Then for every cadlag process H on $[0, T]$ we have*

$$\int_0^\cdot H_s \frac{(X_{s+\epsilon} - X_s)(Y_{s+\epsilon} - Y_s)}{\epsilon} ds \xrightarrow[\epsilon \rightarrow 0]{ucp} \int_0^\cdot H_s d[X, Y]_s .$$

Definition 1.5.

- 1) An (\mathcal{F}_t) -adapted real process A is called **(\mathcal{F}_t) -martingale orthogonal process** if $[A, N] = 0$ for any continuous (\mathcal{F}_t) -local martingale N .
- 2) A real process A is called **(\mathcal{F}_t) -strongly predictable** if there is $\rho > 0$ such that $X_{\cdot+\rho}$ is (\mathcal{F}_t) -adapted.

(\mathcal{F}_t) -strongly predictable processes were introduced in ⁵; the proposition below is a consequence of Corollary 3.11 of ⁵, taking $h \equiv 1$.

Proposition 1.6. *Let A be an (\mathcal{F}_t) -strongly predictable process and M be an (\mathcal{F}_t) -local martingale. Then $[A, M] = 0$. In particular an (\mathcal{F}_t) -strongly predictable process is an (\mathcal{F}_t) -martingale orthogonal process.*

Important subclasses of finite quadratic variation processes are Dirichlet processes. Those processes were introduced in ¹⁶ (using discretization instead of regularization), so that the best denomination for those should be Föllmer-Dirichlet processes.

Definition 1.7. A real continuous process X is called **(\mathcal{F}_t) -Dirichlet process** if X admits a decomposition $X = M + A$ where M is an (\mathcal{F}_t) -local martingale and A is a zero quadratic variation process. For convenience, we suppose $A_0 = 0$.

The decomposition is unique, for instance, if $A_0 = 0$, see Proposition 16 in ²⁴. An (\mathcal{F}_t) -Dirichlet process has finite quadratic variation. In particular an (\mathcal{F}_t) -semimartingale is also an (\mathcal{F}_t) -Dirichlet process and a continuous locally bounded variation process is in fact a zero quadratic variation process.

The concept of (\mathcal{F}_t) -Dirichlet process can be weakened. An extension of such processes consists in the so-called (\mathcal{F}_t) -weak Dirichlet processes, introduced in ¹¹ and ¹⁸, and implicitly in ¹². Other recent developments appear in ^{3,5}. The non-sectorial processes which were constructed in ²⁷ are indeed important examples of weak Dirichlet processes.

(\mathcal{F}_t) -weak Dirichlet processes are generally not (\mathcal{F}_t) -Dirichlet processes but they still maintain a decomposition property.

Definition 1.8. A real continuous process Y is called (\mathcal{F}_t) -**weak Dirichlet process** if Y admits a decomposition $Y = M + A$ where M is an (\mathcal{F}_t) -local martingale and A is an (\mathcal{F}_t) -martingale orthogonal process. For convenience, we will always assume $A_0 = 0$.

The decomposition is unique, see for instance Remark 3.5 in ¹⁸ or again Proposition 16 in ²⁴. In Corollary 3.15 of ⁵, they observe that if the underlying filtration (\mathcal{F}_t) is the natural filtration of a Brownian motion W , then any (\mathcal{F}_t) -adapted process A is an (\mathcal{F}_t) -martingale orthogonal process if and only if $[A, W] = 0$. An (\mathcal{F}_t) -Dirichlet process is also an (\mathcal{F}_t) -weak Dirichlet process, and a zero quadratic variation process is in fact also an (\mathcal{F}_t) -martingale orthogonal process. However, an (\mathcal{F}_t) -weak Dirichlet process is not necessarily a finite quadratic variation process; on the other hand there are (\mathcal{F}_t) -weak Dirichlet processes with finite quadratic variation that are not Dirichlet, see for instance ¹². Let Y be an (\mathcal{F}_t) -weak Dirichlet process with decomposition $Y = W + A$, W being a (\mathcal{F}_t) -Brownian motion and the process A an (\mathcal{F}_t) -martingale orthogonal process; if A has with finite quadratic variation, then Y is also a finite quadratic variation process with $[Y] = [W] + [A]$. In Theorem 5.11 we will provide another class of examples of (\mathcal{F}_t) -weak Dirichlet processes with finite quadratic variation which are not (\mathcal{F}_t) -Dirichlet.

An important property in stochastic calculus involves the conservation of the *semi-martingale* or *Dirichlet process* features via some real transformations. Here are some classical results.

- a) The class of real semimartingales with respect to a given filtration is known to be stable with respect to $C^2(\mathbb{R})$ transformations, i.e. if $f \in C^2(\mathbb{R})$ or difference of convex functions, X is an (\mathcal{F}_t) -semimartingale, then $f(X)$ is still an (\mathcal{F}_t) -semimartingale.
- b) Finite quadratic variation processes are stable under $C^1(\mathbb{R})$ transformations.
- c) Also Dirichlet processes are stable with respect to $C^1(\mathbb{R})$ transformations. If $f \in C^1(\mathbb{R})$ and $X = M + A$ is a real (\mathcal{F}_t) -Dirichlet process with M the (\mathcal{F}_t) -local martingale and A the zero quadratic variation process, then $f(X)$ is still an (\mathcal{F}_t) -Dirichlet process and its decomposition is given by $f(X) = \tilde{M} + \tilde{A}$, where $\tilde{M}_t = f(X_0) + \int_0^t f'(X_s) dM_s$ and $\tilde{A}_t = f(X_t) - \tilde{M}_t$; see ¹ and ²⁵ for details.
- d) In some applications, in particular to control theory (as illustrated in ¹⁷), one often needs to know the nature of the process $(f(t, X_t))$ where $f \in C^{0,1}([0, T] \times \mathbb{R})$ and X is a real semimartingale. It was shown in ¹⁸, Proposition 3.10, that $Y_t := (f(t, X_t))$ is an (\mathcal{F}_t) -weak Dirichlet process if (X_t) is a continuous (\mathcal{F}_t) -weak Dirichlet process with finite quadratic variation, so in particular an (\mathcal{F}_t) -semimartingale. Obviously, Y does not need to be of finite quadratic variation. For example, consider the case when f is deterministic, only depending on time, with infinite quadratic variation.

Let B_1, B_2 be two general Banach spaces. If \mathbb{X} (resp. \mathbb{Y}) is a B_1 (resp. B_2)

valued stochastic process, it is not obvious how to define an exploitable notion (in practice) of covariation for \mathbb{X} and \mathbb{Y} ; this difficulty remains even if $B_1 = B_2 = H$ is a separable Hilbert space and the processes are H -valued martingales. In Definition 3.4 we recall the definitions of ⁸ for χ -covariation (resp. χ -quadratic variation) with respect to a subspace χ of the dual of $B_1 \hat{\otimes}_\pi B_2$, where \mathbb{X} is B_1 -valued and \mathbb{Y} is B_2 -valued. When χ equals the whole space $(B_1 \hat{\otimes}_\pi B_2)^*$, we say that \mathbb{X} and \mathbb{Y} admit a global covariation. ^{20,9} provide two definitions of quadratic variations related to a Banach space valued process \mathbb{X} : the **real and tensor quadratic variations**. In Definition 1.4 of ⁸ we recover in our regularization language those notions, for avoiding confusion with the quadratic variation of real processes, there we use the terminology **scalar** instead of **real**. Propositions 1.6, 1.7 and Corollary 1.8 of ⁸ give elementary properties for those notions. In Proposition 3.14 of ⁸, we show that whenever \mathbb{X} has a scalar and tensor quadratic variation, then it has the global quadratic variation. However, many Banach space valued processes do not admit a global quadratic variation, even though they admit a χ -quadratic variation for some suitable χ , see ⁸, Section 4 for several examples. In this paper, given different classes of stochastic processes \mathbb{X} with values in some Banach space B and a functional $F : B \rightarrow \mathbb{R}$ with some Fréchet regularity, we seek to find the natural sufficient conditions so that $F(\mathbb{X})$ is a real finite quadratic variation process, a Dirichlet or a weak Dirichlet process.

- In Theorem 4.6 we show that if \mathbb{X} is a B -valued process with χ -quadratic variation and $F : B \rightarrow \mathbb{R}$ is Fréchet of class C^1 with some supplementary properties on DF , then $F(\mathbb{X})$ is a real finite quadratic variation process. This constitutes a natural generalization of the previous item b) concerning real valued processes.

A typical Banach space which justifies the introduction of the notion of χ -quadratic variation is $B = C([-\tau, 0])$ for some $\tau > 0$. Recall that, if X is a real continuous process, the $C([-\tau, 0])$ -valued process $X(\cdot)$ defined by $X_t(y) = X_{t+y}$, where $y \in [-\tau, 0]$, is called **window process** (associated with X). If X is an (\mathcal{F}_t) -Dirichlet (resp. (\mathcal{F}_t) -weak Dirichlet), the process $X(\cdot)$ is called **window (\mathcal{F}_t) -Dirichlet** (resp. **(\mathcal{F}_t) -weak Dirichlet**) **process**. For window processes, we obtain more specific results. We introduce here a notation which will be re-defined in Section 2. Fix $N \in \mathbb{N}$ and a vector $a = (a_N, a_{N-1}, \dots, a_1, 0)$ of $N + 1$ elements of $[-\tau, 0]$, $-\tau = a_N < a_{N-1} < \dots < a_1 < a_0 = 0$. Let $\mathcal{D}_a([-\tau, 0])$ denote the Hilbert space of measures μ on $[-\tau, 0]$ which can be written as a sum of Dirac measures concentrated on points a_i , i.e. $\mu(dx) = \sum_{i=0}^N \lambda_i \delta_{a_i}(dx)$, $\lambda_i \in \mathbb{R}$. $\mathcal{D}_0([-\tau, 0])$ denotes the space $\mathcal{D}_a([-\tau, 0])$ when and $N = 0$, $a = (0)$, i.e. the linear space of multiples of the Dirac measure concentrated in 0. The following items are generalizations of properties c), d) valid for real processes. We set $B = C([-\tau, 0])$.

- Let X be an (\mathcal{F}_t) -Dirichlet process, with associated window process $\mathbb{X} = X(\cdot)$ and again $F : B \rightarrow \mathbb{R}$ Fréchet of class C^1 . Theorem 5.11 gives conditions so

that $F(\mathbb{X})$ is a real (\mathcal{F}_t) -weak Dirichlet process. Under a stronger condition, Theorem 5.9 shows that $F(\mathbb{X})$ is a real (\mathcal{F}_t) -Dirichlet process. More precisely Theorem 5.11 (resp. Theorem 5.9) states the following. Let $F : B \rightarrow \mathbb{R}$ be Fréchet of class $C^1(B)$ such that the first derivative $DF(\eta)$ at each point $\eta \in B$, belongs to $\mathcal{D}_a([-\tau, 0]) \oplus L^2([-\tau, 0])$ (resp. $\mathcal{D}_0([-\tau, 0]) \oplus L^2([-\tau, 0])$). We suppose moreover that DF , with values in the mentioned space, is continuous. Then $F(\mathbb{X})$ is a real (\mathcal{F}_t) -weak Dirichlet process (resp. Dirichlet process).

- The conclusions in the previous item are extended to the case when X is an (\mathcal{F}_t) - weak Dirichlet process with finite quadratic variation. Let $F : [0, T] \times B \rightarrow \mathbb{R}$ be time dependent of class $C^{0,1}$. Similarly to the case when B is finite dimensional, ¹⁸, we cannot expect $(F(t, X_t(\cdot)))$ to be a Dirichlet process. In general it will not even have a finite quadratic variation. In Theorem 5.13 we state the following. Suppose that the first derivative $DF(t, \eta)$, at each point $(t, \eta) \in [0, T] \times C([-\tau, 0])$, belongs to $\mathcal{D}_a([-\tau, 0]) \oplus L^2([-\tau, 0])$. We suppose again that DF , with values in the mentioned space, is continuous. Then $(F(t, X_t(\cdot)))$ is at least a weak Dirichlet process.
- If DF does not necessarily live in $\mathcal{D}_a([-\tau, 0]) \oplus L^2([-\tau, 0])$, and in some cases even if $t \mapsto DF(t, \eta)$ for fixed η is only piecewise continuous but it fulfills a technical condition called the *support predictability condition* (see Definition 5.14), it is possible to recover the conclusion of previous statement, see Theorems 5.16 and 5.17.

One of the consequences of the paper is that, under some modest conditions on a functional $F : [0, T] \times B \rightarrow \mathbb{R}$ and on a B -valued process \mathbb{X} which is a window of a semimartingale (with $B = C([-\tau, 0])$), it is possible to characterize $F(t, \mathbb{X}_t)$ through a Fukushima type decomposition, which is unique, and it plays the role of Itô type formula under weak conditions. The Fukushima decomposition presented in Theorems 5.13, 5.16 and 5.17 constitutes a true novelty in stochastic analysis. In fact it does not concern the decomposition of a functional of an infinite dimensional Dirichlet type process \mathbb{X} , considered as the sum of a local martingale plus a zero global quadratic variation process; even though \mathbb{X} is the window of a real Dirichlet (or weak Dirichlet) process, if considered as an infinite dimensional process, it does not have necessarily a Dirichlet structure.

In Section 6, we consider a diffusion X such that $X_t = X_0 + \int_0^t \sigma(r, X_r) dW_r + \int_0^t b(r, X_r) dr$ and $\sigma, b : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ of class $C^{0,1}$ whose partial derivative in the second variable is bounded. Even if σ is possibly degenerate, we give representations of a class of path dependent random variables h which can be expressed as a C^1 functional of the history of X . More precisely, suppose the existence of $\Phi : C([-\tau, 0]) \rightarrow \mathbb{R}$ Fréchet of class C^1 such that $h = \Phi(X(\cdot))$. A first representation result is Proposition 6.11 which is based on Theorem 6.4. When the process X is a standard Brownian motion we allow the functional to be non-smooth, see Section 6.3. In Section 6.4 we consider h of the form $f(X_{t_1}, \dots, X_{t_N})$ $0 < t_1 < \dots < t_N = T$ and $f \in C^2(\mathbb{R}^N)$ with polynomial growth. In this case the representation can be

associated with N PDEs, each-one stated when the time t varies in the subinterval (t_{i-1}, t_i) , for $1 \leq i \leq N$.

The paper is organized as follows. Section 2 contains general notations and some preliminaries. Section 3 provides the definition of χ -covariation and χ -quadratic variation and some related results. In that section we will recall some calculations of χ -covariations and χ -quadratic variations related to different classes of processes. In Section 4, we discuss how a B -valued process having some χ -quadratic variation transforms under the effect of some C^1 Fréchet functional. In Section 5, we concentrate on the case $B = C([-\tau, 0])$ and on generalized Fukushima decompositions of windows of Dirichlet or weak Dirichlet processes. At Section 6, we provide an application to the problem of recovering quasi-explicit representation formulae for real path-dependent square integrable random variables.

2. Preliminaries

In this section we recall some definitions and notations concerning the whole paper. Let A and B be two general sets such that $A \subset B$; $1_A : B \rightarrow \{0, 1\}$ will denote the indicator function of the set A , so $1_A(x) = 1$ if $x \in A$ and $1_A(x) = 0$ if $x \notin A$. We also write $1_A(x) = 1_{\{x \in A\}}$. $(\Omega, \mathcal{F}, \mathbb{P})$ will be a fixed probability space, equipped with a given filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ fulfilling the usual conditions. Let E and F be Banach spaces over the scalar field \mathbb{R} . $L(E; F)$ will designate the Banach space of F -valued bounded linear maps on E with the norm given by $\|\phi\| = \sup\{\|\phi(e)\|_F : \|e\|_E \leq 1\}$. When $F = \mathbb{R}$, the topological dual space of E will be denoted simply by E^* . If ϕ is a linear functional on E , we shall denote the value of ϕ at an element $e \in E$ either by $\phi(e)$ or $\langle \phi, e \rangle$ or even ${}_{E^*}\langle \phi, e \rangle_E$. Throughout the paper the symbols $\langle \cdot, \cdot \rangle$ will always indicate some type of duality that will change depending on the context. We shall denote the space of \mathbb{R} -valued bounded bilinear forms on the product $E \times F$ by $\mathcal{B}i = \mathcal{B}i(E, F)$ with the norm given by $\|\phi\|_{\mathcal{B}i} = \sup\{|\phi(e, f)| : \|e\|_E \leq 1; \|f\|_F \leq 1\}$. If $a < b$ are two real numbers, $C([a, b])$ will denote the Banach linear space of real continuous functions equipped with the uniform norm denoted by $\|\cdot\|_\infty$. If K is a compact subset of \mathbb{R}^n , $\mathcal{M}(K)$ will denote the dual space $C(K)^*$, i.e. the so-called set of finite signed measures on K . Our principal references about functional analysis and Banach spaces topologies are ^{10,2}.

The capital letters $\mathbb{X}, \mathbb{Y}, \mathbb{Z}$ (resp. X, Y, Z) will generally denote Banach valued (resp. real valued) processes indexed by the time variable $t \in [0, T]$ with $T > 0$. A stochastic process \mathbb{X} will also be denoted by $(\mathbb{X}_t)_{t \in [0, T]}$. A B -valued (resp. \mathbb{R} -valued) stochastic process $\mathbb{X} : \Omega \times [0, T] \rightarrow B$ (resp. $\mathbb{X} : \Omega \times [0, T] \rightarrow \mathbb{R}$) is said to be measurable if $\mathbb{X} : \Omega \times [0, T] \rightarrow B$ (resp. $\mathbb{X} : \Omega \times [0, T] \rightarrow \mathbb{R}$) is measurable with respect to the σ -algebras $\mathcal{F} \otimes \mathcal{B}([0, T])$ and $\mathcal{B}(B)$ (resp. $\mathcal{B}(\mathbb{R})$), \mathcal{B} denoting the corresponding Borel σ -algebra. We recall that $\mathbb{X} : \Omega \times [0, T] \rightarrow B$ (resp. \mathbb{R}) is said to be *strongly measurable* (or *measurable in the Bochner sense*) if it is the limit of measurable countably-valued functions. If \mathbb{X} is measurable and cadlag with B sep-

arable, then \mathbb{X} is strongly measurable. If B is finite dimensional, then a measurable process \mathbb{X} is also strongly measurable. As for the case of real processes, if nothing else is mentioned, all the processes indexed by $[0, T]$ will be naturally extended by continuity setting $\mathbb{X}_t = \mathbb{X}_0$ for $t \leq 0$ and $\mathbb{X}_t = \mathbb{X}_T$ for $t \geq T$. We use a similar convention for deterministic functions. A sequence $(\mathbb{X}^n)_{n \in \mathbb{N}}$ of continuous B -valued processes indexed by $[0, T]$, will be said to converge *ucp* (*uniformly convergence in probability*) to a process \mathbb{X} if $\sup_{0 \leq t \leq T} \|\mathbb{X}_t^n - \mathbb{X}_t\|_B$ converges to zero in probability when $n \rightarrow \infty$. The F-space (or Fréchet space) $\mathcal{C}([0, T])$ will denote the linear space of continuous real processes equipped with the ucp topology and the metric $d(\mathbb{X}, \mathbb{Y}) = \mathbb{E} \left[\sup_{t \in [0, T]} |\mathbb{X}_t - \mathbb{Y}_t| \wedge 1 \right]$.

The direct sum of two Banach spaces E_1 and E_2 will be denoted by $E := E_1 \oplus E_2$. E is still a Banach space under the 2-norm defined by $\|e_1 + e_2\|_E := (\|e_1\|_{E_1}^2 + \|e_2\|_{E_2}^2)^{1/2}$. If each of the spaces E_i is a Hilbert space then E coincides with the uniquely determined Hilbert space with scalar product $\langle e, f \rangle_E = \langle e_1 + e_2, f_1 + f_2 \rangle_E = \sum_{i=1}^2 \langle e_i, f_i \rangle_i$, where $\langle \cdot, \cdot \rangle_i$ is the scalar product in E_i .

