Analysis of the factorization method for a general class of boundary conditions

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Abstract

We analyze the factorization method (introduced by Kirsch in 1998 to solve inverse scattering problems at fixed frequency from the farfield operator) for a general class of boundary conditions that generalizes impedance boundary conditions. For instance, when the surface impedance operator is of pseudo-differential type, our main result stipulates that the factorization method works if the order of this operator is different from one and the operator is Fredholm of index zero with non negative imaginary part. We also provide some validating numerical examples for boundary operators of second order with discussion on the choice of the testing function.

1 Introduction

The factorization method is one of the most established inversion method for inverse scattering problems where one is interested in reconstructing the shape of inclusions from the knowledge of the farfield operator at a fixed frequency in the resonant regime [10, 11]. The present work is a contribution to the analysis of the method for a large class of boundary conditions verified on the boundary of the scattering object. We consider here only the scalar case. The analysis of the method for electromagnetic vectorial problems is still an open question.

Let us recall that this method provides a characterization of the obstacle shape using the range of an operator explicitly constructed from the farfield data (available at a fixed frequency, for all observation directions and all plane wave incident directions). Since this characterization (and subsequent algorithm) is independent from the boundary conditions, a natural question would be to specify the class of boundary conditions for which the method works. We remark that as a corollary, one also obtains a uniqueness result for the shape reconstruction without knowledge of the boundary conditions (in the considered class).

Motivated by so-called generalized impedance boundary conditions that model thin layer effects (due to coatings, small roughness, high conductivity, etc...) [16, 2, 7, 8, 9], we consider here the cases of impedance boundary conditions where the impedance corresponds with a boundary operator acting on a Hilbert space with values in its dual. Pseudo-differential boundary operators can be seen as a particular case of this setting. The simplest form (classical impedance boundary conditions) corresponds with a multiplication by a function and has been analyzed in [11]. For non linear inversion methods associated with second order operators we refer to [4, 5].

We first analyze the forward problem using a surface formulation of the scattering problem. This method allows us to show well posedness of the forward scattering problem under weak

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assumptions on the boundary operator. We then analyze the factorization method for this class of operators. We demonstrate in particular that the method is valid for boundary operators "of order" (if we see them as surface pseudo-differential operators) strictly less than one or strictly greater than one, which are of Fredholm type with index zero and having non negative imaginary parts. We observe that in the first case (order < 1), the analysis of the method follows the lines of the case of classical impedance boundary conditions as in [11] where the second case is rather similar to the case of Dirichlet boundary conditions. Our proof however fails in considering boundary operators of order 1 or mixtures of operators of order less than one and operators of order greater than one. We faced here the same difficulty as one encounters in trying to answer the open question related to the factorization method for obstacles with mixture of Neumann and Dirichlet boundary conditions.

For the numerical two dimensional tests, we relied on the use of classical Tikhonov-Morozov regularization strategy to solve the factorization equation. Our numerical experience suggested that this method handles the case of noisy data more easily than the classically used spectral truncation method. We refer however to [13] for tricky numerical investigations on this type of regularization in the case of noisy data. We also use combinations of monopoles and dipoles as test functions for implementing the inversion algorithm. Numerical examples show how this provides better reconstructions as compared with the use of the sole monopole test functions.

The outline of the paper is as follows. The second section is dedicated to the introduction of the forward problem for a general class of impedance boundary conditions and we explain the principle of the factorization method. Section 3 is devoted to the analysis of the forward scattering problem under general assumptions on the boundary operator, with some applicative examples motivated by thin coating models. The analysis of the factorization method is done in section 4. The last section is devoted to numerical implementation of the inversion algorithm and some validating results. An appendix is added for the proof of two technical results.

2 Quick presentation of the forward problem and the factorization method

Let D be an open bounded domain of $\mathbb{R}^d$ for $d = 2, 3$ with Lipschitz boundary Γ. Assume in addition that its complement $\Omega := \mathbb{R}^d \setminus \overline{D}$ is connected. The scattering of an incident wave u_i that is solution to the homogeneous Helmholtz equation in $\mathbb{R}^d$ by a obstacle D gives rise to a field u_s that is solution to

$$\begin{cases} \Delta u_s + k^2 u_s = 0 \text{ in } \Omega, \\ \partial_\nu u_s + Z_k u_s = -\left(\partial_\nu u_i + Z_k u_i\right) \text{ on } \Gamma, \end{cases} \tag{1}$$

and u_s satisfies the radiation condition

$$\lim_{R\to\infty} \int_{|x|=R} |\partial_r u_s - iku_s|^2 = 0. \tag{2}$$

Here ν is the unit outward normal to Γ and Z_k is an impedance surface operator that can possibly depend on the wavenumber k. In the next section we give a precise definition of Z_k (see Definition 3.1) and we prove in section 3 that under few assumptions on Z_k the forward problem is well-posed, i.e. it has a unique solution in a reasonable energy space and this solution depends continuously on the incident wave. In the following, we say that u_s is an outgoing solution to the Helmholtz equation if it satisfies the radiation condition (2). In this case, one can uniquely define the far-field pattern $u^\infty : S^{d-1} \mapsto \mathbb{C}$ associated to the scattered field u_s (see [6] for further details on scattering theory) by

$$u_s(x) = u^\infty(\hat{x})\gamma(d)\frac{e^{ik|x|}}{|x|^{\frac{d-1}{2}}}\left(1 + O\left(\frac{1}{|x|}\right)\right), \ |x| \to \infty,$$

uniformly for $\hat{x} := x/|x| \in S^{d-1}$ where S^{d-1} is the unit sphere of $\mathbb{R}^d$ and $\gamma(2) := \frac{e^{\frac{i\pi}{4}}}{\sqrt{8\pi k}}$ and $\gamma(3) := \frac{1}{4\pi}$. Let us recall that for any bounded set $\mathcal{O}$ of Lipschitz boundary $\partial\mathcal{O}$ such that $D \subset \mathcal{O}$, the following representation formula holds:

$$u^\infty(\hat{x}) = \int_{\partial\mathcal{O}} \left(u_s(y)\frac{\partial e^{-ik\hat{x}\cdot y}}{\partial\nu(y)} - e^{-ik\hat{x}\cdot y}\frac{\partial u_s(y)}{\partial\nu(y)}\right) ds(y), \ \hat{x} \in S^{d-1}.$$

We define $u^\infty(\cdot, \hat{\theta})$ the far field pattern associated to the scattering of an incident plane wave of incident direction $\hat{\theta}$, which is given by $u_i(x) = e^{ik\hat{\theta}\cdot x}$. The inverse problem we consider is then the following: from the knowledge of $\{u^\infty(\hat{x}, \hat{\theta})$ for all $(\hat{x}, \hat{\theta}) \in S^{d-1} \times S^{d-1}\}$ find the obstacle D with minimal *a priori* assumptions on Z_k.

Let us introduce $F : L^2(S^{d-1}) \mapsto L^2(S^{d-1})$, the so-called far field operator defined for $g \in L^2(S^{d-1})$ by

$$(Fg)(\hat{x}) = \int_{S^{d-1}} u^\infty(\hat{x}, \hat{\theta})g(\hat{\theta})d\hat{\theta}, \qquad \text{for } \hat{x} \in S^{d-1}. \tag{3}$$

This operator factorizes in a particular way, there exists two bounded operators G and T, defined in section 4.1, such that

$$F = -GT^*G^*.$$

As we show in Lemma 4.5 the operator G characterizes the obstacle D and we have the equivalence

$$z \in D \Longleftrightarrow \phi_z \in \mathcal{R}(G)$$

for $\phi_z(\hat{x}) := e^{-ik\hat{x}\cdot z}$ and where for any bounded linear operator L defined from an Hilbert space E_1 into another Hilbert space E_2, the space $\mathcal{R}(L) := \{y \in E_2 \mid \exists x \in E_1 \text{ s.t. } y = Lx\}$ is the range of L. As shown in section 4.3, the factorization Theorem applies and gives the equality between the ranges of $F_\#^{\frac{1}{2}}$ and G where $F_\#$ is the self-adjoint and positive operator given by

$$F_\# = |Re(F)| + Im(F).$$

See (6) for a definition of the real and imaginary parts. As a consequence, the far field patterns for all incident plane waves are sufficient to determine the indicator function of the scatterer D thanks to the equivalence

$$z \in D \Longleftrightarrow \phi_z \in \mathcal{R}(F_\#^{\frac{1}{2}}).$$

Remark that this equivalence does not depend on the impedance operator Z_k. Hence, if justified, the factorization method gives a simple way to compute the indicator function independently from the impedance operator.

