Minimal graphs for 2-factor extension

M.-C. Costa * D. de Werra † C. Picouleau ‡

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Abstract

Let $G = (V, E)$ be a simple loopless finite undirected graph. We say that G is *(2-factor) expandable* if for any non-edge uv, $G + uv$ has a 2-factor F that contains uv. We are interested in the following: Given a positive integer $n = |V|$, what is the minimum cardinality of E such that there exists $G = (V, E)$ which is 2-factor expandable? This minimum number is denoted by $Exp_2(n)$. We give an explicit formula for $Exp_2(n)$ and provide 2-factor expandable graphs of minimum size $Exp_2(n)$.

Keywords: 2-factor, minimum expandable graph, reliability.

1 Introduction

Much work in graph theory has concentrated on 2-factors which generalize perfect matchings (see for instance [3]). A characterization of graphs admitting a 2-factor can be found in [6] (Vol A, page 527). It is a challenging problem to characterize 2-factor edge critical graphs, i.e., graphs with no 2-factor but which have a 2-factor by adding any new edge. Since such a characterization seems hard to derive, in this paper we will restrict our attention to finding the minimum number of edges in a graph G on n vertices such that for any two non adjacent vertices u, v the graph $G+uv$ has a 2-factor containing uv. Notice that contrary to the 2-factor edge critical graphs here G may contain a 2-factor or not. These graphs will be called *minimal 2-factor expandable.*

This work is motivated by the following reliability problem: In a complete graph (V, E) on n vertices edges are subject to breakdowns. We want to reinforce a minimal subset F of edges in such a way that any surviving edge in $E - F$ can be expanded with reinforced edges to a 2-factor. A related problem of 1-factor expansion has been studied in [4] where a complete characterization of minimal 1-factor expandable graphs has been derived.

We now give some definitions required to formalize our problem.

*Conservatoire National des Arts et Métiers, CEDRIC laboratory, Paris (France) and ENSTA-ParisTech. Email: marie-christine.costa@ensta-paristech.fr

‡Conservatoire National des Arts et Métiers, CEDRIC laboratory, Paris (France). Email: christophe.picouleau@cnam.fr

†Ecole Polytechnique Fédérale de Lausanne, Lausanne (Switzerland). Email: dominique.dewerra@epfl.ch

We will consider a simple finite graph $G = (V, E)$ with $n \geq 3$ vertices and m edges. A pair u, v of vertices is a *non-edge* if $uv \notin E$.

- A subset $F \subseteq E$ is a 2-*factor* if every vertex v has exactly two edges in F which are incident in v. Equivalently F is a collection of vertex-disjoint cycles covering all vertices.
- G is *2-factor expandable* (or shortly *expandable*) if for every non-edge xy the graph $G_{xy} = (V, E \cup xy)$ has a 2-factor F with $xy \in F$. In such a case we say that the non-edge xy has been *extended* to a 2-factor.
- For any fixed $n \geq 3$ an expandable graph with a minimum number of edges is a *minimum expandable graph* ($meg(n)$). The size of its edge set is denoted by $Exp_2(n)$.

We intend to determine, for any fixed integer $n \geq 3$, the value $Exp_2(n)$ and to exhibit a graph $meg(n)$.

We now give the notations we use later. For any subset $X \subseteq V$ the subgraph induced by X is denoted by $G[X]$. We write $G - X = G[V \setminus X]$ and $G - v$ for $G - \{v\}$. $N(v)$ is the set of neighbors of a vertex v; $d(v) = |N(v)|$ is the degree of v; a *p-vertex* is a vertex of degree p; If $p = 0$ then v is an *isolated* vertex; when $p = 1$ then v is called a *leaf*; if $d(v) = n - 1$, then v is *universal*. The closed neighborhood of v is $N[v] = N(v) \cup \{v\}$. $\delta(G)$ (resp. $\Delta(G)$) is the minimum (resp. maximum) degree of G. An induced path with p edges is called a p-path. By $d(u, v)$ we denote the *distance* between u and v, i.e., the length of a shortest path (number of edges) between u and v in G. C_k (resp. K_k) is the induced cycle (resp. complete graph) on k vertices.

For all definitions related to graphs, see [3].

We state our main result which will be proved in the following sections:

Proposition 1.1 *The minimum size of a 2-factor expandable graph is:*

- $Exp_2(3) = 2, Exp_2(4) = 4, Exp_2(5) = 6, Exp_2(6) = 9, Exp_2(7) = 10, Exp_2(8) = 11, Exp_2(9) = 12$*;*
- $Exp_2(n) = \lceil \frac{11}{8} n \rceil, n \geq 10.$

The paper is organized as follows. In Section 2 some elementary properties of expandable graphs will be stated for later use. Section 3 will be dedicated to the presentation of $meg(n)$ for $3 \leq n \leq 9$. In Section 4 a lower bound for $Exp_2(n)$ will be established for $n \geq 10$, while it will be shown in Section 5 that it is best possible. Variations of the construction for $n = 8p$ will be presented in Section 5.3 to handle the case $n \neq 0 \ mod(8)$, $n \geq 14$. Finally constructions will be given for $10 \leq n \leq 13$ in Section 5.4. Some conclusions and suggestions for further research are presented in Section 6.

2 Properties of expandable graphs

We shall state some basic properties of expandable graphs which will be used to determine $Exp_2(n)$.

Fact 2.1 *If $G = (V, E)$ is not expandable, then no partial graph $G' = (V, E')$, with $E' \subset E$, is expandable.*

Let $G = (V, E)$ be an expandable graph.

Property 2.1 *G is connected.*

Proof: If u and v are in two distinct components, then clearly uv cannot be extended. $\square$

Property 2.2 *If G has a leaf, then for $n = 5$ we have $m \geq 7$ and for $n \geq 6$ we have $m \geq \frac{3}{2}n$.*

Proof: Let u be a 1-vertex of G. If G is expandable, then $G - u$ induces a clique. So for $n = 5$ we have $m \geq 7$ and for $n \geq 6$ we obtain $m \geq \frac{3}{2}n$. $\square$

Property 2.3 *If G contains a universal vertex and $n \geq 5$, then $m \geq \frac{3}{2}(n-1)$.*

Proof: Assume that G has a universal vertex. If there is no leaf, then $\Sigma_{u \in V} d(u) \geq n - 1 + 2(n-1) = 3(n-1)$, else from Property 2.2 $m \geq \frac{3}{2}(n-1)$. $\square$

Property 2.4 *Let G contain a 2-vertex v with $N(v) = \{a, b\}$ and $ab \notin E$. If there is $c \in N(a) \cap N(b), c \neq v$, then $d(c) \geq 4$.*

Proof: Consider any extension of ab: the triangle (a, b, v) is in the 2-factor. Since c is necessarily covered by another cycle, we have $d(c) \geq 4$. $\square$

Property 2.5 *Let G contain two 2-vertices u, v. If $d(u, v) = 4$ with the 4-path $uu'tv'v$, then $d(t) > 3$.*

Proof: $d(u, v) = 4$ implies that $u'v' \notin E$. Now if $d(t) \leq 3$ the non-edge $u'v'$ cannot be extended. $\square$

Property 2.6 *Let $n \geq 7$. If a vertex v is adjacent to two 2-vertices, then v is universal and $m \geq \frac{3}{2}n$.*

Figure 1: The diamond and the bull. In Property 2.7 a black vertex is a 2-vertices, the grey vertex is the head.

Proof: We can assume that $\delta(G) \geq 2$ since, as seen in the proof of Property 2.2, if there is a leaf G has no 2-vertex. v is universal since, otherwise, there is a non-edge vw that can not be extended. If there are exactly two 2-vertices we have $\Sigma_{u \in V} d(u) \geq (n-1) + 4 + 3(n-3) = 4n - 6$ and $m \geq \frac{3}{2}n$ since $n \geq 7$. If a, b, c are three 2-vertices, then any non-edge xy with $x, y \neq a, b, c$ cannot be extended since any 2-factor containing xy would use exactly two edges among va, vb, vc. Thus $G - \{a, b, c\}$ is a clique. So we have $m = 6 + \frac{1}{2}(n-3)(n-4) \geq \frac{3}{2}n$, since $n \geq 7$. □

Before to state our next property we need to define two graphs. The diamond and the bull are shown in Figure 1. For each of these two graphs the grey vertex is called the *head*.