We recall now some basic concepts and results about tensor products of two Banach spaces E and F . For details and a more complete description of these arguments, the reader may refer to ²⁶, the case with E and F Hilbert spaces being particularly exhaustive in ²¹. If E and F are Banach spaces, the Banach space $E \hat{\otimes}_\pi F$ (resp. $E \hat{\otimes}_h F$) denotes the *projective* (resp. *Hilbert*) *tensor product* of the Banach spaces E and F . If E and F are Hilbert spaces the Hilbert tensor product $E \hat{\otimes}_h F$ is a Hilbert space. We recall that $E \hat{\otimes}_\pi F$ is obtained by a completion of the algebraic tensor product $E \otimes F$ equipped with the projective norm π . Let $\{x_i\}_{1 \leq i \leq n} \subset E$ and $\{y_i\}_{1 \leq i \leq n} \subset F$, for a general element $u = \sum_{i=1}^n x_i \otimes y_i$ in $E \otimes F$, $\pi(u) = \inf \{ \sum_{i=1}^n \|x_i\| \|y_i\| : u = \sum_{i=1}^n x_i \otimes y_i \}$. For $e \in E$ and $f \in F$ the symbol $e \otimes f$ (resp. $e \otimes^2$) will denote a basic element of the algebraic tensor product $E \otimes F$ (resp. $E \otimes E$). The space $(E \hat{\otimes}_\pi F)^*$ denotes, as usual, the topological dual of the projective tensor product. There is an isometric isomorphism between the dual space of the projective tensor product and the space of bounded bilinear forms equipped with the usual norm: $(E \hat{\otimes}_\pi F)^* \cong \mathcal{B}i(E, F) \cong L(E; F^*)$. Through the relation

$${}_{(E \hat{\otimes}_\pi F)^*} \langle T, \sum_{i=1}^n x_i \otimes y_i \rangle_{E \hat{\otimes}_\pi F} = T \left(\sum_{i=1}^n x_i \otimes y_i \right) = \sum_{i=1}^n \tilde{T}(x_i, y_i) = \sum_{i=1}^n \bar{T}(x_i)(y_i) \quad (2.1)$$

we associate a bounded bilinear form $\tilde{T} \in \mathcal{B}i(E, F)$, a bounded linear functional T on $E \hat{\otimes}_\pi F$ and an element $\bar{T} \in L(E; F^*)$. In the sequel that identification will be often used without explicit mention. The importance of tensor product spaces and their duals is justified first of all by identification (2.1). For instance the second order Fréchet derivative of a real function defined on a Banach space E belongs to $\mathcal{B}i(E, E)$. We recall another important property.

$$\mathcal{M}([-\tau, 0]^2) \subset (C([-\tau, 0]) \hat{\otimes}_\pi C([-\tau, 0]))^* . \quad (2.2)$$

Let η_1, η_2 be two elements in $C([- \tau, 0])$. The element $\eta_1 \otimes \eta_2$ in the algebraic tensor product $C([- \tau, 0]) \otimes C([- \tau, 0])$ will be identified with the element η in $C([- \tau, 0]^2)$ defined by $\eta(x, y) = \eta_1(x)\eta_2(y)$ for all x, y in $[- \tau, 0]$. So if μ is a measure on $\mathcal{M}([- \tau, 0]^2)$, the pairing duality $\mathcal{M}([- \tau, 0]^2) \langle \mu, \eta_1 \otimes \eta_2 \rangle_{C([- \tau, 0]^2)}$ has to be understood as the following duality pairing:

$$\mathcal{M}([- \tau, 0]^2) \langle \mu, \eta \rangle_{C([- \tau, 0]^2)} = \int_{[- \tau, 0]^2} \eta(x, y) \mu(dx, dy) = \int_{[- \tau, 0]^2} \eta_1(x)\eta_2(y) \mu(dx, dy). \quad (2.3)$$

The spaces $\mathcal{M}([- \tau, 0])$ and $\mathcal{M}([- \tau, 0]^2)$ and their subsets will play a central role. We will introduce some other notations that will be used in the sequel. Let $N \in \mathbb{N}$ and $-\tau = a_N < a_{N-1} < \dots < a_1 < a_0 = 0$ be $N + 1$ fixed points in $[- \tau, 0]$. The symbol a will refer to the vector $(a_N, a_{N-1}, \dots, a_1, 0)$ which identifies $N + 1$ points in $[- \tau, 0]$. Let $0 \leq i, j \leq N$.

- The symbol $\mathcal{D}_i([- \tau, 0])$ (\mathcal{D}_i for short), will denote the one dimensional Hilbert space of multiples of the Dirac measure concentrated at $a_i \in [- \tau, 0]$, i.e. $\mathcal{D}_i([- \tau, 0]) := \{\mu \in \mathcal{M}([- \tau, 0]); s.t. \mu(dx) = \lambda \delta_{a_i}(dx) \text{ with } \lambda \in \mathbb{R}\}$; the space \mathcal{D}_0 will be the space of multiples of the Dirac measure concentrated at 0.
- The symbol $\mathcal{D}_a([- \tau, 0])$ (\mathcal{D}_a for short), will denote the $(N + 1)$ -dimensional Hilbert space of multiples of Dirac measures concentrated at $a_i \in [- \tau, 0]$, $0 \leq i \leq N$,

$$\mathcal{D}_a([- \tau, 0]) := \{\mu \in \mathcal{M}([- \tau, 0]); s.t. \mu(dx) = \sum_{i=0}^N \lambda_i \delta_{a_i}(dx) \text{ with } \lambda_i \in \mathbb{R}\} = \bigoplus_{i=0}^N \mathcal{D}_i. \quad (2.4)$$

- The symbol $\mathcal{D}_{i,j}([- \tau, 0]^2)$ ($\mathcal{D}_{i,j}$ for short), will denote the one dimensional Hilbert space of the multiples of the Dirac measure concentrated at $(a_i, a_j) \in [- \tau, 0]^2$, i.e. $\mathcal{D}_{i,j}([- \tau, 0]^2) := \{\mu \in \mathcal{M}([- \tau, 0]^2); s.t. \mu(dx, dy) = \lambda \delta_{a_i}(dx) \delta_{a_j}(dy) \text{ with } \lambda \in \mathbb{R}\} \cong \mathcal{D}_i \hat{\otimes}_h \mathcal{D}_j$. The space $\mathcal{D}_{0,0}$ will be the space of the Dirac measure concentrated at $(0, 0)$.
- $L^2([- \tau, 0])$ is a Hilbert subspace of $\mathcal{M}([- \tau, 0])$, as well as $L^2([- \tau, 0]^2) \cong L^2([- \tau, 0]) \hat{\otimes}_h^2 L^2([- \tau, 0])$ is a Hilbert subspace of $\mathcal{M}([- \tau, 0]^2)$, both equipped with the norm derived from the usual scalar product.
- $\mathcal{D}_i([- \tau, 0]) \oplus L^2([- \tau, 0])$ is a Hilbert subspace of $\mathcal{M}([- \tau, 0])$. In the particular case when $i = 0$, the space $\mathcal{D}_0([- \tau, 0]) \oplus L^2([- \tau, 0])$ will often be denoted by $\mathcal{D}_0 \oplus L^2$ for short.
- $\mathcal{D}_i([- \tau, 0]) \hat{\otimes}_h L^2([- \tau, 0])$ is a Hilbert subspace of $\mathcal{M}([- \tau, 0]^2)$.
- $\chi^2([- \tau, 0]^2)$, χ^2 for short, is the Hilbert space defined as follows:

$$L^2([- \tau, 0]^2) \oplus \bigoplus_{i=0}^N (L^2([- \tau, 0]) \hat{\otimes}_h \mathcal{D}_i([- \tau, 0])) \\ \oplus \bigoplus_{i=0}^N (\mathcal{D}_i([- \tau, 0]) \hat{\otimes}_h L^2([- \tau, 0])) \oplus \bigoplus_{i,j=0}^N \mathcal{D}_{i,j}([- \tau, 0]^2).$$

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- As a particular case of $\chi^2([-\tau, 0]^2)$ we will denote $\chi^0([-\tau, 0]^2)$, χ^0 for short, the subspace of measures defined as

$$L^2([-\tau, 0]^2) \oplus L^2([-\tau, 0]) \hat{\otimes}_h \mathcal{D}_0([-\tau, 0]) \oplus \mathcal{D}_0([-\tau, 0]) \hat{\otimes}_h L^2([-\tau, 0]) \oplus \mathcal{D}_{0,0}([-\tau, 0]^2).$$

Let B be a Banach space. A function $F : [0, T] \times B \rightarrow \mathbb{R}$, is said to be $C^{1,2}([0, T] \times B)$, or $C^{1,2}$ for short, if the following properties are fulfilled.

- F is once Fréchet continuously differentiable; the partial derivative with respect to t will be denoted by $\partial_t F : [0, T] \times B \rightarrow \mathbb{R}$;
- for any $t \in [0, T]$, $x \mapsto DF(t, x)$ is of class C^1 where $DF : [0, T] \times B \rightarrow B^*$ denotes the Fréchet derivative with respect to the second argument;
- the second order Fréchet derivative with respect to the second argument $D^2F : [0, T] \times B \rightarrow (B \hat{\otimes}_\pi B)^*$ is continuous.

If $B = C([-\tau, 0])$, we remark that DF defined on $[0, T] \times B$ takes values in $B^* \cong \mathcal{M}([-\tau, 0])$. For all $(t, \eta) \in [0, T] \times C([-\tau, 0])$, we will denote by $D_{dx}F(t, \eta)$ the uniquely determined measure such that for every $h \in C([-\tau, 0])$,

$$\mathcal{M}([-\tau, 0]) \langle DF(t, \eta), h \rangle_{C([-\tau, 0])} = DF(t, \eta)(h) = \int_{[-\tau, 0]} h(x) D_{dx}F(t, \eta).$$

Recalling (2.2), if $D^2F(t, \eta) \in \mathcal{M}([-\tau, 0]^2)$ for all $(t, \eta) \in [0, T] \times C([-\tau, 0])$ (which will happen in most of the treated cases), we will denote with $D_{dx dy}^2 F(t, \eta)$, or $D_{dx}D_{dy}F(t, \eta)$, the measure on $[-\tau, 0]^2$ such that following duality holds $\forall g \in C([-\tau, 0]^2)$

$$\mathcal{M}([-\tau, 0]^2) \langle D^2F(t, \eta), g \rangle_{C([-\tau, 0]^2)} = D^2F(t, \eta)(g) = \int_{[-\tau, 0]^2} g(x, y) D_{dx dy}^2 F(t, \eta).$$

A useful notation, that will be used throughout the paper, is the following.

Notation 2.1. Let $F : [0, T] \times C([-\tau, 0]) \rightarrow \mathbb{R}$ be a Fréchet differentiable function, with Fréchet derivative $DF : [0, T] \times C([-\tau, 0]) \rightarrow \mathcal{M}([-\tau, 0])$. For any given $(t, \eta) \in [0, T] \times C([-\tau, 0])$ and $c \in [-\tau, 0]$, we denote by $D^{ac}F(t, \eta)$ the absolutely continuous part of measure $DF(t, \eta)$, and by $D^{\delta c}F(t, \eta) := DF(t, \eta)(\{c\})$. For every $\eta \in C([-\tau, 0])$, we observe that $t \mapsto D^{\delta c}F(t, \eta)$ is a real valued function. We denote $D^\perp F(t, \eta) = DF(t, \eta) - DF(t, \eta)(\{0\})\delta_0$.

Example 2.2. If for example $DF(t, \eta) \in \mathcal{D}_0 \oplus L^2([-\tau, 0])$ for every $(t, \eta) \in [0, T] \times C([-\tau, 0])$, then we will often write $D_{dx}F(t, \eta) = D^{\delta_0}F(t, \eta)\delta_0(dx) + D_x^{ac}F(t, \eta)dx$.

Note that the mention *Fréchet* referred to derivatives of functions defined on Banach spaces, will often be omitted.

3. Notions of χ -covariation between Banach valued processes

Let B_1, B_2 be two Banach spaces. Whenever $B_1 = B_2$ we will denote it simply by B .

Definition 3.1. A Banach subspace $(\chi, \|\cdot\|_\chi)$ continuously injected into $(B_1 \hat{\otimes}_\pi B_2)^*$ will be called a **Chi-subspace** (of $(B_1 \hat{\otimes}_\pi B_2)^*$).

Remark 3.2. Obviously the pairing between $(B_1 \hat{\otimes}_\pi B_2)^*$ and $(B_1 \hat{\otimes}_\pi B_2)^{**}$ is compatible with the pairing between χ and χ^* .

Example 3.3. When $B = C([-\tau, 0])$, typical examples of Chi-subspaces of $(B \hat{\otimes}_\pi B)^*$ are $\mathcal{M}([-\tau, 0]^2)$ equipped with the total variation norm and all Hilbert closed subspaces of $\mathcal{M}([-\tau, 0]^2)$. For instance $L^2([-\tau, 0]^2)$, $\mathcal{D}_i([-\tau, 0]) \hat{\otimes}_h L^2([-\tau, 0])$, $\mathcal{D}_{i,j}([-\tau, 0]^2)$, for $1 \leq i, j \leq N$, $\chi^2([-\tau, 0]^2)$ and $\chi^0([-\tau, 0]^2)$.

We recall now the definition of χ -covariation between a B_1 -valued stochastic process \mathbb{X} and a B_2 -valued stochastic process \mathbb{Y} . We suppose \mathbb{X} to be a continuous B_1 -valued stochastic process and \mathbb{Y} to be a strongly measurable B_2 -valued stochastic process such that $\int_0^T \|\mathbb{Y}_s\|_{B^*} ds < +\infty$ a.s. Recall that $\mathcal{C}([0, T])$ denotes the space of continuous processes equipped with the ucp topology.

Let χ be a Chi-subspace of $(B_1 \hat{\otimes}_\pi B_2)^*$ and $\epsilon > 0$. We denote by $[\mathbb{X}, \mathbb{Y}]^\epsilon$, the following map $[\mathbb{X}, \mathbb{Y}]^\epsilon : \chi \rightarrow \mathcal{C}([0, T])$ defined by

$$\phi \mapsto \left(\int_0^t \chi \langle \phi, \frac{J((\mathbb{X}_{s+\epsilon} - \mathbb{X}_s) \otimes (\mathbb{Y}_{s+\epsilon} - \mathbb{Y}_s))}{\epsilon} \rangle_{\chi^*} ds \right)_{t \in [0, T]}$$

where J is the canonical injection between of the space $B_1 \hat{\otimes}_\pi B_2$ into its bidual $(B_1 \hat{\otimes}_\pi B_2)^{**}$.

With the map $[\mathbb{X}, \mathbb{Y}]^\epsilon$ we naturally associate $\widetilde{[\mathbb{X}, \mathbb{Y}]^\epsilon} : \Omega \times [0, T] \rightarrow \chi^*$. For $\omega \in \Omega$, $\widetilde{[\mathbb{X}, \mathbb{Y}]^\epsilon}(\omega, \cdot)$ is defined by

$$t \mapsto \left(\phi \mapsto \int_0^t \chi \langle \phi, \frac{J((\mathbb{X}_{s+\epsilon}(\omega) - \mathbb{X}_s(\omega)) \otimes (\mathbb{Y}_{s+\epsilon}(\omega) - \mathbb{Y}_s(\omega)))}{\epsilon} \rangle_{\chi^*} ds \right).$$

Definition 3.4. Let B_1, B_2 be two Banach spaces and χ be a Chi-subspace of $(B_1 \hat{\otimes}_\pi B_2)^*$. Let \mathbb{X} (resp. \mathbb{Y}) be a continuous B_1 (resp. strongly measurable B_2) valued stochastic process such that $\int_0^T \|\mathbb{Y}_s\|_{B^*} ds < +\infty$ a.s.. We say that \mathbb{X} **and** \mathbb{Y} **admit a χ -covariation** if the following holds.

H1 For all (ϵ_n) there exists a subsequence (ϵ_{n_k}) such that

$$\begin{aligned} & \sup_k \int_0^T \sup_{\|\phi\|_\chi \leq 1} \left| \langle \phi, \frac{(\mathbb{X}_{s+\epsilon_{n_k}} - \mathbb{X}_s) \otimes (\mathbb{Y}_{s+\epsilon_{n_k}} - \mathbb{Y}_s)}{\epsilon_{n_k}} \rangle \right| ds \\ &= \sup_k \int_0^T \frac{\|(\mathbb{X}_{s+\epsilon_{n_k}} - \mathbb{X}_s) \otimes (\mathbb{Y}_{s+\epsilon_{n_k}} - \mathbb{Y}_s)\|_{\chi^*}}{\epsilon_{n_k}} ds < \infty \text{ a.s.} \end{aligned}$$

H2 (i) There exists a map $\chi \rightarrow \mathcal{C}([0, T])$, denoted by $[\mathbb{X}, \mathbb{Y}]$, such that

$$[\mathbb{X}, \mathbb{Y}]^\epsilon(\phi) \xrightarrow[\epsilon \rightarrow 0_+]{ucp} [\mathbb{X}, \mathbb{Y}](\phi)$$

for every $\phi \in \chi \subset (B_1 \hat{\otimes}_\pi B_2)^*$.

(ii) There is a measurable process $\widetilde{[\mathbb{X}, \mathbb{Y}]} : \Omega \times [0, T] \longrightarrow \chi^*$, such that

- for almost all $\omega \in \Omega$, $\widetilde{[\mathbb{X}, \mathbb{Y}]}(\omega, \cdot)$ is a (cadlag) bounded variation process,
- $\widetilde{[\mathbb{X}, \mathbb{Y}]}(\cdot, t)(\phi) = [\mathbb{X}, \mathbb{Y}](\phi)(\cdot, t)$ a.s. for all $\phi \in \chi$.

In this case, $\widetilde{[\mathbb{X}, \mathbb{Y}]}$ is called χ -covariation (process) of \mathbb{X} and \mathbb{Y} . By a slight abuse of notation, $[\mathbb{X}, \mathbb{Y}]$, will also be called χ -covariation, since $\widetilde{[\mathbb{X}, \mathbb{Y}]}$ is indeed a version of $[\mathbb{X}, \mathbb{Y}]$.

Definition 3.5. Let $\mathbb{X} = \mathbb{Y}$ be a B -valued stochastic process and χ be a Chi-subspace of $(B \hat{\otimes}_\pi B)^*$. The χ -covariation $\widetilde{[\mathbb{X}, \mathbb{X}]}$ (or $[\mathbb{X}, \mathbb{X}]$) will also be denoted by $\widetilde{[\mathbb{X}]}$ (or $[\mathbb{X}]$); it will be called χ -**quadratic variation** of \mathbb{X} and we will say that \mathbb{X} admits a χ -quadratic variation.

Definition 3.6. If the χ -covariation exists for $\chi = (B_1 \hat{\otimes}_\pi B_2)^*$, we say that \mathbb{X} and \mathbb{Y} admit a **global covariation**. Analogously if \mathbb{X} is B -valued and the χ -quadratic variation exists for $\chi = (B \hat{\otimes}_\pi B)^*$, we say that \mathbb{X} admits a **global quadratic variation**.

We recall now Corollary 3.20 from ⁸, which generalizes Proposition 1.4 to the case where the range of the processes is a Banach space.

Proposition 3.7. *Let B_1, B_2 be two Banach spaces and χ be a Chi-subspace of $(B_1 \hat{\otimes}_\pi B_2)^*$. Let \mathbb{X} and \mathbb{Y} be two stochastic processes with values in B_1 and B_2 admitting a χ -covariation; let \mathbb{H} be a continuous measurable process $\mathbb{H} : \Omega \times [0, T] \longrightarrow \mathcal{V}$ where \mathcal{V} is a closed separable subspace of χ . Then for every $t \in [0, T]$*

$$\int_0^t \chi \langle \mathbb{H}(\cdot, s), d\widetilde{[\mathbb{X}, \mathbb{Y}]}^\epsilon(\cdot, s) \rangle_{\chi^*} \xrightarrow{\epsilon \rightarrow 0} \int_0^t \chi \langle \mathbb{H}(\cdot, s), d\widetilde{[\mathbb{X}, \mathbb{Y}]}(\cdot, s) \rangle_{\chi^*}$$

in probability.

We recall some evaluations of χ -covariations and χ -quadratic variations for window processes given in Section 4 of ⁸, in particular we refer to Proposition 4.9 and Corollary 4.10. For the three results below we will consider again an integer $N, 0 < \tau \leq T$ and we make the same conventions about vector $a = (a_N = -\tau, \dots, a_0 = 0)$ as those introduced after (2.3).

Proposition 3.8. *Let X and Y be two real continuous processes with finite quadratic variation.*

- 1) $X(\cdot)$ and $Y(\cdot)$ admit a zero χ -covariation, where $\chi = L^2([-\tau, 0]^2)$.
- 2) $X(\cdot)$ and $Y(\cdot)$ admit zero χ -covariation, where $\chi = L^2([-\tau, 0]) \hat{\otimes}_h \mathcal{D}_i([-\tau, 0])$ and $\mathcal{D}_i([-\tau, 0]) \hat{\otimes}_h L^2([-\tau, 0])$ for every given $i \in \{0, \dots, N\}$.

If moreover the covariation $[X_{\cdot+a_i}, Y_{\cdot+a_j}]$ exists for a given $i, j \in \{0, \dots, N\}$, the following statements hold.

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3) $X(\cdot)$ and $Y(\cdot)$ admit a χ -covariation, where $\chi = \mathcal{D}_{i,j}([-\tau, 0]^2)$ and it equals

$$[X(\cdot), Y(\cdot)](\mu) = \mu(\{a_i, a_j\})[X_{\cdot+a_i}, Y_{\cdot+a_j}], \quad \forall \mu \in \mathcal{D}_{i,j}([-\tau, 0]^2).$$

4) In the case $i = j = 0$, i.e. X and Y admit a covariation $[X, Y]$, then $X(\cdot)$ and $Y(\cdot)$ admit $\chi^0([-\tau, 0]^2)$ -covariation which equals

$$[X(\cdot), Y(\cdot)](\mu) = \mu(\{0, 0\})[X, Y], \quad \forall \mu \in \chi^0.$$

If $[X_{\cdot+a_i}, Y_{\cdot+a_j}]$ exists for all $i, j = 0, \dots, N$, then

5) $X(\cdot)$ and $Y(\cdot)$ admit a $\chi^2([-\tau, 0]^2)$ -covariation which equals

$$[X(\cdot), Y(\cdot)](\mu) = \sum_{i,j=0}^N \mu(\{a_i, a_j\})[X_{\cdot+a_i}, Y_{\cdot+a_j}], \quad \forall \mu \in \chi^2([-\tau, 0]^2).$$

As application of Proposition 3.8 we obtain the corollaries. The symbol N will denominate, without any risk of confusion, a local martingale and an integer.

Corollary 3.9. *Let X be a real (\mathcal{F}_t) -weak Dirichlet process with finite quadratic variation and decomposition $X = M + A$, M being its (\mathcal{F}_t) -local martingale part. Let N be a real (\mathcal{F}_t) -martingale. We set $\chi = \mathcal{D}_{0,0} \oplus \chi_2$ with*

$$\chi_2 = \oplus_{i=1}^N \mathcal{D}_{i,0} \oplus (L^2([-\tau, 0]) \hat{\otimes}_h \mathcal{D}_0). \quad (3.1)$$

We have the following.

1) $X(\cdot)$ and $N(\cdot)$ admit a $\mathcal{D}_{0,0}$ -covariation given, for $\mu \in \mathcal{D}_{0,0}$, by

$$[X(\cdot), N(\cdot)](\mu) = \mu(\{0, 0\})[M, N]. \quad (3.2)$$

2) $X(\cdot)$ and $N(\cdot)$ admit a zero χ_2 -covariation.

3) $X(\cdot)$ and $N(\cdot)$ admit a χ -covariation and for any $\mu \in \chi$, (3.2) holds.

4) $X(\cdot)$ and $N(\cdot)$ admit a χ^0 -covariation given, for $\mu \in \chi^0$, by (3.2).