3 Analysis of the forward problem for a general class of impedance operators

3.1 Mathematical framework

Let $V(\Gamma) \subset L^2(\Gamma)$ equipped with the scalar product $(\cdot, \cdot)_{V(\Gamma)}$ be a Hilbert space and assume that the embedding $C^\infty(\Gamma) \subset V(\Gamma)$ is continuous and dense. For any Hilbert space X we denote by X^* the space of continuous anti-linear forms on X and we define the duality pairings for all $(u,v) \in X \times X^*$ by

$$\langle v, u\rangle_{X^*,X} := v(u), \quad \langle u, v\rangle_{X,X^*} := \overline{v(u)}.$$

From now on, if not needed, we do not specify the spaces for the duality products. We then define an impedance operator on $V(\Gamma)$ as:

Definition 3.1. *For any frequency $k \in \mathbb{C}$ an impedance operator Z_k is a linear bounded operator from $V(\Gamma)$ into $V(\Gamma)^*$.*

In this case, equations (1)-(2) make sense for any u_s in

$$H^V(\Omega) := \{v \in \mathcal{D}'(\Omega) \mid \forall \phi \in \mathcal{D}(\mathbb{R}^d),\ \phi v \in H^1(\Omega) \text{ and } \gamma_0 v \in V(\Gamma)\}$$

where γ_0 is the trace operator on Γ. Problem (1)-(2) can be seen as a particular case of: find $u_s \in H^V(\Omega)$ such that

$$\begin{cases} \Delta u_s + k^2 u_s = 0 \text{ in } \Omega, \\ \partial_\nu u_s + Z_k u_s = f \text{ on } \Gamma, \\ \lim\limits_{R\to\infty} \displaystyle\int_{|x|=R} |\partial_r u_s - iku_s|^2 = 0 \end{cases} \tag{4}$$

with $f := -(\partial_\nu u_i + Z_k u_i)$. In fact, this problem makes sense whenever the right-hand side f is in the dual space $V^{-\frac{1}{2}}(\Gamma)$ of the Hilbert space $V^{\frac{1}{2}}(\Gamma) := V(\Gamma) \cap H^{\frac{1}{2}}(\Gamma)$ equipped with the scalar product $(\cdot,\cdot)_{V^{\frac{1}{2}}(\Gamma)} := (\cdot,\cdot)_{V(\Gamma)} + (\cdot,\cdot)_{H^{\frac{1}{2}}(\Gamma)}$. The aim of this section is to determine conditions on Z_k under which (4) is well posed in the sense that for all $f \in V^{\frac{1}{2}}(\Gamma)$, it has a unique solution in $H^V(\Omega)$ and for any bounded open subset K of Ω, there exists a constant $C_K > 0$ such that

$$\|u_s\|_{V(\Gamma)} + \|u_s\|_{H^1(K)} \leq C_K \|f\|_{V^{-\frac{1}{2}}(\Gamma)}.$$

To do so we first assume that Z_k satisfies

$$\operatorname{Im}(Z_k) \geq 0, \tag{5}$$

which corresponds with the physical assumption that the model does not produce energy. For all Hilbert spaces E, F and all linear and continuous operator $T : E \mapsto F$ its adjoint $T^* : F^* \mapsto E^*$ is defined by $\langle T^* u, v\rangle_{E^*,E} = \langle u, Tv\rangle_{F^*,F}$ for $(u,v) \in F^* \times E$. Its real and imaginary parts are then given by

$$\operatorname{Re}(T) := \frac{T + T^*}{2} \quad \text{and} \quad \operatorname{Im}(T) := \frac{T - T^*}{2i}. \tag{6}$$

Finally, since $C^\infty(\Gamma) \subset V(\Gamma)$ then the embedding of $V^{\frac{1}{2}}(\Gamma)$ into $H^{\frac{1}{2}}(\Gamma)$ is dense, thus we have the following dense and continuous embeddings:

$$V(\Gamma) \subset L^2(\Gamma) \subset V(\Gamma)^*$$

and

$$V^{\frac{1}{2}}(\Gamma) \subset H^{\frac{1}{2}}(\Gamma) \subset L^2(\Gamma) \subset H^{-\frac{1}{2}}(\Gamma) \subset V^{-\frac{1}{2}}(\Gamma).$$

3.2 An equivalent surface formulation of the problem

In order to optimize the hypothesis on Z_k, we propose to rewrite the problem (4) as a surface problem on the boundary Γ. To do so we use the so-called exterior Dirichlet-to-Neumann map $n_e : H^{\frac{1}{2}}(\Gamma) \mapsto H^{-\frac{1}{2}}(\Gamma)$ defined for $g \in H^{\frac{1}{2}}(\Gamma)$ by $n_e h = \partial_\nu u_h$ where $u_h \in H^1_{\text{loc}}(\Omega) := \{v \in \mathcal{D}'(\Omega) \mid \forall \phi \in \mathcal{D}(\mathbb{R}^d),\ \phi v \in H^1(\Omega)\}$ is the unique solution to

$$\begin{cases} \Delta u_h + k^2 u_h = 0 \text{ in } \Omega, \\ u_h = h \text{ on } \Gamma, \\ \lim_{R\to\infty} \int_{|x|=R} |\partial_r u_h - iku_h|^2 = 0. \end{cases} \tag{7}$$

For f in $V^{-\frac{1}{2}}(\Gamma)$, problem (4) then writes: find $u_\Gamma \in V^{\frac{1}{2}}(\Gamma)$ such that

$$Z_k u_\Gamma + n_e u_\Gamma = f \tag{8}$$

and this formulation is equivalent to (4). The surface formulation that we adopt here is similar to the one used in [18] for the study of second order surface operators.

Lemma 3.2. *For f in $V^{-\frac{1}{2}}(\Gamma)$, if $u_s \in H^V(\Omega)$ is a solution of (4) then $\gamma_0 u_s \in V^{\frac{1}{2}}(\Gamma)$ is a solution of (8). Reciprocally if $u_\Gamma \in V^{\frac{1}{2}}(\Gamma)$ is a solution of (8) then the unique solution $u_s \in H^1_{loc}(\Omega)$ of (7) with $h = u_\Gamma$ is a solution of (4).*

Proof. Take $f \in V^{-\frac{1}{2}}(\Gamma)$, if $u_s \in H^V(\Omega)$ is a solution of (4) then $\gamma_0 u_s$ belongs to $V^{\frac{1}{2}}(\Gamma)$. Moreover

$$\partial_\nu u_s = n_e(\gamma_0 u_s)$$

since u_s is an outgoing solution to the Helmholtz equation outside D and therefore $\gamma_0 u_s$ is a solution of (4).

Reciprocally assume that $u_\Gamma \in V^{\frac{1}{2}}(\Gamma)$ is a solution of (8) and define $u_s \in H^1_{\text{loc}}(\Omega)$ the unique solution of (7) with $h = u_\Gamma$. Thus $u_s \in H^V(\Omega)$ is a solution of (4) because we have:

$$\partial_\nu u_s = n_e u_\Gamma = f - Z_k u_s.$$

□

3.3 Existence and uniqueness of solutions

To prove existence and uniqueness of the solution to problem (4) we use the surface formulation (8) and prove that it is of Fredholm type. Then the following Lemma on uniqueness is sufficient to also have existence and continuous dependance of the solution with respect to the right-hand side.

Lemma 3.3. *The operator $Z_k + n_e : V^{\frac{1}{2}}(\Gamma) \mapsto V^{-\frac{1}{2}}(\Gamma)$ is injective.*

Proof. Thanks to Lemma 3.2 it is sufficient to prove uniqueness for the volume equation (4). Let u_s be a solution of (4) with $f = 0$ then we have

$$\langle \partial_\nu u_s, u_s\rangle + \langle Z_k u_s, u_s\rangle = 0.$$

Hypothesis (5) implies:

$$\text{Im}\langle \partial_\nu u_s, u_s\rangle \le 0. \tag{9}$$

Let us denote by B_r an open ball of radius r which contains $\overline{D}$. A Green's formula in $\Omega_r := B_r \backslash \overline{D}$ yields:

$$\operatorname{Im}\left(\int_{\partial B_r} \partial_r u_s \overline{u_s} ds\right) = \operatorname{Im}\langle \partial_\nu u_s, u_s\rangle \leq 0,$$

by (9). Thanks to the Rellich lemma [6], $u_s = 0$ outside B_r and by unique continuation we have $u_s = 0$ in Ω. □

It still remains to give some hypothesis under which $Z_k + n_e$ is of Fredholm type. To do so, we need the following fundamental properties of the Dirichlet-to-Neumann map.

Proposition 3.4. *There exists a bounded operator C_e from $H^{\frac{1}{2}}(\Gamma)$ into $H^{-\frac{1}{2}}(\Gamma)$ which satisfies the following coercivity and sign conditions: $\exists c_e > 0$ such that for all $x \in H^{\frac{1}{2}}(\Gamma)$:*

$$\begin{cases} Re\langle C_e x, x\rangle \geq c_e \|x\|^2_{H^{\frac{1}{2}}(\Gamma)}, \\ Im\langle C_e x, x\rangle \leq 0, \end{cases}$$

and a compact operator K_e from $H^{\frac{1}{2}}(\Gamma)$ into $H^{-\frac{1}{2}}(\Gamma)$ such that:

$$n_e = -C_e + K_e.$$

The proof of this Proposition is classical and is given in the appendix for the reader convenience.

Corollary 3.5. *If one of the following assumptions is fulfilled*

1. *the operator Z_k writes as $-C_Z + K_Z$ where $C_Z : V^{\frac{1}{2}}(\Gamma) \mapsto V^{-\frac{1}{2}}(\Gamma)$ satisfies:*
$$Re\langle C_Z x, x\rangle - Im\langle C_Z x, x\rangle \geq c_z \|x\|^2_{V(\Gamma)} \ \forall x \in V(\Gamma), \tag{10}$$
for $c_z > 0$ independent of x and $K_Z : V^{\frac{1}{2}}(\Gamma) \mapsto V^{-\frac{1}{2}}(\Gamma)$ is a compact operator,
2. *the space $V(\Gamma)$ is compactly embedded into $H^{\frac{1}{2}}(\Gamma)$ and $Z_k : V(\Gamma) \mapsto V(\Gamma)^*$ writes as $T_Z + K_Z$ for some isomorphism $T_Z : V(\Gamma) \mapsto V(\Gamma)^*$ and compact operator $K_Z : V(\Gamma) \mapsto V(\Gamma)^*$,*
3. *the space $H^{\frac{1}{2}}(\Gamma)$ is compactly embedded into $V(\Gamma)$,*

then $Z_k + n_e$ is an isomorphism from $V^{\frac{1}{2}}(\Gamma)$ into $V^{-\frac{1}{2}}(\Gamma)$.

