Path dependent equations driven by Hölder processes

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Abstract

This paper investigates existence results for path-dependent differential equations driven by a Hölder function where the integrals are understood in the Young sense. The two main results are proved via an application of Schauder theorem and the vector field is allowed to be unbounded. The Hölder function is typically the trajectory of a stochastic process.

Key words: Young integration; Path-dependent; Differential equations.
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1 Introduction

The aim of the paper is to discuss existence theorems for a path-dependent equation of the type

$$Y_t = y_0 + \int_0^t F\left(u, Y_u\left(\centerdot\right)\right) dX_u,\ t \in [0,T], \tag{1.1}$$

where X is an α-Hölder continuous n-dimensional process and F is a path-dependent $m \times n$ matrix-valued vector field defined on $[0,T] \times C([0,T]\,;\mathbb{R}^m)$. The trajectories of the unknown process Y are $\mathbb{R}^m$-valued α-Hölder continuous functions; $Y_u\left(\centerdot\right)$ denotes the trajectory of Y until time u, i.e, $Y_u\left(x\right) = Y_x$, if $x \leq u$ and $Y_u\left(x\right) = Y_u$ otherwise. The integral is intended in the Young sense so that F needs of course to verify a Hölder type regularity, see Sec. 3 for details.

Path-dependent (similarly to functional dependent or delay) equations have a long story. To the best of our knowledge the first author who has contributed in this framework in the stochastic case, is [2] motivated by [4], which is a significant contribution in the deterministic case. A relevant monograph in the subject is the one of [17]. Considerations about functional-dependent equations in law also appear in [18]. More recently several studies

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have been performed studying the relation between functional dependent equations and path-dependent PDEs see e.g. [5, 6, 8, 3] in the framework of Banach space valued stochastic calculus and many others in the framework of path-dependent functional Itô calculus, see [13] and references therein for latest developments.

Young integral was introduced first in [19]. A recent paper on the subject is [11] and an excellent monograph recalling Young integral in the perspective of rough paths is [9]. That integral has been implemented for the study of ordinary stochastic differential equations driven by Hölder processes (see [15, 14]) and also SPDEs, see [16, 12]. As far as we know, the present paper is the first one which discusses functional-dependent equations in the framework of Young integral.

The aim of this paper is to discuss existence results under suitable minimal assumptions on F. Contrarily to most of the literature on Young differential equations even in the non-path dependent setting, the authors allow the vector field F to be unbounded. The main results about existence are Theorem 16 and Theorem 18; the latter supposes F to be bounded but with less restrictive Hölder type conditions on F. The path-dependent framework however offers other perspectives of generalization if one assumes a different type on dependence on the past trajectory. For instance in Section 6 we remark that, whenever the dependence of F with respect to the past allows a gap with respect to the present, the construction of a solution can be done iteratively.

2 Preliminaries

In this section we introduce some basic definitions.

Let U and V be Banach spaces and denote by $L = L(V, U)$ the space of continuous linear maps from V to U.

We reserve the symbols X to denote driving paths of our differential equation. Typically $X : [0, T] \to V$ is an α-Hölder continuous. Hence there is a constant (the smallest one is denoted by $\|X\|_\alpha$) such that

$$|X_t - X_s| \leq \|X\|_\alpha |t - s|^\alpha ,$$

for all $s, t \in [0, T]$. As usual we write, $X \in C^\alpha([0, T], V)$ and

$$\|X\|_\alpha = \sup_{s,t\in[0,T],s\neq t} \frac{|X_t - X_s|}{|t - s|^\alpha}.$$

It is sometimes useful to specify the closed interval $I = [t_0, t_1]$ where we evaluate $\|X\|_\alpha$. For this matter, we further define

$$\|X\|_{\alpha;I} := \sup_{s,t\in I,s\neq t} \frac{|X_t - X_s|}{|t - s|^\alpha}.$$

If $t_0 \geq 0$ and $\tau > 0$ we will simply denote

$$\|X\|_{\alpha;t_0,\tau} := \|X\|_{\alpha;[t_0,t_0+\tau]} .$$

Usually we omit the symbol U in $C^\alpha(I,U)$ becoming simply $C^\alpha(I)$. It is well-known that $\|\cdot\|_{\alpha;I}$ induces semi-distance on the vector space $C^\alpha(I)$, though we can endow it with a metric by setting $|||Z-Y|||_{\alpha;I} := \|Z-Y\|_{\alpha;I} + |Z_{t_0} - Y_{t_0}|$, for all $Z, Y \in C^\alpha(I)$. An important closed subset of $C^\alpha(I)$ is obtained by fixing an *initial condition*. That is, for a fixed $a \in U$, we define $C_a^\alpha(I) := \{Y \in C^\alpha(I) \mid Y_{t_0} = a\}$ and this is a closed subset of $C^\alpha(I)$. Moreover, the metric $|||Z-Y|||_{\alpha;I}$ on $C_a^\alpha(I)$ coincides with $\|\cdot\|_{\alpha;I}$, i.e. $|||Z-Y|||_{\alpha;I} = \|Z-Y\|_{\alpha;I}$, for all $Z, Y \in C^\alpha(I)$. For the sake of clarity, when we refer to C^α-topology we mean the topology induced by $|||\cdot|||_{\alpha;I}$, which coincides with $\|\cdot\|_{\alpha;I}$ in the subset $C_a^\alpha(I)$. The set $C(I)$ is the usual Banach space of continuous functions equipped with the sup-norm, $\|Y\|_{\infty;I} := \sup_{t\in I}|Y_t|$.

Since our differential equation involves a vector field acting on the whole trajectory of a path, we make use of the following notation. Given a continuous path $Y : [0,T] \to U$ we denote by the calligraphic version of Y, namely $\mathcal{Y}$, onto $C([0,T], U)$, i.e.

$$\mathcal{Y} : \underset{t}{[0,T]} \underset{\mapsto}{\to} \underset{\mathcal{Y}_t}{C([0,T],U)},$$

with

$$\mathcal{Y}_t(x) := Y_{t\wedge x}. \tag{2.1}$$

We observe that, if Y is α-Hölder continuous then its lift is α-Hölder as well. Indeed, this is shown in the statement below.

Proposition 1. *Let $Y \in C^\alpha([0,T])$. Then, for any $0 \le a \le b \le T$, $\mathcal{Y} \in C^\alpha([a,b], C[0,T])$ we have*

$$\|\mathcal{Y}\|_{\alpha;[a,b]} \le \|Y\|_{\alpha;[0,T]} \tag{2.2}$$

$$\|\mathcal{Y}_t\|_{\infty;[0,T]} \le \|Y\|_{\alpha;[0,T]}. \tag{2.3}$$

Proof. Fix $s \le t$ in $[a,b]$. For any $x \in [0,T]$ we have

$$\begin{aligned} |\mathcal{Y}_t(x) - \mathcal{Y}_s(x)| &\le \begin{cases} |Y_x - Y_x| & , x \in [0,s] \\ |Y_x - Y_s| & , x \in [s,t] \\ |Y_t - Y_s| & , x \in [t,T] \end{cases} \\ &\le \begin{cases} 0 & , x \in [0,s] \\ \|Y\|_\alpha |x-s|^\alpha & , x \in [s,t] \\ \|Y\|_\alpha |t-s|^\alpha & , x \in [t,T] \end{cases} \\ &\le \|Y\|_\alpha |t-s|^\alpha, \end{aligned}$$

so $\|\mathcal{Y}_t - \mathcal{Y}_s\|_{\infty;[0,T]} \le \|Y\|_\alpha |t-s|^\alpha$. This proves $\mathcal{Y} \in C^\alpha([a,b], C([0,T]))$ with $\|\mathcal{Y}\|_{\alpha;[a,b]} \le \|Y\|_{\alpha;[0,T]}$ i.e. equality (2.2).

Now, fix $t \in [a,b]$ and let any $x, y \in [0,T]$. By definition, see (2.1), it follows

$$|\mathcal{Y}_t(x) - \mathcal{Y}_t(y)| \le \|Y\|_\alpha |x-y|^\alpha,$$

hence $\mathcal{Y}_t \in C^\alpha([0,T])$ and equality (2.3) holds. □

3 Young integral

At this level we recall the fundamental inequality characterizing Young integral. For a complete treatment we refer the reader [10, Ch. 6]. For a reference closer to our spirit we recommend [9, pp.47-48;63]. We remark that the second inequality below is a consequence of the first one.