Corollary 3.10. *Let X be a real (\mathcal{F}_t) -Dirichlet process with decomposition $X = M + A$, M being its (\mathcal{F}_t) -local martingale part. Let N be a real (\mathcal{F}_t) -martingale. Then we have the following.*

1) $X(\cdot)$ admits a χ^2 -quadratic variation given by

$$[X(\cdot)](\mu) = \sum_{i=0}^N \mu(\{a_i, a_i\})[M]_{\cdot+a_i}.$$

2) $X(\cdot)$ and $N(\cdot)$ admit a χ^2 -covariation given by

$$[X(\cdot), N(\cdot)](\mu) = \sum_{i=0}^N \mu(\{a_i, a_i\})[M, N]_{\cdot+a_i}.$$

Remark 3.11. More details about Dirichlet processes and their properties will be given in section 5. Examples of finite quadratic variation weak Dirichlet processes are provided in Section 2 of ¹². For an (\mathcal{F}_t) -weak Dirichlet process X the covariations $[X_{\cdot+a_i}, X_{\cdot+a_j}]$ are not a priori determined.

Proof. [Proof of Corollary 3.9].

- 1) By Proposition 1.6 $[X, Y] = [M, N]$, so this point is a consequence of Proposition 3.8 3).
- 2) Keeping in mind the direct sum decomposition of χ_2 given in (3.1), we compute the $\bigoplus_{i=1}^N \mathcal{D}_{0,i}$ -covariation. For every $i = 0, \dots, N$, $[X_{\cdot+a_i}, N] = 0$ because it is the sum of $[M_{\cdot+a_i}, N]$ and $[A_{\cdot+a_i}, N]$ which are zero by Proposition 1.6. By item 3) of Proposition 3.8, it follows that $X(\cdot)$ and $N(\cdot)$ have zero $\bigoplus_{i=1}^N \mathcal{D}_{i,0}$ -covariation. By item 2) of Proposition 3.8, $X(\cdot)$ and $N(\cdot)$ have zero $L^2([-\tau, 0] \hat{\otimes}_h \mathcal{D}_i)$ -covariation for every i . When χ is the direct sum of Chi-spaces χ_j , Proposition 3.16 in ⁸ permits to express the χ -covariation as the sum of χ_j -covariations, which allows to conclude the proof of item 2).
- 3) It follows by 1), 2) and again by Proposition 3.16 in ⁸.
- 4) Since X is a weak Dirichlet process, then $[X, N] = [M, N]$. So this point follows by item 4) of Proposition 3.8. \square

Proof. [Proof of Corollary 3.10].

- 1) If $i \neq j$, it follows that $[X_{\cdot+a_i}, X_{\cdot+a_j}] = 0$ because of Proposition 1.6 and Remark 1.2. If $i = j$, by Remark 1.2 and by definition of quadratic variation we get $[X_{\cdot+a_i}] = [M]_{\cdot+a_i}$. The result follows by item 5) of Proposition 3.8.
- 2) Similarly as in the proof of item 1) we have that $[X_{\cdot+a_i}, N_{\cdot+a_j}] = 0$ if $i \neq j$ and $[X_{\cdot+a_i}, N_{\cdot+a_i}] = [M, N]_{\cdot+a_i}$. The result follows as a consequence of item 5) of Proposition 3.8. \square

Other interesting results about χ -covariation and χ -quadratic variation for a window of a finite quadratic variation process are Proposition 6.4 in ⁷ and Propositions 4.14 and 4.15 in ⁸.

4. Chain rules for χ -quadratic variation and of χ -covariation

If X is a real finite quadratic variation process and $f \in C^1(\mathbb{R})$, we recall that $f(X)$ is again a finite quadratic variation process. Let us suppose now that \mathbb{X} is a process with values in a Banach space B admitting a χ -quadratic variation and let $F : B \rightarrow \mathbb{R}$ be a C^1 functional. Is the real process $F(\mathbb{X})$ a finite quadratic variation process? If yes, what is the expression of that quadratic variation? This section will answer those questions and discuss some related problems to covariations.

The finite dimensional case was the object of Remark 3 in ¹⁴ that we recall below.

Proposition 4.1. *Let $X = (X^1, \dots, X^n)$ be a \mathbb{R}^n -valued process having all its mutual covariations $[X^i, X^j]_t$ and $F, G \in C^1(\mathbb{R}^n)$. Then the covariation $[F(X), G(X)]$ exists and is given by*

$$[F(X), G(X)] = \sum_{i,j=1}^n \int_0^\cdot \partial_i F(X) \partial_j G(X) d[X^i, X^j] \quad (4.1)$$

This includes the case of Proposition 2.1 in ²², setting $n = 2$, $F(x, y) = f(x)$, $G(x, y) = g(y)$, $f, g \in C^1(\mathbb{R})$.

At this level, we recall a notion related to tensor spaces.

Proposition 4.2. *Let E be a Banach space, $S, T : E \rightarrow \mathbb{R}$ be continuous linear forms. There is a unique continuous linear form from $E \hat{\otimes}_\pi E$ to $\mathbb{R} \hat{\otimes}_\pi \mathbb{R} \cong \mathbb{R}$, designated by $S \otimes T$, such that $S \otimes T(e_1 \otimes e_2) = S(e_1) \cdot T(e_2)$ and $\|S \otimes T\| = \|S\| \|T\|$.*

Proof. See Proposition 2.3 in ²⁶. □

Remark 4.3.

- 1) If $T = S$, we will denote $S \otimes S = S \otimes^2$.
- 2) Let B be a Banach space and $F, G : E \rightarrow \mathbb{R}$ of class $C^1(E)$. If x and y are fixed, $DF(x)$ and $DF(y)$ are continuous linear forms on E . We remark that the symbol $DF(x) \otimes DF(y)$ is defined according to Proposition 4.2 and we insist on the fact that $DF(x) \otimes DF(y)$ does not denote (a priori) an element of some tensor product $E^* \otimes E^*$.

When E is a Hilbert space, the map $S \otimes T$ in Proposition 4.2 can be further specified.

Proposition 4.4. *Let E be a Hilbert space, $S, T \in E^*$ and \mathcal{S}, \mathcal{T} the associated elements in E via Riesz identification. $S \otimes T$ can be characterized as the continuous bilinear form*

$$(x, y) \mapsto \langle \mathcal{S}, \mathcal{T} \rangle_E \cdot \langle x, y \rangle_E = \langle \mathcal{S} \otimes \mathcal{T}, x \otimes y \rangle_{E \hat{\otimes}_h E}, \quad \forall x, y \in E.$$

In particular the linear form $S \otimes T$ belongs to $(E \hat{\otimes}_h E)^$ and via Riesz it is identified with the tensor product $\mathcal{S} \otimes \mathcal{T}$. That Riesz identification will be omitted in the sequel.*

Proof. We indicate by ϕ , the element belonging to $(E \hat{\otimes}_h E)^*$ defined, as the image via Riesz on the Hilbert space $(E \hat{\otimes}_h E)$, of $\mathcal{S} \otimes \mathcal{T}$. Since $(E \hat{\otimes}_h E)^* \subset (E \hat{\otimes}_\pi E)^*$, ϕ also belongs to $(E \hat{\otimes}_\pi E)^*$. Moreover we have

$$\|\phi\|_{B_i} = \sup_{\|f\|_E \leq 1, \|g\|_E \leq 1} |\phi(f, g)| = \sup_{\|f\|_E \leq 1} |\langle \mathcal{S}, f \rangle| \sup_{\|g\|_E \leq 1} |\langle \mathcal{T}, g \rangle| = \|\mathcal{S}\|_{E^*} \|\mathcal{T}\|_{E^*}.$$

By the uniqueness statement in Proposition 4.2, ϕ must coincide with $S \otimes T$. □

As application of Proposition 4.4, setting the Hilbert space $E = \mathcal{D}_a \oplus L^2([-\tau, 0])$, we state the following useful result that will be often used in Section 5 devoted to $C([-\tau, 0])$ -valued window processes.

Example 4.5. Let F^1 and F^2 be two functions from $C([-\tau, 0])$ to $\mathcal{D}_a \oplus L^2([-\tau, 0])$ such that $\eta \mapsto F^j(\eta) = \sum_{i=0, \dots, N} \lambda_i^j(\eta) \delta_{a_i} + g^j(\eta)$ with $\eta \in C([-T, 0])$, $\lambda_i^j : C([-\tau, 0]) \rightarrow \mathbb{R}$ and $g^j : C([-\tau, 0]) \rightarrow L^2([-T, 0])$ continuous for $j = 1, 2$. Then for any $\eta_1, \eta_2 \in C([-\tau, 0])$, $(F^1 \otimes F^2)(\eta_1, \eta_2)$ will be identified with the true tensor product $F^1(\eta_1) \otimes F^2(\eta_2)$ which belongs to $\chi^2([-\tau, 0]^2)$. In fact we have

$$\begin{aligned} F^1(\eta_1) \otimes F^2(\eta_2) &= \sum_{i,j=0, \dots, N} \lambda_i^1(\eta_1) \lambda_j^2(\eta_2) \delta_{a_i} \otimes \delta_{a_j} + g^1(\eta_1) \otimes \sum_{i=0, \dots, N} \lambda_i^2(\eta_2) \delta_{a_i} + \\ &+ \sum_{i=0, \dots, N} \lambda_i^1(\eta_1) \delta_{a_i} \otimes g^2(\eta_2) + g^1(\eta_1) \otimes g^2(\eta_2) \end{aligned}$$

We are able now to state a generalization of Proposition 4.1, which allows to calculate the covariation of functionals of processes admitting a χ -covariation.

Theorem 4.6. *Let B be a separable Banach space, χ a Chi-subspace of $(B \hat{\otimes}_\pi B)^*$ and $\mathbb{X}^1, \mathbb{X}^2$ two B -valued continuous stochastic processes admitting a χ -covariation. Let $F^1, F^2 : B \rightarrow \mathbb{R}$ be two functions being of class C^1 . We suppose moreover that the map $DF^1(\cdot) \otimes DF^2(\cdot) : B \times B \rightarrow \chi \subset (B \hat{\otimes}_\pi B)^*$, $(x, y) \mapsto DF^1(x) \otimes DF^2(y)$ is continuous. Then, the covariation between $F^1(\mathbb{X}^1)$ and $F^2(\mathbb{X}^2)$ exists and is given by*

$$[F^1(\mathbb{X}^1), F^2(\mathbb{X}^2)] = \int_0^\cdot \langle DF^1(\mathbb{X}_s^1) \otimes DF^2(\mathbb{X}_s^2), d[\widetilde{\mathbb{X}^1, \mathbb{X}^2}]_s \rangle. \quad (4.2)$$

Remark 4.7. In view of an application of Proposition 3.7 in the proof of Theorem 4.6, we observe the following. Since B is separable and $DF^1(\cdot) \otimes DF^2(\cdot) : B \times B \rightarrow \chi$ is continuous, the process $H_t = DF^1(\mathbb{X}_t^1) \otimes DF^2(\mathbb{X}_t^2)$ takes values in a separable closed subspace \mathcal{V} of χ .

Corollary 4.8. *Let us formulate the same assumptions as in Theorem 4.6. If there is a χ^* -valued stochastic process $\mathbb{H}^{1,2}$ such that $[\mathbb{X}^1, \mathbb{X}^2]_s = \int_0^s \mathbb{H}_u^{1,2} du$ in the Bochner sense then*

$$[F^1(\mathbb{X}^1), F^2(\mathbb{X}^2)]_t = \int_0^t \langle DF^1(\mathbb{X}_s^1) \otimes DF^2(\mathbb{X}_s^2), \mathbb{H}_s^{1,2} \rangle ds, \quad t \in [0, T]. \quad (4.3)$$

Proof. [Proof of Theorem 4.6]. We make use in an essential manner of Proposition 3.7. Without restriction of generality we only consider the case $F^1 = F^2 = F$ and $\mathbb{X}^1 = \mathbb{X}^2 = \mathbb{X}$.

Let $t \in [0, T]$. By definition of the quadratic variation of a real process in Definition 1.1, it will be enough to show that the quantity $\frac{1}{\epsilon} \int_0^t (F(\mathbb{X}_{s+\epsilon}) - F(\mathbb{X}_s))^2 ds$ converges in probability to the right-hand side of (4.3). Using Taylor's expansion we

have

$$\begin{aligned} \frac{1}{\epsilon} \int_0^t (F(\mathbb{X}_{s+\epsilon}) - F(\mathbb{X}_s))^2 ds &= \frac{1}{\epsilon} \int_0^t \left(\langle DF(\mathbb{X}_s), \mathbb{X}_{s+\epsilon} - \mathbb{X}_s \rangle + \right. \\ &\quad \left. + \int_0^1 \langle DF((1-\alpha)\mathbb{X}_s + \alpha\mathbb{X}_{s+\epsilon}) - DF(\mathbb{X}_s), \mathbb{X}_{s+\epsilon} - \mathbb{X}_s \rangle d\alpha \right)^2 ds = \\ &= A_1(\epsilon) + A_2(\epsilon) + A_3(\epsilon), \end{aligned}$$

where

$$A_1(\epsilon) = \frac{1}{\epsilon} \int_0^t \langle DF(\mathbb{X}_s), \mathbb{X}_{s+\epsilon} - \mathbb{X}_s \rangle^2 ds = \int_0^t \langle DF(\mathbb{X}_s) \otimes DF(\mathbb{X}_s), \frac{(\mathbb{X}_{s+\epsilon} - \mathbb{X}_s) \otimes^2}{\epsilon} \rangle ds$$

$$\begin{aligned} A_2(\epsilon) &= \frac{2}{\epsilon} \int_0^t \langle DF(\mathbb{X}_s), \mathbb{X}_{s+\epsilon} - \mathbb{X}_s \rangle \cdot \\ &\quad \cdot \int_0^1 \langle DF((1-\alpha)\mathbb{X}_s + \alpha\mathbb{X}_{s+\epsilon}) - DF(\mathbb{X}_s), \mathbb{X}_{s+\epsilon} - \mathbb{X}_s \rangle d\alpha ds = \\ &= 2 \int_0^t \int_0^1 \langle DF(\mathbb{X}_s) \otimes (DF((1-\alpha)\mathbb{X}_s + \alpha\mathbb{X}_{s+\epsilon}) - DF(\mathbb{X}_s)), \frac{(\mathbb{X}_{s+\epsilon} - \mathbb{X}_s) \otimes^2}{\epsilon} \rangle d\alpha ds \end{aligned}$$

$$\begin{aligned} A_3(\epsilon) &= \frac{1}{\epsilon} \int_0^t \left(\int_0^1 \langle DF((1-\alpha)\mathbb{X}_s + \alpha\mathbb{X}_{s+\epsilon}) - DF(\mathbb{X}_s), \mathbb{X}_{s+\epsilon} - \mathbb{X}_s \rangle d\alpha \right)^2 ds \leq \\ &\leq \frac{1}{\epsilon} \int_0^t \int_0^1 \langle DF((1-\alpha)\mathbb{X}_s + \alpha\mathbb{X}_{s+\epsilon}) - DF(\mathbb{X}_s), \mathbb{X}_{s+\epsilon} - \mathbb{X}_s \rangle^2 d\alpha ds = \\ &= \int_0^t \int_0^1 \langle (DF((1-\alpha)\mathbb{X}_s + \alpha\mathbb{X}_{s+\epsilon}) - DF(\mathbb{X}_s)) \otimes^2, \frac{(\mathbb{X}_{s+\epsilon} - \mathbb{X}_s) \otimes^2}{\epsilon} \rangle d\alpha ds. \end{aligned}$$

According to Remark 4.7 and Proposition 3.7 with $\mathbb{X} = \mathbb{Y}$, it follows

$$A_1(\epsilon) \xrightarrow{\mathbb{P}} \int_0^t \langle DF(\mathbb{X}_s) \otimes DF(\mathbb{X}_s), d[\widetilde{\mathbb{X}}]_s \rangle.$$

It remains to show the convergence in probability of $A_2(\epsilon)$ and $A_3(\epsilon)$ to zero.

The term $DF(\mathbb{X}_s) \otimes (DF((1-\alpha)\mathbb{X}_s + \alpha\mathbb{X}_{s+\epsilon}) - DF(\mathbb{X}_s))$ appearing in $A_2(\epsilon)$ equals

$$DF(\mathbb{X}_s) \otimes DF((1-\alpha)\mathbb{X}_s + \alpha\mathbb{X}_{s+\epsilon}) - DF(\mathbb{X}_s) \otimes DF(\mathbb{X}_s). \quad (4.4)$$

The expression $(DF((1-\alpha)\mathbb{X}_s + \alpha\mathbb{X}_{s+\epsilon}) - DF(\mathbb{X}_s)) \otimes^2$ that appears in $A_3(\epsilon)$ equals

$$\begin{aligned} &DF((1-\alpha)\mathbb{X}_s + \alpha\mathbb{X}_{s+\epsilon}) \otimes^2 - DF((1-\alpha)\mathbb{X}_s + \alpha\mathbb{X}_{s+\epsilon}) \otimes DF(\mathbb{X}_s) \\ &+ DF(\mathbb{X}_s) \otimes DF(\mathbb{X}_s) - DF(\mathbb{X}_s) \otimes DF((1-\alpha)\mathbb{X}_s + \alpha\mathbb{X}_{s+\epsilon}). \end{aligned} \quad (4.5)$$

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By (4.4), $|A_2(\epsilon)|$ is smaller or equal than

$$\begin{aligned}
 & 2 \int_0^t \int_0^1 \left| \langle DF(\mathbb{X}_s) \otimes (DF((1-\alpha)\mathbb{X}_s + \alpha\mathbb{X}_{s+\epsilon}) - DF(\mathbb{X}_s)), \frac{(\mathbb{X}_{s+\epsilon} - \mathbb{X}_s) \otimes^2}{\epsilon} \rangle \right| d\alpha ds \\
 & \leq \int_0^t \int_0^1 \|DF(\mathbb{X}_s) \otimes DF((1-\alpha)\mathbb{X}_s + \alpha\mathbb{X}_{s+\epsilon}) - DF(\mathbb{X}_s) \otimes DF(\mathbb{X}_s)\|_{\chi} \cdot \\
 & \qquad \qquad \qquad \cdot \left\| \frac{(\mathbb{X}_{s+\epsilon} - \mathbb{X}_s) \otimes^2}{\epsilon} \right\|_{\chi^*} d\alpha ds .
 \end{aligned} \tag{4.6}$$

For fixed $\omega \in \Omega$ we denote by $\mathcal{V}(\omega) := \{\mathbb{X}_t(\omega); t \in [0, T]\}$ and $\mathcal{U} = \mathcal{U}(\omega) = \overline{\text{conv}(\mathcal{V}(\omega))}$, i.e. the set \mathcal{U} is the closed convex hull of the compact subset $\mathcal{V}(\omega)$ of B . From (4.6) we deduce

$$|A_2(\epsilon)| \leq \varpi_{DF \otimes DF}^{\mathcal{U} \times \mathcal{U}}(\varpi_{\mathbb{X}}(\epsilon)) \int_0^t \left\| \frac{(\mathbb{X}_{s+\epsilon} - \mathbb{X}_s) \otimes^2}{\epsilon} \right\|_{\chi^*} ds,$$

where $\varpi_{DF \otimes DF}^{\mathcal{U} \times \mathcal{U}}$ is the continuity modulus of the map $DF(\cdot) \otimes DF(\cdot) : B \times B \rightarrow \chi$ restricted to $\mathcal{U} \times \mathcal{U}$ and $\varpi_{\mathbb{X}}$ is the continuity modulus of the continuous process \mathbb{X} . We recall that

$$\varpi_{DF \otimes DF}^{\mathcal{U} \times \mathcal{U}}(\delta) = \sup_{\|(x_1, y_1) - (x_2, y_2)\|_{B \times B} \leq \delta} \|DF(x_1) \otimes DF(y_1) - DF(x_2) \otimes DF(y_2)\|_{\chi}$$

where the space $B \times B$ is equipped with the norm obtained summing the norms of the two components.

Similarly to the proof of Theorem 4.6 $\mathcal{U}(\omega)$ is also compact, so that the function $DF(\cdot) \otimes DF(\cdot)$ on $\mathcal{U}(\omega) \times \mathcal{U}(\omega)$ is uniformly continuous and $\varpi_{DF \otimes DF}^{\mathcal{U} \times \mathcal{U}}$ is a positive, increasing function on \mathbb{R}^+ converging to 0 when the argument converges to zero.

Let (ϵ_n) converging to zero; Condition **H1** in the definition of χ -quadratic variation, implies the existence of a subsequence (ϵ_{n_k}) such that $A_2(\epsilon_{n_k})$ converges to zero a.s. This implies that $A_2(\epsilon) \rightarrow 0$ in probability.