Proof. Assume that the first assumption is fulfilled. Thanks to Proposition 3.4, there exists a compact operator $K_e : H^{\frac{1}{2}}(\Gamma) \mapsto H^{-\frac{1}{2}}(\Gamma)$ and an operator $C_e : H^{\frac{1}{2}}(\Gamma) \mapsto H^{-\frac{1}{2}}(\Gamma)$ with coercive real part such that $n_e = -C_e + K_e$. For all z in $\mathbb{C}$ we have

$$|z| \geq \frac{|\operatorname{Re}(z)| + |\operatorname{Im}(z)|}{\sqrt{2}},$$

and then for all $x \in V^{\frac{1}{2}}(\Gamma)$

$$|\langle (C_e + C_Z)x, x\rangle| \geq \frac{\operatorname{Re}\langle C_e x, x\rangle + \operatorname{Re}\langle C_Z x, x\rangle - \operatorname{Im}\langle C_Z x, x\rangle - \operatorname{Im}\langle C_e x, x\rangle}{\sqrt{2}},$$

and from (10)

$$|\langle (C_e + C_Z)x, x\rangle| \geq \frac{\operatorname{Re}\langle C_e x, x\rangle + c_z\|x\|^2_{V(\Gamma)} - \operatorname{Im}\langle C_e x, x\rangle}{\sqrt{2}}.$$

Proposition 3.4 yields:

$$|\langle (C_e + C_Z)x, x\rangle| \geq \frac{c_e\|x\|^2_{H^{\frac{1}{2}}(\Gamma)} + c_z\|x\|^2_{V(\Gamma)}}{\sqrt{2}},$$

and then $C_e + C_Z : V^{\frac{1}{2}}(\Gamma) \mapsto V^{-\frac{1}{2}}(\Gamma)$ is coercive on $V^{\frac{1}{2}}(\Gamma)$. Furthermore $K_e + K_Z$ is compact from $V^{\frac{1}{2}}(\Gamma)$ into $V^{-\frac{1}{2}}(\Gamma)$ and

$$Z_k + n_e = -(C_e + C_Z) + K_e + K_Z$$

which implies that $Z_k + n_e$ is a Fredholm type operator of index zero. The conclusion is a direct consequence of the uniqueness Lemma 3.3.

If we suppose that the second assumption is fulfilled, then $V(\Gamma) = V^{\frac{1}{2}}(\Gamma)$ and they have equivalent norms. In addition, $n_e : V^{\frac{1}{2}}(\Gamma) \mapsto V^{-\frac{1}{2}}(\Gamma)$ is a compact operator and then $Z_k + n_e : V^{\frac{1}{2}}(\Gamma) \mapsto V^{-\frac{1}{2}}(\Gamma)$ is clearly a Fredholm type operator of index zero.

Finally if we assume that the last assumption is fulfilled then $H^{\frac{1}{2}}(\Gamma) = V^{\frac{1}{2}}(\Gamma)$ and they have equivalent norms. Thanks to Proposition 3.4 there exist an isomorphism $C_e : H^{\frac{1}{2}}(\Gamma) \mapsto H^{-\frac{1}{2}}(\Gamma)$ and a compact operator $K_e : H^{\frac{1}{2}}(\Gamma) \mapsto H^{-\frac{1}{2}}(\Gamma)$ such that $n_e = -C_e + K_e$. Since $H^{\frac{1}{2}}(\Gamma)$ is compactly embedded into $V(\Gamma)$, the operator $Z_k : H^{\frac{1}{2}}(\Gamma) \mapsto H^{-\frac{1}{2}}(\Gamma)$ is compact and so is $K_e + Z_k : H^{\frac{1}{2}}(\Gamma) \mapsto H^{-\frac{1}{2}}(\Gamma)$. Thus $Z_k + n_e : H^{\frac{1}{2}}(\Gamma) \mapsto H^{-\frac{1}{2}}(\Gamma)$ is a Fredholm type operator of index zero and by Lemma 3.3 it is an isomorphism. □

Remark 3.6 (Example of applications). *We denote by ∇_Γ the surface gradient operator, div_Γ the surface divergence operator which is the L^2 adjoint of $-\nabla_\Gamma$ and $\Delta_\Gamma := div_\Gamma\nabla_\Gamma$ the Laplace Beltrami operator on Γ (see [15] for more details on these surface operators). We consider*

$$Z_k = \Delta_\Gamma \nu_k \Delta_\Gamma - div_\Gamma \mu_k \nabla_\Gamma + \lambda_k$$

with $(\nu_k, \mu_k, \lambda_k) \in (L^\infty(\Gamma))^3$ such that the imaginary parts of the bounded functions ν_k, μ_k and λ_k are non negative and such that the real part of ν_k is positive definite. This example is motivated by high order asymptotic models associated with thin coatings or imperfectly conducting obstacles (see [2, 9]). One can easily check that classical variational techniques cannot let us conclude on the analysis of the scattering problem associated with Z_k while the method developed above, namely point 3 of Corollary 3.5, indicates that this problem is well posed with $V(\Gamma) := H^2(\Gamma)$.

4 Factorization of the far field operator

We shall assume here that one of the hypothesis of Corollary 3.5 is fulfilled so that $Z_k + n_e : V^{\frac{1}{2}}(\Gamma) \mapsto V^{-\frac{1}{2}}(\Gamma)$ is an isomorphism and problem (4) is well posed.

4.1 Formal factorization

In this part we give a formal factorization of the far field operator defined by (3) under the form $F = -GT^*G^*$. To achieve this objective, we proceed as for classical impedance boundary conditions. Let G, H_0 and H_1 be defined by $Gf := u^\infty$ where u^∞ is the far field associated with the solution u_s of (4), $H_0 g := \gamma_0 v_g$ and $H_1 g = \gamma_1 v_g$ where for $g \in L^2(S^{d-1})$,

$$v_g(x) := \int_{S^{d-1}} e^{ik\hat{\theta}\cdot x} g(\hat{\theta})\, ds(\hat{\theta}).$$

The operator γ_1 denotes the normal derivative trace operator defined for $v \in \{w \in H^1_{\rm loc}(\Omega)$ s.t. $\Delta w \in L^2(\Omega)\}$ by

$$\gamma_1 v := \partial_\nu v|_\Gamma.$$

We recall that $F : L^2(S^{d-1}) \to L^2(S^{d-1})$ is defined for $g \in L^2(S^{d-1})$ by

$$(Fg)(\hat{x}) = \int_{S^{d-1}} u^\infty(\hat{x}, \hat{\theta}) g(\hat{\theta}) ds(\hat{\theta})$$

where $u^\infty(., \hat{\theta})$ is the far field pattern associated with the incident wave $x \mapsto e^{ik\hat{\theta}\cdot x}$. For $(\hat{\theta}, \hat{x}) \in (S^{d-1})^2$, this far field writes as $u^\infty(\hat{x}, \hat{\theta}) = -[G(Z_k e^{ik\hat{\theta}\cdot x} + \gamma_1 e^{ik\hat{\theta}\cdot x})](\hat{x})$. Hence, by linearity of the forward problem with respect to the incident waves,

$$(Fg)(\hat{x}) = \int_{S^{d-1}} -[G(Z_k e^{ik\hat{\theta}\cdot x} + \gamma_1 e^{ik\hat{\theta}\cdot x})](\hat{x}) g(\hat{\theta}) ds(\hat{\theta}) = -G\left(Z_k v_g + \gamma_1 v_g\right)(\hat{x})$$

and then we get the first step of the formal factorization:

$$F = -G(Z_k H_0 + H_1). \tag{11}$$

Formally, the adjoints of H_0 and H_1 are respectively given for all q and $\hat{x} \in S^{d-1}$ by:

$$(H_0^* q)(\hat{x}) = \int_\Gamma e^{-ik\hat{x}y} q(y) ds(y), \quad (H_1^* q)(\hat{x}) = \int_\Gamma \frac{\partial e^{-ik\hat{x}y}}{\partial \nu(y)} q(y) ds(y).$$

Therefore,

$$(Z_k H_0 + H_1)^* q(\hat{x}) = \int_\Gamma e^{-ik\hat{x}y} Z_k^* q(y) ds(y) + \int_\Gamma \frac{\partial e^{-ik\hat{x}y}}{\partial \nu(y)} q(y) ds(y). \tag{12}$$

We recognize the far field pattern associated to

$$v := \mathrm{SL_k}(Z_k^* q) + \mathrm{DL_k}(q), \tag{13}$$

where $\mathrm{SL_k}$ et $\mathrm{DL_k}$ are the single and double layer potentials associated to the wave number k:

$$\begin{cases} \mathrm{SL_k}(q)(x) = \displaystyle\int_\Gamma G_k(x-y) q(y) ds(y), \ x \in \mathbb{R}^3 \setminus \Gamma, \\ \mathrm{DL_k}(q)(x) = \displaystyle\int_\Gamma \frac{\partial G_k(x-y)}{\partial \nu(y)} q(y) ds(y), \ x \in \mathbb{R}^3 \setminus \Gamma, \end{cases}$$

and G_k is the Green's outgoing function of $\Delta + k^2$ given for $x \in \mathbb{R}^d$ by

$$G_k(x) = \frac{i}{4} H_0^1(k|x|) \ \text{ if } d = 2; \qquad G_k(x) = \frac{e^{ik|x|}}{4\pi|x|} \text{ if } d = 3,$$

where $H_0^1(k|x|)$ is the Hankel function of the first kind and order zero. The function v defined by (13) is a solution of problem (4) with the right hand-side

$$f = Z_k v + \partial_\nu v = Z_k \mathcal{S}_k Z_k^* q + \mathcal{N}_k q + Z_k \mathcal{D}_k q + \mathcal{D}_k' Z_k^* q,$$

where

$$\begin{array}{llll} \mathcal{S}_k & := \gamma_0 \, \mathrm{SL_k}, & \mathcal{D}_k & := \gamma_0 \, \mathrm{DL_k}, \\ \mathcal{D}_k' & := \gamma_1 \, \mathrm{SL_k}, & \mathcal{N}_k & := \gamma_1 \, \mathrm{DL_k}. \end{array}$$