Theorem 2. *Let $t_0 \in [0,T]$, $a \in U$ and let $\tau > 0$ such that $\tau + t_0 \leq T$. Given $W \in C^{\gamma}([t_0, t_0+\tau], L(V,U))$ and $X \in C^{\alpha}([0,T], V)$ with $\alpha+\gamma > 1$, the map*

$$W \mapsto I(W) := a + \int_{t_0}^{\cdot} W_u dX_u \in C^{\alpha}([0,T]), \tag{3.1}$$

is continuous and it satisfies, for $s \leq t$ in $[t_0, t_0+\tau]$,

$$\left| \int_s^t W_u dX_u - W_s (X_t - X_s) \right| \leq k_{\alpha+\gamma} \|W\|_{\gamma;[s,t]} \|X\|_{\alpha;[s,t]} |t-s|^{\alpha+\gamma}, \tag{3.2}$$

where $k_\mu = \frac{1}{1-2^{1-\mu}}$.

If furthermore $\tau \leq 1$, we can write the inequality above as

$$\|I(W)\|_{\alpha;t_0,\tau} \leq (k_{\alpha+\gamma}+1) \|X\|_{\alpha;t_0,\tau} \left(\|W\|_{\gamma;t_0,\tau} + |W_{t_0}| \right). \tag{3.3}$$

4 The vector field F

In this section we formally introduce the driving vector field of the equation. For every $t \in [0,T]$ and $Y \in C([0,T],U)$, $F(t,Y)$ is a linear map acting on V to U. Also we will present some fundamental inequalities regarding composition maps, such as $Y \mapsto F(t, \mathcal{Y}_t)$.

Definition 3. Let

$$F : [0,T] \times C([0,T], U) \to L.$$

We will say that it is **non-anticipating** if it satisfies

$$F(t,Y) = F(t, \mathcal{Y}_t),$$

for all $Y \in C[0,T]$ and $t \in [0,T]$

The requirement above means that $F(t,Y)$ does not depend on what happened on $Y|_{[t,T]}$. It will indeed fulfill the property below.

Remark 4. Given Y,$\tilde{Y} \in C[0,T]$ such that $Y|_{[0,s]} = \tilde{Y}|_{[0,s]}$. Then

$$F(s,Y) = F\left(s, \tilde{Y}\right).$$

We assume Hölder regularity on F as follows. The forthcoming examples will play as motivation for the definition below.

Definition 5. The vector field $F : [0,T] \times C^{\alpha}([0,T],U) \to L$ will be said **(α,β)-Hölder** if there are some non-negative constants $c_{\alpha,\beta}$ and $\tilde{c}_{\alpha,\beta}$,

$$|F(t,Y) - F(s,Y)| \leq c_{\alpha,\beta}\left(1 + \|Y\|^{\beta}_{\alpha,[s,t]}\right)|t-s|^{\alpha\beta}, \ \forall s,t \in [0,T], \quad (4.1)$$

$$\left|F(s,Y) - F\left(s,\tilde{Y}\right)\right| \leq \tilde{c}_{\alpha,\beta}\left(\left\|Y - \tilde{Y}\right\|_{\alpha,[0,s]} + \left|Y_0 - \tilde{Y}_0\right|\right)^{\beta}, \ \forall s \in [0,T], \quad (4.2)$$

for all $Y, \tilde{Y} \in C^{\alpha}([0,T],U)$.

Remark 6.
1. If F is (α,β)-Hölder then (4.2) implies that F is non-anticipating.
2. It is well-known that, for $\alpha' < \alpha$, if Y is α-Hölder then Y is α'-Hölder. This follows from the inequality $\|Y\|_{\alpha';[t_0,t_1]} \leq \|Y\|_{\alpha;[t_0,t_1]}|t_1 - t_0|^{\alpha-\alpha'}$. This remark motivates the lemma below, which will be useful in the next sections.

Lemma 7. *Let* $0 < \alpha' \leq \alpha \leq 1$ *and* $\beta \in (0,1]$. *Suppose that* $F : [0,T] \times C^{\alpha'}([0,T],U) \to L$ *is an* (α',β)*-Hölder continuous vector field. Then* F *is* (α,β)*-Hölder continuous with constants* $c_{\alpha,\beta} := c_{\alpha',\beta}\left(T^{\alpha'\beta} \vee 1\right)$ *and* $\tilde{c}_{\alpha,\beta} := \tilde{c}_{\alpha',\beta}\left(T^{(\alpha-\alpha')\beta} \vee 1\right)$.

Proof. Let $s < t$ belonging to $[0,T]$ and let $Y \in C^{\alpha}([0,T])$. Using the inequality $\|Y\|_{\alpha';[s,t]} \leq \|Y\|_{\alpha;[s,t]}|t-s|^{\alpha-\alpha'}$ and (α',β)-Hölder continuity of F (below we are going to omit the sub-index β related to the constants $c_{\alpha,\beta}$, $\tilde{c}_{\alpha,\beta}$, $c_{\alpha',\beta}$ and $\tilde{c}_{\alpha',\beta}$), we obtain

$$\begin{aligned}
|F(t,Y) - F(s,Y)| &\leq c_{\alpha'}\left(1 + \|Y\|^{\beta}_{\alpha';[s,t]}\right)|t-s|^{\alpha'\beta} \\
&\leq c_{\alpha'}\left(1 + \|Y\|^{\beta}_{\alpha;[s,t]}|t-s|^{(\alpha-\alpha')\beta}\right)|t-s|^{\alpha'\beta} \\
&= c_{\alpha'}\left(|t-s|^{\alpha'\beta} + \|Y\|^{\beta}_{\alpha;[s,t]}|t-s|^{\alpha\beta}\right) \\
&\leq c_{\alpha'}\left(T^{\alpha'\beta} + \|Y\|^{\beta}_{\alpha;[s,t]}|t-s|^{\alpha\beta}\right) \\
&\leq c_{\alpha'}\left(T^{\alpha'\beta} \vee 1\right)\left(1 + \|Y\|^{\beta}_{\alpha;[s,t]}\right)|t-s|^{\alpha'\beta},
\end{aligned}$$

hence equality (4.1) holds with $c_{\alpha} = c_{\alpha'}\left(T^{\alpha'\beta} \vee 1\right)$. It remains to show (4.2). Let $Y, \tilde{Y} \in C^{\alpha}([0,T])$ and $s \in [0,T]$. We have

$$\begin{aligned}
\left|F(s,Y) - F\left(s,\tilde{Y}\right)\right| &\leq \tilde{c}_{\alpha'}\left(\left\|Y - \tilde{Y}\right\|_{\alpha',[0,s]} + \left|Y_0 - \tilde{Y}_0\right|\right)^{\beta} \\
&\leq \tilde{c}_{\alpha'}\left(\left\|Y - \tilde{Y}\right\|_{\alpha,[0,s]} s^{\alpha-\alpha'} + \left|Y_0 - \tilde{Y}_0\right|\right)^{\beta} \\
&\leq \tilde{c}_{\alpha'}\left(T^{(\alpha-\alpha')\beta} \vee 1\right)\left(\left\|Y - \tilde{Y}\right\|_{\alpha,[0,s]} + \left|Y_0 - \tilde{Y}_0\right|\right),
\end{aligned}$$

which proves the claim with $\tilde{c}_{\alpha} = \tilde{c}_{\alpha'}\left(T^{(\alpha-\alpha')\beta} \vee 1\right)$. □

Example 8. B. Dupire (see [7]) introduced a notion of non-anticipating functional in the sense introduced below. Denote $\Lambda := \cup_{s\in[0,T]}\Lambda^s$ and $\Lambda^s := C([0,s])$. We endow Λ with the metric d defined as follows. For any $Z^t, Y^s \in \Lambda$ (the superscript here means that $Z^t \in \Lambda^t$ and $Y^s \in \Lambda^s$) assuming $s \le t$, he defines $d\left(Z^t, Y^s\right) := |t-s| + \left\|Z^t - \mathcal{Y}^s_s\right\|_{\infty;[0,t]}$, where similarly as at (2.1), we set $\mathcal{Y}^s_u(x) := Y^s_{u\wedge x}, u, x \in [0,t]$. Let $f : \Lambda \to \mathbb{R}$ be a β-Hölder continuous functional with respect to d. Then the vector field F defined by

$$F(t,Y) := f\left(Y|_{[0,t]}\right), \ \forall t \in [0,T], \ \forall Y \in C^\alpha([0,T]),$$

is non-anticipating in the sense of Definition 3 and it is (α,β)-Hölder.

Indeed, for $Y \in C^\alpha([0,T])$, we have

$$\begin{aligned}
|F(t,Y) - F(s,Y)| &= \left|f\left(Y|_{[0,t]}\right) - f\left(Y|_{[0,s]}\right)\right| \\
&\le \|f\|_\beta \, d\left(Y|_{[0,t]}, Y|_{[0,s]}\right)^\beta \\
&\le \|f\|_\alpha \left(|t-s| + \|Y - \mathcal{Y}_s\|_{\infty;0,t}\right)^\beta \\
&= \|f\|_\alpha \left(|t-s| + \sup_{x\in[s,t]} |Y_x - Y_s|\right)^\beta \\
&\le \|f\|_\alpha \left(|t-s| + \|Y\|_{\alpha;[s,t]} |t-s|^\alpha\right)^\beta \\
&\le \|f\|_\alpha \left(T^{1-\alpha} + \|Y\|_{\alpha;[s,t]}\right)^\beta |t-s|^{\alpha\beta}.
\end{aligned}$$

This proves (4.1).