With similar arguments, using (4.5), we can show that $A_3(\epsilon) \rightarrow 0$ in probability. We observe in fact that $|A_3(\epsilon)|$ is smaller or equal than

$$\begin{aligned}
 & \int_0^t \int_0^1 \left\| DF((1-\alpha)\mathbb{X}_s + \alpha\mathbb{X}_{s+\epsilon}) \otimes^2 - DF(\mathbb{X}_s) \otimes DF((1-\alpha)\mathbb{X}_s + \alpha\mathbb{X}_{s+\epsilon}) \right\|_{\chi} \cdot \\
 & \qquad \qquad \qquad \cdot \left\| \frac{(\mathbb{X}_{s+\epsilon} - \mathbb{X}_s) \otimes^2}{\epsilon} \right\|_{\chi^*} d\alpha ds \\
 & + \int_0^t \int_0^1 \left\| DF((1-\alpha)\mathbb{X}_s + \alpha\mathbb{X}_{s+\epsilon}) \otimes DF(\mathbb{X}_s) - DF(\mathbb{X}_s) \otimes^2 \right\|_{\chi} \cdot \\
 & \qquad \qquad \qquad \cdot \left\| \frac{(\mathbb{X}_{s+\epsilon} - \mathbb{X}_s) \otimes^2}{\epsilon} \right\|_{\chi^*} d\alpha ds \\
 & \leq 2\varpi_{DF \otimes DF}^{\mathcal{U} \times \mathcal{U}}(\varpi_{\mathbb{X}}(\epsilon)) \int_0^t \left\| \frac{(\mathbb{X}_{s+\epsilon} - \mathbb{X}_s) \otimes^2}{\epsilon} \right\|_{\chi^*} ds .
 \end{aligned}$$

The result is now established. \square

Corollary 4.9. *Let B be a separable Banach space and B_0 be a Banach space such that $B_0 \supset B$ continuously. Let $\chi = (B_0 \hat{\otimes}_\pi B_0)^*$ and \mathbb{X} a continuous B -valued stochastic process admitting a χ -quadratic variation. Let $F^1, F^2 : B \rightarrow \mathbb{R}$ be functions of class C^1 such that DF^i , $i = 1, 2$ are continuous as maps from B to B_0^* . Then the covariation of $F^1(\mathbb{X})$ and $F^2(\mathbb{X})$ exists and it is given by*

$$[F^1(\mathbb{X}), F^2(\mathbb{X})] = \int_0^\cdot \langle DF^1(\mathbb{X}_s) \otimes DF^2(\mathbb{X}_s), d[\widetilde{\mathbb{X}}]_s \rangle.$$

Proof. It is clear that χ is a Chi-subspace of $(B \hat{\otimes}_\pi B)^*$. For any given $x, y \in B$, the characterization of $DF^1(x) \otimes DF^2(y)$ given in Proposition 4.2 and Remark 4.3 imply that the map $DF^1(x) \otimes DF^2(y) : B_0 \hat{\otimes}_\pi B_0 \rightarrow \mathbb{R}$ is continuous. The result follows by Theorem 4.6. \square

Remark 4.10. Suppose that in the statement of Corollary 4.9 that B_0 is a Hilbert space. Then for any $x, y \in B$, $DF(x) \otimes DG(y)$ belongs to $(B_0 \hat{\otimes}_h B_0)^*$ because of Proposition 4.4: it will be associated to a true tensor product in the sense explained in the same proposition.

We discuss rapidly the finite dimensional framework. A detailed analysis can be found in Paragraph 1, Chapter 6 of ⁶. Recall that $\mathbb{R}^n \hat{\otimes}_\pi \mathbb{R}^n$ can be identified with the space of matrices $\mathbb{M}_{n \times n}(\mathbb{R})$. Since $\mathbb{R}^n \hat{\otimes}_\pi \mathbb{R}^n$ is finite dimensional and all norm topologies are equivalent, it is enough to show the identification on $\mathbb{R}^n \otimes \mathbb{R}^n$. Let $(e_i)_{1 \leq i \leq n}$ be the canonical basis of \mathbb{R}^n ; to $u \in \mathbb{R}^n \otimes \mathbb{R}^n$ of the form $u = \sum_{1 \leq i, j \leq n} u_{i,j} e_i \otimes e_j$, we can associate a unique matrix $U = (u_{i,j})_{1 \leq i, j \leq n}$, $U \in \mathbb{M}_{n \times n}(\mathbb{R})$. Conversely to a matrix $U \in \mathbb{M}_{n \times n}(\mathbb{R})$ of the form $U = (U_{i,j})_{1 \leq i, j \leq n}$, we associate the unique element $u \in \mathbb{R}^n \otimes \mathbb{R}^n$ defined by $u = \sum_{1 \leq i, j \leq n} U_{i,j} e_i \otimes e_j$. The dual space $(\mathbb{R}^n \otimes \mathbb{R}^n)^* (\cong L(\mathbb{R}^n; L(\mathbb{R}^n)))$ can also be naturally identified with $\mathbb{M}_{n \times n}(\mathbb{R})$. A matrix $T \in \mathbb{M}_{n \times n}(\mathbb{R})$ of the form $T = (T_{i,j})_{1 \leq i \leq m, 1 \leq j \leq n}$ is associated with the linear form $t : \mathbb{R}^n \otimes \mathbb{R}^n \rightarrow \mathbb{R}$ such that $t(x \otimes y) =_{\mathbb{R}^n} \langle Tx, y \rangle_{\mathbb{R}^n}$. Moreover the duality pairing between an element $t \in (\mathbb{R}^n \hat{\otimes}_\pi \mathbb{R}^n)^*$ and an element $u \in (\mathbb{R}^n \hat{\otimes}_\pi \mathbb{R}^n)$ (or simply $(\mathbb{R}^n \otimes \mathbb{R}^n)$), denoted by $\langle t, u \rangle$, coincides with the trace $Tr(TU)$, whenever U (resp. T) is the $\mathbb{M}_{n \times n}(\mathbb{R})$ matrix associated with u (resp. t).

Example 4.11. Let $\mathbb{X} = (X^1, \dots, X^n)$ be a \mathbb{R}^n -valued stochastic process admitting all its mutual covariations, and $F, G : \mathbb{R}^n \rightarrow \mathbb{R} \in C^1(\mathbb{R}^n)$. We recall that \mathbb{X} admits a global quadratic variation $[\widetilde{\mathbb{X}}]$ which coincides with the tensor element associated with the matrix $([\mathbb{X}^*, \mathbb{X}])_{1 \leq i, j \leq n} := ([X^i, X^j])_{i, j}$, see Proposition 6.2 in ⁶. The application of Theorem 4.6 to this context provides a new proof of Proposition 4.1. According to Proposition 6.2 item 2.(b) in ⁶, the right-hand side of (4.2) equals

$$\int_0^\cdot Tr(DF(\mathbb{X}_s) \otimes DG(\mathbb{X}_s) \cdot d[\mathbb{X}^*, \mathbb{X}]_s)$$

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which coincides with the right-hand side of (4.1).

5. Transformation of window Dirichlet processes and window weak Dirichlet processes

5.1. Some preliminary results on measure theory

We set now $B = C([-τ, 0])$ in order to present some Fukushima type decomposition results about B -valued window Dirichlet and window weak Dirichlet processes. First we need a preliminary result on measure theory.

We start with some notations which appear in ⁹, Chapter 1, Section D, Definition 18. Let E be a Banach space and $g : [0, T] \rightarrow E^*$ be a bounded variation function. Then the real function $\|g\|_{E^*} : [0, T] \rightarrow \mathbb{R}$ has also bounded variation. If $f : [0, T] \rightarrow E$ is $d\|g\|$ strongly measurable, then the Bochner integral $\int_0^T \langle f(s), dg(s) \rangle_{E^*}$ is well-defined provided that

$$\int_0^T \|f(s)\|_E d\|g\|_{E^*}(s) < +\infty. \quad (5.1)$$

We denote by $L_E^1(g)$ the linear space of $d\|g\|$ -strongly measurable functions $f : [0, T] \rightarrow E$ verifying (5.1).

Lemma 5.1. *Let E be the topological direct sum $E_1 \oplus E_2$ where E_1, E_2 are Banach spaces equipped with norms $\|\cdot\|_{E_i}$. We denote by P_i the projectors $P_i : E \rightarrow E_i$, $i \in 1, 2$.*

Let $\tilde{g} : [0, T] \rightarrow E^$ and we define $\tilde{g}_i : [0, T] \rightarrow E_i^*$ setting $\tilde{g}_i(t)(\eta) := \tilde{g}(t)(\eta)$ for all $\eta \in E_i$, i.e. the restriction of $\tilde{g}(t)$ to E_i^* . We suppose \tilde{g} continuous with bounded variation. Let $f : [0, T] \rightarrow E$ measurable with projections $f_i := P_i(f)$ defined from $[0, T]$ to E_i .*

Then the following statements hold.

1) *f in $L_E^1(\tilde{g})$ if and only if f_i in $L_{E_i}^1(\tilde{g}_i)$, $i = 1, 2$ and yields*

$$\int_0^t \langle f(s), d\tilde{g}(s) \rangle_{E^*} = \int_0^t \langle f_1(s), d\tilde{g}_1(s) \rangle_{E_1^*} + \int_0^t \langle f_2(s), d\tilde{g}_2(s) \rangle_{E_2^*}. \quad (5.2)$$

2) *If $\tilde{g}_2(t) \equiv 0$ and $f_1 \in L_{E_1}^1(\tilde{g}_1)$ then*

$$\int_0^t \langle f(s), d\tilde{g}(s) \rangle_{E^*} = \int_0^t \langle f_1(s), d\tilde{g}_1(s) \rangle_{E_1^*}.$$

Remark 5.2.

- Let $0 \leq a < b \leq T$. We can easily show that

$$\begin{aligned} \|\tilde{g}(b) - \tilde{g}(a)\|_{E^*} &\leq \|\tilde{g}_1(b) - \tilde{g}_1(a)\|_{E_1^*} + \|\tilde{g}_2(b) - \tilde{g}_2(a)\|_{E_2^*} \\ \|\tilde{g}_i(b) - \tilde{g}_i(a)\|_{E_i^*} &\leq \|\tilde{g}(b) - \tilde{g}(a)\|_{E^*}, \quad i = 1, 2. \end{aligned}$$

- Consequently \tilde{g} is continuous with bounded variation if and only if \tilde{g}_1, \tilde{g}_2 are continuous with bounded variation.

Proof.

1) By the hypothesis $\tilde{g} : [0, T] \rightarrow E^*$ has bounded variation. If $f : [0, T] \rightarrow E$ belongs to L_E^1 , then $f_i = P_i(f) : [0, T] \rightarrow E_i$, $i = 1, 2$ belong to $L_{E_i}^1(g)$ by the property $\|P_i f\|_{E_i} \leq \|f\|_E$.

We prove (5.2) for a simple (or finitely valued measurable) function $f : [0, T] \rightarrow E$ of the form $f(s) = \sum_{j=1}^N 1_{A_j}(s) f_j$ with 1_{A_j} indicator functions of Borel subsets A_j of $[0, T]$ and $f_j \in E$. We write $f_j = f_{1j} + f_{2j}$ with $f_{ij} = P_i f_j$, $i = 1, 2$, so

$$\begin{aligned} \int_0^T \langle f(s), d\tilde{g}(s) \rangle_{E^*} &= \sum_{j=1}^N \int_{A_j} \langle f_j, d\tilde{g}(s) \rangle_{E^*} = \sum_{j=1}^N \langle f_j, \int_{A_j} d\tilde{g}(s) \rangle_{E^*} \\ &= \sum_{j=1}^N \langle f_{1j}, d\tilde{g}_1(A_j) \rangle_{E_1^*} + \sum_{j=1}^N \langle f_{2j}, d\tilde{g}_2(A_j) \rangle_{E_2^*} \\ &= \int_0^T \langle f_1(s), d\tilde{g}_1(s) \rangle_{E_1^*} + \int_0^T \langle f_2(s), d\tilde{g}_2(s) \rangle_{E_2^*}. \end{aligned}$$

The general case follows by usual approximation arguments. Now, by Bochner theorem, $L_E^1(g)$ is the space of $d\|g\|$ -strongly measurable functions such there is a sequence of simple functions (f_n) with $\lim_{n \rightarrow \infty} \int_0^T \|f_n - f\| d\|g\| = 0$. A general function f in $L_E^1(\tilde{g})$ is a sum $f_1 + f_2$, $f_i \in L_{E_i}^1(\tilde{g}_i)$ for $i = 1, 2$. Both f_1 and f_2 can be approximated by simple functions.

2) It follows directly by 1). □

Now recall the Chi-spaces which appears in Section 2, as for example, $\mathcal{D}_{i,j}$ for $0 \leq i, j \leq N$. Next we have the following useful consequence of Lemma 5.1.

Proposition 5.3. *Let $E = E_1 \oplus E_2$ with $E_1 = \mathcal{D}_{i,j}([-\tau, 0]^2)$ and E_2 being a Banach subspace of $\mathcal{M}([-\tau, 0]^2)$.*

- 1) Let $\tilde{g} : [0, T] \rightarrow E^*$ such that $\tilde{g}(t)|_{E_2} \equiv 0$.
- 2) We suppose that the function $g_1 : [0, T] \rightarrow \mathbb{R}$ defined by $g_1(t) := \langle \delta_{(a_i, a_j)}, \tilde{g}(t) \rangle_{E^*}$ is continuous with bounded variation.
- 3) Let $f : [0, T] \rightarrow E$ such that $t \rightarrow f(t)(\{a_i, a_j\}) \in L^1(d|g_1|)$.

Then

$$\int_0^t \langle f(s), d\tilde{g}(s) \rangle_{E^*} = \int_0^t f(s)(\{a_i, a_j\}) dg_1(s). \quad (5.3)$$

Remark 5.4.

- 1) E_1^* is a one-dimensional space generated by the multiples of $\delta_{(a_i, a_j)}^*$ defined such that $\langle \delta_{(a_i, a_j)}, \delta_{(a_i, a_j)}^* \rangle_{E_1^*} = 1$.

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- 2) We define $\tilde{g}_1 : [0, T] \rightarrow E_1^*$ by $\tilde{g}_1(t) = g_1(t) \delta_{(a_i, a_j)}^*$. By item 1) it follows $\tilde{g}_1(t)(f) = \tilde{g}(t)(f)$ for every $f \in E_1, t \in [0, T]$. By Remark 5.2, for $a, b \in [0, T]$, with $a < b$, we have $\|\tilde{g}(b) - \tilde{g}(a)\|_{E_1^*} = \|\tilde{g}_1(b) - \tilde{g}_1(a)\|_{E_1^*} = |g_1(b) - g_1(a)|$, so that g_1 is continuous with bounded variation if and only if \tilde{g} is continuous with bounded variation.

Proof. [Proof of Proposition 5.3].

We apply Lemma 5.1 2). Since $P_1(f) = f(\{a_i, a_j\})\delta_{(a_i, a_j)}$, it follows $\int_0^t \langle f(s), d\tilde{g}(s) \rangle_{E_1^*} = \int_0^t \langle f(s)(\{a_i, a_j\})\delta_{(a_i, a_j)}, d\tilde{g}_1(s) \rangle_{E_1^*}$. But

$$E_1 \langle \delta_{(a_i, a_j)}, \tilde{g}_1(t) \rangle_{E_1^*} = E_1 \langle \delta_{(a_i, a_j)}, \delta_{(a_i, a_j)}^* \rangle_{E_1^*} g_1(t) = g_1(t),$$

and so previous expression equals the right-hand side of (5.3) simply because of Theorem 30 in Chapter 1, paragraph 2 of ⁹. \square

Remark 5.5. Let E be a Banach subspace of $\mathcal{M}([-\tau, 0]^2)$ containing $\mathcal{D}_{i,j}([-\tau, 0]^2)$. A typical example of application of Proposition 5.3 is given by $E_1 = \mathcal{D}_{i,j}([-\tau, 0]^2)$ and $E_2 = \{\mu \in E \mid \mu(\{a_i, a_j\}) = 0\}$. Any $\mu \in E$ can be decomposed into $\mu_1 + \mu_2$, where $\mu_1 = \mu(\{a_i, a_j\})\delta_{(a_i, a_j)}$, which belongs to E_1 , and $\mu_2 \in E_2$.

In the proof of item 3) in the proposition below we will use Proposition 5.3 considering \tilde{g} , as the χ -covariation of two processes $X(\cdot)$ and $Y(\cdot)$, for fixed $\omega \in \Omega$.

Proposition 5.6. *Let $i, j \in \{0, \dots, N\}$ and let χ_2 be a Banach subspace of $\mathcal{M}([-\tau, 0]^2)$ such that $\mu(\{a_i, a_j\}) = 0$ for every $\mu \in \chi_2$. We set $\chi = \mathcal{D}_{i,j}([-\tau, 0]^2) \oplus \chi_2$. Let X, Y be two real continuous processes such that $(X_{\cdot+a_i}, Y_{\cdot+a_j})$ admits their mutual covariations and such that $X(\cdot)$ and $Y(\cdot)$ admit a zero χ_2 -covariation. Then the following properties hold.*

- 1) χ is a Chi-subspace of $(B \hat{\otimes}_\pi B)^*$, with $B = C([-\tau, 0])$.
- 2) $X(\cdot)$ and $Y(\cdot)$ admit a χ -covariation of the type

$$[X(\cdot), Y(\cdot)] : \chi \longrightarrow \mathcal{C}([0, T]), \quad [X(\cdot), Y(\cdot)](\mu) = \mu(\{a_i, a_j\})[X_{\cdot+a_i}, Y_{\cdot+a_j}].$$

- 3) For every χ -valued process \mathbb{Z} with a.s. bounded paths (for instance cadlag) we have

$$\int_0^\cdot \langle \mathbb{Z}_s, d[X(\cdot), Y(\cdot)]_s \rangle = \int_0^\cdot \mathbb{Z}_s(\{a_i, a_j\}) d[X_{\cdot+a_i}, Y_{\cdot+a_j}]_s. \quad (5.4)$$

Proof.

- 1) By Remark 3.2 item 2 ⁸, χ is a closed subspace of $\mathcal{M}([-\tau, 0]^2)$. The claim follows again by Remark 3.2 item 2 in ⁸.
- 2) We denote here $\chi_1 = \mathcal{D}_{i,j}([-\tau, 0]^2)$; χ_1 and χ_2 are closed subspaces of $\mathcal{M}([-\tau, 0]^2)$. By Proposition 3.8, item 3) $X(\cdot)$ and $Y(\cdot)$ admit a χ_1 -covariation. Proposition 3.16 in ⁸ implies that $X(\cdot)$ and $Y(\cdot)$ admit a χ -covariation which

can be determined from the χ_1 -covariation and the χ_2 -covariation. More precisely, for μ in χ with decomposition $\mu_1 + \mu_2$, $\mu_1 \in \chi_1$ and $\mu_2 \in \chi_2$, with a slight abuse of notations, we have

$$\begin{aligned} [X(\cdot), Y(\cdot)](\mu) &= [X(\cdot), Y(\cdot)](\mu_1) + [X(\cdot), Y(\cdot)](\mu_2) = [X(\cdot), Y(\cdot)](\mu_1) \\ &= \mu_1(\{a_i, a_j\})[X_{\cdot+a_i}, Y_{\cdot+a_j}] = \mu(\{a_i, a_j\})[X_{\cdot+a_i}, Y_{\cdot+a_j}]. \end{aligned}$$

- 3) Since both sides of (5.4) are continuous processes, it is enough to show that they are equal a.s. for every fixed $t \in [0, T]$. This follows for almost all $\omega \in \Omega$ using Proposition 5.3 where $f = \mathbb{Z}(\omega)$ and $\tilde{g} = [X(\cdot), Y(\cdot)](\omega)$. We remark that here $\tilde{g}_1 = [X_{\cdot+a_i}, Y_{\cdot+a_j}](\omega)$ and $g_1 = [X_{\cdot+a_i}, Y_{\cdot+a_j}](\omega)$. \square

Remark 5.7. Proposition 5.6 will be used in the sequel especially in the case $a_i = a_j = 0$.

Remark 5.8. Under the same assumptions as Proposition 5.6, if \mathbb{Z} takes values in $\mathcal{D}_{i,j}$, then

$$\int_0^\cdot \langle \mathbb{Z}_s, d[X(\cdot), Y(\cdot)]_s \rangle = \int_0^\cdot \mathbb{Z}_s(\{a_i, a_j\}) d[X_{\cdot+a_i}, Y_{\cdot+a_j}]_s. \quad (5.5)$$

In fact, the left-hand side equals $\int_0^\cdot \mathbb{Z}_s(a_i, a_j) \langle \delta_{a_i, a_j}, d[X(\cdot), Y(\cdot)]_s \rangle$. That expression equals the right-hand side of (5.5) because of item 3) in Proposition 3.8 and Theorem 30, Chapter 1, par 2 of⁹.

5.2. On some generalized Fukushima decompositions

We are ready now to show some decomposition results.

Theorem 5.9. *Let X be a real continuous (\mathcal{F}_t) -Dirichlet process with decomposition $X = M + A$, where M is an (\mathcal{F}_t) -local martingale and A is a zero quadratic variation process with $A_0 = 0$. Let $F : C([-\tau, 0]) \rightarrow \mathbb{R}$ be a Fréchet differentiable function such that the range of DF is $\mathcal{D}_0([-\tau, 0]) \oplus L^2([-\tau, 0])$. Moreover we suppose that $DF : C([-\tau, 0]) \rightarrow \mathcal{D}_0([-\tau, 0]) \oplus L^2([-\tau, 0])$ is continuous. Then $F(X(\cdot))$ is an (\mathcal{F}_t) -Dirichlet process with local martingale part equal to*

$$\bar{M} = F(X_0(\cdot)) + \int_0^\cdot D^{\delta_0} F(X_s(\cdot)) dM_s, \quad (5.6)$$

where from Notation 2.1 we recall that $D^{\delta_0} F(\eta) = DF(\eta)(\{0\})$.

Remark 5.10. The Itô integral in (5.6) makes sense because $(D^{\delta_0} F(X_t(\cdot)))$ is (\mathcal{F}_t) -adapted.