Thus we can deduce that

$$(Z_k H_0 + H_1)^* = GT, \tag{14}$$

with

$$T := Z_k \mathcal{S}_k Z_k^* + \mathcal{N}_k + Z_k \mathcal{D}_k + \mathcal{D}_k' Z_k^*$$

and from (11) we finally get

$$F = -GT^*G^*. \tag{15}$$

4.2 Appropriate function setting

Thanks to (15), the operator F has the form required by the factorization Theorem [11, Theorem 2.15] that allows us to prove that D is characterized by the range of $F_{\#}^{\frac{1}{2}}$. We just have to find two Hilbert spaces X and Y such that the operators:

$$G : X \longmapsto Y,$$
$$T^* : X^* \longmapsto X,$$

are bounded. The space X has to be a reflexive Banach space such that $X^* \subset U \subset X$ with continuous and dense embedding for some Hilbert space U. The space Y being the data space, the choice $Y = L^2(S^{d-1})$ seems appropriate. For the space U we take $L^2(\Gamma)$, and given that G maps a right-hand side f to the far field pattern of the solution to (4), $V^{-\frac{1}{2}}(\Gamma)$ is a natural choice for X. Thus T has to be bounded from $V^{\frac{1}{2}}(\Gamma)$ onto $V^{-\frac{1}{2}}(\Gamma)$ which is false in general since $Z_k \mathcal{S}_k Z_k^* : V^{\frac{1}{2}}(\Gamma) \mapsto V^{-\frac{1}{2}}(\Gamma)$ is not necessarily bounded. It is in particular the case for the impedance operator $Z_k = \Delta_\Gamma$ which is continuous from $V(\Gamma) = H^1(\Gamma)$ into $H^{-1}(\Gamma)$. To overcome this difficulty, let us introduce the following Hilbert space:

$$\Lambda(\Gamma) := \left\{ u \in V^{\frac{1}{2}}(\Gamma),\ Z_k^* u \in H^{-\frac{1}{2}}(\Gamma) \right\}$$

associated to the scalar product

$$(u, v)_{\Lambda(\Gamma)} = (u, v)_{V^{\frac{1}{2}}(\Gamma)} + (Z_k^* u, Z_k^* v)_{H^{-\frac{1}{2}}(\Gamma)}.$$

We denote by $\| \cdot \|_{\Lambda(\Gamma)}$ the norm associated with its scalar product and this norm is equivalent to the norm

$$||| \cdot |||^2_{\Lambda(\Gamma)} := \| \cdot \|^2_{H^{\frac{1}{2}}(\Gamma)} + \|Z_k^* \cdot \|^2_{H^{-\frac{1}{2}}(\Gamma)}.$$

Indeed, since $Z_k + n_e : V^{\frac{1}{2}}(\Gamma) \mapsto V^{-\frac{1}{2}}(\Gamma)$ is an isomorphism, $(Z_k + n_e)^* : V^{\frac{1}{2}}(\Gamma) \mapsto V^{-\frac{1}{2}}(\Gamma)$ is also an isomorphism and there exists $C > 0$ such that for all v in $\Lambda(\Gamma)$

$$\|v\|_{H^{\frac{1}{2}}(\Gamma)} \leq C \|(Z_k + n_e)^* v\|_{V^{-\frac{1}{2}}(\Gamma)} \leq C \|(Z_k + n_e)^* v\|_{H^{-\frac{1}{2}}(\Gamma)} \leq C \left(\|v\|_{H^{\frac{1}{2}}(\Gamma)} + \|Z_k^* v\|_{H^{-\frac{1}{2}}(\Gamma)} \right),$$

and then there exists $C > 0$ such that for all $v \in \Lambda(\Gamma)$:

$$\frac{1}{C} |||v|||_{\Lambda(\Gamma)} \leq \|v\|_{\Lambda(\Gamma)} \leq C |||v|||_{\Lambda(\Gamma)}.$$

Remark 4.1. *If we have the inclusion $H^{\frac{1}{2}}(\Gamma) \subset V(\Gamma)$ then:*

$$V^{\frac{1}{2}}(\Gamma) = H^{\frac{1}{2}}(\Gamma) \quad \textit{and} \quad \Lambda(\Gamma) = H^{\frac{1}{2}}(\Gamma).$$

With this space, we recover some symmetry for the spaces on which T is defined and the factorization (15) makes sense as proven in the remainder of this section. But first of all let us indicate an important property of $\Lambda(\Gamma)$.

Lemma 4.2. *The space $\Lambda(\Gamma)$ is dense in $V^{\frac{1}{2}}(\Gamma)$ and*

$$\Lambda(\Gamma) \subset V^{\frac{1}{2}}(\Gamma) \subset H^{\frac{1}{2}}(\Gamma) \subset L^2(\Gamma) \subset H^{-\frac{1}{2}}(\Gamma) \subset V^{-\frac{1}{2}}(\Gamma) \subset \Lambda(\Gamma)^*.$$

Moreover, if $V(\Gamma)$ is compactly embedded in $H^{\frac{1}{2}}(\Gamma)$ then $\Lambda(\Gamma)$ is compactly embedded in $V^{\frac{1}{2}}(\Gamma)$.

The proof of this lemma is straightforward and is given in the appendix for the reader convenience.

Proposition 4.3. *The operator* $Z_k + n_e$ *can be extended to an isomorphism from* $H^{\frac{1}{2}}(\Gamma)$ *into* $\Lambda(\Gamma)^*$.

Proof. First, let us prove that $(Z_k+n_e)^*$ is an isomorphism from $\Lambda(\Gamma)$ into $H^{-\frac{1}{2}}(\Gamma)$. By definition of $\Lambda(\Gamma)$ we have:

$$\begin{aligned}\Lambda(\Gamma) &= \left\{u \in V^{\frac{1}{2}}(\Gamma),\ Z_k^* u \in H^{-\frac{1}{2}}(\Gamma)\right\},\\ &= \left\{u \in V^{\frac{1}{2}}(\Gamma),\ (Z_k + n_e)^* u \in H^{-\frac{1}{2}}(\Gamma)\right\},\\ &= ((Z_k + n_e)^*)^{-1}\left(H^{-\frac{1}{2}}(\Gamma)\right). \end{aligned} \tag{16}$$

Since $(Z_k + n_e)^*$ is assumed to be an isomorphism from $V^{\frac{1}{2}}(\Gamma)$ into $V^{-\frac{1}{2}}(\Gamma)$ and $H^{-\frac{1}{2}}(\Gamma) \subset V^{-\frac{1}{2}}(\Gamma)$, (16) implies that $(Z_k + n_e)^* : \Lambda(\Gamma) \mapsto H^{-\frac{1}{2}}(\Gamma)$ is bijective. By definition of the norm on $\Lambda(\Gamma)$ we get the continuity of $(Z_k + n_e)^*$ defined from $\Lambda(\Gamma)$ into $H^{-\frac{1}{2}}(\Gamma)$. Then $(Z_k + n_e)^* : \Lambda(\Gamma) \mapsto H^{-\frac{1}{2}}(\Gamma)$ is an isomorphism and its adjoint $(Z_k + n_e) : H^{\frac{1}{2}}(\Gamma) \mapsto \Lambda(\Gamma)^*$ is also an isomorphism. □

Then the following Lemma is straightforward and completes the factorization of F.

Lemma 4.4. G *and* T *enjoy the following properties,*

- *the operator* G *is bounded from* $\Lambda(\Gamma)^*$ *to* $L^2(S^{d-1})$,
- *the operator* T *is bounded from* $\Lambda(\Gamma)$ *to* $\Lambda(\Gamma)^*$.

Proof. Let us introduce the bounded operator $A^\infty : H^{\frac{1}{2}}(\Gamma) \mapsto L^2(S^{d-1})$ defined by $A^\infty h = u^\infty$ where u^∞ is the far field pattern associated to the scattered field $u_h \in H^1_{\text{loc}}(\Omega)$ solution to the Dirichlet problem (7). We have:

$$G = A^\infty (Z_k + n_e)^{-1} \tag{17}$$

and Proposition 4.3 proves the first point.

Let us recall the definition of T:

$$T = Z_k \mathcal{S}_k Z_k^* + \mathcal{N}_k + Z_k \mathcal{D}_k + \mathcal{D}'_k Z_k^*.$$

The definition of $\Lambda(\Gamma)$ implies that $Z_k : H^{\frac{1}{2}}(\Gamma) \mapsto \Lambda(\Gamma)^*$ and $Z_k^* : \Lambda(\Gamma) \mapsto H^{-\frac{1}{2}}(\Gamma)$ are two bounded operators. Thus using that $\mathcal{S}_k : H^{-\frac{1}{2}}(\Gamma) \mapsto H^{\frac{1}{2}}(\Gamma)$, $\mathcal{N}_k : H^{\frac{1}{2}}(\Gamma) \mapsto H^{-\frac{1}{2}}(\Gamma)$, $\mathcal{D}_k : H^{-\frac{1}{2}}(\Gamma) \mapsto H^{\frac{1}{2}}(\Gamma)$ and $\mathcal{D}'_k : H^{-\frac{1}{2}}(\Gamma) \mapsto H^{-\frac{1}{2}}(\Gamma)$ are bounded (see [14, chapter 7]) we obtain the second point. □

We show in the next section how to use the factorization (15) to determine the obstacle D.

4.3 Characterization of the obstacle using $F_\sharp$

First of all, D is characterized by the range of G as stated in the next Lemma.

Lemma 4.5. *Let* $\phi_z \in L^2(S^{d-1})$ *be defined for* $(z, \hat{x}) \in \mathbb{R}^d \times S^{d-1}$ *by* $\phi_z(\hat{x}) = e^{-ik\hat{x}z}$ *then we have*

$$z \in D \Longleftrightarrow \phi_z \in \mathcal{R}(G).$$

Proof. If $z \in D$, then for $f(x) = G_k(x-z)$ we have

$$\phi_z = -G[(Z_k + \partial_\nu)f]$$

since ϕ_z is the far-field of the outgoing solution $G_k(x-z)$ to the Helmholtz equation outside D. Hence $\phi_z \in \mathcal{R}(G)$.