It remains to prove (4.2). Given $Y, Z \in C^\alpha([0,T])$ and $s \in [0,T]$,

$$\begin{aligned}
|F(s,Y) - F(s,Z)| &= \left|f\left(Y|_{[0,s]}\right) - f\left(Z|_{[0,s]}\right)\right| \\
&\le \|f\|_\beta \left\{|s-s| + \|Y - Z\|_{\infty;[0,s]}\right\}^\beta \\
&\le \|f\|_\beta \left\{0 + \|Y - Z\|_{\alpha;[0,s]} s^\alpha + |Y_0 - Z_0|\right\}^\beta .
\end{aligned}$$

Example 9. *Young integral functional.* Let $\alpha, \gamma \in (0,1]$ with $\alpha + \gamma > 1$. Fix a function g in $C^\gamma([0,T])$. Define the vector field F_g by setting $F_g(t,Y) := \int_0^t g_u dY_u$, for each $Y \in C^\alpha([0,T])$. Then F_g is $(\alpha,1)$-Hölder continuous. Indeed, using Young integral inequality (3.2) with W (and X) instead of g (and Y), for any $s,t \in [0,T]$ and $Y \in C^\alpha([0,T])$, it follows that

$$\begin{aligned}
|F_g(t,Y) - F_g(s,Y)| &= \left|\int_s^t g_u dY_u\right| \\
&\le k_{\alpha+\gamma} \|g\|_{\gamma;[s,t]} \|Y\|_{\alpha;[s,t]} |t-s|^{\alpha+\gamma} + |g_s| |Y_t - Y_s| \\
&\le \left\{k_{\alpha+\gamma} \|g\|_{\gamma;[0,T]} T^\gamma + \|g\|_{\infty;[0,T]}\right\} \|Y\|_{\alpha;[s,t]} |t-s|^\alpha ,
\end{aligned}$$

which proves (4.1).

Now, given any $Y, Z \in C^\alpha([0,T])$ and $s \in [0,T]$, we apply previous inequality with $Y - Z$ instead of Y and for $s = 0$. Thus, we obtain

$$|F_g(s,Y) - F_g(s,Z)| = \left|\int_0^s g_u d(Y-Z)_u\right|$$

$$\leq \left\{ k_{\alpha+\gamma} \|g\|_{\gamma;[0,T]} T^{\gamma} + \|g\|_{\infty;[0,T]} \right\} \|Y - Z\|_{\alpha;[0,s]} T^{\alpha},$$

which proves (4.2).

Example 10. *Young Integral functional (continued).* More generally, we consider the vector field $F(t, Y) := h\left(t, \int_0^t g_u^1 dY_u, \ldots, \int_0^t g_u^N dY_u\right)$, where $g^i \in C^{\gamma_i}([0,T])$ with $\alpha + \gamma_i > 1, 1 \leq i \leq N$, and $h : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ such there is a constant $K > 0$ with

$$|h(t,b) - h(s,a)| \leq K \left\{ |t-s|^{\alpha\beta} + \max_{i=1,\ldots,N} |b_i - a_i|^{\beta} \right\}, \ \forall t \in [0,T], \forall a, b \in \mathbb{R}^N.$$

Then F is (α, β)-Hölder. Indeed, it is easy to see that F satisfies the inequalities

$$|F(t,Y) - F(s,Y)| \leq K \left(1 + \max_{i=1,\ldots,N} c_i \|Y\|_{\alpha;[s,t]}^{\beta} \right) |t-s|^{\alpha\beta},$$
$$|F(s,Y) - F(s,Z)| = |F(s, Y-Z)| \leq K \max_{i=1,\ldots,N} c_i \|Y - Z\|_{\alpha;[0,s]}^{\beta} T^{\alpha\beta},$$

for any $Y, Z \in C^{\alpha}$ and $s, t \in [0,T]$ where $c_i := \left(k_{\alpha+\gamma_i} \|g_i\|_{\gamma_i} T^{\gamma_i} + \|g_i\|_{\infty} \right)^{\beta}$.

Remark 11. In the proposition below, we will use a simple technique involving Hölder norms inequalities. It is called **geometric interpolation** (in contrast to the linear one, $a \leq \theta a + (1-\theta) b \leq b$) which states that, whenever $W \in C^{\alpha}$ and $\theta \in (0,1)$, then

$$\|W\|_{\alpha\theta} \leq \|W\|_0^{1-\theta} \|W\|_{\alpha}^{\theta}, \tag{4.3}$$

recalling the notation $\|W\|_0 = \sup_{s,t} |W_t - W_s|$. The proof of (4.3) is a consequence of the equality $\frac{|W_t - W_s|}{|t-s|^{\alpha\theta}} = |W_t - W_s|^{1-\theta} \left(\frac{|W_t - W_s|}{|t-s|^{\alpha}} \right)^{\theta}$. In particular we get

$$\|W\|_{\alpha\theta} \leq 2^{1-\theta} \|W\|_{\infty}^{1-\theta} \|W\|_{\alpha}^{\theta}. \tag{4.4}$$

Proposition 12. *Let $F : [0,T] \times C^{\alpha}([0,T], U) \to L$ be a non-anticipating and (α, β)-Hölder continuous vector field.*

Fix $t_0, t_1 \in [0,T]$. Given $Y, Z \in C^{\alpha}([0,t_1], U)$ the inequality

$$\|F(\cdot, Y)\|_{\alpha\beta;[t_0,t_1]} \leq c_{\alpha,\beta} \left(1 + \|Y\|_{\alpha;[t_0,t_1]}^{\beta} \right) \tag{4.5}$$

holds. Moreover, if $\|Y\|_{\alpha;[t_0,t_1]}, \|Z\|_{\alpha;[t_0,t_1]} \leq R$, for any $\theta \in (0,1)$, it follows

$$\|F(\cdot, Y) - F(\cdot, Z)\|_{\alpha\beta\theta;[t_0,t_1]} \leq 2\tilde{c}_{\alpha,\beta}^{1-\theta} c_{\alpha;\beta}^{\theta} \left(1 + R^{\beta} \right)^{\theta} \left(\|Y - Z\|_{\alpha;0,t_1} + \left| Y_0 - \tilde{Y}_0 \right| \right)^{\beta(1-\theta)}, \tag{4.6}$$

where $c_{\alpha,\beta}$ and $\tilde{c}_{\alpha,\beta}$ are the constants introduced in Definition 5.

Proof. For $s < t$ in $[t_0, t_1]$, and $Y \in C^{\alpha}([0, t_1])$, (4.5) follows directly from the definition of (α, β)-Hölder continuous.

We prove now (4.6). We set $I := [t_0, t_1]$. Given $Y, Z \in C^{\alpha}([0, t_1])$ with $\|Y\|_{\alpha;I}, \|Z\|_{\alpha;I} \le R$, we write $W_t := F(t, Y) - F(t, Z)$, $t \in I$. Fix an arbitrary $\theta \in (0, 1)$; using (4.4) we obtain

$$\|W\|_{\alpha\beta\theta;I} \le 2^{1-\theta} \|W\|_{\infty;I}^{1-\theta} \|W\|_{\alpha\beta;I}^{\theta}, \tag{4.7}$$

so it remains to bound $\|W\|_{\infty;I}$ and $\|W\|_{\alpha\beta;I}$.

On the one hand, since F is (α, β)-Hölder continuous, we have

$$\begin{aligned} \|W\|_{\infty;I} &= \sup_{t\in[t_0,t_1]} |F(t, Y) - F(t, Z)| \\ &\le \tilde{c}_{\alpha,\beta} \left(\|Y - Z\|_{\alpha,[0,t_1]} + |Y_0 - Z_0| \right)^{\beta}. \end{aligned} \tag{4.8}$$

On the other hand, using (4.5) and recalling $\|Y\|_{\alpha;I}, \|Z\|_{\alpha;I} \le R$, it follows

$$\begin{aligned} \|W\|_{\alpha\beta;I} &\le \|F(t, Y)\|_{\alpha\beta;I} + \|F(t, Z)\|_{\alpha\beta;I} \\ &\le c_{\alpha,\beta} \left(1 + \|Y\|_{\alpha;I}^{\beta}\right) + c_{\alpha,\beta} \left(1 + \|Z\|_{\alpha;I}^{\beta}\right) \\ &\le 2c_{\alpha,\beta} \left(1 + R^{\beta}\right). \end{aligned} \tag{4.9}$$

Finally inequality (4.6) follows substituting (4.8) and (4.9) into (4.7), recalling that $W_t := F(t, Y) - F(t, Z), t \in [0, T]$.