Proof. We need to show that $[\bar{A}] = 0$ where $\bar{A} := F(X(\cdot)) - \bar{M}$. For simplicity of notations, in this proof we will denote $\alpha_0(\eta) = D^{\delta_0} F(\eta)$. By the linearity of the covariation of real processes, we have $[\bar{A}] = A_1 + A_2 - 2A_3$ where

$$A_1 = [F(X(\cdot))], \quad A_2 = \left[\int_0^\cdot \alpha_0(X_s(\cdot)) dM_s \right], \quad A_3 = \left[F(X(\cdot)), \int_0^\cdot \alpha_0(X_s(\cdot)) dM_s \right].$$

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Since X is a finite quadratic variation process, by Proposition 3.8 4), its window process $X(\cdot)$ admits a $\chi^0([-\tau, 0]^2)$ -quadratic variation. Moreover by Example 4.5 and Remark 4.10 the map $DF \otimes DF : C([-\tau, 0]) \times C([-\tau, 0]) \rightarrow \chi^0([-\tau, 0]^2)$ is a continuous map. Applying Theorem 4.6 and (5.4) of Proposition 5.6 we obtain that A_1 equals

$$\int_0^\cdot \langle DF(X_s(\cdot)) \otimes DF(X_s(\cdot)), d[\widetilde{X(\cdot)}]_s \rangle = \int_0^\cdot \alpha_0^2(X_s(\cdot)) d[X]_s = \int_0^\cdot \alpha_0^2(X_s(\cdot)) d[M]_s .$$

The term A_2 is the quadratic variation of a local martingale; by Remark 1.2 item 2) we get $A_2 = \int_0^\cdot \alpha_0^2(X_s(\cdot)) d[M]_s$. It remains to prove that $A_3 = \int_0^\cdot \alpha_0^2(X_s(\cdot)) d[M]_s$. We define $G : C([-\tau, 0]) \rightarrow \mathbb{R}$ by $G(\eta) = \eta(0)$. We observe that $\bar{M} = G(\bar{M}(\cdot))$ where $\bar{M}(\cdot)$ denotes as usual the window process associated to \bar{M} . G is Fréchet differentiable and $DG(\eta) = \delta_0$, therefore DG is continuous from $C([-\tau, 0])$ to $\mathcal{D}_0([-\tau, 0]) \oplus L^2([-\tau, 0])$. Moreover by Example 4.5 we know that $DF \otimes DG : C([-\tau, 0]) \times C([-\tau, 0]) \rightarrow \chi^0([-\tau, 0]^2)$ is continuous. Corollary 3.9 item 2) says that the $\chi^0([-\tau, 0]^2)$ -covariation between $X(\cdot)$ and $\bar{M}(\cdot)$ exists and it is given by $[X(\cdot), \bar{M}(\cdot)](\mu) = \mu(\{0, 0\})[X, \bar{M}]$. But we have $[X, \bar{M}] = [M, \bar{M}] + [A, \bar{M}] = [M, \bar{M}]$. By Remark 1.2 item 2) and the usual properties of stochastic calculus we have

$$[X, \bar{M}] = \left[M, \int_0^\cdot \alpha_0(X_s(\cdot)) dM_s \right] = \int_0^\cdot \alpha_0(X_s(\cdot)) d[M]_s . \quad (5.7)$$

Finally, applying again Theorem 4.6, relation (5.4) in Proposition 5.6 and (5.7) we obtain

$$\begin{aligned} A_3 &= [F(X(\cdot)), G(\bar{M}(\cdot))] = \int_0^\cdot \langle DF(X_s(\cdot)) \otimes DG(\bar{M}_s(\cdot)), d[\widetilde{X(\cdot), \bar{M}(\cdot)}]_s \rangle \\ &= \int_0^\cdot \alpha_0(X_s(\cdot)) d[X, \bar{M}]_s = \int_0^\cdot \alpha_0^2(X_s(\cdot)) d[M]_s . \end{aligned}$$

The result is now established. \square

Theorem 5.9 admits a slight generalization which involves the space \mathcal{D}_a as defined at equation (2.4), but the final process is no longer a Dirichlet process, only a weak Dirichlet one.

Theorem 5.11. *Let X be a real continuous (\mathcal{F}_t) -Dirichlet process with decomposition $X = M + A$, M being a local martingale and A a zero quadratic variation process with $A_0 = 0$. Let $F : C([-\tau, 0]) \rightarrow \mathbb{R}$ be a Fréchet differentiable function such that $DF : C([-\tau, 0]) \rightarrow \mathcal{D}_a([-\tau, 0]) \oplus L^2([-\tau, 0])$ is continuous. We have the following.*

- 1) $F(X(\cdot))$ is an (\mathcal{F}_t) -weak Dirichlet process with decomposition $F(X(\cdot)) = \bar{M} + \bar{A}$, where \bar{M} is the local martingale defined by $\bar{M} := F(X_0(\cdot)) + \int_0^\cdot D^{\delta_0} F(X_s(\cdot)) dM_s$ and \bar{A} is the (\mathcal{F}_t) -martingale orthogonal process, corresponding to Definition 1.8.

2) $F(X(\cdot))$ is a finite quadratic variation process and

$$[F(X(\cdot))] = \sum_{i=0, \dots, N} \int_0^t (D^{\delta_{a_i}} F(X_s(\cdot)))^2 d[M]_{s+a_i}. \quad (5.8)$$

3) The process \bar{A} is a finite quadratic variation process and

$$[\bar{A}]_t = \sum_{i=1, \dots, N} \int_0^t (D^{\delta_{a_i}} F(X_s(\cdot)))^2 d[M]_{s+a_i}. \quad (5.9)$$

4) In particular $\{F(X_t(\cdot)); t \in [0, -a_1]\}$ is a Dirichlet process with local martingale part \bar{M} .

Proof. In this proof $\alpha_i(\eta)$ will denote $D^{\delta_{a_i}} F(\eta) = DF(\eta)(\{a_i\})$ if $\eta \in C([-\tau, 0])$.

1) To show that $F(X(\cdot))$ is an (\mathcal{F}_t) -weak Dirichlet process, we need to show that $[F(X(\cdot)) - \int_0^\cdot \alpha_0(X_s(\cdot)) dM_s, N]$ is zero for every (\mathcal{F}_t) -continuous local martingale N . Again we define $G : C([-\tau, 0]) \rightarrow \mathbb{R}$ by $G(\eta) = \eta(0)$. It holds $N_t = G(N_t(\cdot))$. We remark that the function G is Fréchet differentiable with $DG : C([-\tau, 0]) \rightarrow \mathcal{D}_0([-\tau, 0])$ continuous and $DG(\eta) \equiv \delta_0$.

In view of applying Corollary 3.9, we set $\chi := \mathcal{D}_{0,0} \oplus \chi_2$ where $\chi_2 = \bigoplus_{i=1}^N \mathcal{D}_{i,0} \oplus (L^2([-\tau, 0]) \hat{\otimes}_h \mathcal{D}_0)$. In particular for every $\mu \in \chi_2$ we have $\mu(\{0, 0\}) = 0$. $X(\cdot)$ and $N(\cdot)$ admit a χ -covariation by Corollary 3.9 3) On the other hand $DF \otimes DG : C([-\tau, 0]) \times C([-\tau, 0]) \rightarrow \chi$ and it is a continuous map. By Theorem 4.6 we have $[F(X(\cdot)), N]_t = [F(X(\cdot)), G(N(\cdot))]_t = \int_0^t (DF(X_s(\cdot)) \otimes \delta_0, d[\widetilde{X(\cdot), N(\cdot)}]_s)$. By (5.4) in Proposition 5.6 it follows that

$$\begin{aligned} [F(X(\cdot)), N]_t &= \int_0^t (D^{\delta_0} F(s, X_s(\cdot)) \otimes \delta_0)(\{0, 0\}) d[X, N]_s \\ &= \int_0^t D^{\delta_0} F(s, X_s(\cdot)) d[M, N]_s. \end{aligned}$$

By Remark 1.2 item 2) and usual properties of stochastic calculus, it yields

$$\left[\int_0^\cdot \alpha_0(X_s(\cdot)) dM_s, N \right]_t = \int_0^t \alpha_0(X_s(\cdot)) d[M, N]_s$$

and the result follows.

2) By Example 4.5 we know that $DF \otimes DF : C([-\tau, 0]) \times C([-\tau, 0]) \rightarrow \chi^2([-\tau, 0]^2)$ and it is a continuous linear map. We decompose $DF(\eta) = \sum_{i=1}^N \alpha_i(\eta) \delta_{a_i} + g(\eta)$ where $g : C([-\tau, 0]) \rightarrow L^2([-\tau, 0])$ so that $DF(\eta) \otimes DF(\eta)$ equals

$$\sum_{i,j=0}^N \alpha_i(\eta) \alpha_j(\eta) \delta_{a_i} \otimes \delta_{a_j} + \sum_{i=0}^N \alpha_i(\eta) \delta_{a_i} \otimes g(\eta) + \sum_{j=0}^N \alpha_j(\eta) g(\eta) \otimes \delta_{a_j} + g(\eta) \otimes g(\eta).$$

Applying Theorem 4.6, relation (5.4) in Proposition 5.6 and obvious bilinearity arguments, it follows that $[F(X(\cdot))]_t$ equals

$$\int_0^t \langle DF(X_s(\cdot)) \otimes DF(X_s(\cdot)), d[\widetilde{X_s(\cdot)}] \rangle = \sum_{i,j=0}^N \int_0^t \langle \mathbb{Z}_s^{i,j}, d[\widetilde{X_s(\cdot)}] \rangle + \int_0^t \langle \mathbb{Z}_s, d[\widetilde{X_s(\cdot)}] \rangle \quad (5.11)$$

where $\mathbb{Z}_s^{i,j} = \alpha_i(X_s(\cdot))\alpha_j(X_s(\cdot)) \delta_{a_i} \otimes \delta_{a_j}$ and $\mathbb{Z}_s = DF(X_s(\cdot)) \otimes DF(X_s(\cdot)) - \sum_{i,j=0}^N \mathbb{Z}_s^{i,j}$. A wise application of Proposition 5.6 and Remark 5.8 show that (5.11) equals

$$\sum_{i,j=0,\dots,N} \int_0^t \alpha_i(X_s(\cdot))\alpha_j(X_s(\cdot))d[X_{\cdot+a_i}, X_{\cdot+a_j}]_s = \sum_{i=0,\dots,N} \int_0^t \alpha_i^2(X_s(\cdot))d[M]_{s+a_i}.$$

The last equality is a consequence of Proposition 1.6 and of the definition of weak Dirichlet process. Finally (5.8) is proved.

- 3) Through the bilinearity of the covariation of real processes we have $[\bar{A}] = [F(X(\cdot))] + [\bar{M}] - 2[F(X(\cdot)), \bar{M}]$. The first bracket is equal to (5.8) and the second term gives

$$\left[\int_0^\cdot \alpha_0(X_s(\cdot))dM_s \right] = \int_0^t \alpha_0^2(X_s(\cdot))d[M]_s.$$

Setting $N_t = \int_0^t \alpha_0(X_s(\cdot))dM_s$, (5.10) gives

$$\left[F(X(\cdot)), \int_0^\cdot \alpha_0(X_s(\cdot))dM_s \right] = \int_0^t \alpha_0^2(X_s(\cdot))d[M]_s$$

and (5.9) follows.

- 4) It is an easy consequence of (5.9) since $(\bar{A}_t)_{t \in [0, -a_1[}$ is a zero quadratic variation process. \square

Remark 5.12.

- 1) Theorem 5.11 gives a class of examples of (\mathcal{F}_t) -weak Dirichlet processes with finite quadratic variation which are not necessarily (\mathcal{F}_t) -Dirichlet processes.
- 2) An example of $F : C([-\tau, 0]) \rightarrow \mathbb{R}$ Fréchet differentiable such that $DF : C([-\tau, 0]) \rightarrow \mathcal{D}_a([-\tau, 0]) \oplus L^2([-\tau, 0])$ continuously is, for instance, $F(\eta) = f(\eta(a_0), \dots, \eta(a_N))$, with $f \in C^1(\mathbb{R}^N)$. We have $DF(\eta) = \sum_{i=0}^N \partial_i f(\eta(a_0), \dots, \eta(a_N)) \delta_{a_i}$.
- 3) Let $c \in [-\tau, 0[$ and W be a classical (\mathcal{F}_t) -Brownian motion, the process X defined as $X_t := W_{t+c}$ is an (\mathcal{F}_t) -weak Dirichlet process that is not (\mathcal{F}_t) -Dirichlet. This follows from Theorem 5.11, point 2) and 3) taking $F(\eta) = \eta(c)$. In particular point 3) implies that the quadratic variation of the martingale orthogonal process is $[\bar{A}]_t = (t+c)^+$. This result was also proved directly in Proposition 4.11 in ⁴.

We examine now the structure of the real process obtained applying a real-valued map of class C^1 to a window of weak Dirichlet process.

Theorem 5.13. *Let X be an (\mathcal{F}_t) -weak Dirichlet process with finite quadratic variation where M is the local martingale part. Let $F : [0, T] \times C([- \tau, 0]) \rightarrow \mathbb{R}$ continuous. We suppose moreover that $(t, \eta) \mapsto DF(t, \eta)$ exists with values in $\mathcal{D}_a([- \tau, 0]) \oplus L^2([- \tau, 0])$ and $DF : [0, T] \times C([- \tau, 0]) \rightarrow \mathcal{D}_a([- \tau, 0]) \oplus L^2([- \tau, 0])$ is continuous. Then $(F(t, X_t(\cdot)))_{t \in [0, T]}$ is an (\mathcal{F}_t) -weak Dirichlet process with martingale part*

$$\bar{M}_t^F := F(0, X_0(\cdot)) + \int_0^t D^{\delta_0} F(s, X_s(\cdot)) dM_s . \quad (5.12)$$

Proof. In this proof we will denote the real processes \bar{M}^F simply by \bar{M} and χ will stand for the following Chi-subspace $\chi := (\mathcal{D}_a([- \tau, 0]) \oplus L^2([- \tau, 0])) \hat{\otimes}_h \mathcal{D}_0([- \tau, 0])$. We need to show that for any (\mathcal{F}_t) -continuous local martingale N

$$[F(\cdot, X(\cdot)) - \bar{M}, N] \equiv 0.$$

Since the covariation of semimartingales coincides with the classical covariation, see Remark 1.2 item 2), it follows

$$[\bar{M}, N]_t = \int_0^t D^{\delta_0} F(s, X_s(\cdot)) d[M, N]_s .$$

It remains to check that, for every $t \in [0, T]$,

$$[F(\cdot, X(\cdot)), N]_t = \int_0^t D^{\delta_0} F(s, X_s(\cdot)) d[M, N]_s .$$

For this, for fixed $t \in [0, T]$, we will evaluate the limit in probability of

$$\int_0^t \left(F(s + \epsilon, X_{s+\epsilon}(\cdot)) - F(s, X_s(\cdot)) \right) \frac{N_{s+\epsilon} - N_s}{\epsilon} ds \quad (5.13)$$

if it exists. (5.13) can be written as the sum of the two terms

$$\begin{aligned} I_1(t, \epsilon) &= \int_0^t \left(F(s + \epsilon, X_{s+\epsilon}(\cdot)) - F(s + \epsilon, X_s(\cdot)) \right) \frac{N_{s+\epsilon} - N_s}{\epsilon} ds, \\ I_2(t, \epsilon) &= \int_0^t \left(F(s + \epsilon, X_s(\cdot)) - F(s, X_s(\cdot)) \right) \frac{N_{s+\epsilon} - N_s}{\epsilon} ds . \end{aligned}$$

First we prove that $I_1(t, \epsilon)$ converges to $\int_0^t D^{\delta_0} F(s, X_s(\cdot)) d[M, N]_s$.

If $G : C([- \tau, 0]) \rightarrow \mathbb{R}$ is the function $G(\eta) = \eta(0)$, then G is of class C^1 and $DG(\eta) = \delta_0$ for all $\eta \in C([- \tau, 0])$ so that $DG : C([- \tau, 0]) \rightarrow \mathcal{D}_0([- \tau, 0])$ is continuous. In particular we have the equality $\eta(0) = G(\eta(\cdot)) = \langle \delta_0, \eta \rangle$. We express

$$\begin{aligned} I_1(t, \epsilon) &= \int_0^t \langle DF(s + \epsilon, X_s(\cdot)), (X_{s+\epsilon}(\cdot) - X_s(\cdot)) \rangle \frac{N_{s+\epsilon} - N_s}{\epsilon} ds + R_1(t, \epsilon) \\ &= \int_0^t \langle DF(s + \epsilon, X_s(\cdot)), (X_{s+\epsilon}(\cdot) - X_s(\cdot)) \rangle \frac{\langle \delta_0, N_{s+\epsilon}(\cdot) - N_s(\cdot) \rangle}{\epsilon} ds + R_1(t, \epsilon), \end{aligned} \quad (5.14)$$

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and $R_1(t, \epsilon)$ equals

$$\begin{aligned}
 &= \int_0^t \left[\int_0^1 \langle DF(s + \epsilon, (1 - \alpha)X_s(\cdot) + \alpha X_{s+\epsilon}(\cdot)) - DF(s + \epsilon, X_s(\cdot)), \right. \\
 &\quad \left. (X_{s+\epsilon}(\cdot) - X_s(\cdot)) \rangle d\alpha \right] \cdot \frac{\langle \delta_0, N_{s+\epsilon}(\cdot) - N_s(\cdot) \rangle}{\epsilon} ds = \\
 &= \int_0^t \int_0^1 \langle DF(s + \epsilon, (1 - \alpha)X_s(\cdot) + \alpha X_{s+\epsilon}(\cdot)) \otimes \delta_0 - DF(s + \epsilon, X_s(\cdot)) \otimes \delta_0, \\
 &\quad \frac{(X_{s+\epsilon}(\cdot) - X_s(\cdot)) \otimes (N_{s+\epsilon}(\cdot) - N_s(\cdot))}{\epsilon} \rangle d\alpha ds.
 \end{aligned} \tag{5.15}$$

We will show that $R_1(\cdot, \epsilon)$ converges ucp to zero, when $\epsilon \rightarrow 0$. Since χ is a Hilbert space, making the proper Riesz identification for $t \in [0, T]$, $\eta_1, \eta_2 \in C([-\tau, 0])$, the map $DF(t, \eta_1) \otimes DG(\eta_2)$ coincides with the tensor product $DF(t, \eta_1) \otimes \delta_0$, see Proposition 4.4. As in Example 4.5 the map $DF \otimes \delta_0 : [0, T] \times C([-\tau, 0])$ takes values in the separable space $\chi^2([-\tau, 0]^2)$ and it is a continuous map. In particular its image is contained in χ which is a Hilbert subspace of $\chi^2([-\tau, 0]^2)$.

We denote by $\mathcal{U} = \mathcal{U}(\omega)$ the closed convex hull of the compact subset \mathcal{V} of $C([-\tau, 0])$ defined, for every ω , by

$$\mathcal{V} = \mathcal{V}(\omega) := \{X_t(\omega); t \in [0, T]\}. \tag{5.16}$$

Being $\mathcal{U}(\omega) = \overline{\text{conv}(\mathcal{V}(\omega))}$ compact, the function $DF(\cdot, \cdot) \otimes \delta_0$ on $[0, T] \times \mathcal{U}$ is uniformly continuous and we designate by $\varpi_{DF(\cdot, \cdot) \otimes \delta_0}^{[0, T] \times \mathcal{U}}$ the continuity modulus of the map $DF(\cdot, \cdot) \otimes \delta_0$ restricted to $[0, T] \times \mathcal{U}$ and by ϖ_X the continuity modulus of the continuous process X . $\varpi_{DF(\cdot, \cdot) \otimes \delta_0}^{[0, T] \times \mathcal{U}}$ is, as usual, a positive, increasing function on \mathbb{R}^+ , converging to zero when the argument converges to zero. So we have

$$\sup_{t \in [0, T]} |R_1(t, \epsilon)| \leq \int_0^T \varpi_{DF(\cdot, \cdot) \otimes \delta_0}^{[0, T] \times \mathcal{U}}(\varpi_X(\epsilon)) \left\| \frac{(X_{s+\epsilon}(\cdot) - X_s(\cdot)) \otimes (N_{s+\epsilon}(\cdot) - N_s(\cdot))}{\epsilon} \right\|_{\chi^*} ds. \tag{5.17}$$

By Corollary 3.9, item 3), $X(\cdot)$ and $N(\cdot)$ admit a χ -covariation. In particular using condition **H1** and (5.17) the claim $R_1(\cdot, \epsilon) \xrightarrow[\epsilon \rightarrow 0]{ucp} 0$ follows.

On the other hand, the first addend in (5.14) can be rewritten as

$$\int_0^t \langle DF(s, X_s(\cdot)) \otimes \delta_0, \frac{(X_{s+\epsilon}(\cdot) - X_s(\cdot)) \otimes (N_{s+\epsilon}(\cdot) - N_s(\cdot))}{\epsilon} \rangle ds + R_2(t, \epsilon) \tag{5.18}$$

where $R_2(\cdot, \epsilon) \xrightarrow[\epsilon \rightarrow 0]{ucp} 0$ arguing similarly as for $R_1(t, \epsilon)$.

In view of the application of Proposition 3.7, since $DF \otimes \delta_0 : [0, T] \times C([-\tau, 0]) \rightarrow \chi$ is continuous, we observe that the range of the process $H_s = DF(s, X_s(\cdot)) \otimes \delta_0$ is obviously included in the separable space χ which is a closed subspace of $\chi^2([-\tau, 0]^2)$. Using the bilinearity and Proposition 3.7, the integral in (5.18) converges then in

probability to

$$\int_0^t \langle DF(s, X_s(\cdot)) \otimes \delta_0, d[\widetilde{X(\cdot)}, \widetilde{N(\cdot)}]_s \rangle. \quad (5.19)$$

As in Theorem 5.9, item 1), we decompose χ in the following direct sum $\mathcal{D}_{0,0} \oplus \chi_2$ where we recall that $\chi_2 = \oplus_{i=1}^N \mathcal{D}_{i,0} \oplus (L^2([-\tau, 0]) \hat{\otimes}_h \mathcal{D}_0)$. By Corollary 3.9 2), $X(\cdot)$ and $N(\cdot)$ admit a zero χ_2 -covariation. By (5.4) in Proposition 5.6 it follows that (5.19) equals

$$\int_0^t (D^{\delta_0} F(s, X_s(\cdot)) \otimes \delta_0)(\{0, 0\}) d[X, N]_s = \int_0^t D^{\delta_0} F(s, X_s(\cdot)) d[M, N]_s.$$

We will show now that $I_2(\cdot, \epsilon) \xrightarrow[\epsilon \rightarrow 0]{ucp} 0$.