Property 2.7 *Let $G = (V, E)$ be a connected graph with $n \geq 10$. If G contains two vertex disjoint induced subgraphs H_1 and H_2 such that:*

- *H_i is either a diamond or a bull with head h_i, $i = 1, 2$;*
- *the three vertices inducing the triangle containing the head are 3-vertices;*
- *any other vertex is a 2-vertex in G;*
- *$h_1 h_2 \in E$;*

then G is not expandable.

Proof: If H_1, H_2 are two diamonds, then G is the graph shown in Figure 2, but $n = 8$. So we can suppose that H_1 is a bull. There is a non-edge xy in H_1 such that x is 3-vertex $x \neq h_1$ and y is a 2-vertex. Then since H_2 is a bull or a diamond, xy cannot be extended. □

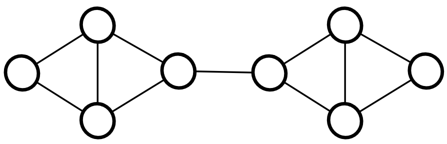

Figure 2: Two diamonds with $h_1 h_2 \in E$.

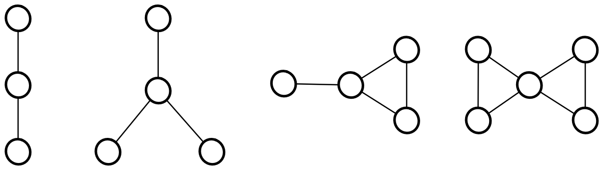

Figure 3: P_3, the claw, the paw, the butterfly (from left to right).

3 Meg(n) for $3 \leq n \leq 9$

We will compute $Exp_2(n)$ for small values of n.

- $Exp_2(3) = 2$: Trivially P_3 the path on three vertices (Figure 3) is a $meg(3)$.
- $Exp_2(4) = 4$: The paw (see Figure 3) is expandable. If $G = (V, E)$ is a $meg(4)$ with $|E| < 4$, then from Property 2.1 G is a tree. So G is either P_4 or the claw. None of those is expandable.
- $Exp_2(5) = 6$: The butterfly (see Figure 3) is expandable. Suppose there is $G = (V, E)$ a $meg(5)$ with $m \leq 5$. From Property 2.2, $\delta(G) \geq 2$. So $G = C_5$ which is not expandable.

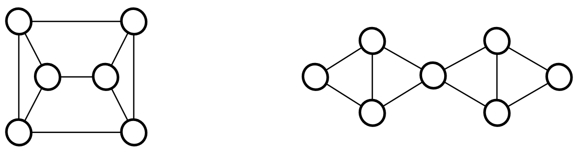

Figure 4: G_6 and G_7 two minimal expandable graphs with 6 and 7 vertices.

- $Exp_2(6) = 9$: The graph G_6 (see Figure 4) is expandable. Following Fact 2.1 it is sufficient to suppose that there is $G = (V, E)$ an expandable graph with $|E| = 8$. If G has a leaf, then from Property 2.2 $|E| \geq 9$, a contradiction. So $\delta(G) \geq 2$. Let n_2 be the number of 2-vertices. We have $2 \leq n_2 \leq 4$.
  Let $n_2 = 2$. Let $d(v_i) = 2, 1 \leq i \leq 2$ and $d(v_i) = 3, 3 \leq i \leq 6$. Suppose that $v_1v_2 \in E$. W.l.o.g. $v_1v_3, v_2v_4 \in E$. If $v_3v_4 \notin E$, then v_3v_4 cannot be extended. So $v_3v_4 \in E$. W.l.o.g. $v_3v_5, v_4v_6 \in E$ but v_5, v_6 cannot be 3-vertices. So $v_1v_2 \notin E$. If $v_i \in N(v_1) \cap N(v_2)$, then v_i is universal by Property 2.6, which is impossible. Thus w.l.o.g. $N(v_1) = \{v_3, v_4\}, N(v_2) = \{v_5, v_6\}$. So $G[V - \{v_1, v_2\}] = C_4$. W.l.o.g. $v_3v_4, v_5v_6, v_3v_5, v_4v_6 \in E$, but $v_3v_6 \notin E$ cannot be extended.
  Let $n_2 = 3$. W.l.o.g. $d(v_1) = d(v_2) = d(v_3) = 2, d(v_4) = d(v_5) = 3, d(v_6) = 4$. We have $|N(v_6) \cap \{v_1, v_2, v_3\}| \geq 2$, so v_6 is universal a contradiction.
  Let $n_2 = 4$. W.l.o.g. $d(v_1) = d(v_2) = d(v_3) = d(v_4) = 2$. If $d(v_5) = d(v_6) = 4$, then $|N(v_6) \cap \{v_1, v_2, v_3, v_4\}| \geq 2$, so v_6 is universal a contradiction. So we have $d(v_5) = 3, d(v_6) = 5$ but $|N(v_5) \cap \{v_1, v_2, v_3, v_4\}| \geq 2$ and v_5 is universal a contradiction.

- $Exp_2(7) = 10$: The graph G_7 (see Figure 4) is expandable. Following Fact 2.1 it is sufficient to suppose that there is $G = (V, E)$ an expandable graph with $|E| = 9$. Since $|E| < \frac{3}{2}n$, from Property 2.2 we have $\delta(G) \geq 2$. Let n_2 be the number of 2-vertices. From Property 2.6 two 2-vertices have no common neighbor, so $n_2 \leq 3$. If $n_2 \leq 2$ we have $|E| \geq 10$. So $n_3 = 3$, w.l.o.g. v_1, v_2, v_3 are the 2-vertices. Since $|E| = 9$, we have that v_4, v_5, v_6, v_7 are 3-vertices. Using Property 2.6 again, w.l.o.g. $v_1v_2, v_1v_4, v_2v_5, v_3v_6, v_3v_7 \in E$. If $v_6v_7 \notin E$ then $v_4 \in N(v_6) \cap N(v_7)$ and from Property 2.4 $d(v_4) > 3$, a contradiction. So $v_6v_7 \in E$, and w.l.o.g. $v_4v_6, v_5v_7, v_4v_5 \in E$ but v_4v_7 cannot be extended.

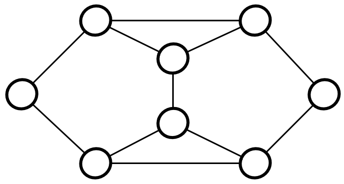

Figure 5: The graph G_8 is a $meg(8)$.

- $Exp_2(8) = 11$: One can check that G_8 (see Figure 5) or the graph shown in Figure 2 are expandable. Following Fact 2.1 it is sufficient to suppose that there is $G = (V, E)$ an expandable graph with $|E| = 10$. From Properties 2.2 and 2.6 $\delta(G) \geq 2$ and each 2-vertex has at least one proper neighbor of degree at least three. Thus there are at most four 2-vertices. Since $|E| = 10$, there are four 2-vertices, says v_1, v_2, v_3, v_4, and four 3-vertices v_5, v_6, v_7, v_8 each of them linked to exactly one 2-vertex. W.l.o.g. $v_1v_2, v_3v_4, v_1v_5, v_2v_6, v_3v_7, v_4v_8 \in E$. From Property 2.4 $v_5v_6, v_7v_8 \notin E$. So $v_5v_7, v_5v_8, v_6v_7, v_6v_8 \in E$, but v_5v_6 cannot be extended.

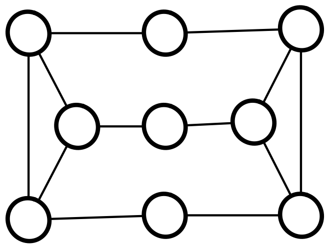

Figure 6: The graph G_9 is a $meg(9)$.

- $Exp_2(9) = 12$: One can check that G_9 (see Figure 6) is expandable. Following Fact 2.1 it is sufficient to suppose that there is $G = (V, E)$ an expandable graph with $|E| = 11$. Property 2.2 implies that $\delta(G) \geq 2$ so there are at least five 2-vertices. From Property 2.6 each 2-vertex has a proper neighbor of degree at least three, which is impossible.

4 A lower bound for $Exp_2(n)$, $n \geq 10$

We will first derive a lower bound of $Exp_2(n)$ for $n \geq 10$. It will be shown in the next section that it is best possible.

Lemma 4.1 *If $G = (V, E)$ with $n \geq 10$ is expandable, then $m \geq \lceil \frac{11}{8} n \rceil$.*

Proof: From Property 2.1 G is connected. Since $n \geq 10$, we have $\lceil \frac{3n-1}{2} \rceil \geq \lceil \frac{11}{8} n \rceil$. If there is a 1-vertex then from Property 2.2, $m \geq \frac{3}{2} n$. If there is a universal vertex, then from Property 2.3, $m \geq \lceil \frac{3n-1}{2} \rceil$. If $\delta(G) \geq 3$, then $m \geq \frac{3}{2} n$. If two 2-vertices have a common neighbor, then from Property 2.6, $m \geq \frac{3}{2} n$.