□

5 The Existence Results

In this section we present the main results of this paper. We introduce formally the equation and its statements regarding existence of solutions. We state a version of Schauder fixed point theorem that we are using.

Theorem 13. *Let M be a non-empty, closed, bounded, convex subset of a Banach space, and suppose $S : M \to M$ is a continuous operator which maps M into a compact subset of M. Then M has a fixed point.*

Proof. See [1, Th. 2.2]. □

The next two lemmas will will help us gluing Hölder functions.

Lemma 14. *Let I and J denote two compact intervals of $\mathbb{R}$ such that $I \cap J$ is non-empty. Let $Y : I \cup J \to U$ a path such that $Y \in C^{\alpha}(I)$ and $Y \in C^{\alpha}(J)$. Then $Y \in C^{\alpha}(I \cup J)$ and*

$$\|Y\|_{\alpha;I\cup J} \le 2 \left(\|Y\|_{\alpha;I} + \|Y\|_{\alpha;J} \right).$$

Proof. See [11, Lemma 3]. □

Lemma 15. *Fix $\tau > 0$. Let $0 =: t_0 < t_1 < \cdots < t_{N+1} := T$ be a partition of $[0, T]$, where every sub-interval has length τ, i.e., $t_i - t_{i-1} = \tau$, $\forall i = 1, \ldots, N$. Let $Y : [0, T] \to U$ such that*

$$\|Y\|_{\alpha;t_i,\tau} \le R,$$

for all $i = 1, \ldots, N$. Then

$$\|Y\|_{\alpha;[0,T]} \le 4R \left(1 \vee T^{1-\alpha}\tau^{\alpha-1}\right).$$

Proof. Let $s, t \in [0,T]$. If $|t-s| \le \tau$ then s and t belong to (at most) two consecutive interval $[t_i, t_{i+1}]$. Hence, $|Y_t - Y_s| \le 2(R+R)|t-s|^\alpha$, see Lemma 14 above. Otherwise, $t_K < s \le t_{K+1} < \cdots < t < t_{K+r+1}$ with $\tau r < t - s \le \tau(r+1)$ for some $r \ge 1$. Then

$$\begin{aligned}
|Y_t - Y_s| &\le \sum_{j=1}^{r+1} \left|Y_{t\wedge t_{K+j}} - Y_{s\vee t_{K+j-1}}\right| \\
&\le R\sum_{j=1}^{r} |t\wedge t_{K+j} - s\vee t_{K+j-1}|^\alpha \\
&\le R\tau^\alpha (r+1) \\
&= R\tau^{\alpha-1}\tau(r+1) \\
&\le 2R\tau^{\alpha-1}\tau r \\
&\le 2R\tau^{\alpha-1}|t-s| \\
&\le 2R\tau^{\alpha-1}T^{1-\alpha}|t-s|^\alpha .
\end{aligned}$$

In conclusion $\|Y\|_{\alpha;[0,T]} \le 4R\left(1 \vee T^{1-\alpha}\tau^{\alpha-1}\right)$. □

Now we state and prove the first existence theorem for global solutions in time. We insist on the fact that our assumptions do not imply that F is bounded. This particular case will be investigated in the subsequent Theorem 18.

Theorem 16. *Let U and V be finite dimensional linear spaces. Let $\alpha > \frac{1}{2}$ and $X : [0,T] \to V$ an α-Hölder path. Let $\beta \in (0,1)$ such $\alpha\beta + \alpha > 1$. Let $F : [0,T] \times C^{\alpha'}([0,T], U) \to L(V,U)$ for some $\alpha' < \alpha$ with the property F is also (α', β)-Hölder continuous.*
Given an initial condition $y_0 \in U$, there is a solution $Y \in C^\alpha([0,T], U)$ for the equation

$$Y_t = y_0 + \int_0^t F(u, Y)\, dX_u,\ t \in [0,T]. \tag{5.1}$$

Remark 17. 1. A more general framework of (5.1) is a path-dependent equation with initial condition at t_0 instead of 0, for $t \in [t_0, t_1]$ with $t_1 \in [t_0, T]$. In that case the initial condition will be a function $\eta \in C^\alpha[0, t_0]$.

In correspondence to this we introduce $Z^\eta : [0, t_1] \to U$ setting

$$Z_t^\eta := \begin{cases} \eta_t; & t \in [0, t_0] \\ Z_t; & t \in [t_0, t_1]. \end{cases} \tag{5.2}$$

The equation of our interest is

$$\begin{cases} Y_t = \eta_{t_0} + \int_{t_0}^t F(u, Y^\eta)\, dX_u, & t \in [0, t_1], \\ Y_s = \eta_s, & s \in [0, t_0]. \end{cases} \tag{5.3}$$

We remark that (5.1) is a particular case of (5.3) setting $t_0 = 0$ and $t_1 = T$.

2. The strategy employed in the proof will be first to construct a solution of (5.1) replacing T with a *small* time τ. Then given t_0, which will be of the type $t_0 = k\tau$ for $k = 0, 1, \ldots,$, and $\eta \in C^{\alpha}([0, t_0])$ we will inductively construct a solution of (5.3) with $t_1 = t_0 + \tau$.
   For the general induction step we consider the so called *solution map*, i.e. a functional $S_\eta : M \to C^{\alpha'}([t_0, t_1])$, where M is a suitable subset of $C^{\alpha'}([t_0, t_1])$ which will be introduced later, so that $S_\eta(M) \subset M$, defined by

$$S_\eta\left(Z\right)_t := \eta_{t_0} + \int_{t_0}^{t} F\left(u, Z^\eta\right) dX_u,\ \ t \in [t_0, t_0 + \tau] \tag{5.4}$$

   and $Z \in M$. We will prove that S_η has a fixed point through Schauder's Theorem 13), which of course solves (5.3) in $[t_0, t_1]$. This would imply the existence of a solution on the whole time interval $[0, T]$, by patching solutions together.

Proof. **Step 1.** *We can assume without loss of generality that*

$$\alpha'\beta + \alpha' > 1, \tag{5.5}$$

and moreover, that there is $\theta \in (0, 1)$ such that F is $\left(\alpha'\theta^2, \beta\right)$-Hölder continuous and

$$\alpha'\theta^2\beta + \alpha'\theta^2 > 1, \tag{5.6}$$

as well.
For this we are going to fabricate constants $\tilde{\alpha}$ larger than α' and θ such that F is $\left(\tilde{\alpha}\theta^2, \beta\right)$-Hölder continuous and $(\tilde{\alpha}, \theta)$ fulfill (5.5) and (5.6) with α' replaced by $\tilde{\alpha}$.
Indeed, since by hypothesis, α is strictly greater than $\frac{1}{2}$ and because of the inequality $\alpha\beta + \alpha > 1$, we can first choose $\tilde{\alpha} \in \left(\frac{1}{\beta+1}, \alpha\right) \cap \left(\frac{1}{2}, \alpha\right)$. $\theta \in (0, 1)$ such that $\theta^2\tilde{\alpha} \in \left(\frac{1}{\beta+1}, \tilde{\alpha}\right) \cap \left(\frac{1}{2}, \tilde{\alpha}\right)$, which is possible by a similar reasoning. Now, we have $\alpha' < \theta^2\tilde{\alpha} < \tilde{\alpha} < \alpha$ so Lemma 7 guarantees that F is $\left(\tilde{\alpha}\theta^2, \beta\right)$-Hölder continuous, with $\tilde{\alpha}$ and θ fulfilling the inequalities (5.5) and (5.6) with α' replaced with $\tilde{\alpha}$.

Morally this step consists in restricting the domain of the vector field F into a suitable smaller set $C^{\tilde{\alpha}}$, i.e., $C^{\alpha} \subset C^{\tilde{\alpha}} \subset C^{\alpha'}$. The choice of a suitable $\tilde{\alpha}$ does not play any role regarding the space where the solution Y lives, as we can see in the next step.

Step 2. *Looking for a solution of* (5.3) *in* $C^{\alpha}([t_0, t_1])$, *it is enough to show that there is a solution* $Y \in C^{\alpha'}([t_0, t_1])$.