Reciprocally, assume that $z \notin D$ and let us prove by contradiction that $\phi_z \notin \mathcal{R}(G)$. Assume that there exists $f \in \Lambda(\Gamma)^*$ such that

$$\phi_z = Gf.$$

Then the solution to (4) coincides with the near field associated with ϕ_z, which is $G_k(x-z)$, in the domain $\Omega \setminus \{z\}$. This contradicts the singular behavior of G_k at point 0 that prevents this function to be locally H^1 in the neighborhood of z. □

Now we will prove that the factorization (15) satisfies the hypothesis of the factorization Theorem (see [11, Theorem 2.15]). We mainly have to prove that G is injective with dense range, that $-T^*$ satisfied good sign properties and that its imaginary part is strictly positive injective.

Lemma 4.6. *The operator $G : \Lambda(\Gamma)^* \mapsto L^2\left(S^{d-1}\right)$ is injective and compact with dense range.*

Proof. Thanks to Proposition 4.3, $(Z_k + n_e)^{-1}$ is an isomorphism from $\Lambda(\Gamma)^*$ into $H^{\frac{1}{2}}(\Gamma)$. Moreover from [11, Lemma 1.13] we get that $A^\infty : H^{\frac{1}{2}}(\Gamma) \mapsto L^2(S^{d-1})$ is injective and compact with dense range. Then using factorization (17) we obtain that $G : \Lambda(\Gamma)^* \mapsto L^2\left(S^{d-1}\right)$ is also injective compact with dense range. □

Lemma 4.7. *If one of the two assumptions is fulfilled:*

1. *the space $V(\Gamma)$ is compactly embedded into $H^{\frac{1}{2}}(\Gamma)$,*
2. *the space $H^{\frac{1}{2}}(\Gamma)$ is compactly embedded into $V(\Gamma)$,*

then the real part of rT writes as

$$C_T + K_T,$$

with $r = 1$ if assumption 1. is fulfilled and $r = -1$ if assumption 2. is fulfilled and $C_T : \Lambda(\Gamma) \mapsto \Lambda(\Gamma)^*$ is a self-adjoint and coercive operator i.e.

$$\exists c_T > 0,\ \langle C_T x, x\rangle \geq c_T \|x\|^2_{\Lambda(\Gamma)}\ \forall x \in \Lambda(\Gamma).$$

Moreover $K_T : \Lambda(\Gamma) \mapsto \Lambda(\Gamma)^*$ is compact and so is $Im(T) : \Lambda(\Gamma) \mapsto \Lambda(\Gamma)^*$.

Proof. The operators $\mathcal{S}_i : H^{-\frac{1}{2}}(\Gamma) \mapsto H^{\frac{1}{2}}(\Gamma)$ and $-\mathcal{N}_i : H^{\frac{1}{2}}(\Gamma) \mapsto H^{-\frac{1}{2}}(\Gamma)$ are self-adjoint and coercive (see proof of Lemma 1.14 and Theorem 1.26 of [11]), i.e., there exists two strictly positive constants $c_{\mathrm{SL_i}}$ and $c_{\mathrm{DL_i}}$ such that:

$$\begin{cases} \langle \mathcal{S}_i x, x\rangle \geq c_{\mathrm{SL_i}} \|x\|^2_{H^{-\frac{1}{2}}(\Gamma)} & \forall x \in H^{-\frac{1}{2}}(\Gamma), \\ \langle -\mathcal{N}_i x, x\rangle \geq c_{\mathrm{DL_i}} \|x\|^2_{H^{\frac{1}{2}}(\Gamma)} & \forall x \in H^{\frac{1}{2}}(\Gamma). \end{cases}$$

Assume that $V(\Gamma)$ is compactly embedded into $H^{\frac{1}{2}}(\Gamma)$. From the coercivity results on $\mathcal{S}_i$ and $\mathcal{N}_i$, we get that $C_T := Z_k \mathcal{S}_i Z_k^* - \mathcal{N}_i : \Lambda(\Gamma) \mapsto \Lambda(\Gamma)^*$ is self-adjoint and coercive. Now let us prove that:

$$\mathcal{K} := Z_k(\mathcal{S}_k - \mathcal{S}_i)Z_k^* + \mathcal{N}_k + \mathcal{N}_i + Z_k \mathcal{D}_k + \mathcal{D}_k' Z_k^*,$$

is compact as an operator from $\Lambda(\Gamma)$ onto $\Lambda(\Gamma)^*$. Since $(\mathcal{S}_k - \mathcal{S}_i) : H^{-\frac{1}{2}}(\Gamma) \mapsto H^{\frac{1}{2}}(\Gamma)$ is compact we obtain that $Z_k(\mathcal{S}_k - \mathcal{S}_i)Z_k^* : \Lambda(\Gamma) \mapsto \Lambda(\Gamma)^*$ is compact. The operators

$$\begin{cases} \mathcal{N}_k + \mathcal{N}_i : H^{\frac{1}{2}}(\Gamma) \mapsto H^{-\frac{1}{2}}(\Gamma), \\ Z_k \mathcal{D}_k : H^{\frac{1}{2}}(\Gamma) \mapsto \Lambda(\Gamma)^*, \\ \mathcal{D}'_k Z_k^* : \Lambda(\Gamma) \mapsto H^{-\frac{1}{2}}(\Gamma), \end{cases}$$

are bounded operators and using that $\Lambda(\Gamma)$ is compactly embedded into $H^{\frac{1}{2}}(\Gamma)$ (Lemma 4.2) we can conclude that $\mathcal{K} : \Lambda(\Gamma) \mapsto \Lambda(\Gamma)^*$ is compact and so are its imaginary and real parts.

Now assume that $H^{\frac{1}{2}}(\Gamma)$ is compactly embedded into $V(\Gamma)$. Thanks to Remark 4.1 we have that $\Lambda(\Gamma) = H^{\frac{1}{2}}(\Gamma)$ and then $Z_k, Z_k^* : H^{\frac{1}{2}}(\Gamma) \mapsto H^{-\frac{1}{2}}(\Gamma)$ are compact operator. Define $C_T = -\mathcal{N}_i$, we recall that this operator is self adjoints and coercive and that $\mathcal{D}_k : H^{\frac{1}{2}}(\Gamma) \mapsto H^{\frac{1}{2}}(\Gamma)$ and $\mathcal{D}'_k : H^{-\frac{1}{2}}(\Gamma) \mapsto H^{-\frac{1}{2}}(\Gamma)$ are bounded operator. Furthermore $\mathcal{N}_k - \mathcal{N}_i : H^{\frac{1}{2}}(\Gamma) \mapsto H^{-\frac{1}{2}}(\Gamma)$ is compact and then the operator

$$\mathcal{K} := Z_k \mathcal{S}_k Z_k^* + Z_k \mathcal{D}_k + \mathcal{D}'_k Z_k^* + \mathcal{N}_k - \mathcal{N}_i,$$

is compact as an operator from $H^{\frac{1}{2}}(\Gamma)$ onto $H^{-\frac{1}{2}}(\Gamma)$ and thus the same goes for its real and imaginary parts. □

We still need to prove that the imaginary part of $-T^*$ is strictly positive on $\overline{\mathcal{R}(G^*)}$. To do so, we have to avoid some special values of k for which this may not be true.

Definition 4.8. *We say that k^2 is an eigenvalue of $-\Delta$ associated with the impedance operator Z_k if there is a nonzero $u \in \{v \in H^1(D),\ \gamma_0 v \in V^{\frac{1}{2}}(\Gamma)\}$ solution to*

$$\begin{cases} \Delta u + k^2 u = 0 \ \textit{in } D, \\ \partial_\nu u + Z_k u = 0 \ \textit{on } \Gamma. \end{cases}$$

Lemma 4.9. *If k^2 is not an eigenvalue of $-\Delta$ associated with the impedance operator Z_k^* then $-Im(T^*)$ is strictly positive on $\overline{\mathcal{R}(G^*)}$.*

Proof. First of all, we can prove similarly to [11, Theorem 2.5] that

$$\mathrm{Im}(F) = k|\gamma(d)|^2 F^* F + R_{Z_k}, \tag{18}$$

where $\gamma(2) = \frac{e^{i\pi/4}}{\sqrt{8\pi k}}$ and $\gamma(3) = \frac{1}{4\pi}$, and $R_{Z_k} : L^2(S^{d-1}) \mapsto L^2(S^{d-1})$ is such that for all $(g, h) \in (L^2(S^{d-1}))^2$

$$(R_{Z_k} g, h)_{L^2(S^{d-1})} = \mathrm{Im}\langle Z_k V_g, V_h \rangle.$$

The functions $V_g := v_g^s + v_g$ and $V_h := v_h^s + v_h$ are the total fields associated to the scattering of the Herglotz incident waves $u_i = v_g$ and $u_i = v_h$ respectively, namely v_g^s (resp. v_h^s) satisfies (1)-(2) with $u_i = v_g$ (resp. v_h). For all $x \in \mathcal{R}(G^*)$ with $x = G^* y$ we have

$$\mathrm{Im}\langle -T^* x, x \rangle = \mathrm{Im}\langle -T^* G^* y, G^* y \rangle = \mathrm{Im}\langle F y, y \rangle$$

and by using (18) together with the fact that R_{Z_k} is non negative we get:

$$\mathrm{Im}\langle -T^* x, x \rangle \geq k|\gamma(d)|^2 \|Fy\|^2 \geq k|\gamma(d)|^2 \|GT^* G^* y\|^2 \geq k|\gamma(d)|^2 \|GT^* x\|^2. \tag{19}$$

Since G and T^* are two bounded operators inequality (19) holds for all $x \in \overline{\mathcal{R}(G^*)}$. Hence we simply have to prove that GT^* is injective to obtain the result of the Lemma.