So from now on we examine the case where $\delta(G) = 2$, there is no universal vertex and the following condition.

Cond. 4.2 *For any two 2-vertices u, v we have $N_G(u) \cap N_G(v) = \emptyset$.*

Let $W = \{v \in V : d(v) = 2\}$. If $|W| = 1$ then $m \geq \lceil \frac{3n-1}{2} \rceil$. So we assume now that $|W| \geq 2$. Let $W_1 = \{v \in W : N_G(v) \cap W \neq \emptyset\}$ that is the subset of 2-vertices adjacent to another 2-vertex.

We will use a discharging procedure where a weight $w(v)$ is assigned to every vertex $v \in V$. At each step of the process some $w(v)$'s are modified but $\Sigma_{v \in V} w(v) = \Sigma_{v \in V} d(v)$ is invariant during the procedure. At the beginning we take $w(v) = d(v)$ for every vertex $v \in V$. We will show that at the end of the procedure $w(v) \geq \frac{11}{4}$ for every vertex $v \in V$, so when the procedure will be completed we will have $\Sigma_{v \in V} w(v) = \Sigma_{v \in V} d(v) = 2m \geq 3n - \lfloor \frac{n}{4} \rfloor = \lceil \frac{11}{4} n \rceil$.

We treat the vertices according to the following sequence of four steps $1, \ldots, 4$. At the beginning the vertices with $w(v) < \frac{11}{4}$ are 2-vertices. During the procedure these vertices are charged while k-vertices, $k \geq 3$, are discharged. At each step a 2-vertex v with $w(v) < \frac{11}{4}$ is charged and u ($u \neq v$) is defined as a 2-vertex such that $d(v, u) = \min\{d(v, x) : x \in W\}$. Notice that from Cond. 4.2 we have $d(v, u) \neq 2$. The vertices of W_1 are treated in step 1. The vertices of $W \setminus W_1$ are treated in steps $2, 3, 4$. Let $N(v) = \{v', v''\}$ and $N(u) = \{u', u''\}$.

1. For all $v \in W_1$ ($d(v, u) = 1$): Let $u = v'$. If $d(v'') \geq 4$, let $w(v) \leftarrow w(v) + \frac{3}{4} = \frac{11}{4}$ and $w(v'') \leftarrow w(v'') - \frac{3}{4}$. Else $N_G(v'') = \{v, y, y'\}$ with $u \neq y, y'$. Let $w(v) \leftarrow w(v) + \frac{3}{4} = \frac{11}{4}$ and $w(v'') \leftarrow w(v'') - \frac{1}{4}, w(y) \leftarrow w(y) - \frac{1}{4}, w(y') \leftarrow w(y') - \frac{1}{4}$.
2. For all $v \in W \setminus W_1$ ($d(v, u) \geq 3$, $d(v') \geq 3$ and $d(v'') \geq 3$): Let $w(v) \leftarrow w(v) + \frac{1}{2} = \frac{10}{4}$ and $w(v') \leftarrow w(v') - \frac{1}{4}, w(v'') \leftarrow w(v'') - \frac{1}{4}$.
3. For all $v \in W \setminus W_1$:
   - If there is $y \in N_G(v)$ such that $d(y) \geq 4$, let $w(v) \leftarrow w(v) + \frac{1}{4} = \frac{11}{4}$ and $w(y) \leftarrow w(y) - \frac{1}{4}$; (*Step 3.a*)
   - else

– if there is y such that $d(v, y) = 2$ with $d(y) \geq 5$ or $d(y) = 4$ and $N_G(y) \cap W_1 = \emptyset$, let $w(v) \leftarrow w(v) + \frac{1}{4} = \frac{11}{4}$ and $w(y) \leftarrow w(y) - \frac{1}{4}$; (*Step 3.b*)

– else, let $\mu = vv'...u'u$ be a shortest path between u and v: If $d(v, u) \geq 4$, let $y \in N_G(v'), y \in \mu$. Let $w(v) \leftarrow w(v) + \frac{1}{4} = \frac{11}{4}$ and $w(y) \leftarrow w(y) - \frac{1}{4}$. If $d(v, u) = 3$, let $y \in N_G(v')$ with $y \notin \mu$. If $d(y) \geq 4$ or $d(y) = 3$ and $w(y) = 3$, let $w(v) \leftarrow w(v) + \frac{1}{4} = \frac{11}{4}$ and $w(y) \leftarrow w(y) - \frac{1}{4}$. (*Step 3.c*)

4. For all $v \in W \setminus W_1$ such that $w(v) = \frac{10}{4}$, let y be such that $d(v, y) \leq 3$ and $w(y) \geq 3$: let $w(v) \leftarrow w(v) + \frac{1}{4} = \frac{11}{4}$ and $w(y) \leftarrow w(y) - \frac{1}{4}$.

Remark 4.1 *Notice that there may be several 2-vertices u such that $d(v, u) = \min\{d(v, x) : x \in W\}$. We can choose anyone.*

We summarize the main facts of the procedure: After step 1 all vertices $v \in W_1$ have a charge $w(v) = \frac{11}{4}$: it is important to note that from Properties 2.4 and 2.6, in the case where $d(v'') = d(y) = d(y') = 3$, u cannot be adjacent to v'', y, y'. After step 3, a vertex $v \in W \setminus W_1$ has a charge $w(v) = \frac{11}{4}$ if there is $x \in V$ verifying one of the following conditions:

a) x is a k-vertex with $k \geq 5$ and $d(v, x) \leq 2$;

b) x is a 4-vertex and $d(v, x) = 1$;

c) x is a 4-vertex, $d(v, x) = 2$ and $N_G(x) \cap W_1 = \emptyset$.

Now we prove that when the discharging procedure is completed we have $w(x) \geq \frac{11}{4}$ for every vertex x of G. First we prove that any vertex x with $d(x) \geq 3$ has a charge $w(x) \geq \frac{11}{4}$ at the end of the procedure. Then we prove that any 2-vertex x has a charge $w(x) = \frac{11}{4}$ at the end of the procedure.

Claim 4.1 *At the end of step 3 any vertex x with $d(x) \geq 3$ verifies $w(x) \geq \frac{11}{4}$. Moreover if $d(x) \geq 5$ or $d(x) \geq 4$ and $N_G(x) \cap W = \emptyset$, then $w(x) \geq 3$.*

Proof: Let x be a discharged vertex. At the end of step 3, x was discharged only to charge 2-vertices at distance 1 or 2.

From Cond. 4.2, x has at most one 2-vertex in its neighborhood, and any neighbor of x has at most one 2-vertex in its own neighborhood so x has at most $d(x)$ 2-vertices at distance exactly 2. x sends a charge $\frac{1}{4}$ or $\frac{3}{4}$ to a neighbor, and/or a charge $\frac{1}{4}$ to a 2-vertex at distance 2.

- $d(x) \geq 5$. From above we have $w(x) \geq 5 - (\frac{3}{4} + d(x)\frac{1}{4}) \geq 3$.
- $d(x) = 4$. Assume that $N_G(x) \cap W = \emptyset$. Since x sends a charge $\frac{1}{4}$ to at most 4 2-vertices, we have $w(x) \geq 4 - 4 \times \frac{1}{4} = 3$.

  Now $N_G(x) \cap W \neq \emptyset$. Let $N_G(x) \cap W = \{v\}$.

  Assume that $v \notin W_1$. x sends a charge $\frac{1}{4}$ to v at step 2 and another charge $\frac{1}{4}$ at step 3.a. Moreover during steps 3.b and 3.c, x can send a charge $\frac{1}{4}$ to a

2-vertex at distance 2: since $v \notin W_1$, there are at most $d(x)-1$ such 2-vertices. Thus we have $w(x) \geq 4 - (2 \times \frac{1}{4} + 3 \times \frac{1}{4}) = \frac{11}{4}$.