Indeed, if $Y \in C^{\alpha'}([t_0, t_1])$ solves (5.3), then by (3.2) in Theorem 2 and (4.1), it follows

$$\begin{aligned}
|Y_t - Y_s| &= \left| \int_s^t F\left(u, Y\right) dX_u \right| \\
&\le k_{\alpha'\beta+\alpha} c \left(1 + \|Y\|_{\alpha';[s,t]}^{\beta}\right) \|X\|_{\alpha;[0,T]} |t - s|^{\alpha'\beta+\alpha} + |F\left(s, Y\right)| \, \|X\|_{\alpha;[0,T]} |t - s|^{\alpha} \\
&\le \left\{ k_{\alpha'\beta+\alpha} c \left(1 + \|Y\|_{\alpha';[0,t_1]}^{\beta}\right) T^{\alpha'\beta} + \|F\left(\cdot, Y\right)\|_\infty \right\} \|X\|_{\alpha;[0,T]} |t - s|^{\alpha},
\end{aligned}$$

where $c = c_{\alpha',\beta}$ as defined in (4.2).

Step 3. *Discussion about set the $M \subset C^{\alpha'}([t_0, t_0+\tau])$, anticipated in the Remark 17 item 2.* M will be of the type

$$M := M^{\alpha'}_{t_0,\tau,R,a} := \left\{ Z \in C^{\alpha'}([t_0, t_0+\tau]) \mid Z_{t_0} = a, \text{ and } \|Z\|_{\alpha';t_0,\tau} \leq R \right\}, \tag{5.7}$$

for fixed $R, \tau > 0$ and $a \in U$. We will indeed set $a = \eta_{t_0} \in U$, and R, τ will be suitable parameters, see (5.12) and (5.13), in order to guarantee that $S_\eta\left(M^{\alpha'}_{t_0,\tau,R,\eta_{t_0}}\right) \subset M^{\alpha'}_{t_0,\tau,R,\eta_{t_0}}$.

Step 4. *Let $R > 0$, $t_0 \in [0, T)$, $\tau \in (0, 1]$, $a \in U$, which will be arbitrary in this step, the set $M^{\alpha'}_{t_0,\tau,R,a}$ is compact in $C^{\theta\alpha'}$-topology.* This is a standard result, however though we present its proof for the sake of completeness.

We recall the set $M^{\alpha'}_{t_0,\tau,R,a}$ is a subset of $C^{\theta\alpha'}([t_0, t_0+\tau])$ with a fixed initial condition, hence $C^{\theta\alpha'}$-topology in $M^{\alpha'}_{t_0,\tau,R,a}$ is induced by $\|\cdot\|_{\alpha'\theta;t_0,\tau}$. Now we prove the claim, let (Z^n) be a sequence in $M^{\alpha'}_{t_0,\tau,R,a}$, $n \in \mathbb{N}$. This is an equicontinuous family in $[t_0, t_0+\tau]$, since $|Z^n_t - Z^n_s| \leq R\,|t-s|^{\alpha'}$, $s, t \in [t_0, t_0+\tau]$ for each n. Also, the sequence is uniformly bounded since $|Z^n_t| \leq |Z^n_0| + R\,|t-0|^{\alpha'} = |a| + R\,|t|^{\alpha'}$. Hence, by the classical Arzelà-Ascoli Theorem, there is $Z \in C([t_0, t_0+\tau])$ such that, for a subsequence (also denoted by Z^n), $\|Z^n - Z\|_{\infty;t_0,t_0+\tau} \to 0$, as $n \to \infty$. So, in particular $Z_{t_0} = a$, on the one hand. On the other hand, it is easy to see that $\|Z\|_{\alpha';t_0,\tau} \leq R$, since

$$\begin{aligned} |Z_t - Z_s| &= \lim_{n\to\infty} |Z^n_t - Z^n_s| \\ &\leq \lim_{n\to\infty} R\,|t-s|^{\alpha'}. \end{aligned}$$

Therefore $Z \in M^{\alpha'}_{t_0,\tau,R,a}$.

In order to show $\|Z^n - Z\|_{\theta\alpha';t_0,\tau} \underset{n\to\infty}{\to} 0$, we use the geometric interpolation, see Remark 11, so

$$\begin{aligned} \|Z^n - Z\|_{\theta\alpha'} &\leq 2^{1-\theta}\,\|Z^n - Z\|^{1-\theta}_{\infty;t_0,\tau}\,\|Z^n - Z\|^{\theta}_{\alpha';t_0,\tau} \\ &\leq 2^{1-\theta}\,\|Z^n - Z\|^{1-\theta}_{\infty;t_0,\tau}\left(\|Z^n\|_{\alpha';t_0,\tau} + \|Z\|_{\alpha';t_0,\tau}\right)^{\theta} \\ &\leq 2^{1-\theta}\,\|Z^n - Z\|^{1-\theta}_{\infty;t_0,\tau}\,(2R)^{\theta} \end{aligned}$$

and we observe that the right-hand side converges to zero as n goes to ∞.

Step 5. *Let $R > 0$, $t_0 \in [0, T)$, $\tau \in (0, 1]$, $a \in U$ and $\eta \in C^{\alpha}([t_0, t_0+\tau])$, which will be arbitrary in this step. Let θ as introduced in Step 1. Then the map $S_\eta : M^{\alpha'}_{t_0,\tau,R,a} \to C^{\alpha'}([t_0, t_0+\tau])$ is continuous under the $C^{\alpha'\theta}([t_0, t_0+\tau])$-topology. We recall that $\alpha'\theta^2\beta + \alpha'\theta^2 > 1$, see* (5.6). Since $M^{\alpha'}_{t_0,\tau,R,a}$ is compact, it is a closed subset of $C^{\alpha'\theta}([t_0, t_0+\tau])$.

Fix an arbitrary $W \in M^{\alpha'}_{t_0,\tau,R,a}$. We will show $\|S_\eta(Z) - S_\eta(W)\|_{\theta\alpha';t_0,\tau} \to 0$ as $\|Z - W\|_{\theta\alpha';t_0,\tau} \to 0$, $Z \in M^{\alpha'}_{t_0,\tau,R,a}$.

We use now Young integral inequality (3.3) with α replaced with $\alpha'\theta$ and γ replaced with $\alpha'\theta^2\beta$. We observe that the sum $\mu := \alpha'\theta^2\beta + \alpha'\theta^2$ which is strictly larger than 1 so Theorem 2 can be applied and by (3.3)

$$\|S_\eta(Z) - S_\eta(W)\|_{\theta\alpha';t_0,\tau} = \left\| \int_{t_0}^{\cdot} F(u, Z^\eta) - F(u, W^\eta)\, dX_u \right\|_{\theta\alpha';t_0,\tau}$$

$$\leq \left(k_\mu + 1\right) \|X\|_{\theta\alpha';t_0,\tau} \left\|F\left(\cdot, Z^\eta\right) - F\left(\cdot, W^\eta\right)\right\|_{\theta^2\alpha'\beta;t_0,\tau}. \tag{5.8}$$

We recall that F is $(\alpha'\theta^2, \beta)$-Hölder continuous, see Step 1.

From Proposition 12, see (4.6), using $\alpha'\theta$ (and $t_0 + \tau$) instead of α (and t_1), it follows that

$$\begin{aligned} \left\|F\left(\cdot, Z^\eta\right) - F\left(\cdot, W^\eta\right)\right\|_{\theta^2\alpha'\beta;t_0,\tau} &\leq 2\tilde{c}^{1-\theta}_{\alpha'\theta^2,\beta} c^{\theta}_{\alpha'\theta^2,\beta} \left(1 + R^\beta\right)^\theta \|Z^\eta - W^\eta\|^{\beta(1-\theta)}_{\theta\alpha';0,t_0+\tau} \\ &= 2\tilde{c}^{1-\theta}_{\alpha'\theta^2,\beta} c^{\theta}_{\alpha'\theta^2,\beta} \left(1 + R^\beta\right)^\theta \|Z - W\|^{\beta(1-\theta)}_{\theta\alpha';t_0,\tau}. \end{aligned} \tag{5.9}$$

From (5.8) and (5.9) we conclude that S_η is continuous with respect to $C^{\alpha'\theta}$-topology.

Step 6. *We prove now that $S_\eta\left(M^{\alpha'}_{t_0,\tau,R,y_0}\right) \subset M^{\alpha'}_{t_0,\tau,R,y_0}$ in the case when $t_0 = 0$, with $\eta : \{0\} \to U$, $\eta_0 = y_0$ for any $y_0 \in U$ and suitable $R, \tau > 0$ introduced below.* We will extend this property at Step 8. for $t_0 = N\tau$, $N = 1, 2, \ldots$.

