Definition 1. *We say that u satisfies the outgoing radiation condition if and only if there exist* $(u_n^\pm)_n$ *such that* $\forall x \in \mathcal{C}$, $p \in \mathbb{N}$, $N \in \mathbb{N}$

$$u(x_1 \pm p, x_2) = \sum_{n \in I(\omega)} \sum_{\substack{\xi \in \Xi_n(\omega) \\ \pm\lambda_n'(\xi) > 0}} u_n^\pm \, \varphi_n(x;\xi)\, e^{\imath p \xi} \; + \; \mathcal{O}_{H^1}\left(p^{-N}\right)$$

Theorem 3. *Suppose* $\omega \notin \sigma_0 \cup \tilde{\sigma}_0$. *Let u be a solution of*

$$\begin{cases} -\triangle u - \omega^2\, n_p^2\, u = 0 & \text{in } \Omega \\ \partial_\nu u = 0 & \text{on } \partial\Omega \end{cases}$$

which satisfies the outgoing radiation condition. Then $u = 0$.

4. Conclusions

This analysis is one of the main tool to solve inverse problems in locally perturbed periodic waveguide when the data are far field measurements of scattering problems.

One challenging perspective of this work is to extend these results to periodic problems in domains which are periodic and infinite in at least 2 directions.

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Finite Element Heterogeneous Multiscale Method for the Wave Equation: Long Time Effects

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(joint work with Assyr Abdulle, and Christian Stohrer)

1. Introduction

For limited time the propagation of waves in a highly oscillatory medium is well-described by the non-dispersive homogenized wave equation. With increasing time, however, the true solution deviates from the classical homogenization limit, as a large secondary wave train develops unexpectedly. Here, we propose a new finite element heterogeneous multiscale method (FE-HMM), which captures not only the short-time macroscale behavior of the wave field but also those secondary long-time dispersive effects.

2. Long-Time Wave Propagation

Let $\Omega \subset \mathbb{R}^n$ be a domain and $T > 0$. We consider the wave equation

$$(1) \qquad \begin{cases} \partial_{tt} u^\varepsilon - \nabla \cdot (a^\varepsilon \nabla u^\varepsilon) = F & \text{in } \Omega \times (0,T), \\ u^\varepsilon(x,0) = f(x) & \text{in } \Omega, \\ \partial_t u^\varepsilon(x,0) = g(x) & \text{in } \Omega, \end{cases}$$

where $a^\varepsilon(x) \in (L^\infty(\Omega))^{d\times d}$ is symmetric, uniformly elliptic, and bounded. Here $\varepsilon > 0$ represents a small scale in the problem, which we cannot afford to fully resolve and thus characterizes the multiscale nature of $a^\varepsilon(x)$. With appropriate Dirichlet or periodic boundary conditions, the solution u^ε is uniquely determined for every $\varepsilon > 0$.

2.1. **Classical homogenization.** According to classical homogenization theory, u^ε converges to the solution u^0 of the "homogenized" wave equation as $\varepsilon \to 0$,

$$\partial_{tt} u^0 - \nabla \cdot (a^0 \nabla u^0) = F,$$

yet the homogenized tensor (or squared velocity field) a^0 can only rarely be computed explicitly. Although u^0 approximates u^ε for short times in the L^2-norm, it becomes increasingly inadequate at later times $T \sim \varepsilon^{-2}$, since it neglects microscopic dispersive effects that accumulate over time, as shown in Figure 1. Here we consider (1) in $\Omega = (-1,1)$ with periodic boundary conditions, let $u(x,0)$ be a Gaussian pulse with zero initial velocity and set

$$(2) \qquad a^\varepsilon = \sqrt{2} + \sin\Big(2\pi \frac{x}{\varepsilon}\Big) \ \text{ with } \varepsilon = \frac{1}{50}.$$

In Figure 1, the reference solution of (1)–(2) corresponds to a direct numerical simulation (DNS), where the micro-scale is fully resolved. After one revolution ($T = 2$), the homogenized and the DNS solution coincide. After fifty revolutions ($T = 100$), however, the DNS displays dispersive effects, which the homogenized solution fails to capture.