The operators $A^\infty : H^{\frac{1}{2}}(\Gamma) \mapsto L^2(S^{d-1})$ and $(Z_k + n_e)^{-1} : \Lambda(\Gamma)^* \mapsto H^{\frac{1}{2}}(\Gamma)$ are injective, therefore, (17) implies that G is also injective. Hence if T^* is injective, GT^* will also be injective. Let us prove that in fact T and then T^* are isomorphisms. From Lemma 4.7, the operator T is of Fredholm type and index zero, hence we simply have to prove that it is injective to deduce that it is an isomorphism. By (14), the injectivity of T is equivalent to the injectivity of $(Z_k H_0 + H_1)^*$, take $q \in \Lambda(\Gamma)$ such that $(Z_k H_0 + H_1)^* q = 0$, expression (12) leads to

$$\int_\Gamma e^{-ik\hat{x}y} Z_k^* q(y) ds(y) + \int_\Gamma \frac{\partial e^{-ik\hat{x}y}}{\partial \nu(y)} q(y) ds(y) = 0.$$

The left-hand side of this expression is the far field pattern associated to

$$v_+ := \mathrm{SL_k}(Z_k^* q) + \mathrm{DL_k}(q) = 0 \quad \text{ in } \Omega.$$

Let us also define

$$v_- := \mathrm{SL_k}(Z_k^* q) + \mathrm{DL_k}(q) \quad \text{ in } D.$$

Rellich's Lemma implies that $v_+ = 0$ outside D and thanks to the jump conditions for the single and double layer potentials (see [14] for example) we have on Γ:

$$\begin{cases} v_- = v_- - v_+ = -q, \\ \partial_\nu v_- = \partial_\nu v_- - \partial_\nu v_+ = Z_k^* q. \end{cases}$$

The assumption on k^2 implies that $v_- = 0$ inside D and therefore $q = 0$. We finally get that T^* is an isomorphism, which concludes the proof. □

We are now in position to state the main Theorem of this paper.

Theorem 4.10. *If one of the two assumptions is fulfilled:*

1. *the space $V(\Gamma)$ is compactly embedded into $H^{\frac{1}{2}}(\Gamma)$,*
2. *the space $H^{\frac{1}{2}}(\Gamma)$ is compactly embedded into $V(\Gamma)$,*

then provided that k^2 is not an eigenvalue of $-\Delta$ associated to the impedance condition Z_k^* we have:

$$\mathcal{R}(G) = \mathcal{R}(F_\#^{\frac{1}{2}}) \tag{20}$$

and

$$z \in D \Longleftrightarrow \phi_z \in \mathcal{R}(F_\#^{\frac{1}{2}}). \tag{21}$$

Proof. Under these assumptions the results of Lemmas 4.6, 4.7 and 4.9 hold which means that the factorization of the far field operator satisfies all the requirements of [11, Theorem 2.15]. Thus $\mathcal{R}(G) = \mathcal{R}(F_\#^{\frac{1}{2}})$ and we can conclude using Lemma 4.5. □

Remark 4.11. *Identity* (20) *and the decomposition $G = A^\infty(Z_k + n_e)^{-1}$ implies that we can replace the family $(\phi_z)_{z\in\mathbb{R}^d}$ by any family of functions $(\psi_z)_{z\in\mathbb{R}^d}$ satisfying:*

$$z \in D \Longleftrightarrow \psi_z \in \mathcal{R}(A^\infty). \tag{22}$$

An example of these functions is given by the far field pattern of a dipole located at point z with polarization p defined by $\psi_{z,p}(\hat{x}) = p \cdot \hat{x}\phi_z(\hat{x})$. These are the farfields associated with $x \mapsto p \cdot \nabla_x G_k(x - z)$.

Remark 4.12. *In the intermediate case, when none of the compact embeddings ($V(\Gamma) \subset H^{\frac{1}{2}}(\Gamma)$ or $H^{\frac{1}{2}}(\Gamma) \subset V(\Gamma)$) hold, the principal part of operator T fails to be positive. As a matter of fact we are not able to link the range of G with the range of $F_{\#}^{\frac{1}{2}}$.*

To complement this work we should verify that most of the time k^2 is not an eigenvalue of $-\Delta$ associated to the impedance operator Z_k^*.

Theorem 4.13. *Assume that Z_k^* depends analytically on $k \in \mathbb{C}$. Under one of the assumptions of Theorem 4.10, the set of eigenvalues of $-\Delta$ associated to the impedance operator Z_k^* is discrete and the only possible accumulation point is $+\infty$.*

Proof. In order to prove that, we use the Analytic Fredholm Theorem ([6, Theorem 8.19]). If $-k^2$ is not a Dirichlet eigenvalue in D we can define the interior Dirichlet-to-Neumann map $n_i(k) : H^{\frac{1}{2}}(\Gamma) \mapsto H^{-\frac{1}{2}}(\Gamma)$ for $g \in H^{\frac{1}{2}}(\Gamma)$ by $n_i(k)g = \partial_\nu u$ where u is the unique solution to the Helmholtz' equation in D with $u = g$ on Γ. If $Z_k^* + n_i(k)$ is injective, then $-k^2$ is not an eigenvalue associated with Z_k^*. In the remaining of the proof, we show that $Z_k^* + n_i(k)$ fails to be injective only for k in a discrete set of $(0, +\infty)$.

Let k_0 be a complex number such that: $k_0^2 = i$, and assume that the embedding of $V(\Gamma)$ into $H^{\frac{1}{2}}(\Gamma)$ is compact. Then,

$$Z_k^* + n_i(k) = T_k + n_i(k) - n_i(k_0), \tag{23}$$

where $T_k : V^{\frac{1}{2}}(\Gamma) \mapsto V^{-\frac{1}{2}}(\Gamma)$ is defined by

$$T_k := Z_k^* + n_i(k_0).$$

Let us prove that T_k is an isomorphism. Following the lines of the proof of points 2 and 3 of Corollary 3.5 one easily observe that T_k is Fredholm with index 0. Let us prove that $T_k : V^{\frac{1}{2}}(\Gamma) \mapsto V^{-\frac{1}{2}}(\Gamma)$ is injective. Take $g \in V^{\frac{1}{2}}(\Gamma)$ such that $T_k g = 0$ and $u_g \in H^1(D)$ such that

$$\begin{cases} \Delta u_g + k_0^2 u_g = 0 \text{ in } D, \\ u_g = g \text{ on } \Gamma. \end{cases}$$

Then $\partial_\nu u_g = n_i(k_0)g = -Z_k^* u_g$ and Green's formula yields:

$$\int_D |\nabla u_g|^2 dx - k_0^2 \int_D |u_g|^2 dx + \langle Z_k^* u_g, u_g \rangle = 0.$$

By taking the imaginary part of this last equation we have:

$$\text{Im}(k_0^2) \int_D |u_g|^2 dx - \text{Im}\langle Z_k^* u_g, u_g \rangle = 0.$$

Since we supposed that $\text{Im}(Z_k) \geq 0$ and $k_0^2 = i$, we get

$$\int_D |u_g|^2 dx = 0,$$

which gives $g = 0$ and T_k is an isomorphism. From (23) we have:

$$Z_k + n_i(k) = T_k(I + T_k^{-1}(n_i(k) - n_i(k_0)))$$

and then $Z_k + n_i(k) : V^{\frac{1}{2}}(\Gamma) \mapsto V^{-\frac{1}{2}}(\Gamma)$ is injective if and only if the operator

$$I + T_k^{-1}(n_i(k) - n_i(k_0)) : V^{\frac{1}{2}}(\Gamma) \mapsto V^{\frac{1}{2}}(\Gamma),$$

is injective. The parameter dependent family of operators $n_i(k)$ depends analytically on k over the complement of the Dirichlet eigenvalue in D and we supposed that Z_k^* depends analytically on k then $T_k^{-1}(n_i(k) - n_i(k_0))$ depends analytically on k. Since this operator is bounded from $H^{\frac{1}{2}}(\Gamma) \mapsto V^{\frac{1}{2}}(\Gamma)$ this operator is a compact operator from $V^{\frac{1}{2}}(\Gamma) \mapsto V^{\frac{1}{2}}(\Gamma)$. Thus we can apply the the analytic Fredholm Theorem to

$$I + T_k^{-1}(n_i(k) - n_i(k_0)).$$

For $k = k_0$ this operator is injective and then one can conclude that $I + T_k^{-1}(n_i(k) - n_i(k_0))$ is injective except for a discrete a discrete set of k. Thus the set of the eigenvalues of $-\Delta$ associated to the impedance condition Z_k^* is discrete with $+\infty$ for only possible accumulation point.

□

5 Numerical tests for second order surface operators

Motivated by models for thin coatings we consider in this section impedance operators of the form

$$Z_k = \text{div}_\Gamma \mu \nabla_\Gamma - \lambda \tag{24}$$

where $(\mu, \lambda) \in (L^\infty(\Gamma))^2$ depend analytically on k with non positive imaginary parts and where the real part of μ is positive definite or negative definite. Then, Theorems 4.10 and 4.13 apply and except for a discrete set of wave numbers k, the following equation

$$F_\#^{\frac{1}{2}} g_z = \phi_z, \tag{25}$$

has a solution if and only if $z \in D$ whenever ϕ_z is a valid right-hand side to apply Theorem 4.10 (see Remark 4.11). We shall give in the following some two dimensional numerical experiments that show how this test behaves when the far-field data are given by an obstacle characterized by a generalized impedance boundary condition of the form (24).