We set

$$K := \left(k_{\alpha'\beta+\alpha'} + 1\right) \|X\|_{\alpha;0,T} 2c_F \left(1 + T^{\alpha'\beta}\right) \tag{5.10}$$

and $c_F := \max\left\{|F(0,0)|; c_{\alpha',\beta}; \tilde{c}_{\alpha',\beta}\right\}$. We can assume $K > 0$, otherwise either $\|X\|_{\alpha;0,T} = 0$ or $F = 0$, thus the constant function $Y_t := y_0$ solves (5.1). Let $\varepsilon \in \left(0, \frac{K}{2}\right)$ be fixed and τ is defined by

$$\tau := \left(\frac{\varepsilon}{K}\right)^{\frac{1}{\alpha-\alpha'}}, \tag{5.11}$$

so that, $0 < \tau < 1$ and

$$K\tau^{\alpha-\alpha'} = \varepsilon. \tag{5.12}$$

Let $R > 0$ big enough such that

$$\varepsilon\left(1 + 5\left(1 \vee T^{1-\alpha'}\tau^{\alpha'-1}\right)^\beta R^\beta\right) \leq R, \tag{5.13}$$

$$|y_0| \leq R, \tag{5.14}$$

which is always possible since $\beta < 1$. Indeed, given a function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(R) = c + dR^\beta$, $c, d > 0$, the limit of $\frac{g(R)}{R}$ when $R \to \infty$ is zero.

From now on in this step we set $S := S_\eta$. We prove now that $S\left(M^{\alpha'}_{0,\tau,R,y_0}\right) \subset M^{\alpha'}_{0,\tau,R,y_0}$. Indeed, given $Z \in M^{\alpha'}_{0,\tau,R,y_0}$, from Young integral inequality (3.3), it follows that

$$\|S(Z)\|_{\alpha';0,\tau} \leq \left(k_{\alpha'\beta+\alpha'} + 1\right) \|X\|_{\alpha';0,\tau} \left(\|F(\cdot, Z)\|_{\alpha'\beta;0,\tau} + |F(0, Z)|\right). \tag{5.15}$$

Since F is (α', β)-Hölder continuous, by Proposition 12 it follows that

$$\|F(\cdot, Z)\|_{\alpha'\beta;0,\tau} \leq c_{\alpha',\beta}\left(1 + \|Z\|^\beta_{\alpha';0,\tau}\right). \tag{5.16}$$

Moreover by (4.2), it also holds that

$$\begin{aligned}|F(0,Z)| &\le |F(0,0)| + \tilde{c}_{\alpha',\beta}\,|Z_0 - 0|^{\beta} \\ &= |F(0,0)| + \tilde{c}_{\alpha',\beta}\,|y_0|^{\beta}. \qquad (5.17)\end{aligned}$$

Plugging (5.16),(5.17) into (5.15), using $\|X\|_{\alpha';0,\tau} \le \|X\|_{\alpha;0,T}\,\tau^{\alpha-\alpha'}$ and $|y_0|, \|Z\|_{\alpha';0,\tau} \le R$, also recalling $c_F = \max\left\{|F(0,0)|; c_{\alpha',\beta}; \tilde{c}_{\alpha',\beta}\right\}$, definitions from τ and R, (see (5.12), (5.13)), we have

$$\begin{aligned}\|S(Z)\|_{\alpha';0,\tau} &\le \left(k_{\alpha'\beta+\alpha'}+1\right)\|X\|_{\alpha;0,T}\,\tau^{\alpha-\alpha'}\left(c\left(1+R^{\beta}\right) + |F(0,0)| + \tilde{c}R^{\beta}\right) \\ &\le \left(k_{\alpha'\beta+\alpha'}+1\right)\|X\|_{\alpha;0,T}\,\tau^{\alpha-\alpha'}\left(2c_F\left(1+R^{\beta}\right)\right) \\ &\le K\tau^{\alpha-\alpha'}\left(1+R^{\beta}\right) \\ &= \varepsilon\left(1+R^{\beta}\right) \\ &\le R.\end{aligned}$$

This proves Step 6.

Let $R > 0$ as in (5.13) and (5.14) together with τ selected in (5.11) until the end of the proof.

Step 7. *There is a solution $Y \in C^{\alpha}([0,\tau])$ for (5.1) replacing T with τ, with $Y_0 = y_0$ and $\|Y\|_{\alpha';0,\tau} \le R$. This constitutes the first stage of a statement which will be proved by induction in Step 9. below.*
This simply follows from Steps 4., 5., 6. which allow us to use Theorem 13 and finally Step 2.

From Step 6. the map $S : M^{\alpha'}_{0,\tau,R,y_0} \to M^{\alpha'}_{0,\tau,R,y_0}$ is well-defined and Step 5. shows us it is continuous under $C^{\alpha'\theta}$-topology. Since $M^{\alpha'}_{0,\tau,R,y_0}$ is compact under $C^{\alpha'\theta}$-topology, see Step 4., Schauder's Theorem 13 claims that there is a fixed point for the map S, denoted by $Y \in M^{\alpha'}_{0,\tau,R,a}$. In other words, there is $Y \in M^{\alpha'}_{0,\tau,R,a}$, such that

$$Y_t = S(Y)_t = y_0 + \int_0^t F(u,Y)\,dX_u,\ t \in [0,\tau].$$

Finally, from Step 2., we conclude that $Y \in C^{\alpha}[0,\tau]$.

Step 8. *Now we prove the general statement announced in step 6. Let $t_0 = N\tau$ for some $N = 1,2,\ldots$. Assume that $\eta \in C^{\alpha}([0,t_0])$ such that $\|\eta\|_{\alpha';k\tau,\tau} \le R$ for $k = 0,1,\ldots,N-1$, we have*

$$S_{\eta}\left(M^{\alpha'}_{t_0,\tau,R,\eta_{t_0}}\right) \subset M^{\alpha'}_{t_0,\tau,R,\eta_{t_0}}. \qquad (5.18)$$

Indeed, let $Z \in M^{\alpha'}_{t_0,\tau,R,\eta_{t_0}}$. From Young integral inequality (3.3) using $\alpha'\beta$ (and α') instead of δ (and α),

$$\|S_{\eta}(Z)\|_{\alpha';t_0,\tau} \le \left(k_{\alpha'\beta+\alpha'}+1\right)\|X\|_{\alpha';t_0,\tau}\left\{\|F(\cdot,Z^{\eta})\|_{\alpha'\beta;t_0,\tau} + |F(t_0,Z^{\eta})|\right\}. \qquad (5.19)$$

Since F is (α', β)-Hölder continuous, by (4.5) from Proposition 12 with α' instead of α and noting that $\|Z\|_{\alpha';t_0,\tau} \leq R$ it follows

$$\begin{aligned} \|F(\cdot, Z^\eta)\|_{\alpha'\beta;t_0,\tau} &\leq c_{\alpha',\beta}\left(1 + \|Z^\eta\|^\beta_{\alpha';t_0,\tau}\right) \\ &\leq c_{\alpha',\beta}\left(1 + \|Z\|^\beta_{\alpha';t_0,\tau}\right) \\ &\leq c_{\alpha',\beta}\left(1 + R^\beta\right). \end{aligned} \tag{5.20}$$

Regarding $|F(t_0, Z^\eta)|$, we split it as

$$\begin{aligned} |F(t_0, Z^\eta)| &= |F(t_0, \eta)| \\ &\leq |F(t_0, \eta) - F(0, \eta)| + |F(0, \eta)| \\ &=: A + B. \end{aligned} \tag{5.21}$$

On the one hand, by (4.5) from Proposition 12, Lemma 15 together with hypothesis $\|\eta\|_{\alpha';k\tau,\tau} \leq R$, recalling again that $c_F = \max\left\{|F(0,0)|; c_{\alpha',\beta}; \tilde{c}_{\alpha',\beta}\right\}$ and also that $t_0 = N\tau \leq T$ and $4^\beta < 5$, it follows that

$$\begin{aligned} A &= |F(N\tau, \eta) - F(0, \eta)| \\ &\leq c_{\alpha',\beta}\left(1 + \|\eta\|^\beta_{\alpha';0,N\tau}\right)(N\tau)^{\alpha'\beta} \\ &\leq c_{\alpha',\beta}\left(1 + \left(4R\left(1 \vee T^{1-\alpha'}\tau^{\alpha'-1}\right)\right)^\beta\right)T^{\alpha'\beta} \\ &\leq c_F T^{\alpha'\beta}\left(1 + 5R^\beta\left(1 \vee T^{1-\alpha'}\tau^{\alpha'-1}\right)^\beta\right). \end{aligned} \tag{5.22}$$