2.2. **Effective dispersive equation.** Various formal asymptotic arguments were derived to elucidate that peculiar inherently dispersive long-time behavior of waves propagating through a strongly heterogeneous periodic medium [1]. An effective equation that captures those dispersive effects was recently derived in [2] for the one-dimensional case when a^ε is ε-periodic:

$$(3) \qquad \partial_{tt}(u^{\text{eff}} - \varepsilon^2\, b\, \partial_{xx} u^{\text{eff}}) - a^0 \partial_{xx} u^{\text{eff}} = F.$$

Again, a^0 denotes the homogenized effective coefficient from classical homogenization theory whereas $b > 0$ denotes a distinct constant. As shown in Figure 1, u^ε and u^{eff} essentially coincide both at early and later times.

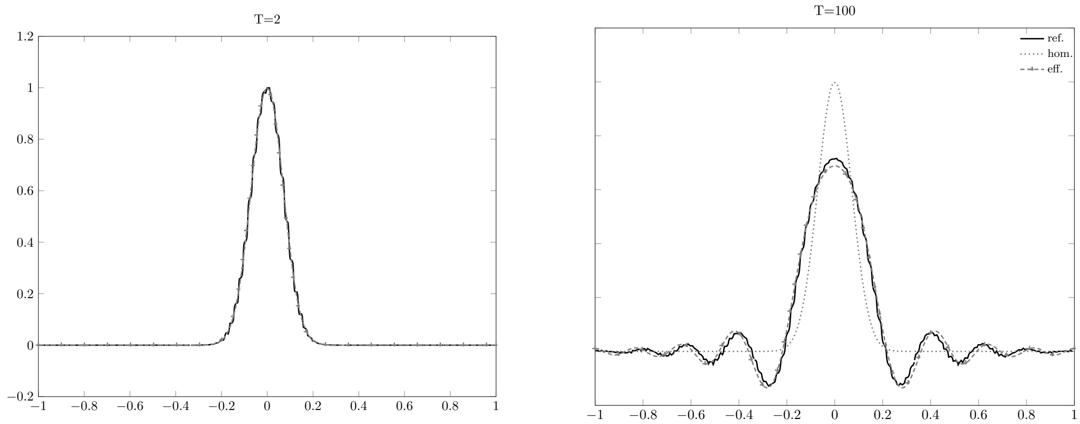

Figure 1. Reference (ref.), homogenized (hom.) and effective (eff.) solution: short-time $T = 2$ (left), and long-time $T = 100$ (right).

3. FE Heterogeneous Multiscale Method

In [3], the FE-HMM for elliptic problems [4] was extended to the time dependent wave equation. It was shown to converge to u^0 at finite times, yet it also fails to capture long-time dispersive effects in the true solution. To incorporate those dispersive effects, we not only need an effective bilinear form but also an effective inner product, akin to the weak formulation of (3). Both require the numerical solutions of micro problems on sampling domains K_δ of size δ proportional to ε. An alternative HMM scheme, based on the finite difference approximation of an effective flux, was proposed in [5]. Since it is based on an effective model [2], which is ill-posed, appropriate regularization techniques need to be implemented.

We now give a brief description of our new FE-HMM scheme. First, we generate a macro triangulation $\mathcal{T}_H$ and choose an appropriate macro FE space $S(\Omega, \mathcal{T}_H)$. By macro we mean that $H \gg \varepsilon$ is allowed. Within each macro element $K \in \mathcal{T}_H$ we choose two quadrature formulas $\{x_{K,j}, \omega_{K,j}\}$ and $\{x^L_{K,j}, \omega^L_{K,j}\}$. The HMM solution u_H is given by the following variational problem:

$$
(4) \qquad \begin{cases} \text{Find } u_H : [0,T] \to S(\Omega, \mathcal{T}_H) \text{ such that} \\ (\partial_{tt} u_H, v_H)_Q + B_H(u_H, v_H) = (F, v_H) \\ \text{for all } v_H \in S(\Omega, \mathcal{T}_H) \text{ and,} \\ \quad u_H(0) = f_H,\ \partial_t u_H(0) = g_H \text{ in } \Omega, \end{cases}
$$