Now we assume that $v \in W_1$. Let $x = v', u = v''$ with $d(u) = 2$. By step 1 x sends a charge $\frac{3}{4}$ to v. Moreover x can send a charge $\frac{1}{4}$ to some 2-vertex $z, z \neq u$, with $d(x,z) = 2$. Assume that x charges three such 2-vertices. Let r, s, t be these vertices and $r''rr'x, s''ss'x, t''tt'x$ be the three corresponding paths. So $N_G(x) = \{v, r', s', t'\}$ with $d(r'), d(s'), d(t') \geq 3$. From Cond. 4.2, $r', s', t', r'', s'', t''$ are pairwise distinct. If $d(r') \geq 4$ or $d(r'') \geq 4$ then r is not charged from x and so $w(x) \geq \frac{11}{4}$. So we can assume that $d(r') = 3$ and $d(r'') \in \{2, 3\}$. If $d(r'') = 2$, then from Cond. 4.2 $r'' \neq u$, so $r''rr'xv$ is a 4-path between the 2-vertices r'' and v, and by Property 2.5 $d(r') = 4$, a contradiction. Thus $d(r'') = d(s'') = d(t'') = 3$, and r, s and t are not charged at step 1. Since r is charged by x, this can only be done at step 3.c (case $d(v,u) = 4$): Here x plays the role of y, with $d(x) = 3$ and $w(x) = 3$, and r corresponds to v in the path μ, also $x \notin \mu$. So μ is a 3-path $rr'z'z$ with $z \in W$. Since $d(v) = 4$, $z' \notin N_G(x)$, i.e., $xz' \notin E$. If $z = u$, then xz' cannot be extended. Thus there is a 4-path $zz'r'xv$ and from Property 2.5 $d(r') \geq 4$, a contradiction. So x charges v and at most two 2-vertices and thus $w(x) \geq 4 - (\frac{3}{4} + 2(\frac{1}{4})) = \frac{11}{4}$.

- $d(x) = 3$. Assume that x charges two 2-vertices u and v at step 1. We have $u, v \in W_1$ and since $d(x) = 3$, u and v cannot be adjacent. So there are a 2-vertex s adjacent to u and a 2-vertex t adjacent to v. If $x \in N_G(u)$, then from Cond. 4.2 $d(v,x) = 2$. Hence there is a path $suxv'v$ with $\{xv, xs, uv, uv', vs\} \notin E$ and by Cond. 4.2 again $sv' \notin E$. Thus $suxv'v$ is a 4-path and by Property 2.5 we have $d(x) \geq 4$, a contradiction. It follows that $d(u,x) = d(v,x) = 2$ and there is a path $suu'xv'vt$. By step 1 u', v' are 3-vertices. If $u'v' \notin E$ it cannot be extended, so $u'v' \in E$. But now uv' cannot be extended. Hence x charges at most one 2-vertex at step 1.

  Assume that x charges a 2-vertex t at step 2 and a 2-vertex v at step 1. We have $t \in W \setminus W_1, v \in W_1$, $t \in N_G(x)$ and from Cond. 4.2 $d(v,x) = 2$. So there is a path $uvv''xt$ with $u \in W_1$. From Cond. 4.2 it is a 4-path and by Property 2.5, $d(v'') \geq 4$. But from step 1 x cannot charge v. So a vertex x cannot charge a 2-vertex in step 2 and a 2-vertex in step 1. By Cond. 4.2 x charges at most one 2-vertex in step 2. If follows that x charges at most one 2-vertex in steps 1 and 2.

  So before step 3 we have $w(x) \geq \frac{11}{4}$. Since $d(x) = 3$, during step 3, x can be discharged only if $w(x) \geq 3$. Thus after step 3 we have $w(x) \geq \frac{11}{4}$.

Hence after step 3 every vertex x with $d(x) \geq 3$ has a charge $w(x) \geq \frac{11}{4}$. □

Since in step 4 a vertex x is discharged only if $w(x) \geq 3$, we have the following:

At the end of the procedure any vertex x with $d(x) \geq 3$ verifies $w(x) \geq \frac{11}{4}$.

Now we consider the 2-vertices.

Claim 4.2 *At the end of the procedure any vertex $v \in W$ verifies $w(v) = \frac{11}{4}$.*

Proof: After step 1 the vertices $v \in W_1$ satisfy $w(v) = \frac{11}{4}$.

Now let $v \in W \setminus W_1$. At the end of step 2 we have $w(v) = \frac{10}{4}$.

- $d(v, u) \geq 5$. From **a), b), c)** if there exists x a k-vertex with $k \geq 4$ and $d(v, x) \leq 2$, then we have $w(v) = \frac{11}{4}$ after step 3.b. Else let y be defined as in step 3.c. Any 2-vertex is at distance at least three from y. So y is not discharged for another 2-vertex before the treatment of v. Thus $w(y) = 3$ and step 3.c can be applied to v and then $w(v) = \frac{11}{4}$.

- $d(v, u) = 4$. There is a 4-path $vv'xu'u$ and x has no adjacent 2-vertex. From Property 2.5 $d(x) \geq 4$ and from **c)** we have $w(v) = \frac{11}{4}$.

- $d(v, u) = 3$. From Step 3.a, if v has a neighbor y with $d(y) \geq 4$ then $w(v) = \frac{11}{4}$. From now any neighbor of v is a 3-vertex. There is a path $v''vv'u'uu''$ such that $vv'u'u$ is a 3-path. From Cond. 4.2 u', u'', v', v'' are pairwise distinct. Let $N_G(v') = \{v, u', x\}$. If $x = v''$, then $v''u'$ cannot be a non-edge else it cannot be extended, so $v''u' \in E$ but the non-edge vu' cannot be extended, thus $x \neq v''$.

  Assume that $w(v) = \frac{10}{4}$ after step 3. From step 3.c, x is a 3-vertex and $w(x) = \frac{10}{4}$. Moreover v', v'' are 3-vertices (from step 3.a), $3 \leq d(u') \leq 4$ and if $d(u') = 4$, then $N_G(u') \cap W_1 \neq \emptyset$ (from step 3.b).

  We assume that there is a 2-vertex $t \in N_G(x)$ (x may discharge for t in step 2). Let $N_G(t) = \{x, t''\}$. If xu' is a non-edge it cannot be extended. So $xu' \in E$.

  Assume that $d(u') = 3$. If $t = u$ then $x = u''$ and $v'u$ cannot be extended. Hence $t \neq u$. From Cond. 4.2 $t'' \neq u'', v''$. We have $v''u'', v''t'' \in E$, else they cannot be extended. Moreover $u''t'' \in E$ else it cannot be extended. Since $n \geq 10$, it exists $y \notin \{v, v', v'', u, u', u'', x, t, t''\}$. Since v'' is a 3-vertex, $yv'' \notin E$ but $v''y$ cannot be extended.

  Hence $d(u') = 4$ and $u \in W_1$. Since t is a 2-vertex and $d(v, t) = 3$, t can play the role of u in step 3.c., and u' the role of y (see Remark 4.1). Then $w(v) = \frac{11}{4}$, a contradiction.

  So we have $N_G(x) \cap W = \emptyset$. Let $N_G(x) = \{v', x', x''\}$. We assume that a 2-vertex $z, z \neq v$ is adjacent to x' or x''. W. l.o.g. $z \in N_G(x')$. Since $v \notin W_1$, we have that vz is a non-edge. From Cond. 4.2 $vx', v'z \notin E$ and since $x \neq v''$, $vx \notin E$. If $x' \neq u'$, then $v'x' \notin E$, $vv'xx'z$ is a 4-path and by Property 2.5 $d(x) \geq 4$, a contradiction. Thus $x' = u'$. Since $x' \neq x''$, $N(x'') \cap W = \emptyset$, the 2-vertex z is unique and $z = u$. Moreover $xu' \in E$.

  Since $w(v) = \frac{10}{4}$, x has discharged $\frac{1}{4}$ to u, u being the unique 2-vertex apart v at distance 2 from x. It follows that u' is a 3-vertex, hence $B = G[\{u, u', v, v', x\}]$ is a bull with x for head. Moreover $w(u) = \frac{11}{4}$ and u is charged from x.

  We assume that $w(x') = \frac{11}{4}$. From Claim 4.1 x' is a 3-vertex and it exists a 2-vertex z such that $d(z, x') = 2$, z is charged by x'. So there is a 2-path $zz'x'$

with $d(z') = 3$. Since u is charged by x and $w(v) = \frac{11}{4}$, we have that $z \neq u, v$. From Cond. 4.2 we have $z' \neq u'', v''$. Since $d(z') = 3$, we have $z'v'' \notin E$ or $z'u'' \notin E$ but this non-edge cannot be extended.

So $w(x') \geq 3$ after step 3. Since $d(v, x') = 3$, it follows that $w(v) = \frac{11}{4}$ and $w(x') = \frac{11}{4}$ after step 4. From Property 2.7 x' cannot be the head of another bull B' with the same conditions as in B. Hence step 4 can be applied for every vertex v such that $w(x) = \frac{10}{4}$ after step 3.