Stochastic Fubini theorem implies

$$I_2(t, \epsilon) = \int_0^t \xi(\epsilon, r) dN_r$$

where

$$\xi(\epsilon, r) = \frac{1}{\epsilon} \int_{0 \vee (r-\epsilon)}^r [F(s+\epsilon, X_s(\cdot)) - F(s, X_s(\cdot))] ds.$$

Proposition 2.26, chapter 3 of ¹⁹ says that $I_2(\cdot, \epsilon) \xrightarrow[\epsilon \rightarrow 0]{ucp} 0$ if

$$\int_0^T \xi^2(\epsilon, r) d[N]_r \xrightarrow[\epsilon \rightarrow 0]{} 0 \quad (5.20)$$

in probability. We fix $\omega \in \Omega$ and we show that the convergence in (5.20) holds even a.s. We denote by $\varpi_F^{[0,T] \times \mathcal{U}}$ the continuity modulus of the application F restricted to the compact set $[0, T] \times \mathcal{U}$. For every $r \in [0, T]$ we have

$$|\xi(\epsilon, r)| \leq \sup_{r \in [0, T]} |F(r+\epsilon, X_r(\cdot)) - F(r, X_r(\cdot))| \leq \varpi_F^{[0,T] \times \mathcal{U}}(\epsilon)$$

which converges to zero for ϵ going to zero since the function F on $[0, T] \times \mathcal{U}$ is uniformly continuous on the compact set and $\varpi_F^{[0,T] \times \mathcal{U}}$ is, as usual, a positive, increasing function on \mathbb{R}^+ converging to zero when the argument converges to zero. By Lebesgue's dominated convergence theorem we finally obtain (5.20). \square

5.3. Fukushima decomposition and support predictability condition

If DF does not necessarily live in some $\mathcal{D}_a([-T, 0]) \oplus L^2([-T, 0])$ space, it is still possible to express a variant of Theorem 5.13. The price to pay is a new condition required for DF which will be called *support predictability condition*. It is described below.

Definition 5.14. Let $0 \leq a < b \leq T$. A function $F : [a, b] \times C([-\tau, 0]) \rightarrow \mathbb{R}$ such that $F(t, \cdot)$ is differentiable for any $t \in [a, b]$ is said to fulfill the **support**

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predictability condition if the following holds. For every compact K of $C([-\tau, 0])$, we have

$$\int_a^b \left[\sup_{\eta \in K} \frac{1}{\epsilon} \int_{(-\epsilon) \vee (-\tau)}^0 |D_{dr}^\perp F|(t, \eta) \right] dt = O(\epsilon), \quad (5.21)$$

where we recall that $D_{dr}^\perp F = D_{dr} F - DF(\{0\})\delta_0(dr)$.

Remark 5.15. Suppose that $F(t, \cdot)$ is differentiable for any $t \in [a, b]$.

- 1) Suppose the existence of $\rho > 0$ such that $D^\perp F(t, \eta)$ has support in $[-\tau, -\rho]$ for any $t \in [a, b]$, $\eta \in C([-\tau, 0])$. Then F fulfills the support predictability condition; in fact quantity (5.21) vanishes for ϵ small.
- 2) Suppose that $D^\perp F(t, \eta)$ is absolutely continuous for every $t \in [a, b]$. We denote $(D_r^\perp F(t, \eta), r \in [-\tau, 0])$, the corresponding density. If for any compact K of $C([-\tau, 0])$ there is $\rho_1 > 0$ such that $t \mapsto \sup_{r \in [-\rho_1, 0], \eta \in K} |D_r^\perp F(t, \eta)|$ belongs to $L^1([a, b])$, then F fulfills the support predictability condition. This is for instance verified if $(r, t, \eta) \mapsto D_r^\perp F(t, \eta)$ is continuous.
- 3) Let $D^\perp F = G_0 + G_1$ and suppose that G_1 verifies the same assumption as $D^\perp F$ in item 2). Then support predictability condition is verified whenever (5.21) holds with $D^\perp F$ replaced by G_0 .

As announced a variant of Theorem 5.13 is given below.

Theorem 5.16. *Let $0 \leq a < b \leq T$ and X be an (\mathcal{F}_t) -weak Dirichlet process with finite quadratic variation and decomposition $X = M + A$, M local martingale. Let $F : [a, b] \times C([-\tau, 0]) \rightarrow \mathbb{R}$ continuous such that*

- i) $F(t, \cdot)$ is differentiable for every $t \in [a, b]$,*
- ii) $(t, \eta) \mapsto D^\perp F(t, \eta)$ is bounded on each compact of $[a, b] \times C([-\tau, 0])$,*
- iii) F fulfills the support predictability condition,*
- iv) $(t, \eta) \mapsto D^{\delta_0} F(t, \eta)$ is continuous on $[a, b] \times C([-\tau, 0])$ and it admits a continuous extension on $[a, b] \times C([-\tau, 0])$.*

Then $F(\cdot, X(\cdot))$ is an (\mathcal{F}_t) -weak Dirichlet process with martingale part

$$\tilde{M}_t^F = F(a, X_a(\cdot)) + \int_a^t D^{\delta_0} F(s, X_s(\cdot)) dM_s, \quad t \in [a, b].$$

Proof. Without restriction of generality we will suppose $a = 0$ and $b = T$. The proof follows from a modification of the one of Theorem 5.13. (5.13) was expressed as $I_1(t, \epsilon) + I_2(t, \epsilon)$. $I_1(t, \epsilon)$ is the sum of $I_{11}(t, \epsilon)$ and $I_{12}(t, \epsilon)$ where

$$I_{11}(t, \epsilon) = \int_0^t D^{\delta_0} F(s + \epsilon, X_s(\cdot)) \frac{(X_{s+\epsilon} - X_s)(N_{s+\epsilon} - N_s)}{\epsilon} ds,$$

Covariation and extended Fukushima decompositions for Banach space valued processes.

$$I_{12}(t, \epsilon) = \int_0^t \int_0^1 [D^{\delta_0} F(s + \epsilon, (1 - \alpha)X_s(\cdot) + \alpha X_{s+\epsilon}(\cdot)) - D^{\delta_0} F(s + \epsilon, X_s(\cdot))] d\alpha \frac{(X_{s+\epsilon} - X_s)(N_{s+\epsilon} - N_s)}{\epsilon} ds,$$

$$I_{13}(t, \epsilon) = \int_0^t \int_0^1 \mathcal{M}_{([- \tau, 0])} \langle D^\perp F(s + \epsilon, (1 - \alpha)X_s(\cdot) + \alpha X_{s+\epsilon}(\cdot)), (X_{s+\epsilon}(\cdot) - X_s(\cdot)) \rangle_{C([- \tau, 0])} d\alpha \frac{(N_{s+\epsilon} - N_s)}{\epsilon} ds.$$

We have $I_{11}(t, \epsilon) = J_{11}(t, \epsilon) + R_{11}(t, \epsilon)$, where

$$J_{11}(t, \epsilon) = \int_0^t D^{\delta_0} F(s, X_s(\cdot)) \frac{(X_{s+\epsilon} - X_s)(N_{s+\epsilon} - N_s)}{\epsilon} ds$$

and $\sup_{t \leq T} |R_{11}(t, \epsilon)| \xrightarrow{\epsilon \rightarrow 0} 0$ in probability because $D^{\delta_0} F$ is continuous by item iv) and therefore uniformly continuous on each compact. In fact (X, N) has all its mutual covariations, so by Proposition 1.4 and the fact that $[X, N] = [M, N]$, clearly,

$$J_{11}(t, \epsilon) \xrightarrow[\epsilon \rightarrow 0]{ucp} \int_0^t D^{\delta_0} F(s, X_s(\cdot)) d[M, N]_s.$$

$I_{12}(t, \epsilon)$ behaves similarly to $R_1(t, \epsilon)$ in (5.15), so it converges ucp to zero. The term $I_{13}(t, \epsilon)$ can be rewritten as

$$I_{13}(t, \epsilon) = \int_0^t \int_0^1 \int_{-\tau}^0 D_{dr}^\perp F(s + \epsilon, (1 - \alpha)X_s(\cdot) + \alpha X_{s+\epsilon}(\cdot)) (X_{s+r+\epsilon} - X_{s+r}) d\alpha \frac{N_{s+\epsilon} - N_s}{\epsilon} ds$$

and it decomposes into $J_{13}(t, \epsilon) + R_{13}(t, \epsilon)$ where

$$J_{13}(t, \epsilon) = \int_0^t Z_s(\epsilon) \frac{N_{s+\epsilon} - N_s}{\epsilon} ds$$

with

$$Z_s(\epsilon) = \int_0^1 \int_{-\tau}^{-\epsilon} D_{dr}^\perp F(s + \epsilon, (1 - \alpha)X_s(\cdot) + \alpha X_{s+\epsilon}(\cdot)) (X_{s+r+\epsilon} - X_{s+r}) d\alpha$$

and

$$\begin{aligned} R_{13}(t, \epsilon) &= \int_0^t \int_0^1 \int_{-\epsilon}^0 D_{dr}^\perp F(s + \epsilon, (1 - \alpha)X_s(\cdot) + \alpha X_{s+\epsilon}(\cdot)) (X_{s+r+\epsilon} - X_{s+r}) d\alpha \frac{N_{s+\epsilon} - N_s}{\epsilon} ds \\ &= \int_\epsilon^{t+\epsilon} \int_0^1 \int_{-\epsilon}^0 D_{dr}^\perp F(s, (1 - \alpha)X_{s-\epsilon}(\cdot) + \alpha X_s(\cdot)) (X_{s+r} - X_{s+r-\epsilon}) d\alpha \frac{N_s - N_{s-\epsilon}}{\epsilon} ds. \end{aligned}$$

Again stochastic Fubini theorem allows to get

$$J_{13}(t, \epsilon) = \int_0^t \xi(u, \epsilon) dN_u, \quad \text{with} \quad \xi(u, \epsilon) = \frac{1}{\epsilon} \int_{(u-\epsilon)^+}^u Z_s(\epsilon) ds.$$

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Proposition 2.26, chapter 3 of ¹⁹ states again that $J_{13}(\cdot, \epsilon) \xrightarrow[\epsilon \rightarrow 0]{ucp} 0$ if

$$\int_0^T \xi^2(u, \epsilon) d[N]_u \xrightarrow[\epsilon \rightarrow 0]{} 0 \quad (5.22)$$

in probability. We have

$$\begin{aligned} |Z_s(\epsilon)| &\leq \int_{(u-\epsilon)^+}^u \frac{1}{\epsilon} \int_0^1 \int_{-\tau}^{-\epsilon} \|D_{dr}^\perp F(s + \epsilon, (1 - \alpha)X_s(\cdot) + \alpha X_{s+\epsilon}(\cdot))\| |X_{s+r+\epsilon} - X_{s+r}| d\alpha ds \\ &\leq \varpi_X(\epsilon) \int_0^1 \|D_{dr}^\perp F(s + \epsilon, (1 - \alpha)X_s(\cdot) + \alpha X_{s+\epsilon}(\cdot))\|_{Var([- \tau, 0])} d\alpha \\ &\leq \varpi_X(\epsilon) \sup_{(t, \eta) \in [0, T] \times \mathcal{U}} \|D_{dr}^\perp F\|_{Var([- \tau, 0])}(t, \eta) \end{aligned}$$

where $\mathcal{U} = \mathcal{U}(\omega) = \overline{\text{conv}(\mathcal{V})(\omega)}$ where $\mathcal{V}(\omega)$ was defined in (5.16). Previous expression is bounded because of item ii) in the assumptions and since \mathcal{U} is a compact subset of the infinite dimensional space $C([- \tau, 0])$. So $\xi^2(u, \epsilon) \leq \varpi_X^2(\epsilon) \sup_{(t, \eta) \in [0, T] \times \mathcal{U}} \|D_{dr}^\perp F\|_{Var([- \tau, 0])}^2$ since

$$|\xi(u, \epsilon)| = \left| \frac{1}{\epsilon} \int_{(u-\epsilon)^+}^u Z_s(\epsilon) ds \right| \leq \varpi_X(\epsilon) \sup_{(t, \eta) \in [0, T] \times \mathcal{U}} \|D_{dr}^\perp F\|_{Var([- \tau, 0])}(t, \eta).$$

Finally the left-hand side of (5.22) is bounded by

$$\varpi_X^2(\epsilon) \sup_{(t, \eta) \in [0, T] \times \mathcal{U}} \|D_{dr}^\perp F\|_{Var([- \tau, 0])}^2 [N]_T$$

which converges to zero a.s. for $\epsilon \rightarrow 0$.

It remains to control $R_{13}(t, \epsilon)$. This term is bounded by

$$\varpi_N(\epsilon) \varpi_X(\epsilon) \int_0^{T+\epsilon} \frac{1}{\epsilon} \left[\sup_{\eta \in \mathcal{U}(\omega)} \int_{-\epsilon}^0 |D_{dr} F|(t, \eta) \right] dt$$

The result follows since F fulfills the support predictability condition. \square

We present now a slight generalization of Theorem 5.16.

Theorem 5.17. *Let X be an (\mathcal{F}_t) -weak Dirichlet process with finite quadratic variation and decomposition $X = M + A$, M local martingale. Let $F : [0, T] \times C([- \tau, 0]) \rightarrow \mathbb{R}$ continuous, fulfilling assumptions i), ii) and iii) of Theorem 5.16 for $a = 0$ and $b = T$. Suppose the existence of $0 = a_0 < a_1 < \dots < a_N = T$ such that $(t, \eta) \mapsto D^{\delta_0} F(t, \eta)$ is continuous on $]a_i, a_{i+1}] \times C([- \tau, 0])$ admitting a continuous extension on $[a_i, a_{i+1}] \times C([- \tau, 0])$, for $0 \leq i \leq (N - 1)$. Then $(F(t, X_t(\cdot)))_{t \in [0, T]}$ is an (\mathcal{F}_t) -weak Dirichlet process with local martingale part*

$$\tilde{M}_t^F = F(0, X_0(\cdot)) + \int_0^t D^{\delta_0} F(s, X_s(\cdot)) dM_s.$$

Proof. Let N be an (\mathcal{F}_t) -local martingale. Since

$$\left[\tilde{M}^F, N \right]_t = \int_0^t D^{\delta_0} F(s, X_s(\cdot)) d[M, N]_s, \quad t \in [0, T],$$

it will be enough to show that

$$[(F(t, X_t(\cdot))), N]_t = \int_0^t D^{\delta_0} F(s, X_s(\cdot)) d[M, N]_s, \quad t \in [0, T]. \quad (5.23)$$

We observe that $F|_{[a_i, a_{i+1}] \times C([- \tau, 0])}$ verifies the assumptions of Theorem 5.16 with $a = a_i$, $b = a_{i+1}$. Consequently for $t \in]a_i, a_{i+1}]$, $i \in \{0, \dots, (N-1)\}$, $(F(t, X_t(\cdot)))_{t \in [a_i, a_{i+1}]}$ is an (\mathcal{F}_t) -weak Dirichlet process with local martingale part

$$\tilde{M}_t^i = F(a_i, X_{a_i}(\cdot)) + \int_{a_i}^t D^{\delta_0} F(s, X_s(\cdot)) dM_s, \quad t \in [a_i, a_{i+1}[.$$

(5.23) follows by summation. \square

We discuss now some consequences related to the martingale representation.

5.4. About some martingale representations

Suppose that X is an (\mathcal{F}_t) -weak Dirichlet process with finite quadratic variation with decomposition $X = M + A$, M local martingale. Let $h \in L^1(\Omega)$. We are interested in sufficient conditions so that $h = h_0 + \int_0^T \xi_s dM_s$ where (ξ_s) is an explicit predictable process, $h_0 \in \mathbb{R}$. The two results below are a consequence respectively of Theorems 5.13 and 5.17. They settle the basis for such a representation.

$\mathcal{D}_a \oplus L^2$ will denote here $\mathcal{D}_a([- \tau, 0]) \oplus L^2([- \tau, 0])$.

Proposition 5.18. *Let $F : [0, T] \times C([- \tau, 0]) \rightarrow \mathbb{R}$ continuous such that $(s, \eta) \mapsto DF(s, \eta)$ exists with values in $\mathcal{D}_a \oplus L^2$ and $DF : [0, T] \times C([- \tau, 0]) \rightarrow \mathcal{D}_a \oplus L^2$ is continuous. If moreover*

$$\mathbb{E}[h | \mathcal{F}_t] = F(t, X_t(\cdot)) \quad a.s. \quad \forall t \in [0, T[\quad (5.24)$$

then

$$h = F(0, X_0(\cdot)) + \int_0^T D^{\delta_0} F(s, X_s(\cdot)) dM_s.$$

Remark 5.19. We observe that $F(0, X_0(\cdot)) = \mathbb{E}[h | \mathcal{F}_0]$.

Proof. Since F verifies the assumptions of Theorem 5.13, then $F(\cdot, X(\cdot))$ is an (\mathcal{F}_t) -weak Dirichlet process with martingale part given by

$$M_t^F = F(0, X_0(\cdot)) + \int_0^t D^{\delta_0} F(s, X_s(\cdot)) dM_s,$$

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according to (5.12). By (5.24), $F(\cdot, X(\cdot))$ is obviously an (\mathcal{F}_t) -martingale being a conditional expectation with respect to the filtration (\mathcal{F}_t) . By the uniqueness of the decomposition of (\mathcal{F}_t) -weak Dirichlet processes, it follows

$$F(t, X_t(\cdot)) = F(0, X_0(\cdot)) + \int_0^t D^{\delta_0} F(s, X_s(\cdot)) dM_s .$$

In particular the (\mathcal{F}_t) -martingale orthogonal process is zero. Since $h = F(T, X_T(\cdot))$ and $F(0, X_0(\cdot)) = \mathbb{E}[h|\mathcal{F}_0]$ the result follows. \square

Corollary 5.20. *Let X be an (\mathcal{F}_t) -weak Dirichlet process with finite quadratic variation with decomposition $X = M + A$, M local martingale.*

Let $0 < a_1 < \dots < a_N = T$. Let $h \in L^1(\Omega)$. Let $F : [0, T] \times C([-\tau, 0]) \rightarrow \mathbb{R}$ verifying the following properties.

- a) $F(s, \cdot)$ is differentiable $\forall s \in [0, T]$.
- b) F fulfills the support predictability condition.
- c) $(s, \eta) \mapsto D^{\delta_0} F(s, \eta)$ is continuous on $]a_i, a_{i+1}[$, $0 \leq i \leq N - 1$ and it admits a continuous extension on $[a_i, a_{i+1}]$.
- d) $(s, \eta) \mapsto D^\perp F(s, \eta)$ is bounded for each compact, with respect to the total variation norm.
- e) $F(s, X_s(\cdot)) = \mathbb{E}[h|\mathcal{F}_s]$.

Then

$$h = F(0, X_0(\cdot)) + \int_0^T D^{\delta_0} F(s, X_s(\cdot)) dM_s . \quad (5.25)$$

Proof. By Theorem 5.17 $(F((t, X_t(\cdot)))_{t \in [0, T]}$ is an (\mathcal{F}_t) -weak Dirichlet process. On the other hand by construction $(F((t, X_t(\cdot)))_{t \in [0, T]}$ is an (\mathcal{F}_t) -martingale. The result follows again by uniqueness of the decomposition of an (\mathcal{F}_t) -weak Dirichlet process. \square

Remark 5.21. Suppose $h \in L^2(\Omega)$. Indeed previous corollary can be stated requiring the same assumptions a), b), c), d), e) but on F restricted to $[0, b] \times C([-\tau, 0])$ for every $b \in]a_{N-1}, T[$. In fact, the square integrable martingale $\mathbb{E}[h|\mathcal{F}_t]$ can be decomposed, according to Kunita-Watanabe's theorem, into the sum

$$F(0, X_0(\cdot)) + \int_0^t \xi_s dM_s + N_t , \quad t \in [0, T],$$

where N is an (\mathcal{F}_t) -local martingale strongly orthogonal to M , i.e. $[N, M] = 0$; moreover we know that $\int_0^T \xi_s^2 d[M]_s < +\infty$ a.s. Since $(F(t, X_t(\cdot)))_{t \in [0, b]}$ is also an (\mathcal{F}_t) -weak Dirichlet process by Theorem 5.17, the uniqueness of the decomposition implies that

$$\int_0^t D^{\delta_0} F(s, X_s(\cdot)) dM_s = \int_0^t \xi_s dM_s + N_t , \quad t \in [0, b]$$

and so $N_s = 0, s \in [0, b]$ and $\xi_s \equiv D^{\delta_0} F(s, X_s(\cdot))$. This implies (5.25).

6. Consequences on quasi explicit representations of path dependent random variables

6.1. General considerations

This section has illustrative aspects. We test our method on the representation of a random variable which depends on the path of a diffusion process. This is a toy model for future investigations in verification theorems related to control theory on functional dependent equations. Let $(W_t)_{t \in [0, T]}$ be a standard Brownian motion with respect to some usual filtration (\mathcal{F}_t) . In this section η (resp. γ) will denote an element in $C([-T, 0])$ (resp. $C([0, T])$), so that τ will be equal to T . Let $(X_t)_{t \in [0, T]}$ be a real continuous process; with X we will also denote the whole trajectory of X in $C([0, T])$. a will stand for a grid $0 = a_0 < a_1 < \dots < a_N = T$, $N \in \mathbb{N}$, not anymore in $[-T, 0]$ as before. \mathcal{D}_a will symbolize a similar type of space as in Section 2.

We aim at implementing Corollary 5.20 when X is solution of a SDE of the type

$$X_t = X_0 + \int_0^t \sigma(r, X_r) dW_r + \int_0^t b(r, X_r) dr, \quad (6.1)$$

where $\sigma, b : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ are continuous, $\partial_x \sigma$ and $\partial_x b$ exist and they are bounded. We remark that σ is possibly degenerate. With $\partial_x \sigma$ (resp. $\partial_x b$) we will denote the derivative of σ (resp. b) with respect to the second argument. In the present context X is an (\mathcal{F}_t) -semimartingale, therefore an (\mathcal{F}_t) -weak Dirichlet, with decomposition $X = M + A$, M the local martingale given by $M_t = X_0 + \int_0^t \sigma(r, X_r) dW_r$.

The idea is to evaluate

$$h := \Phi(X)$$

with $\Phi : C([0, T]) \rightarrow \mathbb{R}$ is continuous such that $h \in L^1(\Omega)$.