where the initial data f_H and g_H are suitable approximations of f and g in $S(\Omega, \mathcal{T}_H)$ whereas the effective bilinear form B_H and the effective inner product $(\cdot,\cdot)_Q$ are defined as follows. The FE-HMM bilinear form is given by

$$
B_H(v_H, w_H) = \sum_{K \in \mathcal{T}_H} \sum_{j=1}^{J} \frac{\omega_{K,j}}{|K_\delta|} \int_{K_\delta} a^\varepsilon(x) \nabla v_h(x) \cdot \nabla w_h(x)\, dx,
$$

and the FE-HMM inner product by

$$(v_H, w_H)_Q = (v_H, w_H)_H + (v_H, w_H)_M.$$

Here,

$$(v_H, w_H)_H = \sum_{K \in \mathcal{T}_H} \sum_{j=1}^{J_L} \omega_{K,j}^L v_H(x_{K,j}^L) w_H(x_{K,j}^L).$$

Note that $(\cdot, \cdot)_H$ corresponds to a standard approximation of the L^2-inner product by numerical quadrature, whereas the long-time correction is given by

$$(v_H, w_H)_M = \sum_{K \in \mathcal{T}_H} \sum_{j=1}^{J} \frac{\omega_{K,j}}{|K_\delta|} \int_{K_\delta} (v_h(x) - v_{H,\mathrm{lin}}(x))(w_h(x) - v_{H,\mathrm{lin}}(x))\, dx.$$

In the above, the micro solution v_h (resp. w_h) is given by

$$\left\{ \begin{array}{l} \text{Find } v_h \text{ such that } (v_h - v_{H,\mathrm{lin}}) \in S(K_\delta, \mathcal{T}_h) \text{ and} \\ \displaystyle\int_{K_\delta} a^\varepsilon(x) \nabla v_h(x) \cdot \nabla z_h(x)\, dx = 0, \\ \text{for all } z_h \in S(K_\delta, \mathcal{T}_h). \end{array} \right. \tag{5}$$

Here $S(K_\delta, \mathcal{T}_h)$ is a micro FE space on the sampling domain K_δ with micro triangulation $\mathcal{T}_h$, and $v_{H,\mathrm{lin}}$ denotes the linearization of v_H at the quadrature point $x_{K,j}$,

$$v_{H,\mathrm{lin}}(x) = v_H(x_{K,j}) + (x - x_{K,j}) \cdot \nabla v_H(x_{K,j}).$$

Since B_H is elliptic and bounded and $(\cdot, \cdot)_Q$ is a true inner product, the FE-HMM is well-defined for all $H, h > 0$.

For every quadrature node $x_{K,j}$, we must solve the associated micro problem (5) whose solution is then used both for B_H and $(\cdot, \cdot)_Q$. By choosing two different quadrature formulas for $(\cdot, \cdot)_H$ and $(\cdot, \cdot)_M$, the number of micro problems required, and hence the computational cost, remains the same as for the FE-HMM from[3].

4. Numerical Experiments

We again apply our FE-HMM, defined in (4), to (1)–(2) as in Figure 1. We use cubic FE at the macro- and the micro-scale, with mesh sizes $H = 1/75$ and $h = \varepsilon/20 = 1/1000$. Note that linear or quadratic finite elements could also be used. For time-stepping we use a standard Leap-Frog scheme, with $\Delta t = H/10$. As shown in Figure 2, the new FE-HMM succeeds in capturing, the long-time effects in the true solution. In contrast, the solution of the FE-HMM of [3] is unable to capture those dispersive effects, since this solution was proven to converge to the homogenized solution, u^0, as $\varepsilon \to 0$ on finite time intervals.