Hence the procedure terminates with $w(v) = \frac{11}{4}$ for every 2-vertex. □

So after the procedure $w(x) \geq \frac{11}{4}$ for every $x \in V$. □

5 Meg(n) for $n \geq 10$

5.1 A basic module

To build the minimum expandable graphs we define their components. Figure 7 gives the component $\mathcal{H}$.

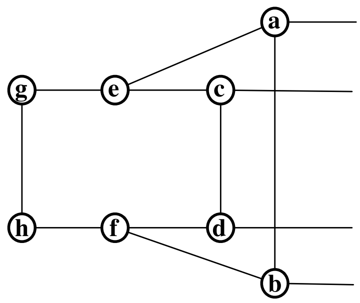

Figure 7: The subgraph $\mathcal{H}$ with two 2-vertices and six 3-vertices.

The graph $\mathcal{H}(2)$ is as follows (see Figure 8): $\mathcal{H}(2)$ contains 2 copies $\mathcal{H}_1, \mathcal{H}_2$ of $\mathcal{H}$. The vertices of $\mathcal{H}_i$ are denoted by $a_i, b_i, \ldots, h_i, 1 \leq i \leq 2$. The edges between $\mathcal{H}_1$ and $\mathcal{H}_2$ are $a_1b_2, b_1a_2, c_1c_2, d_1d_2$. Notice that since $\mathcal{H}_2$ is a copy of $\mathcal{H}_1$, there are symmetries in $\mathcal{H}(2)$. In each module, a_i, c_i, e_i, g_i is symmetric to b_i, d_i, f_i, h_i and a_i (resp. b_i) and c_i (resp. d_i) play identical roles. As a consequence, if there is a 2-factor containing, for instance, the non-edge a_1f_1 then, by *symmetry*, there is also a 2-factor containing c_1f_1 as well as a 2-factor containing b_1e_1 or $d_1e_1, a_2f_2, b_2e_2, d_2e_2, c_2f_2$. Also, a_1c_2 plays the same role as a_1d_2.

The graph $\mathcal{H}(p), p \geq 3$, is built from $\mathcal{H}(p-1)$ as follows (see Fig. 9): add one copy $\mathcal{H}_p$ of $\mathcal{H}$ to $\mathcal{H}(p-1)$. The vertices of $\mathcal{H}_i = (\mathcal{V}_i, \mathcal{E}_i)$ are denoted by $a_i, b_i, \ldots, h_i, 1 \leq i \leq p$. Remove the two edges b_1a_{p-1} and d_1d_{p-1} of $\mathcal{H}_(p-1)$. Add the four edges $b_1a_p, a_{p-1}b_p, d_1d_p, c_pd_{p-1}$. The modules $\mathcal{H}_i$ are arranged around a cycle and numbered clockwise from 1 to p. Notice that there is a symmetry on each side of $\mathcal{H}_1$ between $\mathcal{H}_i$ and $\mathcal{H}_{p-i+2}$, for $i = 2, ..., \lfloor p/2 \rfloor + 1$. As for $\mathcal{H}(2)$, a_i (resp. b_i) and

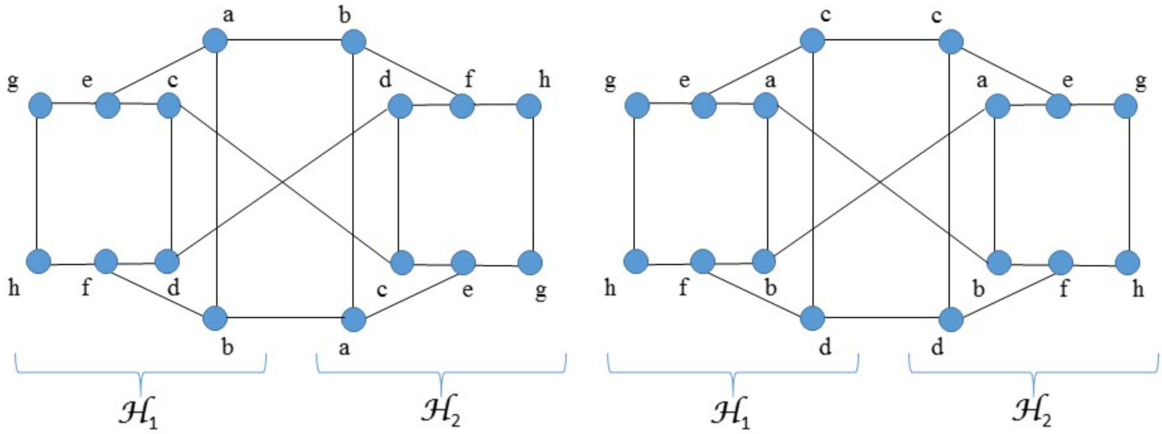

Figure 8: Two representations of $\mathcal{H}(2)$. (The vertices index are omitted).

c_i (resp. d_i) play identical roles: this can be seen by looking at Figure 9. If there is a 2-factor containing, for instance $a_1 f_2$, then, there is also a 2-factor containing $b_1 f_5$ as well as a 2-factor containing $c_1 f_2$ or $d_1 f_5$. Notice also that $a_p b_1$ and $d_p d_1$, the two edges linking $\mathcal{H}_p$ to $\mathcal{H}_1$ play the same role.

In the following, we shall shorten many proofs by referring to all these properties as *symmetries*.

Remark 5.1 *The graph $\mathcal{H}_p$ has a 2-factor: for instance, take the cycle $(a_1, b_2, a_2, ..., a_p, b_1, a_1)$ and the p cycles $(c_i, e_i, g_i, h_i, f_i, d_i, c_i)$, $i = 1, ..., p$.*

Remark 5.2 *The subgraph induced by $\mathcal{V}_i \cup \mathcal{V}_{i+1}$, $1 < i < p$, is hamiltonian. A hamiltonian cycle is $(a_i, b_i, f_i, h_i, g_i, e_i, c_i, d_i, c_{i+1}, d_{i+1}, f_{i+1}, h_{i+1}, g_{i+1}, e_{i+1}, a_{i+1}, b_{i+1}, a_i)$.*

5.2 Meg(n) for $n = 8p$, $p \geq 2$

We use a recurrence to prove that $\mathcal{H}(p)$ is a $meg(8p)$.

Property 5.1 *$\mathcal{H}(2)$ is a $meg(16)$.*

Proof: $\mathcal{H}(2)$ contains $m = 22 = \lceil \frac{11}{8} \times 16 \rceil$ edges.

We show that $\mathcal{H}(2)$ is expandable. Let $xy \notin E$. We give, first a chain $(x, \ldots, y)$, and then, possibly, a set of cycles that provide a 2-factor of $\mathcal{H}(2)$.

- $xy = a_1 c_1$: $(a_1, e_1, g_1, h_1, f_1, b_1, a_2, b_2, f_2, h_2, g_2, e_2, c_2, d_2, d_1, c_1)$; by symmetry $b_1 d_1, b_2 d_2, a_2 c_2$ can be extended;
- $xy = a_1 d_1$: $(d_1, c_1, e_1, g_1, h_1, f_1, b_1, a_2, b_2, a_1), (d_2, c_2, e_2, g_2, h_2, f_2, d_2)$; by symmetry $b_1 c_1, b_2 c_2, a_2 d_2$ can be extended;

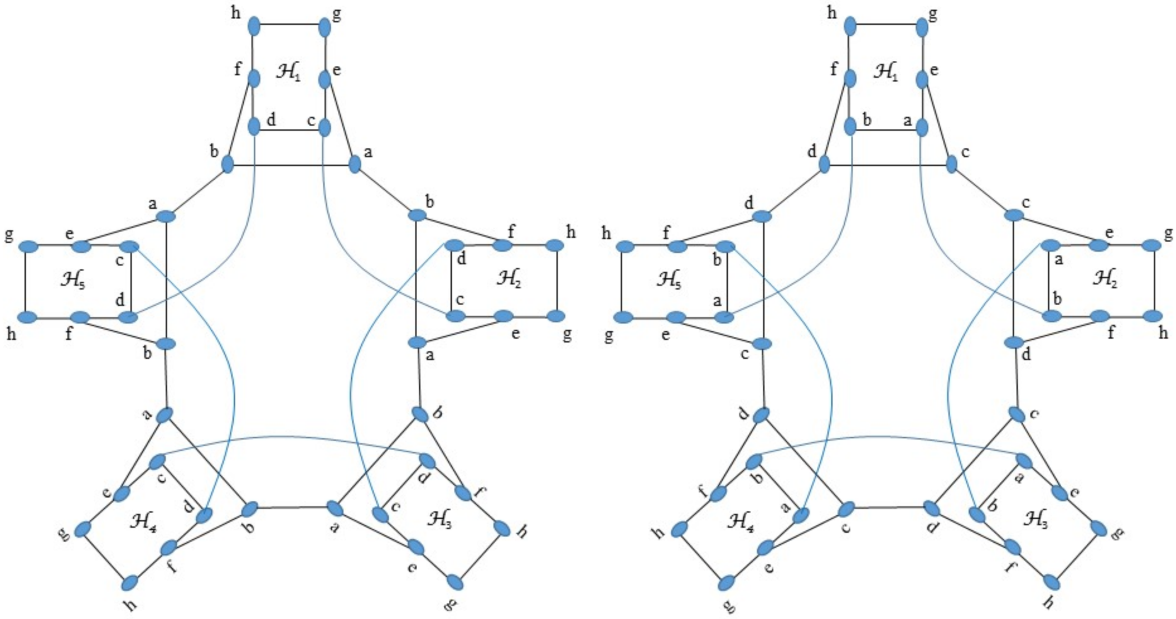

Figure 9: Two representation of $\mathcal{H}(5)$. (The vertex indices are omitted).