It is well-known that the following flow property holds:

$$X_t = Y_t^{s, X_s(\cdot)}, \quad (6.2)$$

where, for $(s, \eta) \in [0, T] \times C([- \tau, 0])$, $Y = Y^{s, \eta}$ is the process with range in $C([0, T])$, defined as the solution of

$$Y_t = \begin{cases} \eta(t - s) & t \in [0, s] \\ \eta(0) + \int_s^t \sigma(r, Y_r) dW_r + \int_s^t b(r, Y_r) dr & t \in [s, T]. \end{cases} \quad (6.3)$$

For fixed ω a.s., $Y^{s, \eta}$ also symbolizes a function in $C([0, T])$. We start with some basic estimates.

Proposition 6.1. *For every $q \geq 1$, there is a constant $C(q)$ such that*

$$\sup_{s \in [0, T]} \mathbb{E} [\|Y^{s, \eta}\|_\infty^q] \leq C(q) \|\eta\|_\infty^q$$

Proof. Since σ, b have linear growth, by Burkholder-Davis-Gundy inequality, for every $q \geq 1$, and by Gronwall lemma, there is a constant $C(q)$ such that

$$\sup_{0 \leq s \leq t \leq T} \mathbb{E} [|Y_t^{s, \eta}|^q] \leq C(q) |\eta(0)|^q.$$

This implies the result. \square

The process $Y = Y^{s,\eta}$ given in (6.3) also solves

$$Y_t = \eta(0) + \int_s^t \sigma(r, Y_r^{s,\eta}) d\bar{W}_r + \int_s^t b(r, Y_r^{s,\eta}) dr$$

where $\bar{W} = W_{s+} - W_s$ is independent of $\mathcal{F}_t = \sigma(Y_r, r \leq t)$. Taking into account (6.2), for $s \leq t$, we get

$$M_s := \mathbb{E}[\Phi(X)|\mathcal{F}_s] = u(s, X_s(\cdot))$$

where $u : [0, T] \times C([-T, 0]) \rightarrow \mathbb{R}$

$$u(s, \eta) = \mathbb{E}[\Phi(Y^{s,\eta})] \quad (6.4)$$

and $Y^{s,\eta}$ is given by (6.3)

Remark 6.2.

- 1) We remark that $Y^{s,\eta} : \Omega \rightarrow C([0, T])$.
- 2) It is possible to show that $u \in C^0([0, T] \times C([-T, 0]))$.

We would like to examine some situations where $F := u$ fulfills the conditions for the application of Corollary 5.20. In particular Theorem 6.4 below allows to verify the condition a) of that corollary. Checking the other assumptions will be the object of Proposition 6.11. In this section the natural candidate for the function u is given in (6.4). We remark that, whenever u is Fréchet differentiable with respect to the second argument, we denote by D that derivative and $D^{\delta_0}u(t, \eta) = Du(t, \eta)\{0\}$ and $D_{dr}^\perp u(t, \eta)$ is singular with respect to δ_0 . We introduce now useful notation.

Notation 6.3. If X is a continuous real (\mathcal{F}_t) -semimartingale, the symbol $\mathcal{E}\{X\}$ denotes the Doléans exponential of X , in particular $\mathcal{E}_t\{X\} = \exp\{X_t - X_0 - \frac{1}{2}[X]_t\}$.

Theorem 6.4. *Let X be a diffusion process of type (6.1) and $\Phi : C([0, T]) \rightarrow \mathbb{R}$ of class C^1 such that $D\Phi$ has polynomial growth. For $s \in [0, T]$ and $\eta \in C([-T, 0])$, we set $u : [0, T] \times C([-T, 0]) \rightarrow \mathbb{R}$ by $u(s, \eta) = \mathbb{E}[\Phi(Y^{s,\eta})]$ where $Y^{s,\eta}$ belongs to $C([0, T])$ and it is defined by (6.3).*

Then $u \in C^0([0, T] \times C([-T, 0]))$. Moreover for every $s \in [0, T]$, $u(s, \cdot)$ belongs to $C^1(C([-T, 0]))$ and $Du(s, \eta)$, with $s \in [0, T]$ and $\eta \in C([-T, 0])$, is characterized by

$$D_{dr}u(s, \eta) = D_{dr}^\perp u(s, \eta) + D^{\delta_0}u(s, \eta)\delta_0(dr) \quad (6.5)$$

with

$$D_{dr}^\perp u(s, \eta) = \mathbb{E}\left[D_{s+dr}\Phi(Y^{s,\eta}) 1_{[-s, 0]}(r)\right] \quad \text{and}$$

$$D^{\delta_0}u(s, \eta) = \mathbb{E}\left[\int_{[s, T]} D_{d\rho}\Phi(Y^{s,\eta}) \mathcal{E}_\rho \left\{ \int_s^\cdot \partial_x \sigma(\xi, Y_\xi^{s,\eta}) dW_\xi + \int_s^\cdot \partial_x b(\xi, Y_\xi^{s,\eta}) d\xi \right\}\right]. \quad (6.6)$$

In particular item a) of Corollary 5.20 is verified.

Proof. We recall that $Y : [0, T] \times C([-T, 0]) \times \Omega \longrightarrow C([0, T])$ is a.s. continuous. We suppose for a moment that $\partial_x \sigma$ and $\partial_x b$ are Hölder continuous. In this case it is possible to show that $\eta \mapsto Y^{s, \eta}$ is of class $C^1(C([-T, 0]); C([0, T]))$ a.s. The first order Fréchet derivative with respect to the second argument η will be the map $DY : [0, T] \times C([-T, 0]) \times \Omega \longrightarrow \mathcal{L}(C([-T, 0]), C([0, T]))$, in particular for fixed s and η , $DY^{s, \eta} : C([-T, 0]) \longrightarrow C([0, T])$ is a linear continuous functional.

Remark 6.5. If we fix $t \in [0, T]$ then it holds $Y_t : [0, T] \times C([-T, 0]) \times \Omega \longrightarrow \mathbb{R}$; in this case the first order Fréchet derivative with respect to the second argument η will be $DY_t : [0, T] \times C([-T, 0]) \times \Omega \longrightarrow (C([-T, 0]))^* = \mathcal{M}([-T, 0])$. In particular if $f \in C([-T, 0])$,

$$\mathcal{M}([-T, 0]) \langle DY_t^{s, \eta}(\omega), f \rangle_{C([-T, 0])} = \int_{[-T, 0]} f(r) D_{dr} Y_t^{s, \eta}(\omega) \quad \square$$

We go on with the proof of Theorem 6.4. Computing the derivative $DY_t^{s, \eta}$ gives

$$D_{dr} Y_t^{s, \eta} = \begin{cases} \delta_{t-s}(dr) & t \leq s \\ \delta_0(dr) + \int_s^t \partial_x \sigma(\xi, Y_\xi^{s, \eta}) D_{dr} Y_\xi^{s, \eta} dW_\xi + \int_s^t \partial_x b(\xi, Y_\xi^{s, \eta}) D_{dr} Y_\xi^{s, \eta} d\xi & t \geq s. \end{cases}$$

Consequently, for $t \geq s$, it follows

$$D_{dr} Y_t^{s, \eta} = \begin{cases} \delta_{t-s}(dr) = \delta_{dr}(t-s) & t \leq s \\ \delta_0(dr) \mathcal{E}_t \left\{ \int_s^\cdot \partial_x \sigma(\xi, Y_\xi^{s, \eta}) dW_\xi + \int_s^\cdot \partial_x b(\xi, Y_\xi^{s, \eta}) d\xi \right\} & t \geq s. \end{cases} \quad (6.7)$$

We remind from Notation 6.3 that

$$\mathcal{E}_t \left\{ \int_s^\cdot \partial_x \sigma(\xi, Y_\xi^{s, \eta}) dW_\xi + \int_s^\cdot \partial_x b(\xi, Y_\xi^{s, \eta}) d\xi \right\}$$

equals

$$\exp \left\{ \int_s^t \partial_x \sigma(\xi, Y_\xi^{s, \eta}) dW_\xi - \frac{1}{2} \int_s^t (\partial_x \sigma)^2(\xi, Y_\xi^{s, \eta}) d\xi + \int_s^t \partial_x b(\xi, Y_\xi^{s, \eta}) d\xi \right\}.$$

Moreover, by usual integration theory for every $t \in [0, T]$, $u(s, \cdot)$ is of class $C^1(C([-T, 0]))$ and

$$D_{dr} u(s, \eta) = \mathbb{E} \left[\int_{[0, T]} D_{d\rho} \Phi(Y^{s, \eta}) D_{dr} Y_\rho^{s, \eta} \right].$$

Taking into account (6.7) and omitting the technical details we obtain

$$\begin{aligned}
 D_{dr}u(s, \eta) &= \mathbb{E} \left[\int_{[0, s[} D_{d\rho} \Phi(Y^{s, \eta}) D_{dr} Y_{\rho}^{s, \eta} \right] + \mathbb{E} \left[\int_{[s, T]} D_{d\rho} \Phi(Y^{s, \eta}) D_{dr} Y_{\rho}^{s, \eta} \right] \\
 &= \mathbb{E} \left[\int_{[0, s[} D_{d\rho} \Phi(Y^{s, \eta}) \delta_{\rho-s}(dr) \right] \\
 &+ \mathbb{E} \left[\int_{[s, T]} D_{d\rho} \Phi(Y^{s, \eta}) \delta_0(dr) \mathcal{E}_{\rho} \left\{ \int_s^{\cdot} \partial_x \sigma(\xi, Y_{\xi}^{s, \eta}) dW_{\xi} + \int_s^{\cdot} \partial_x b(\xi, Y_{\xi}^{s, \eta}) d\xi \right\} \right] \\
 &= \mathbb{E} [D_{s+dr} \Phi(Y^{s, \eta})] 1_{[-s, 0[}(r) + \\
 &+ \delta_0(dr) \mathbb{E} \left[\int_{[s, T]} D_{d\rho} \Phi(Y^{s, \eta}) \mathcal{E}_{\rho} \left\{ \int_s^{\cdot} \partial_x \sigma(\xi, Y_{\xi}^{s, \eta}) dW_{\xi} + \int_s^{\cdot} \partial_x b(\xi, Y_{\xi}^{s, \eta}) d\xi \right\} \right].
 \end{aligned}$$

Finally we obtain (6.5) and (6.6).

If $\partial_x \sigma$ and $\partial_x b$ are not Hölder continuous, Y will not be a.s. differentiable in η , but only in $L^2(\Omega)$, i.e. in quadratic mean. However the two expressions in (6.6) still remain valid. We omit the details. \square

Remark 6.6. In fact, the regularity assumption on σ and $b : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ can be partly reduced, see for instance the technique developed in ¹³. Indeed, under the hypotheses given in Section 1.2 and 6 of ¹³, the expression of $DY^{s, \eta}$ given in (6.7) remains valid.

6.2. Some remarkable representations of random variables

In this section we discuss a basic example related to a significant particular case of Φ for which we specify the explicit form of u by Theorem 6.4.

Φ will involve the dependence of an argument in $L^2([-T, 0])$ via an intermediate functional f . ∇f or $r \mapsto \nabla_r f$ will designate the derivative with respect to $L^2([-T, 0])$. The symbol $D_{\rho}^{ac} \Phi(\gamma)$ denotes the absolute continuous component of the first order Fréchet derivative of Φ and if $c \in [0, T]$, $D^{\delta_c} \Phi(\gamma) = D\Phi(\gamma)\{c\}$.

Example 6.7. Let $a_0 = 0 < a_1 < \dots < a_N = T$.

Let $\Phi(\gamma) = f(\gamma(a_1), \dots, \gamma(a_N), \gamma)$ with $f : \mathbb{R}^N \times L^2([0, T]) \rightarrow \mathbb{R}$, $f \in C^1(\mathbb{R}^N \times L^2([0, T]) \rightarrow \mathbb{R})$ such that all the derivatives have polynomial growth. In this case $D\Phi$ has the following particular form

$$D_{dr} \Phi(\gamma) = \sum_{i=1}^N D^{\delta_{a_i}} \Phi(\gamma) \delta_{a_i}(dr) + (D_r^{ac} \Phi)(\gamma) dr, \quad (6.8)$$

$$\text{with } \begin{cases} D^{\delta_{a_i}} \Phi(\gamma) = \partial_i f(\gamma(a_1), \dots, \gamma(a_N), \gamma) \\ (D_r^{ac} \Phi)(\gamma) dr = (\nabla_r f)(\gamma(a_1), \dots, \gamma(a_N), \gamma) dr \end{cases}$$

where the $r \mapsto D_r^{ac} \Phi(\gamma)$ stands for the absolute continuous part of the measure

$D\Phi(\gamma)$. In particular $D\Phi \in \mathcal{D}_a([0, T]) \oplus L^2([0, T])$.

By (6.5) and (6.6) in Theorem 6.4, it yields

$$\begin{aligned}
 D^{\delta_0} u(s, \eta) &= \mathbb{E} \left[\sum_{a_i \geq s} D^{\delta_{a_i}} \Phi(Y^{s, \eta}) \mathcal{E}_{a_i} \left\{ \int_s^\cdot \partial_x \sigma(\xi, Y_\xi^{s, \eta}) dW_\xi + \int_s^\cdot \partial_x b(\xi, Y_\xi^{s, \eta}) d\xi \right\} \right] \\
 &+ \mathbb{E} \left[\int_s^T \nabla_\rho \Phi(Y^{s, \eta}) \mathcal{E}_\rho \left\{ \int_s^\cdot \partial_x \sigma(\xi, Y_\xi^{s, \eta}) dW_\xi + \int_s^\cdot \partial_x b(\xi, Y_\xi^{s, \eta}) d\xi \right\} d\rho \right] \\
 &= \mathbb{E} \left[\sum_{a_i \geq s} \partial_i f(\eta(a_1 - s), \dots, \eta(a_{i-1} - s), Y_{a_i}^{s, \eta}, \dots, Y_{a_N}^{s, \eta}, Y^{s, \eta}) \right. \\
 &\quad \left. \mathcal{E}_{a_i} \left\{ \int_s^\cdot \partial_x \sigma(\xi, Y_\xi^{s, \eta}) dW_\xi + \int_s^\cdot \partial_x b(\xi, Y_\xi^{s, \eta}) d\xi \right\} \right] \\
 &+ \mathbb{E} \left[\int_s^T \nabla_\rho f(\eta(a_1 - s), \dots, \eta(a_{i-1} - s), Y_{a_i}^{s, \eta}, \dots, Y_{a_N}^{s, \eta}, Y^{s, \eta}) \right. \\
 &\quad \left. \mathcal{E}_\rho \left\{ \int_s^\cdot \partial_x \sigma(\xi, Y_\xi^{s, \eta}) dW_\xi + \int_s^\cdot \partial_x b(\xi, Y_\xi^{s, \eta}) d\xi \right\} d\rho \right] \tag{6.9}
 \end{aligned}$$

and

$$\begin{aligned}
 D_{dr}^\perp u(s, \eta) &= \mathbb{E} \left[\sum_{a_i < s} D^{\delta_{a_i}} \Phi(Y^{s, \eta}) \delta_{a_i - s}(dr) \right] + \mathbb{E} \left[\int_0^s \nabla_\rho \Phi(Y^{s, \eta}) \delta_{\rho - s}(dr) d\rho \right] \\
 &= \mathbb{E} \left[\sum_{a_i < s} \partial_i f(\eta(a_1 - s), \dots, \eta(a_{i-1} - s), Y_{a_i}^{s, \eta}, \dots, Y_{a_N}^{s, \eta}, Y^{s, \eta}) \delta_{a_i - s}(dr) \right] \\
 &+ \mathbb{E} \left[1_{[-s, 0]}(r) \nabla_{r+s} f(\eta(a_1 - s), \dots, \eta(a_{i-1} - s), Y_{a_i}^{s, \eta}, \dots, Y_{a_N}^{s, \eta}, Y^{s, \eta}) dr \right]. \tag{6.10}
 \end{aligned}$$

∇f denotes the Fréchet derivative of f with respect to the last argument; of course $\nabla \cdot f$ can be seen as a function $r \mapsto \nabla_r f(\cdot)$ in $L^2([0, T])$ (for any argument of f).

Remark 6.8. Let $j \in \{1, \dots, N\}$.

1) If $s \in]a_{j-1}, a_j]$ we have

$$\begin{aligned}
 D^{\delta_0} u(s, \eta) &= \mathbb{E} \left[\sum_{i=j}^N D^{\delta_{a_i}} \Phi(Y^{s, \eta}) \mathcal{E}_{a_i} \left\{ \int_s^\cdot \partial_x \sigma(\xi, Y_\xi^{s, \eta}) dW_\xi + \int_s^\cdot \partial_x b(\xi, Y_\xi^{s, \eta}) d\xi \right\} \right] \\
 &+ \mathbb{E} \left[\int_s^T \nabla_\rho \Phi(Y^{s, \eta}) \mathcal{E}_\rho \left\{ \int_s^\cdot \partial_x \sigma(\xi, Y_\xi^{s, \eta}) dW_\xi + \int_s^\cdot \partial_x b(\xi, Y_\xi^{s, \eta}) d\xi \right\} d\rho \right] \tag{6.11}
 \end{aligned}$$

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and

$$D_{dr}^\perp u(s, \eta) = \mathbb{E} \left[\sum_{i=1}^{j-1} D^{\delta_{a_i}} \Phi(Y^{s, \eta}) \delta_{a_i - s}(dr) \right] \\ + \mathbb{E} \left[\int_0^s \nabla_\rho \Phi(Y^{s, \eta}) \delta_{\rho - s}(dr) d\rho \right].$$

In particular for fixed $\eta \in C([-T, 0])$, $s \mapsto D^{\delta_0} u(s, \eta)$ is continuous on $]a_{j-1}, a_j[$, is left-continuous and it admits a continuous extension to $[a_{j-1}, a_j]$.

- 2) If $s \in [0, a_1]$, the equation (6.11) holds with $j = 1$ and $s \mapsto D^{\delta_0} u(s, \eta)$ is continuous on $[0, a_1]$.
- 3) We remark that u is not necessarily of class $C^{0,1}([0, T] \times C([-T, 0]))$ except if $N = 1$.

Below we state a particular case of Example 6.7 where Φ only depends on the maturity and on the whole trajectory in L^2 .

Example 6.9. (Particular case)

Suppose that $\Phi(\gamma) = f(\gamma(T), \gamma)$ for $f : \mathbb{R} \times L^2([-T, 0]) \rightarrow \mathbb{R}$ of class C^1 with polynomial growth derivatives. (6.9) and (6.10) give

$$D^{\delta_0} u(s, \eta) = \mathbb{E} \left[\partial_1 f(Y_T^{s, \eta}, Y^{s, \eta}) \mathcal{E}_T \left\{ \int_s^\cdot \partial_x \sigma(\xi, Y_\xi^{s, \eta}) dW_\xi + \int_s^\cdot \partial_x b(\xi, Y_\xi^{s, \eta}) d\xi \right\} \right] \\ + \mathbb{E} \left[\int_s^T \nabla_\rho f(Y_T^{s, \eta}, Y^{s, \eta}) \mathcal{E}_\rho \left\{ \int_s^\cdot \partial_x \sigma(\xi, Y_\xi^{s, \eta}) dW_\xi + \int_s^\cdot \partial_x b(\xi, Y_\xi^{s, \eta}) d\xi \right\} d\rho \right],$$

$$D_{dr}^\perp u(s, \eta) = \mathbb{E} [1_{[-s, 0]}(r) \nabla_{r+s} f(Y_T^{s, \eta}, Y^{s, \eta}) dr].$$

In this case u is of class C^1 for every $(s, \eta) \in [0, T] \times C([-T, 0])$, $Du(s, \eta) \in \mathcal{D}_0 \oplus L^2([-T, 0])$ and $Du : [0, T] \times C([-T, 0]) \rightarrow \mathcal{D}_0 \oplus L^2([-T, 0])$ is continuous.

Next example is a particular case of Example 6.7 where Φ only depends pointwise on a finite number of points; in this case $\nabla_r f \equiv 0$.

Example 6.10. (Particular case)

Let $\Phi(\gamma) = f(\gamma(a_1), \dots, \gamma(a_N))$ with $f : \mathbb{R}^N \rightarrow \mathbb{R}$. In this case (6.8) reduces to $D_{dr} \Phi(\gamma) = \sum_{i=1}^N \partial_i f(\gamma(a_1), \dots, \gamma(a_N)) \delta_{a_i}(dr)$. (6.9) reduces to $D^{\delta_0} u(s, \eta)$ being equal to

$$\mathbb{E} \left[\sum_{a_i \geq s (i \geq 1)} \partial_i f(Y_{a_1}^{s, \eta}, \dots, Y_{a_N}^{s, \eta}) \mathcal{E}_{a_i} \left\{ \int_s^\cdot \partial_x \sigma(\xi, Y_\xi^{s, \eta}) dW_\xi + \int_s^\cdot \partial_x b(\xi, Y_\xi^{s, \eta}) d\xi \right\} \right].$$

We recall that, if $s \in [a_{N-1}, a_N[$ then $s \geq a_i$ for all $i = 1, \dots, (N-1)$ and $Y_{a_i}^{s, \eta} = \eta(a_i - s)$, by definition (6.3).

The result below is a fundamental step for obtaining a quasi-explicit representation of integrable random variables. In fact we verify the validity of the assumptions of Corollary 5.20 .

Proposition 6.11. *Let $\Phi : C([0, T]) \rightarrow \mathbb{R}$ be of the type $\Phi(\gamma) = f(\gamma(a_1), \dots, \gamma(a_N), \gamma)$ with $f : \mathbb{R}^N \times L^2([0, T]) \rightarrow \mathbb{R}$ of class C^1 with partial derivatives having polynomial growth. We set $u(s, \eta) = \mathbb{E}[\Phi(Y^{s, \eta})]$ for $s \in [0, T]$ and $\eta \in C([-T, 0])$. Then*

$$h = u(0, X_0(\cdot)) + \int_0^T D^{\delta_0} u(s, X_s(\cdot)) \sigma(s, X_s) dW_s, \quad (6.12)$$

where $D^{\delta_0} u(s, \eta)$ is given in (6.9).