On the other hand, since F is (α', β)-Hölder continuous using $5^\beta \leq 5$ and the hypothesis $\|\eta\|_{\alpha';k\tau,\tau} \leq R$, $|y_0| \leq R$ it follows that

$$\begin{aligned} B &\leq |F(0,0)| + |F(0,\eta) - F(0,0)| \\ &\leq F(0,0) + \tilde{c}_{\alpha',\beta}\left(\|\eta - 0\|_{\alpha';0,N\tau} + |\eta_0 - 0|\right)^\beta \\ &\leq c_F\left(1 + \left(\|\eta\|_{\alpha';0,N\tau} + |y_0|\right)^\beta\right) \\ &\leq c_F\left(1 + \left(4R\left(1 \vee T^{1-\alpha'}\tau^{\alpha'-1}\right) + R\right)^\beta\right) \\ &\leq c_F\left(1 + \left(5R\left(1 \vee T^{1-\alpha'}\tau^{\alpha'-1}\right)\right)^\beta\right) \\ &\leq c_F\left(1 + 5R^\beta\left(1 \vee T^{1-\alpha'}\tau^{\alpha'-1}\right)^\beta\right), \end{aligned} \tag{5.23}$$

where we have used Lemma 15 in the fourth inequality. Hence, from (5.22) and (5.23) we conclude

$$|F(t_0, Z^\eta)| \leq c_F\left(1 + T^{\alpha'\beta}\right)\left(1 + 5R^\beta\left(1 \vee T^{1-\alpha'}\tau^{\alpha'-1}\right)^\beta\right). \tag{5.24}$$

Now, plugging (5.20) and (5.24) into (5.19), using $\|X\|_{\alpha';t_0,\tau} \leq \|X\|_{\alpha;0,T}\,\tau^{\alpha-\alpha'}$ and recalling the definitions of τ and R given in (5.12), (5.13), it yields

$$\begin{aligned}\|S_\eta(Z)_t\|_{\alpha';0,\tau} &\leq \left(k_{\alpha'\beta+\alpha'}+1\right)\|X\|_{\alpha;0,T}\,\tau^{\alpha-\alpha'}\left\{2c_F\left(1+T^{\alpha'\beta}\right)\left(1+5R^\beta\left(1\vee T^{1-\alpha'}\tau^{\alpha'-1}\right)^\beta\right)\right\}\\ &= \varepsilon\left(1+5R^\beta\left(1\vee T^{1-\alpha'}\tau^{\alpha'-1}\right)^\beta\right)\leq R.\end{aligned}$$

This proves (5.18).

Step 9. *There is a solution $Y \in C^\alpha([0,(N+1)\tau])$ for the (5.1) replacing T with $(N+1)\tau$, with $Y_0 = y_0$ and $\|Y\|_{\alpha';k\tau,\tau} \leq R$, each $k = 0,1,\dots,N$. This constitutes the induction stage which we announced in Step 7.*

Indeed, the case $N = 0$ was proved in Step 7. Now, assume there is a solution $\eta \in C^\alpha([0,N\tau])$ (replacing T with $N\tau$) of (5.1) with $\|\eta\|_{\alpha';k\tau,\tau} \leq R$, each $k = 0,1,\dots,N-1$. The solution η fulfills the conditions of the Step 8. with $t_0 = N\tau$, hence $S_\eta\left(M^{\alpha'}_{t_0,\tau,R,\eta_{t_0}}\right) \subset M^{\alpha'}_{t_0,\tau,R,\eta_{t_0}}$. Reasoning as at Step 7., the map $S_\eta : M^{\alpha'}_{t_0,\tau,R,\eta_{t_0}} \to M^{\alpha'}_{t_0,\tau,R,\eta_{t_0}}$ has a fixed point denoted by W, which in particular solves (5.3):

$$W_t = S_\eta(W)_t = \eta_{N\tau} + \int_{N\tau}^t F(u,W^\eta)\,dX_u, \quad t \in [N\tau,(N+1)\tau], \tag{5.25}$$

where we remind that the notation W^η was introduced in (5.2).

Finally, we define $Y := W^\eta \in C^\alpha([0,(N+1)\tau])$ which trivially extends η. We show below that Y solves equation (5.1). Indeed, on the one hand for each $t \in [0,N\tau]$, recalling $F(u,\eta) = F(u,Y)$ for $u \in [0,N\tau]$ and that η is a solution in the interval $[0,N\tau]$, we have

$$\begin{aligned}Y_t &= W_t^\eta = \eta_t\\ &= y_0 + \int_0^t F(u,\eta)\,dX_u\\ &= y_0 + \int_0^t F(u,Y)\,dX_u.\end{aligned}$$

On the other hand, arguing as above and using (5.25), we have, for $t \in [N\tau,(N+1)\tau]$,

$$\begin{aligned}Y_t &= W_t^\eta = W_t\\ &= \eta_{N\tau} + \int_{N\tau}^t F(u,W^\eta)\,dX_u\\ &= y_0 + \int_0^{N\tau} F(u,\eta)\,dX_u + \int_{N\tau}^t F(u,W^\eta)\,dX_u\\ &= y_0 + \int_0^{N\tau} F(u,Y)\,dX_u + \int_{N\tau}^t F(u,Y)\,dX_u\\ &= y_0 + \int_0^t F(u,Y)\,dX_u.\end{aligned}$$

This concludes that Y is a solution to (5.1) on the interval $[0,N\tau+\tau]$. Moreover, $\|Y\|_{\alpha';k\tau,\tau} = \|W^\eta\|_{\alpha';k\tau,\tau} \leq R$ for $k = 0,1,\dots,N$. Indeed for

$k = 0, 1, \ldots, N-1$, this holds by assumption and for $k = N$ it comes from (5.18). Hence this establishes the induction step and it concludes Step 9. $\square$

The theorem below shows that when F is bounded the coefficient β is also allowed to be 1. We remark that in Theorem 16 we have required that $\beta < 1$.

Theorem 18. *Let U and V be finite dimensional linear spaces. Let $\alpha, \beta \in (0,1]$ such that $\alpha > \frac{1}{2}$ and $\alpha\beta + \alpha > 1$. Let $X : [0,T] \to V$ be an α-Hölder path and $F : [0,T] \times C^{\alpha'}([0,T], U) \to L(V,U)$ be a bounded and (α', β)-Hölder continuous vector field for some $\alpha' < \alpha$. Given an initial condition $y_0 \in U$, there is a solution $Y \in C^{\alpha}([0,T], U)$ for the equation*

$$Y_t = y_0 + \int_0^t F(u, Y)\, dX_u, \; t \in [0,T]. \tag{5.26}$$

Proof. This proof is simpler than the one of Theorem 16 (where F is not bounded) and we will explain here the significant changes. We start re-introducing the objects. First, as explained in Step 1. of Theorem 16 we can assume that $\frac{1}{2} < \alpha' < \alpha$ and also that there is $\theta \in (0,1)$ such that F is $(\alpha'\theta^2, \beta)$-Hölder continuous and the inequalities (5.5) together with (5.6) are still in force.

Second, we re-define the parameters K, ε and τ. So, differently from (5.10) in Step 6., we set

$$K := \left(k_{\alpha'\beta+\alpha'} + 1\right) \|X\|_{\alpha;0,T} \left(2c_{\alpha',\beta} + \|F\|_{\infty}\right).$$

We fix $\varepsilon \in \left(0, \frac{K}{2} \wedge 1\right)$ and define again τ as in (5.11) so that $\tau \in (0,1)$ and (5.12) still holds.

For an arbitrary $t_0 \in [0,T)$ and $\eta \in C^{\alpha}([0,t_0])$ with $\eta_0 = y_0$, similarly as in the proof of Theorem 16, we will find a solution of (5.3) with $t_1 = t_0 + \tau$. This can be done performing the same program of Steps 2. to 5. in the proof of Theorem 16. The notation $M^{\alpha'}_{t_0,\tau,1,a}$ will denote the same ball as in (5.7). We define Z^{η} as in (5.2) and $S_{\eta} : M^{\alpha'}_{t_0,\tau,1,\eta_{t_0}} \to C^{\alpha'}([t_0, t_0+\tau])$ as in (5.4).

In order to show that S_{η} has a fixed point (which therefore solves (5.3)) we need to prove first that

$$S_{\eta}\left(M^{\alpha'}_{t_0,\tau,1,\eta_{t_0}}\right) \subset M^{\alpha'}_{t_0,\tau,1,\eta_{t_0}}. \tag{5.27}$$

This will replace Steps 6. and 8. of the proof of Theorem 16 which will merge, since it will not be necessary to distinguish $t_0 = 0$ from $t_0 = N\tau,\; N \geq 1$.