5.1 Numerical setup and regularization procedure

Here we identify the unit sphere S^1 of $\mathbb{R}^2$ with the interval $]0, 2\pi[$. The data set we consider is not the operator F itself but a discrete version that we represent by the matrix $\mathbb{U}_n := (u_{ij}^n)_{1\le i\le n, 1\le j\le n}$ where u_{ij}^n is an approximation of $u^\infty\left(\frac{2\pi(i-1)}{n}, \frac{2\pi(j-1)}{n}\right)$. For $\hat{\theta} \in S^1$, $u^\infty(., \hat{\theta})$ is the far field associated to u_s which is the unique solution of (1)-(2) with u_i defined for $x \in \mathbb{R}^2$ by $u_i(x) := e^{ik\hat{\theta}\cdot x}$.

In practice, the matrix $\mathbb{U}_n$ contains the values of a synthetic far field computed with a finite element method. We also contaminate these data by adding some random noise and build a noisy far field matrix $\mathbb{U}_n^\delta := (u_{ij}^{n,\delta})_{1\le i\le n, 1\le j\le n}$ where

$$u_{ij}^{n,\delta} = u_{ij}^n(1 + \eta(X_1^{ij} + iX_2^{ij}))$$

for $\eta > 0$ and $(X_k^{ij})_{1\le i\le n, 1\le j\le n, k\in\{1,2\}}$ are uniform random variables on $[-1, 1]$. We denote by $\Phi_z^n \in \mathbb{C}^n$ the vector defined $1 \le i \le n$, $\Phi_z^n(i) = \phi_z(\frac{2\pi i}{n})$ where ϕ_z is a valid right-hand side to apply Theorem 4.10 (see Remark 4.11). We define the real symmetric matrix $(\mathbb{U}_n^\delta)_\# := |\mathrm{Re}(\mathbb{U}_n^\delta)| + |\mathrm{Im}(\mathbb{U}_n^\delta)|$ and denote by $(e_i^n, \lambda_i^n)_{1\le i\le n}$ its eigenvectors and eigenvalues. Since this matrix is positive its eigenvalues are positive, we assume in addition that they are non zero. Then, Theorem 4.10 and the Picard criterion suggest to use the function

$$z \mapsto \left(\sum_{j=1}^{n} \frac{1}{\lambda_j^n} |(e_j^n, \phi_z^n)|^2 \right)^{-1} \tag{26}$$

as an indicator function for the obstacle D since it should have small values for any z outside the obstacle D and greater values for z inside D. The far field operator F is compact, the values λ_j^n are very small for large j in $\mathbb{N}$ and then the computation of this indicator function is unstable. Thus we prefer to use the function:

$$w_n(z) := \left(\sum_{j=1}^{n} \frac{\lambda_j^n}{(\alpha_n + \lambda_j^n)^2} |(e_j^n, \phi_z^n)|^2 \right)^{-1}, \tag{27}$$

where α_n is the regularization coefficient. We remark that $w_n(z)$ is nothing but the inverse of the squared norm of the unique solution to the discrete problem with Tikhonov regularization

$$(\alpha_n I + (\mathbb{U}_n^\delta)_\#) g_n(z) = (\mathbb{U}^\delta)_\#^{\frac{1}{2}} \phi_z^n.$$

The regularization coefficient α_n is chosen by using the Morozov discrepancy principle which is: choose α_n as the unique solution of find $\alpha \in (0, \delta_n \lambda_1^n]$ such that

$$\|(\mathbb{U}_n^\delta)_\#^{\frac{1}{2}} g_n(z) - \phi_z^n\|_{L^2(S^1)} = \delta_n \|g_n(z)\|_{L^2(S^1)}$$

or equivalently

$$\sum_{j=1}^{n} \frac{\alpha^2 - \delta_n^2 |\lambda_j^n|}{|\alpha + \lambda_j^n|^2} |(e_j^n, \phi_z^n)|^2 = 0,$$

where $\delta_n > 0$ is a bound on the noise on the operator $F_\#^{1/2}$ and the right hand side ϕ_z. It has been proven in [10] (see also [17]) that the function $w_n(z)$ is an indicator function of D when n tends to infinity and the noise δ_n to 0.

5.2 Numerical experiments

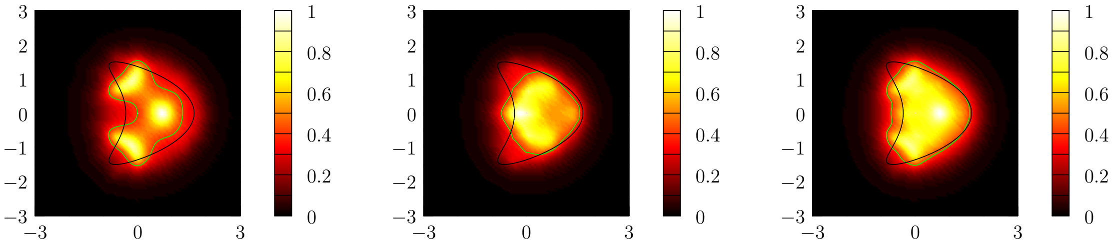
Let $w_n(z)$ be the function given by (27) whith the test function $\phi_z = \psi_z / \|\psi_z\|_{L^2}$ with $\phi_z(\hat{x}) = e^{-ik\hat{x}\cdot z}$ and $w^\theta_{n,\mathrm{dipole}}(z)$ the function given by (27) when $\phi_z = \psi_z / \|\psi_z\|_{L^2}$ with $\psi_z(\hat{x}) = p(\theta) \cdot \hat{x} e^{-ik\hat{x}\cdot z}$, with $p(\theta) = (\cos(\theta), \sin(\theta))^t$. We choose to take 4 directions of the polarization p and we define

$$w_{n,\mathrm{dipole}}(z) := \min_{\theta \in \{0, \pi/4, 3\pi/4, \pi\}} w^\theta_{n,\mathrm{dipole}}(z).$$

Remark that thanks to the linearity of the farfield equation, taking θ in the interval $[0, \pi]$ is equivalent to take it in $[0, 2\pi]$. We finally plot the values of

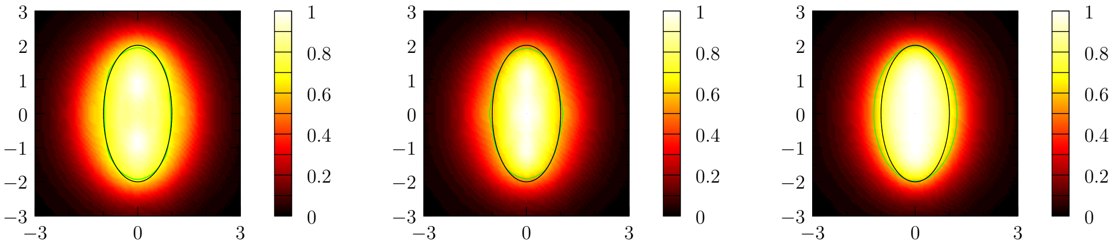
$$W_n(z) := w_n(z) + w_{n,\mathrm{dipole}}(z)$$

in order to determine the indicator function of the scatterer. We refer to [3, 1] for a discussion on this choice of indicator function based on the analysis of impedance boundary conditions for



(a) w_n (b) $w_{n,\text{dipole}}$ (c) W_n

Figure 1: Comparison of different right-hand sides, with $\mu = 0.1$, $\lambda = 0$, $k = 2$ and no noise.



(a) ellipse, $\mu = 0.01$ (b) ellipse, $\mu = 0.1$ (c) ellipse, $\mu = 10$

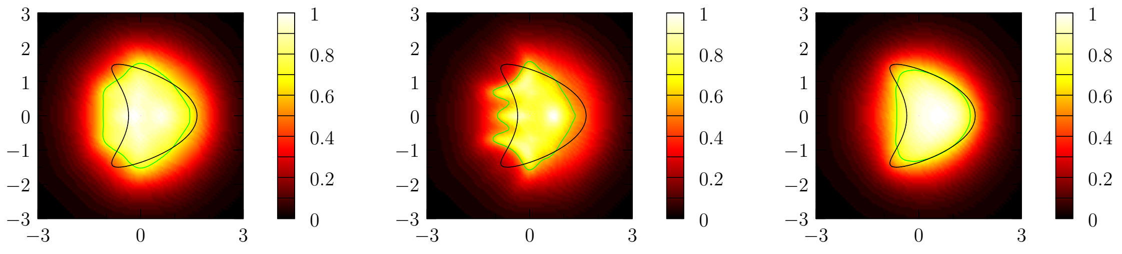
Figure 2: Reconstruction of a convex geometry for several μ, $k = 2$ with 1% of noise.

cracks. Numerical illustration of the importance of this choice is given by Figure 1 where we compare the plots of w_n, $w_{n,\text{dipole}}$ and W_n for $\mu = 0.1$. The reconstruction provided by W_n is much better than the two others. The searched domain (the black line on the plot) is given by

$$\partial D = \{(\cos(t) + 0.65\cos(2t), 1.5\sin(t)),\ 0 \leq t \leq 2\pi\}. \tag{28}$$

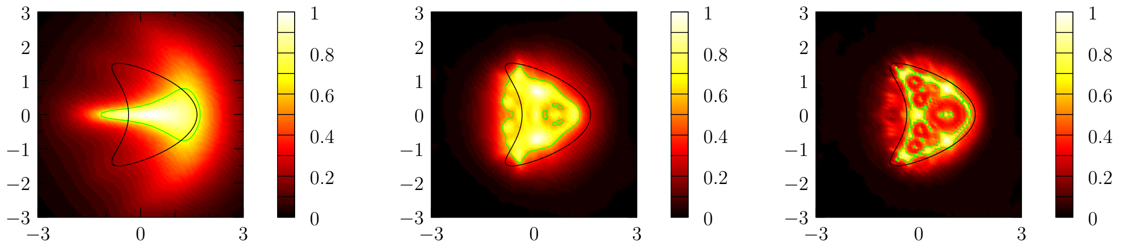
The green line represents the isoline of the function $W_n(z)$ that visually fits better the unknown shape ∂D. The size of D is of the order of one wavelength when the wavenumber k is equal to 2. The sampling domain is the square $[-3, 3]^2$ which is discretized with 80×80 points z. Finally, we take $n = 50$, which means that we send 50 incident waves uniformly distributed on the unit circle and we observe the far field at all these directions. For noisy data, we take $\eta = 1\%$. We restrict ourselves to the cases $\lambda = 0$ but the results for $\lambda \neq 0$ are similar.