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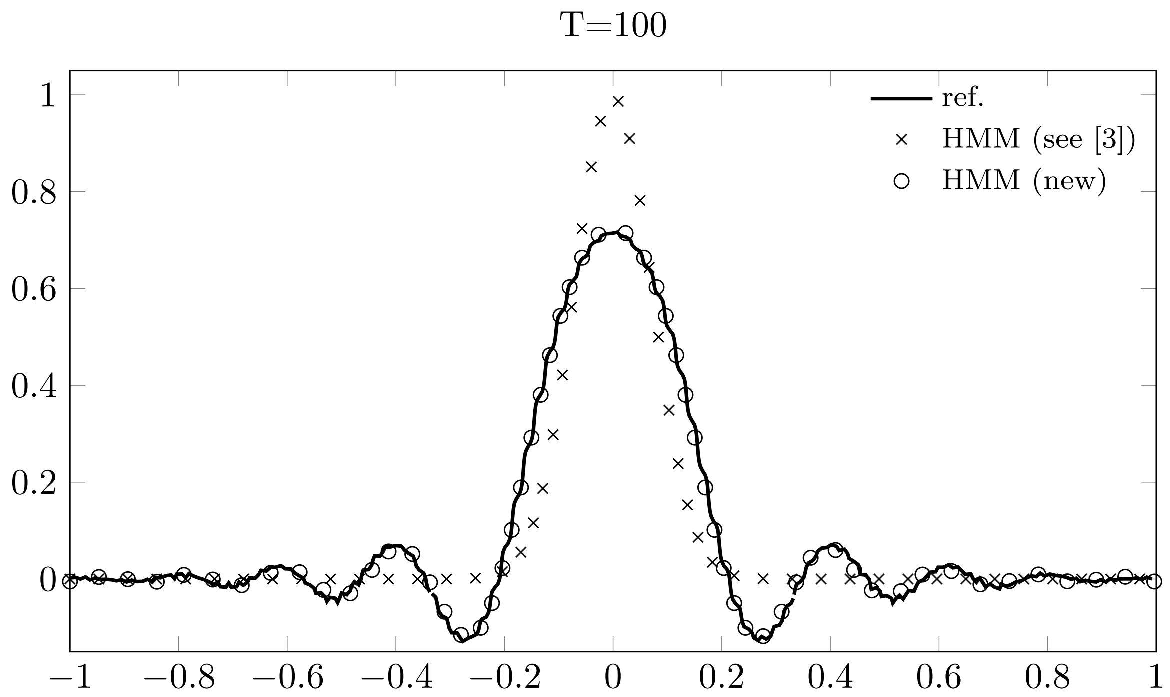

FIGURE 2. Reference solution (ref.), FE-HMM from [3] and new FE-HMM at time $T = 100$.

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Stabilized Galerkin Methods for Magnetic Advection

HOLGER HEUMANN

(joint work with Ralf Hiptmair)

The behavior of electromagnetic fields in the stationary flow field of a conducting fluid can be modelled by the (non-dimensional) advection-diffusion equation [9, Section 5]

$$(1) \qquad \underbrace{\mathbf{curl}\,\nu\,\mathbf{curl}\,\mathbf{A}}_{\text{diffusion}} + \underbrace{\alpha\mathbf{A}}_{\text{dissipation}} + \underbrace{\mathbf{curl}\,\mathbf{A}\times\boldsymbol{\beta} + \mathbf{grad}\,(\mathbf{A}\cdot\boldsymbol{\beta})}_{\text{advection}} = \mathbf{f} \quad \text{in } \Omega\,.$$

Here $\Omega \subset \mathbb{R}^3$ is a bounded domain scaled such that $\operatorname{diam}(\Omega) \approx 1$, and the vector field $\mathbf{A} = \mathbf{A}(\boldsymbol{x})$ stands for the magnetic vector potential. The fluid velocity is $\boldsymbol{\beta} = \boldsymbol{\beta}(\boldsymbol{x})$, of which we assume $\boldsymbol{\beta} \in \boldsymbol{W}^{1,\infty}(\Omega)$ and a scaling that achieves $\max_{\boldsymbol{x}} |\boldsymbol{\beta}(\boldsymbol{x})| \approx 1$. The coefficient $\nu = \nu(\boldsymbol{x}) \geq 0$ controls the strength of magnetic diffusion, whereas the conductivity of the fluid enters through the bounded scalar