- $xy = a_1f_1$: $(f_1, h_1, g_1, e_1, c_1, d_1, d_2, c_2, e_2, g_2, h_2, f_2, b_2, a_2, b_1, a_1)$; by symmetry $d_1e_1, c_1f_1, b_1e_1, c_2f_2, d_2e_2, a_2f_2, b_2e_2$ can be extended;

- $xy = a_1g_1$: $(g_1, h_1, f_1, b_1, a_2, b_2, f_2, h_2, g_2, e_2, c_2, d_2, d_1, c_1, e_1, a_1)$; by symmetry $c_1g_1, b_1h_1, d_1h_1, b_2h_2, d_2h_2, a_2g_2, c_2g_2$ can be extended;

- $xy = a_1a_2$: $(a_2, b_1, f_1, h_1, g_1, e_1, c_1, d_1, d_2, c_2, e_2, g_2, h_2, f_2, b_2, a_1)$; by symmetry b_1b_2, c_1d_2, d_1c_2 can be extended.

- $xy = a_1c_2$: $(c_2, d_2, d_1, c_1, e_1, g_1, h_1, f_1, b_1, a_2, e_2, g_2, h_2, f_2, b_2, a_1)$; by symmetry $b_1d_2, b_1c_2, a_1d_2, b_2c_1, d_1b_2, a_2c_1, a_2d_1$ can be extended.

- $xy = d_1g_1$: $(g_1, h_1, f_1, b_1, a_2, b_2, a_1, e_1, c_1, c_2, e_2, g_2, h_2, f_2, d_2, d_1)$; by symmetry $b_1g_1, c_1h_1, a_1h_1, c_2h_2, a_2h_2, d_2g_2, b_2g_2$ can be extended;

- $xy = e_1a_2$: $(e_1, g_1, h_1, f_1, b_1, a_1, b_2, f_2, h_2, g_2, e_2, a_2), (c_1, d_1, d_2, c_2, c_1)$; by symmetry $b_2f_1, b_1f_2, a_1e_2, d_2e_1, c_2f_1, d_1e_2, c_1f_2$ can be extended;

- $xy = e_1b_2$: $(e_1, g_1, h_1, f_1, d_1, c_1, c_2, d_2, f_2, h_2, g_2, e_2, a_2, b_1, a_1, b_2)$; by symmetry $a_2f_1, a_1f_2, b_1e_2, c_2e_1, d_2f_1, c_1e_2, d_1f_2$ can be extended;

- $xy = e_1f_1$: $(e_1, g_1, h_1, f_1), (a_1, b_1, a_2, e_2, g_2, h_2, f_2, b_2, a_1), (c_1, d_1, d_2, c_2, c_1)$; by symmetry e_2f_2 can be extended;

- $xy = e_1d_2$: $(e_1, g_1, h_1, f_1, d_1, c_1, c_2, d_2), (a_1, b_1, a_2, e_2, g_2, h_2, f_2, b_2, a_1)$; by symmetry $a_2e_1, c_2f_1, b_2f_1, c_1f_2, d_1e_2, a_1e_2, b_1f_2$ can be extended;

- $xy = e_1f_2$: $(e_1, g_1, h_1, f_1, b_1, a_1, b_2, a_2, e_2, g_2, h_2, f_2), (c_1, d_1, d_2, c_2, c_1)$; by symmetry e_1e_2, f_1f_2, e_2f_1 can be extended;

- $xy = e_1h_2$: $(e_1, g_1, h_1, f_1, d_1, c_1, c_2, d_2, f_2, b_2, a_1, b_1, a_2, e_2, g_2, h_2)$; by symmetry $e_1g_2, f_1h_2, f_1g_2, e_2g_1, f_2h_1, f_2g_1, e_2h_1$ can be extended;

- $xy = f_1g_1$: $(g_1, h_1, f_1), (b_1, a_1, e_1, c_1, d_1, d_2, c_2, e_2, g_2, h_2, f_2, b_2, a_2, b_1)$; by symmetry e_1h_1, f_2g_2, e_2h_2 can be extended;

- $xy = g_1b_2$: $(g_1, h_1, f_1, d_1, d_2, c_2, c_1, e_1, a_1, b_1, a_2, e_2, g_2, h_2, f_2, b_2)$; by symmetry $c_2g_1, a_2h_1, d_2h_1, a_1h_2, d_1h_2, b_1g_2, c_1g_2$ can be extended;

- $xy = g_1d_2$: $(g_1, h_1, f_1, d_1, d_2), (a_1, e_1, c_1, c_2, e_2, g_2, h_2, f_2, b_2, a_2, b_1, a_1)$; by symmetry $a_2g_1, c_2h_1, b_2h_1, c_1h_2, b_1h_2, d_1g_2, a_1g_2$ can be extended;

- $xy = g_1h_2$: $(g_1, h_1, f_1, d_1, d_2, f_2, b_2, a_2, b_1, a_1, e_1, c_1, c_2, e_2, g_2, h_2)$; by symmetry g_1g_2, h_1h_2, h_1g_2 can be extended;

□

Property 5.2 *$\mathcal{H}(3)$ is a meg(24).*

Proof: $\mathcal{H}(3)$ contains $m = 33 = \lceil \frac{11}{8} \times 24 \rceil$ edges.

We show that $\mathcal{H}(3)$ is expandable. Let $xy \notin E$.

Case 1. $x, y \notin \mathcal{H}_3$. Looking at the 2-factors given for $\mathcal{H}(2)$, we observe that they all contain at least one of the two edges b_1a_2, d_1d_2 and then we can build a 2-factor containing xy in $\mathcal{H}(3)$ from a 2-factor F containing xy in $\mathcal{H}(2)$. If $b_1a_2 \in F$ and $d_1d_2 \notin F$, we substitute b_1a_2 for $b_1a_3b_3a_2$ and we add the cycle $(c_3, d_3, f_3, h_3, g_3, e_3, c_3)$ to obtain a 2-factor in $\mathcal{H}(3)$. The case $b_1a_2 \notin F$ and $d_1d_2 \in F$ is symmetric. If $b_1a_2 \in F$ and $d_1d_2 \in F$, we substitute b_1a_2 for $b_1a_3b_3a_2$ and d_1d_2 for $d_1d_3f_3h_3g_3e_3c_3d_2$ to obtain a 2-factor in $\mathcal{H}(3)$.

Case 2. $x, y \notin \mathcal{H}_2$: by symmetry this is equivalent to Case 1.

Case 3. $x \in \mathcal{H}_2$, $y \in \mathcal{H}_3$. For each xy, we give, first a chain $(x, \ldots, y)$, and then, possibly, a set of cycles that provide a 2-factor of $\mathcal{H}(3)$.