Figures 2 and 3 show the influence of the generalized impedance coefficient μ on two different geometries: the first one is convex and is an ellipse of semi-minor axis 1 and semi-major axis 2, the second is the non-convex obstacle (Kite shape) given by (28). For large values and small values of μ ($\mu = 10$ and $\mu = 0.01$) the reconstruction is quite satisfactory (see Figures 2(c), 3(c), 2(a) and 3(a)) while for an intermediate value ($\mu = 0.1$) the reconstruction is poorer (see Figures 2(b) and 3(b)). Let us also mention that the noiseless reconstruction (Figure 1(c)) for $\mu = 0.1$ is quite accurate which means that in this case the reconstruction is sensitive to the noise. A possible explanation is that the test functions ϕ_z we use are not well-suited to this case. Figure 4 shows the influence of the frequency, these figures should be compared to the Figure 3(b) and, as we would expect, we increase the precision of the reconstruction by increasing the frequency.



(a) $\mu = 0.01$ (b) $\mu = 0.1$ (c) $\mu = 10$

Figure 3: Reconstruction of a non-convex geometry for several μ, $k = 2$ with 1% of noise.



(a) $k = 1$ (b) $k = 4$ (c) $k = 8$

Figure 4: Reconstruction of a non-convex geometry for several k, $\mu = 0.1$ with 1% of noise.

Appendix

Proof of Proposition 3.4. Let us denote by B_r an open ball of radius r which contains $\overline{D}$. The exterior Dirichlet-to-Neumann map on ∂B_r, $S_r : H^{\frac{1}{2}}(\partial B_r) \mapsto H^{-\frac{1}{2}}(\partial B_r)$ is defined for $g \in H^{\frac{1}{2}}(\partial B_r)$ by $S_r g = \partial_r u^e|_{\partial B_r}$ where u^e is the outgoing solution of the Helmholtz' equation outside B_r and $u^e = g$ on ∂B_r. Thanks to the appendix of [12] for $d = 2$ and [15, Theorem 2.6.4] for $d = 3$ we recall that the operator S_r satisfies

$$\mathrm{Re}\langle S_r u, u\rangle \leq 0, \ \forall u \in H^{\frac{1}{2}}(\partial B_r), \tag{29}$$

$$\mathrm{Im}\langle S_r u, u\rangle \geq 0, \ \forall u \in H^{\frac{1}{2}}(\partial B_r). \tag{30}$$

Let us denote $\Omega_r := (\mathbb{R}^3 \setminus \overline{D}) \cap B_r$ and define the bounded operator $A_r : H^{\frac{1}{2}}(\Gamma) \mapsto H^1(\Omega_r)$ for $g \in H^{\frac{1}{2}}(\Gamma)$ by $A_r g := u_r$ where u_r is the unique solution of:

$$\begin{cases} \Delta u_r + k^2 u_r = 0 \text{ in } \Omega_r, \\ u_r = g \text{ on } \Gamma, \\ \partial_r u_r = S_r u_r \text{ on } \partial B_r. \end{cases} \tag{31}$$

Let us introduce $C_e : H^{\frac{1}{2}}(\Gamma) \mapsto H^{-\frac{1}{2}}(\Gamma)$ and $K_e : H^{\frac{1}{2}}(\Gamma) \mapsto H^{-\frac{1}{2}}(\Gamma)$ defined by

$$\langle C_e f, g\rangle = (A_r f, A_r g)_{H^1(\Omega_r)} - \langle S_r f, g\rangle,$$

and

$$\langle K_e f, g\rangle = (k^2 + 1)\int_{\Omega_r} A_r f \overline{A_r g} dx,$$

for all $(f, g) \in H^{\frac{1}{2}}(\Gamma) \times H^{\frac{1}{2}}(\Gamma)$. Using Green's formula and the Helmholtz' equation satisfied by $A_r f$ and $A_r g$ in Ω_r, we prove that

$$n_e = -C_e + K_e.$$

Let us prove that C_e has a coercive real part and that K_e is compact. Remark that

$$\mathrm{Re}\langle C_e f, f\rangle = \|A_r f\|^2_{H^1(\Omega_r)} - \mathrm{Re}\langle S_r f, f\rangle, \tag{32}$$

but $\mathrm{Re}\langle S_r f, f\rangle \leq 0$ by (29). The range of A_r is given by:

$$\mathcal{R}(A_r) = \{u \in H^1(\Omega_r), \ \Delta u + k^2 u = 0 \text{ and} S_r u = \partial_\nu u \text{ on } \partial B_r\},$$

and this space is a closed subspace of $H^1(\Omega_r)$. Hence, since A_r is injective there exists $C > 0$ such that

$$\|A_r f\|_{H^1(\Omega_r)} \geq C\|f\|_{H^{\frac{1}{2}}(\Gamma)}, \ \forall f \in H^{\frac{1}{2}}(\Gamma).$$

In regard of (32), we deduce that C_e has a coercive real part. We also have

$$\mathrm{Im}\langle C_e f, g\rangle = -\mathrm{Im}\langle S_r f, g\rangle \leq 0$$

by (30).

To conclude remark that $A_r : H^{\frac{1}{2}}(\Gamma) \mapsto H^1(B_r)$ is bounded, and since the embedding of $H^1(\Omega_r)$ into $L^2(\Omega_r)$ is compact, $K_e : H^{\frac{1}{2}}(\Gamma) \mapsto H^{-\frac{1}{2}}(\Gamma)$ is compact. □

Proof of lemma 4.2. Let us prove that this space is dense into $V^{\frac{1}{2}}(\Gamma)$. Take $x \in V^{\frac{1}{2}}(\Gamma)$ and $\epsilon > 0$. The space $L^2(\Gamma)$ is densely embedded in $V^{-\frac{1}{2}}(\Gamma)$ and then there exists $y_\epsilon \in L^2(\Gamma)$ such that:

$$\|(Z_k + n_e)^* x - y_\epsilon\|_{V^{-\frac{1}{2}}(\Gamma)} \leq \frac{\epsilon}{\| \left((Z_k + n_e)^{-1}\right)^* \|_{\mathcal{L}\left(V^{-\frac{1}{2}}(\Gamma), V^{\frac{1}{2}}(\Gamma)\right)}}$$

where for any Hilbert space E, $\mathcal{L}(E, E^*)$ is the space of linear and bounded applications from E to E^*. Thus we have for $z_\epsilon := \left((Z_k + n_e)^{-1}\right)^* y_\epsilon$

$$\|x - z_\epsilon\|_{V^{\frac{1}{2}}(\Gamma)} \leq \| \left((Z_k + n_e)^{-1}\right)^* \|_{\mathcal{L}\left(V^{-\frac{1}{2}}(\Gamma), V^{\frac{1}{2}}(\Gamma)\right)} \|(Z_k + n_e)^* x - y_\epsilon\|_{V^{-\frac{1}{2}}(\Gamma)},$$

and we get

$$\|x - z_\epsilon\|_{V^{\frac{1}{2}}(\Gamma)} \leq \epsilon.$$

But $z_\epsilon \in V^{\frac{1}{2}}(\Gamma)$ since $y_\epsilon \in L^2(\Gamma)$ and we have

$$Z_k^* z_\epsilon = (Z_k + n_e)^* z_\epsilon - n_e^* z_\epsilon = y_\varepsilon - n_e^* z_\epsilon \in H^{\frac{1}{2}}(\Gamma).$$

Hence we obtain the density result.

Assume that $V(\Gamma)$ is compactly embedded into $H^{\frac{1}{2}}(\Gamma)$ and let us prove that $\Lambda(\Gamma)$ is also compactly embedded into $V^{\frac{1}{2}}(\Gamma)$. Let $(x_n)_{n\in\mathbb{N}}$ be bounded a sequence in $\Lambda(\Gamma)$. We have

$$\begin{aligned}\|(Z_k + n_e)^* x_n\|^2_{H^{-\frac{1}{2}}(\Gamma)} &\leq \left(\|Z_k^* x_n\|_{H^{-\frac{1}{2}}(\Gamma)} + \|n_e^* x_n\|_{H^{-\frac{1}{2}}(\Gamma)}\right)^2, \\ &\leq 2 \max\left(1, \|n_e\|_{\mathcal{L}\left(H^{\frac{1}{2}}(\Gamma), H^{-\frac{1}{2}}(\Gamma)\right)}\right) \left(\|Z_k^* x_n\|^2_{H^{-\frac{1}{2}}(\Gamma)} + \|x_n\|^2_{H^{\frac{1}{2}}(\Gamma)}\right),\end{aligned}$$

and then $(Z_k + n_e)^* x_n$ is bounded in $H^{-\frac{1}{2}}(\Gamma)$. Using that $H^{-\frac{1}{2}}(\Gamma)$ is compactly embedded into $V^{-\frac{1}{2}}(\Gamma)$ we get that (up to a subsequence) $(Z_k + n_e)^* x_n$ converges into $V^{-\frac{1}{2}}(\Gamma)$. Thus using that $((Z_k + n_e)^*)^{-1} : V^{-\frac{1}{2}}(\Gamma) \mapsto V^{\frac{1}{2}}(\Gamma)$ is bounded we get that x_n (up to a subsequence) converges into $V^{\frac{1}{2}}(\Gamma)$.

□

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