Proof. As anticipated, we verify the assumptions of Corollary 5.20 for $F = u$. u is of course continuous. e) is fulfilled by construction. a) was the object of Theorem 6.4. In Example 6.7 we have given the expression of $Du(s, \cdot)$ for any $s \in [0, T]$. (6.9) and (6.10) give the explicit expression respectively for $D^{\delta_0} u(s, \eta)$ and $D_{dr}^\perp u(s, \eta)$. c) follows from those explicit expressions. d) follows by the fact that $(t, \eta) \mapsto D^\perp u(t, \eta)$ is bounded on each compact of $[a_i, a_{i+1}] \times C([-T, 0])$, $1 \leq i \leq N-1$. It remains to check b), i.e. that u fulfills the support predictability condition. Since

$$(r, s, \eta) \mapsto \mathbb{E} [1_{[-s, 0]}(r) \nabla_{r+s} f(Y_{a_1}^{s, \eta}, \dots, Y_{a_N}^{s, \eta}, Y^{s, \eta})]$$

is bounded on each compact, in order to verify the support predictability condition we only need to show that for each compact $K \subset C([-T, 0])$,

$$\int_0^T \left[\sup_{\eta \in K} \frac{1}{\epsilon} \int_{-\epsilon}^0 g(dr, s, \eta) \right] ds = O(\epsilon) \quad (6.13)$$

where

$$g(dr, s, \eta) = \sum_{a_j < s} \mathbb{E} [\partial_j f(Y_{a_1}^{s, \eta}, \dots, Y_{a_N}^{s, \eta}, Y^{s, \eta})] \delta_{a_j - s}(dr).$$

For $\epsilon < \min_{j \in \{1, \dots, N\}} \{a_j - a_{j-1}\}$, the left-hand side of (6.13) gives

$$\begin{aligned} & \sum_{i=0}^{N-1} \int_{a_i}^{a_{i+1}} \left[\sup_{\eta \in K} \frac{1}{\epsilon} \int_{-\epsilon}^0 \sum_{a_j < s} \mathbb{E} [\partial_j f(Y_{a_1}^{s, \eta}, \dots, Y_{a_N}^{s, \eta}, Y^{s, \eta})] \delta_{a_j - s}(dr) \right] ds \\ &= \sum_{i=0}^{N-1} \int_{a_i}^{a_{i+1}} \left[\sup_{\eta \in K} \frac{1}{\epsilon} \sum_{a_j < s} \mathbb{E} [\partial_j f(Y_{a_1}^{s, \eta}, \dots, Y_{a_N}^{s, \eta}, Y^{s, \eta})] \int_{-\epsilon}^0 \delta_{a_j - s}(dr) \right] ds. \end{aligned} \quad (6.14)$$

We remark that

$$\int_{-\epsilon}^0 \delta_{a_j - s}(dr) = \begin{cases} 1 & \text{if } (a_j - s) \in [-\epsilon, 0] \Leftrightarrow s \in [a_j, a_j + \epsilon] \\ 0 & \text{otherwise} \end{cases}.$$

So in the second sum of (6.14), for $a_j < s$ only remains the term $a_j = a_i$ and (6.14) gives

$$\sum_{i=0}^{N-1} \int_{a_i}^{a_i + \epsilon} \left[\sup_{\eta \in K} \frac{1}{\epsilon} \mathbb{E} [\partial_i f(Y_{a_1}^{s, \eta}, \dots, Y_{a_N}^{s, \eta}, Y^{s, \eta})] \right] ds.$$

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Since all the derivatives of f have polynomial growth, previous expression is bounded by

$$C(N, f, T) \sup_{\eta \in K, s \in [0, T]} \{1 + \mathbb{E} [\|Y^{s, \eta}\|_{\infty}^q]\}$$

where $C(N, f, T)$ is some constant and q is some positive real. The conclusion follows by Proposition 6.1. So u fulfills the support predictability condition. \square

6.3. About the representation of non-smooth random variables

The next example is again essentially illustrative. It will be developed and treated in a more general context in a paper in preparation. It will be possible to represent, still using Corollary 5.20, random variables of the type

$$h := \Phi(X), \quad \text{where} \quad \Phi(\gamma) = f \left(\int_0^T \varphi_1(r) d\gamma(r), \dots, \int_0^T \varphi_N(r) d\gamma(r) \right),$$

where f is continuous with polynomial growth and $\varphi_1, \dots, \varphi_N$ of class $C^1([0, T])$. Now $\int_0^T \varphi(r) d\gamma(r)$ can be naturally defined by an integration by parts as $\varphi(T)\gamma(T) - \varphi(0)\gamma(0) - \int_0^T \gamma(r) d\varphi(r)$. More general formulations can be performed even with less regularity using specific approximation techniques. Here we aim only at obtaining a representation directly, without approximations. For simplicity, we consider X to be a diffusion process of the type (6.1) with $\sigma \equiv 1$ and $b \equiv 0$, so that X is a Brownian motion W and $Y^{s, \eta}$ introduced in (6.3) is a Brownian flow.

In this exemplifying subsection we only suppose $N = 1$. Let $h = f \left(\int_0^T \varphi(r) dW_r \right)$, $\varphi \in C^1([0, T])$, $\varphi(T) \neq 0$, $f : \mathbb{R} \rightarrow \mathbb{R}$ measurable with polynomial growth. We define $u(s, \eta) = \mathbb{E} [\Phi(Y^{s, \eta})]$ as in (6.4). It is possible to show the assumptions of Theorem 5.16 setting $a = 0$ and $b < T$. We suppose for a moment that $f \in C_b^1(\mathbb{R})$. We come back to Theorem 6.4 which says that for every $s \in [0, T]$, $u(s, \cdot)$ is of class C^1 and

$$D_{dr} u(s, \eta) = D^{\delta_0} u(s, \eta) \delta_0(dr) + D_{dr}^{\perp} u(s, \eta).$$

Indeed we have

$$D_{dr} \Phi(\gamma) = f' \left(\int_0^T \varphi(r) d\gamma(r) \right) \left(\varphi(T) \delta_T(dr) - \varphi(0) \delta_0(dr) - \dot{\varphi}(r) 1_{[0, T]}(r) dr \right).$$

This, together with (6.6), gives

$$\begin{aligned} D^{\delta_0} u(s, \eta) &= \mathbb{E} [Z_{\varphi}] [\varphi(s) - \varphi(0) 1_{\{0\}}(s)] \\ D_{dr}^{\perp} u(s, \eta) &= \mathbb{E} [Z_{\varphi}] [(-\varphi(0) \delta_{-s}(dr) - \dot{\varphi}(s+y)) 1_{[-s, 0]}(y) dy], \end{aligned} \quad (6.15)$$

where

$$Z_{\varphi} = f' \left(\int_0^T \varphi(r) dY^{s, \eta}(r) \right) = f' \left(\eta(0) \varphi(s) - \varphi(0) \eta(-s) - \int_0^s \eta(r-s) d\varphi(r) + \int_s^T \varphi(r) dW_r \right);$$

previous equality holds because of integration by parts and expression (6.3). We denominate by h the random variable $\Phi(Y^{s,\eta})$. Denoting D^m the Malliavin derivative we obtain

$$D_r^m(h) = D_r^m(\Phi(Y^{s,\eta})) = Z_\varphi \varphi(r) 1_{]s,T[}(r),$$

then

$$\langle D^m(h), \varphi \rangle_{L^2([0,T])} = \int_0^T D_r^m(h) \varphi(r) dr = Z_\varphi \int_s^T \varphi^2(r) dr$$

and so

$$\mathbb{E}[Z_\varphi] = \frac{1}{\int_s^T \varphi^2(r) dr} \mathbb{E}[\langle D^m(h), \varphi \rangle]. \quad (6.16)$$

Consequently, inserting (6.16) in (6.15) we obtain

$$D^{\delta_0} u(s, \eta) = \frac{\varphi(s) - \varphi(0) 1_{\{0\}}(s)}{\int_s^T \varphi^2(r) dr} \mathbb{E}[\langle D^m(h), \varphi \rangle] \quad (6.17)$$

and

$$D_{dy}^\perp u(s, \eta) = \frac{(-\varphi(0) \delta_{-s}(dr) - \dot{\varphi}(s+y)) 1_{[-s,0[}(y) dy}{\int_s^T \varphi^2(r) dr} \mathbb{E}[\langle D^m(h), \varphi \rangle]. \quad (6.18)$$

By the well-known integration by parts on the Wiener space, it follows

$$\mathbb{E}[\langle D^m(h), \varphi \rangle] = \mathbb{E}[h \cdot \delta(\varphi)] = \mathbb{E} \left[\Phi(Y^{s,\eta}) \cdot \int_0^T \varphi(r) dW_r \right].$$

Consequently, for $s \in [0, T[$

$$D^{\delta_0} u(s, \eta) = \frac{\varphi(s) - \varphi(0) 1_{\{0\}}(s)}{\int_s^T \varphi^2(r) dr} \mathbb{E} \left[\Phi(Y^{s,\eta}) \cdot \int_0^T \varphi(r) dW_r \right] \quad (6.19)$$

$$D_{dy}^\perp u(s, \eta) = \frac{(-\varphi(0) \delta_{-s}(dr) - \dot{\varphi}(s+y)) 1_{[-s,0[}(y) dy}{\int_s^T \varphi^2(r) dr} \mathbb{E} \left[\Phi(Y^{s,\eta}) \cdot \int_0^T \varphi(r) dW_r \right]. \quad (6.20)$$

Remark 6.12. We remark that in (6.19) and (6.20) the derivative of f does not appear. At this point we admit a technical point not to overcharge the proof. Even if f is not of class C^1 , $u(t, \cdot)$ is still of class C^1 for every $s \in [0, T[$ and (6.19) and (6.20) still hold.

As promised, we verify the assumptions of Theorem 5.16. We first observe that

$$(s, \eta) \mapsto \mathbb{E} \left[\Phi(Y^{s,\eta}) \cdot \int_0^T \varphi(r) dW_r \right] \quad (6.21)$$

is continuous, therefore bounded on each compact of $[0, T] \times C([-T, 0])$. Given a compact subset K of $C([-T, 0])$, we denote

$$C(K) := \sup_{\eta \in K} \mathbb{E} \left[\Phi(Y^{s,\eta}) \cdot \int_0^T \varphi(r) dW_r \right] .$$

Assumptions i), ii) and iv) are clearly verified taking into account (6.21), (6.17) and (6.18). It remains to check the support predictability condition. Let K be a compact of $C([0, T])$. We observe that

$$\sup_{s \in [0, b], y \in [-s, 0], \eta \in K} \left| \frac{\dot{\varphi}(s+y) 1_{[-s, 0]}(y)}{\int_s^T \varphi^2(r) dr} \mathbb{E} \left[\Phi(Y^{s,\eta}) \cdot \int_0^T \varphi(r) dW_r \right] \right|$$

is finite, Remark 5.15, item 3) implies that it will be enough to show that

$$\int_0^b \sup_{\eta \in K} \frac{1}{\epsilon} \int_{(-\epsilon) \vee (-T)}^0 \left| \frac{\varphi(0) \delta_{-s}(dr) 1_{[-s, 0]}(y) dy}{\int_s^T \varphi^2(r) dr} \mathbb{E} \left[\Phi(Y^{s,\eta}) \cdot \int_0^T \varphi(r) dW_r \right] \right| ds = O(\epsilon) . \quad (6.22)$$

Let $\epsilon < b$. The left-hand side of (6.22) is bounded by

$$\frac{C(K)}{\epsilon} \int_0^\epsilon \frac{\varphi(0)}{\int_s^T \varphi^2(r) dr} ds \leq \frac{C(K)}{\int_b^T \varphi^2(r) dr} \|\varphi\|_\infty < +\infty .$$

So also assumption iii) of Theorem 5.16, i.e. the support predictability condition is verified. The conclusion follows by Remark 5.21.

6.4. Link to a finite dimensional PDE

We come back to the assumptions on coefficients σ and b of Section 6. They characterize again a diffusion process X as solution of (6.1). We link now Corollary 5.20 and Proposition 6.11 with a well-known result of representation related to hedging theory in mathematical finance. We consider a *contingent claim* defined by $h = \Phi(\gamma) = f(\gamma(a_1), \dots, \gamma(a_N))$, i.e.

$$h = f(X_{a_1}, \dots, X_{a_N}) , \quad 0 < a_1 < \dots < a_N = T$$

with the usual convention $a_0 = 0$. We consider here the case f of class C^2 with polynomial growth but σ may become degenerate.

Proposition 6.13. *Let X be a diffusion process of type (6.1). Let $N \geq 2$ and $f : \mathbb{R}^N \rightarrow \mathbb{R}$ of class C^2 with polynomial growth. We suppose $\sigma : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ of class $C^{0,2}$, such that $\partial_x \sigma, \partial_{xx}^2 \sigma$ are bounded.*

There exist functions

$$\nu^i : \mathbb{R}^{i-1} \times [a_{i-1}, a_i] \times \mathbb{R} \rightarrow \mathbb{R} , \quad 1 \leq i \leq N ,$$

such that for $y_1, \dots, y_{i-1} \in \mathbb{R}$,

$$\nu^i(s, y) := \nu^i(y_1, \dots, y_{i-1}; s, y)$$

solves

$$\begin{cases} \partial_s \nu^i(s, y) + \frac{1}{2} \sigma^2(s, y) \partial_{yy}^2 \nu^i(s, y) + b(s, y) \partial_y \nu^i(s, y) = 0 & s \in]a_{i-1}, a_i[\\ \nu^i(a_i, y) = \nu^{i+1}(y_1, \dots, y_{i-1}, y; a_i, y) & i < N \\ \nu^N(T, y) = f(y_1, \dots, y_{N-1}, y) & i = N \end{cases} \quad (6.23)$$

such that

$$f(X_{a_1}, \dots, X_{a_N}) = H_0 + \int_0^T \xi_s dX_s \quad (6.24)$$

and, for $1 \leq i \leq N$,

$$\begin{aligned} \xi_s &= \partial_y \nu^i(X_{a_1}, \dots, X_{a_{i-1}}; s, X_s) \quad , \quad s \in]a_{i-1}, a_i[\\ H_0 &= \nu^1(0, X_0) \end{aligned} \quad (6.25)$$

In particular $(s, y) \mapsto \nu^i(s, y) \in C^{1,2}(\]a_{i-1}, a_i[\times \mathbb{R}) \cap C^0([a_{i-1}, a_i] \times \mathbb{R})$.

Remark 6.14. Let X be a finite quadratic variation process such that $[X]_t = \int_0^t \sigma^2(s, X_s) ds$. An example is of course our basic process X introduced in (6.1) but there are plenty of other examples. Let $f : \mathbb{R}^N \rightarrow \mathbb{R}$ be continuous with linear growth. If there are functions ν^1, \dots, ν^N as in Proposition 6.13, then the representation (6.24), with (6.25), holds replacing dX_s with the forward integral d^-X_s . If X is a martingale we recall that $d^-X_s = dX_s$. This was proved in ⁴, Proposition 4.30. The aim of Proposition 6.13 is to construct effectively such functions ν^1, \dots, ν^N .

Proof. [Proof of Proposition 6.13]. According to Corollary 5.20, representation (6.24) holds with $H_0 = u(0, X_0(\cdot))$ and $\xi_s = D^{\delta_0} u(s, X_s(\cdot))$ where $u(s, \eta) := \mathbb{E}[f(Y_{a_1}^{s, \eta}, \dots, Y_{a_N}^{s, \eta})]$. In fact, $\Phi(\gamma) = f(\gamma(a_1), \dots, \gamma(a_N))$ is C^1 Fréchet differentiable with polynomial growth.

We denote by $(Z_t^{s, y})$ the flow such that $Z = Z^{s, y}$ verifies

$$Z_t = y + \int_s^t \sigma(r, Z_r) dW_r + \int_s^t b(r, Z_r) dr \quad , \quad t \geq s .$$

In particular we have $Y^{s, \eta} = Z^{s, \eta(0)}$.

Let $1 \leq i \leq N$. Clearly, if $s \in [a_{i-1}, a_i]$,

$$\begin{aligned} u(s, \eta) &= \mathbb{E} \left[f \left(\eta(a_1 - s), \dots, \eta(a_{i-1} - s), Z_{a_i}^{s, \eta(0)}, \dots, Z_{a_N}^{s, \eta(0)} \right) \right] \\ &= \nu^i(\eta(a_1 - s), \dots, \eta(a_{i-1} - s); s, \eta(0)) \end{aligned}$$

where

$$\nu^i(s, y) = \mathbb{E} [f(y_1, \dots, y_{i-1}, Z_{a_i}^{s, y}, \dots, Z_{a_N}^{s, y})] \quad . \quad (6.26)$$

Now ν^i is continuous. We keep in mind the representation (6.12) in Proposition 6.11. In particular (6.24) holds with $H_0 = u(0, X_0(\cdot))$ and $\xi_s = D^{\delta_0} u(s, X_s(\cdot))$. We

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recall the expression of $D^{\delta_0}u$ calculated in Example 6.7 in a more general situation. We had

$$D^{\delta_0}u(s, \eta) \mathbb{1}_{]a_{i-1}, a_i[}(s) = \mathbb{E} \left[\sum_{a_j \geq s} f \left(\eta(a_1 - s), \dots, \eta(a_{i-1} - s), Z_{a_i}^{s, \eta(0)}, \dots, Z_{a_N}^{s, \eta(0)} \right) \mathcal{E}_{a_j} \left\{ \int_s^\cdot \partial_x \sigma \left(r, Z_r^{s, \eta(0)} \right) dW_r + \int_s^\cdot \partial_x b \left(r, Z_r^{s, \eta(0)} \right) dr \right\} \right].$$

In fact by usual integration theory it is not difficult to show that for any $s \in]a_{i-1}, a_i[$, $y_1, \dots, y_{i-1} \in \mathbb{R}$ fixed, $y \mapsto \nu^i(s, y)$ is of class C^2 . We observe that $u(0, X_0(\cdot)) = \nu^1(0, X_0)$. Moreover $s \mapsto \partial_y \nu^i(s, y)$ and $s \mapsto \partial_{yy}^2 \nu^i(s, y)$ are continuous on $]a_{i-1}, a_i[$. In particular if $s \in]a_{i-1}, a_i[$

$$\partial_y \nu^i(s, y) = \mathbb{E} \left[\sum_{j \geq i} \partial_j f \left(y_1, \dots, y_{i-1}, Z_{a_i}^{s, y}, \dots, Z_{a_N}^{s, y} \right) \partial_y Z_{a_j}^{s, y} \right].$$

Since, at least in $L^2(\Omega)$,

$$\partial_y Z_{a_j}^{s, y} = \mathcal{E}_{a_j} \left\{ \int_s^\cdot \partial_x \sigma \left(r, Z_r^{s, y} \right) dW_r \right\},$$

it follows that, for $s \in]a_{i-1}, a_i[$,

$$D^{\delta_0}u(s, \eta) = \partial_y \nu^i \left(\eta(a_1 - s), \dots, \eta(a_{i-1} - s); s, \eta(0) \right)$$

and so (6.25) is established.

It remains to prove (6.23).

We remark that we can evaluate the second order derivative with respect to y . It gives

$$\begin{aligned} \partial_{yy}^2 \nu^i(s, y) &= \mathbb{E} \left[\sum_{k, j \geq i} \partial_{kj}^2 f \left(y_1, \dots, y_{i-1}, Z_{a_i}^{s, y}, \dots, Z_{a_N}^{s, y} \right) \partial_y Z_{a_j}^{s, y} \partial_y Z_{a_k}^{s, y} \right] \\ &+ \mathbb{E} \left[\sum_{j \geq i} \partial_j f \left(y_1, \dots, y_{i-1}, Z_{a_i}^{s, y}, \dots, Z_{a_N}^{s, y} \right) \partial_{yy}^2 Z_{a_j}^{s, y} \right] \end{aligned}$$

where $\partial_{yy}^2 Z_{a_j}^{s, y}$ could be calculated explicitly.

It remains to provide the partial derivative with respect to s . Let $s, s+h \in]a_{i-1}, a_i[$ and suppose $h > 0$. Since $Z_{a_i}^{s, y} = Z_{a_i}^{s+h, Z_{s+h}^{s, y}}$, we easily obtain that

$$u(s, y) = \mathbb{E} \left[u \left(s+h, Z_{s+h}^{s, y} \right) \right].$$

So, by Itô formula, $\frac{1}{h} (\nu^i(s+h, y) - \nu^i(s, h))$ equals

$$\begin{aligned} \frac{1}{h} (u(s+h, y) - u(s, y)) &= \frac{1}{h} \mathbb{E} \left[u(s+h, y) - u(s+h, Z_{s+h}^{s, y}) \right] \\ &= \frac{1}{h} \mathbb{E} \left[- \int_s^{s+h} \partial_y u(s+h, Z_r^{s, y}) dZ_r^{s, y} - \frac{1}{2} \int_s^{s+h} \sigma^2 \left(r, Z_r^{s, y} \right) \partial_{yy}^2 u(s+h, Z_r^{s, y}) dr \right] \end{aligned}$$

where $dZ_r^{s,y} = \sigma(r, Z_r^{s,y}) dW_r + b(r, Z_r^{s,y}) dr$.

Similar arguments allow to discuss the limit when $h \rightarrow 0$. Letting h go to zero we get

$$\begin{aligned} \partial_s \nu^i(s, y) &= \lim_{h \rightarrow 0} \frac{u(s+h, y) - u(s, y)}{h} = -\frac{1}{2} \sigma^2(s, y) \partial_{yy}^2 u(s, y) - b(s, y) \partial_y u(s, y) \\ &= -\frac{1}{2} \sigma^2(s, y) \partial_{yy}^2 \nu^i(s, y) - b(s, y) \partial_y \nu^i(s, y). \end{aligned}$$

We have finally established the first line of (6.23). The second and third conditions in (6.23) are verified by inspection using (6.26). \square

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