Indeed, given $Z \in M^{\alpha'}_{t_0,\tau,1,\eta_{t_0}}$, from Young integral inequality (3.3), replacing γ there by $\alpha'\beta$ as for (5.15), it follows that

$$\|S_{\eta}(Z)\|_{\alpha';t_0,\tau} \leq \left(k_{\alpha'\beta+\alpha'} + 1\right) \|X\|_{\alpha';t_0,\tau} \left(\|F(\cdot, Z^{\eta})\|_{\alpha'\beta;t_0,\tau} + |F(t_0, Z^{\eta})|\right). \tag{5.28}$$

Since F is (α', β)-Hölder continuous, it follows from Proposition 12, see (4.5) with α' (resp. Z^{η}) instead of α (resp. Y) that

$$\|F(\cdot, Z^{\eta})\|_{\alpha'\beta;t_0,\tau} \leq c_{\alpha',\beta}\left(1 + \|Z^{\eta}\|^{\beta}_{\alpha';t_0,\tau}\right)$$

$$
\begin{aligned}
&= c_{\alpha',\beta}\left(1+\|Z\|^{\beta}_{\alpha';t_0,\tau}\right)\\
&\le c_{\alpha',\beta}\left(1+1^{\beta}\right)\\
&\le 2c_{\alpha',\beta}.
\end{aligned} \tag{5.29}
$$

Also, since F is bounded it holds

$$
\begin{aligned}
|F(t_0, Z^{\eta})| &= |F(t_0,\eta)|\\
&\le \|F\|_{\infty}.
\end{aligned} \tag{5.30}
$$

We substitute (5.29) and (5.30) into (5.28). Using $\|X\|_{\alpha';t_0,\tau} \le \|X\|_{\alpha;0,T}\,\tau^{\alpha-\alpha'}$, $\varepsilon \le 1$ and recalling that τ satisfies (5.10), it follows that

$$
\begin{aligned}
\|S_{\eta}(Z)\|_{\alpha';t_0,\tau} &\le \left(k_{\alpha'\beta+\alpha'}+1\right)\|X\|_{\alpha;0,T}\,\tau^{\alpha-\alpha'}\left(2c_{\alpha',\beta}+\|F\|_{\infty}\right)\\
&= K\tau^{\alpha-\alpha'}\\
&= \varepsilon\\
&\le 1,
\end{aligned}
$$

henceforth $S_{\eta}\left(M^{\alpha'}_{t_0,\tau,1,\eta_{t_0}}\right) \subset M^{\alpha'}_{t_0,\tau,1,\eta_{t_0}}$. This concludes the proof of (5.27).

The sequel of the proof consists in treating simultaneously the corresponding Steps 7. and 9. of the proof of Theorem 16. We claim that $S_{\eta} : M^{\alpha'}_{t_0,\tau,1,\eta_{t_0}} \to M^{\alpha'}_{t_0,\tau,1,\eta_{t_0}}$ has a fixed point.

Indeed, we already know that $M^{\alpha'}_{t_0,\tau,1,\eta_{t_0}}$ is a compact set with respect $\alpha'\theta$-Hölder topology (see Step 4. in the proof of Theorem 16) and S_{η} is continuous with respect $\alpha'\theta$-Hölder topology (see Step 5. in the proof of Theorem 16). So S_{η} verifies the hypothesis of Schauder's Theorem 13, hence there is a fixed point for S_{η}, which we denote by $W \in C^{\alpha'}([t_0, t_0+\tau])$; W also belongs to $C^{\alpha}([t_0, t_0+\tau])$ by Step 2. in the proof of Theorem 16. In other words,

$$
W_t = \eta_{t_0} + \int_{t_0}^{t} F(u, W^{\eta})\, dX_u, t \in [t_0, t_0+\tau]. \tag{5.31}
$$

Finally, we can conclude the proof showing that there is a solution $Y \in C^{\alpha}([0,T])$ to (5.1) proceeding similarly as after (5.25).

□

6 On a particular path-dependent structure of the vector field.

We conclude the paper showing that, in some cases, the solution to our path-dependent equation can be constructed directly. In this case the method leads us to an existence (and even uniqueness) statement under weaker assumptions than in Theorems 16 and 18. This happens when the past dependence structure of the vector F allows a gap (of size $\delta > 0$) between the past and the present, see Definition 19 below for the precise meaning.

Definition 19. Let $F : [0,T] \times C([0,T], U) \to L$ be an vector field. We will say that it is **δ-non-anticipating** if it satisfies

$$F(t,Y) = F\left(t, \mathcal{Y}_{(t-\delta)_+}\right), \tag{6.1}$$

for all $Y \in C([0,T], U)$ and $t \in [0,T]$.

Under this assumption we can construct a solution step by step on intervals $[0,\delta], [0,2\delta], \ldots$ and so on only supposing, for instance, that for each $Z \in C^\alpha([0,T])$, $t \mapsto F(t,Z)$ is γ-Hölder continuous with $\alpha + \gamma > 1$. The theorem below holds even when U and V are generic Banach spaces.

Theorem 20. *Let $F : [0,T] \times C^\alpha([0,T], U) \to L$ be a δ-non-anticipating vector field for some $\delta > 0$. Suppose that for each $Z \in C^\alpha([0,T], U)$, $t \mapsto F(t,Z)$ is γ-Hölder continuous with $\alpha + \gamma > 1$. Let $X \in C^\alpha([0,T], V)$. Then for each $y_0 \in U$ there is a unique solution $Y \in C^\alpha([0,T])$ for the equation*

$$Y_t = y_0 + \int_0^t F(u,Y)\, dX_u,\ t \in [0,T]. \tag{6.2}$$

Proof. We start discussing existence. Without restriction of generality we can suppose that $T = N\delta$ for some integer N. We denote by Y^0 the constant function $t \mapsto y_0 \in U$ on $[0,T]$. By recurrence arguments we construct a sequence $Y^1, \ldots, Y^N$ in $C^\alpha([0,T])$ verifying, for $n = 1, \ldots, N$,

$$Y_t^n := y_0 + \int_0^{t \wedge n\delta} F\left(u, Y^{n-1}\right) dX_u,\ t \in [0,T]. \tag{6.3}$$

Although we need to define Y_t^n only for $t \in [0, n\delta]$, we have chosen to extend it to the whole interval $t \in [0,T]$ since this simplifies the formulation of some arguments during this proof.

The expression (6.3) is well-defined via Theorem 2 with $W_t = F\left(t, Y^{n-1}\right)$. Indeed, suppose that (6.3) holds replacing integer n with $n-1$. Since $Y^{n-1} \in C^\alpha([0,T])$, the hypothesis implies that the path $t \in [0,T] \mapsto F\left(t, Y^{n-1}\right)$ is γ-Hölder continuous hence we can define $Y_t^n := y_0 + \int_0^{t\wedge n\delta} F\left(u, Y^{n-1}\right) dX_u$, for $t \in [0,T]$.

At this point our aim is to prove that $Y := Y^N$ solves equation (6.2). For this we will show that for $n = 1, \ldots, N$

$$Y_t^n = y_0 + \int_0^{t \wedge n\delta} F(u, Y^n)\, dX_u,\ t \in [0,T]. \tag{6.4}$$

Taking into account the construction (6.3), it will be enough to show by recurrence that, for every $n = 1, \ldots, N$

$$F(u, Y^n) = F(u, Y^{n-1}),\ u \in [0, n\delta]. \tag{6.5}$$

Suppose for a moment that, for every $n = 1, \ldots, N$,

$$Y_t^n = Y_t^{n-1},\ t \in [0, (n-1)\delta]. \tag{6.6}$$

Then (6.5) will follow. Indeed let $u \in [0, n\delta]$, for some n; then $t := (u-\delta)_+$ belongs to $[0,(n-1)\delta]$; thus from (6.6), recalling $\mathcal{Y}_r(x) := Y_{r\wedge x}$ it follows

$$\mathcal{Y}^n_{(u-\delta)_+} = \mathcal{Y}^{n-1}_{(u-\delta)_+}, u \in [0, n\delta]. \tag{6.7}$$

So, using (6.1) and taking in account (6.7), yields

$$F\left(u, Y^{n}\right)=F\left(u, \mathcal{Y}_{(u-\delta)_{+}}^{n}\right)=F\left(u, \mathcal{Y}_{(u-\delta)_{+}}^{n-1}\right)=F\left(u, Y^{n-1}\right).$$

It remains to show (6.6); we will do it by induction on n. The case $n=1$ is an obvious consequence of (6.3) evaluated for $t=0$. We suppose now that (6.6) holds replacing n with $n-1$.

Let $t \in [0,(n-1)\delta]$ for an integer n with $n \geq 2$. By (6.1), the construction (6.3), the induction hypothesis related to (6.6) and the recurrence (6.5) with $n-1$ replacing n, we obtain

$$\begin{aligned} Y_t &= y_0+\int_0^{t \wedge n\delta} F\left(u, Y^{n-1}\right) dX_u \\ &= y_0+\int_0^{t \wedge(n-1)\delta} F\left(u, Y^{n-1}\right) dX_u \\ &= y_0+\int_0^{t \wedge(n-1)\delta} F\left(u, Y^{n-2}\right) dX_u \\ &= Y_t^{n-1}. \end{aligned}$$

This concludes the proof of the induction step in (6.6). The existence part of the theorem is finally established.

Uniqueness follows easily by an obvious induction argument.

□

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