- $xy = a_2a_3$: $(a_2, b_3, f_3, h_3, g_3, e_3, a_3); (a_1, b_1, f_1, h_1, g_1, e_1, c_1, d_1, d_3, c_3, d_2, c_2, e_2, g_2, h_2, f_2, b_2, a_1)$; by symmetry b_2b_3, d_2d_3, c_2c_3 can be extended;

- $xy = a_2c_3$: $(a_2, b_3, a_3, e_3, g_3, h_3, f_3, d_3, d_1, c_1, e_1, g_1, h_1, f_1, b_1, a_1, b_2, f_2, h_2, g_2, e_2, c_2, d_2, c_3)$; by symmetry d_2b_3 can be extended;

- $xy = a_2d_3$: $(a_2, b_3, f_3, h_3, g_3, e_3, a_3, b_1, a_1, b_2, f_2, h_2, g_2, e_2, c_2, d_2, c_3, d_3)$, $(c_1, d_1, f_1, h_1, g_1, e_1, c_1)$; by symmetry b_2c_3, c_2b_3, d_2a_3 can be extended;

- $xy = a_2e_3$: $(a_2, b_3, a_3, b_1, a_1, b_2, f_2, h_2, g_2, e_2, c_2, d_2, c_3, d_3, f_3, h_3, g_3, e_3)$, $(c_1, d_1, f_1, h_1, g_1, e_1, c_1)$; by symmetry f_2b_3, d_2f_3, e_2c_3 can be extended;

- $xy = a_2f_3$: $(a_2, b_3, a_3, e_3, g_3, h_3, f_3), (a_1, b_1, f_1, h_1, g_1, e_1, c_1, d_1, d_3, c_3, d_2, c_2, e_2, g_2, h_2, f_2, b_2, a_1)$; by symmetry e_2b_3, f_2c_3, d_2e_3 can be extended;

- $xy = a_2g_3$: $(a_2, b_3, a_3, e_3, c_3, d_2, c_2, e_2, g_2, h_2, f_2, b_2, a_1, b_1, f_1, h_1, g_1, e_1, c_1, d_1, d_3, f_3, h_3, g_3)$; by symmetry h_2b_3, g_2c_3, d_2h_3 can be extended;

- $xy = a_2h_3$: $(a_2, b_3, f_3, d_3, c_3, d_2, c_2, e_2, g_2, h_2, f_2, b_2, a_1, b_1, a_3, e_3, g_3, h_3)$, $(c_1, d_1, f_1, h_1, g_1, e_1, c_1)$; by symmetry g_2b_3, h_2c_3, d_2g_3 can be extended;

- $xy = b_2a_3$: $(b_2, f_2, h_2, g_2, e_2, a_2, b_3, f_3, h_3, g_3, e_3, a_3), (c_1, d_1, d_3, c_3, d_2, c_2, c_1)$, $(a_1, b_1, f_1, h_1, g_1, e_1, a_1)$; by symmetry c_2d_3 can be extended;

- $xy = b_2d_3$: $(b_2, a_2, b_3, a_3, b_1, a_1, e_1, g_1, h_1, f_1, d_1, c_1, c_2, e_2, g_2, h_2, f_2, d_2, c_3, e_3, g_3, h_3, f_3, d_3)$; by symmetry c_2a_3 can be extended;

- $xy = b_2e_3$: $(b_2, a_2, b_3, a_3, b_1, a_1, e_1, g_1, h_1, f_1, d_1, c_1, c_2, e_2, g_2, h_2, f_2, d_2, c_3, d_3, f_3, h_3, g_3, e_3)$; by symmetry f_2a_3, e_2d_3, c_2f_3 can be extended;

- $xy = b_2f_3$: $(b_2, a_1, e_1, g_1, h_1, f_1, b_1, a_3, b_3, a_2, e_2, g_2, h_2, f_2, d_2, c_2, c_1, d_1, d_3, c_3, e_3, g_3, h_3, f_3)$; by symmetry e_2a_3, f_2d_3, c_2e_3 can be extended;

- $xy = b_2g_3$: $(b_2, a_1, e_1, g_1, h_1, f_1, b_1, a_3, e_3, c_3, d_3, d_1, c_1, c_2, d_2, f_2, h_2, g_2, e_2, a_2, b_3, f_3, h_3, g_3)$; by symmetry h_2a_3, g_2d_3, c_2h_3 can be extended;

- $xy = b_2h_3$: $(b_2, a_2, b_3, f_3, d_3, c_3, d_2, f_2, h_2, g_2, e_2, c_2, c_1, d_1, f_1, h_1, g_1, e_1, a_1, b_1, a_3, e_3, g_3, h_3)$; by symmetry g_2a_3, h_2d_3, c_2g_3 can be extended;

- $xy = e_2e_3$: $(e_2, g_2, h_2, f_2, b_2, a_2, b_3, a_3, b_1, a_1, e_1, g_1, h_1, f_1, d_1, c_1, c_2, d_2, c_3, d_3, f_3, h_3, g_3, e_3)$; by symmetry f_2f_3 can be extended;
- $xy = e_2f_3$: $(e_2, g_2, h_2, f_2, d_2, c_2, c_1, d_1, d_3, c_3, e_3, g_3, h_3, f_3)$; $(a_1, e_1, g_1, h_1, f_1, b_1, a_3, b_3, a_2, b_2, a_1)$; by symmetry f_2e_3 can be extended;
- $xy = e_2g_3$: $(e_2, g_2, h_2, f_2, b_2, a_2, b_3, a_3, e_3, c_3, d_2, c_2, c_1, d_1, d_3, f_3, h_3, g_3)$; $(a_1, b_1, f_1, h_1, g_1, e_1, a_1)$; by symmetry h_2f_3, f_2h_3, g_2e_3 can be extended;
- $xy = e_2h_3$: $(e_2, g_2, h_2, f_2, b_2, a_2, b_3, f_3, d_3, c_3, d_2, c_2, c_1, d_1, f_1, h_1, g_1, e_1, a_1, b_1, a_3, e_3, g_3, h_3)$; by symmetry g_2f_3, f_2g_3, h_2e_3 can be extended;
- $xy = g_2g_3$: $(g_2, h_2, f_2, b_2, a_1, b_1, f_1, h_1, g_1, e_1, c_1, d_1, d_3, f_3, h_3, g_3)$, $(a_2, e_2, c_2, d_2, c_3, e_3, a_3, b_3, a_2)$; by symmetry h_2h_3 can be extended;
- $xy = g_2h_3$: $(g_2, h_2, f_2, b_2, a_1, b_1, a_3, e_3, g_3, h_3)$, $(a_2, e_2, c_2, d_2, c_3, d_3, f_3, b_3, a_2)$, $(c_1, d_1, f_1, h_1, g_1, e_1, c_1)$; by symmetry h_2g_3 can be extended;

□

Fact 5.1 *Let $xy \notin \mathcal{H}(p)$, $p \geq 3$. If $x, y \neq h_1$, then any 2-factor containing xy contains a_pb_1 or d_pd_1.*

Proof: Let F be a 2-factor containing xy and assume that none of a_pb_1 and d_pd_1 are in F. To cover b_1 and d_1, F must contain the edges b_1a_1, b_1f_1, d_1c_1 and d_1f_1. Clearly these 4 edges cannot be completed in a 2-factor if x and $y \neq h_1$. □

Property 5.3 *$\mathcal{H}(p)$ is a $meg(8p), p \geq 4$.*

Proof: $\mathcal{H}(p)$ contains $m = 11p = \lceil \frac{11}{8} \times 8p \rceil$ edges.

The proof is by induction. From Properties 5.1 and 5.2, $\mathcal{H}(2)$ and $\mathcal{H}(3)$ are expandable. So for $p \geq 3$ we assume that $\mathcal{H}(p)$ and $\mathcal{H}(p-1)$ are expandable.

We now prove that for any $p \geq 3$, $\mathcal{H}(p+1)$ is expandable.

Let $xy \notin \mathcal{H}(p+1)$. We examine several cases.

Case 1. $x, y \notin \mathcal{V}_1 \cup \mathcal{V}_{p+1}$. Let F be a 2-factor containing xy in $\mathcal{H}(p)$. From Fact 5.1, $a_pb_1 \in F$ or $d_pd_1 \in F$. If F contains a_pb_1 but not d_pd_1, then a_pb_1 is replaced by $a_pb_{p+1}a_{p+1}b_1$; adding the cycle $(c_{p+1}, d_{p+1}, f_{p+1}, h_{p+1}, g_{p+1}, e_{p+1}, c_{p+1})$ we have a 2-factor in $\mathcal{H}(p+1)$. By symmetry, the case where $d_pd_1 \in \mathcal{H}(p)$ and $a_pb_1 \notin \mathcal{H}(p)$ is equivalent. If both a_pb_1 and d_pd_1 are in F, then a_pb_1 is substituted for $a_pb_{p+1}a_{p+1}b_1$ and d_pd_1 for $d_pc_{p+1}e_{p+1}g_{p+1}h_{p+1}f_{p+1}d_{p+1}, d_1$ to obtain a 2-factor in $\mathcal{H}(p+1)$.

Case 2. $x \in \mathcal{V}_1$.

Case 2.1. $y \notin \mathcal{V}_p \cup \mathcal{V}_{p+1}$. From our assumption xy can be extended in $\mathcal{H}(p-1)$. If the 2-factor in $\mathcal{H}(p-1)$ contains $a_{p-1}b_1$ but not $d_{p-1}d_1$, then $a_{p-1}b_1$ is substituted for $a_{p-1}b_pa_pb_{p+1}a_{p+1}b_1$; adding the two cycles $(c_j, d_j, f_j, h_j, g_j, e_j, c_j)$, $p \leq j \leq p+1$, we have a 2-factor in $\mathcal{H}(p+1)$. By symmetry, the case where the 2-factor in $\mathcal{H}(p-1)$ contains $d_{p-1}d_1$ but not $a_{p-1}b_1$ is equivalent. If both $a_{p-1}b_1$ and $d_{p-1}d_1$ are in the 2-factor of $\mathcal{H}(p-1)$, then $a_{p-1}b_1$ is substituted for $a_{p-1}b_pa_pb_{p+1}a_{p+1}b_1$ and $d_{p-1}d_1$ for $d_{p-1}c_pe_pg_ph_pf_pd_pc_{p+1}e_{p+1}g_{p+1}h_{p+1}f_{p+1}d_{p+1}d_1$ to obtain a 2-factor in $\mathcal{H}(p+1)$. Finally, if $a_{p-1}b_1, d_{p-1}d_1 \notin F$, we add to F the hamiltonian cycle on $\mathcal{V}_{p+1} \cup \mathcal{V}_p$ (see Remark 5.2) to obtain a 2-factor in $\mathcal{H}(p+1)$.

Case 2.2. $y \in \mathcal{V}_p \cup \mathcal{V}_{p+1}$. By symmetry, $y \in \mathcal{V}_{p+1}$ is equivalent to $y \in \mathcal{V}_2$ and, if $p \geq 4$, $y \in \mathcal{V}_p$ is equivalent to $y \in \mathcal{V}_3$ seen just before.

It remains the case $p = 3$, $y \in \mathcal{V}_3$: we remove from $\mathcal{H}(4)$ the modules $\mathcal{H}_2$ and $\mathcal{H}_4$ and we obtain $\mathcal{H}'(2)$ isomorphic to $\mathcal{H}(2)$. Looking at the 2-factors obtained in $\mathcal{H}(2)$ for $xy \notin E$, $x \in \mathcal{V}_1, y \in \mathcal{V}_2$, we see that they all contain exactly three edges among $a_1b_2, b_1a_2, c_1c_2, d_1d_2$. Thus in $\mathcal{H}'(2)$, any 2-factor F containing $xy \notin E$, $x \in \mathcal{V}_1, y \in \mathcal{V}_3$ contains also exactly three edges among $a_1b_3, b_1a_3, c_1c_3, d_1d_3$ or, equivalently, they all miss exactly one edge among $a_1b_3, b_1a_3, c_1c_3, d_1d_3$. To obtain a 2-factor in $\mathcal{H}(4)$, we make the following substitutions: if $a_1b_3 \notin F$, we replace c_1c_3 by $c_1c_2d_2c_3$, d_1d_3 by $d_1d_4c_4d_3$, b_1a_3 by $b_1a_4e_4g_4h_4f_4b_4a_3$ and we add the cycle $(b_2, a_2, e_2, g_2, h_2, f_2, b_2)$; if $b_1a_3 \notin F$, we replace c_1c_3 by $c_1c_2d_2c_3$, d_1d_3 by $d_1d_4c_4d_3$, a_1b_3 by $a_1b_2f_2h_2g_2e_2a_2b_3$ and we add the cycle $(a_4, e_4, g_4, h_4, f_4, b_4, a_4)$; if $c_1c_3 \notin F$, we replace b_1a_3 by $b_1a_4b_4a_3$, a_1b_3 by $a_1b_2a_2b_3$, d_1d_3 by $d_1d_4f_4h_4g_4e_4c_4d_3$ and we introduce the cycle $(c_2, e_2, g_2, h_2, f_2, d_2, c_2)$; finally, if $d_1d_3 \notin F$, we replace b_1a_3 by $b_1a_4b_4a_3$, a_1b_3 by $a_1b_2f_2h_2g_2e_2a_2b_3$, c_1c_3 by $c_1c_2d_2c_3$ and we introduce the cycle $(c_4, d_4, f_4, h_4, g_4, e_4, c_4)$.

Case 3. $x \in \mathcal{V}_{p+1}$.

Case 3.1. $y \in \mathcal{V}_i$, $i \neq 2$. If $y \in \mathcal{V}_1$, by symmetry, this is equivalent to the case $x \in \mathcal{V}_1$, $y \in \mathcal{V}_{p+1}$ treated in Case 2.

If $y \in \mathcal{V}_i$, $i = 3, ..., p-1$, these cases are equivalent to cases $x \in \mathcal{V}_2$, $y \in \mathcal{V}_{p-i+3}$ which were treated in Case 1.

If $y \in \mathcal{V}_p$, this is equivalent to $x \in \mathcal{V}_2$, $y \in \mathcal{V}_3$ treated in Case 1.

If $y \in \mathcal{V}_{p+1}$, this is equivalent to $x, y \in \mathcal{V}_2$ treated in Case 1.

Case 3.2 $y \in \mathcal{V}_2$, $p \geq 4$. There are two adjacent modules $\mathcal{H}_i$ and $\mathcal{H}_{i+1}$ with $i \neq 2$ and $i+1 \neq p+1$. We remove these modules and add the edges $a_{i-1}b_{i+2}$ and $d_{i-1}c_{i+2}$. We obtain a graph $\mathcal{H}'(p-1)$ isomorphic to $\mathcal{H}(p-1)$ which has a 2-factor F containing xy by assumption. If $a_{i-1}b_{i+2} \in F$ but $d_{i-1}c_{i+2} \notin F$, then $a_{i-1}b_{i+2}$ is substituted for $a_{i-1}b_ia_ib_{i+1}a_{i+1}b_{i+2}$; adding the two cycles $(c_j, d_j, f_j, h_j, g_j, e_j, c_j)$, $i \leq j \leq i+1$, we have a 2-factor in $\mathcal{H}(p+1)$. By symmetry, the case where $d_{i-1}c_{i+2} \in F$ and $a_{i-1}b_{i+2} \notin F$ is equivalent. If both $a_{i-1}b_{i+2}$ and $d_{i-1}c_{i+2}$ are in the 2-factor of $\mathcal{H}'(p-1)$, then $a_{i-1}b_{i+2}$ is substituted for $a_{i-1}b_ia_ib_{i+1}a_{i+1}b_{i+2}$ and $d_{i-1}c_{i+2}$ for $d_{i-1}c_i, e_ig_ih_if_id_ic_{i+1}e_{i+1}g_{i+1}h_{i+1}f_{i+1}d_{i+1}c_{i+2}$ to obtain a 2-factor in $\mathcal{H}(p+1)$. Finally, if neither $a_{i-1}b_{i+2}$ nor $d_{i-1}c_{i+2}$ are in the 2-factor of $\mathcal{H}'(p-1)$, then adding the hamiltonian cycle covering $\mathcal{V}_i \cup \mathcal{V}_{i+1}$ (see Remark 5.2) we obtain a 2-factor in $\mathcal{H}(p+1)$.

Case 3.3 $y \in \mathcal{V}_2$, $p = 3$, $x \in \mathcal{V}_4$. We remove the module $\mathcal{H}_3$ from $\mathcal{H}(4)$ and add the edges a_2b_4 and d_2c_4: we get a graph $\mathcal{H}'(3)$ isomorphic to $\mathcal{H}(3)$ in which xy can be extended. Let F be a 2-factor of $\mathcal{H}'(3)$ containing xy.

Looking at the 2-factors given for $\mathcal{H}(3)$ to complete an edge $xy \notin \mathcal{H}(3)$ with $x \in \mathcal{V}_2$ and $y \in \mathcal{V}_3$, we see that they all contain a_2b_3 or d_2c_3. Equivalently, F contains a_2b_4 or d_2c_4.

If F contains a_2b_4 but not d_2c_4, then a_2b_4 is substituted for $a_2b_3a_3b_4$; adding the cycle $(c_3, d_3, f_3, h_3, g_3, e_3, c_3)$ we have a 2-factor in $\mathcal{H}(4)$. By symmetry, the case where $\mathcal{H}'(3)$ contains d_2c_4 but not a_2b_4 is equivalent. If both a_2b_4 and d_2c_4 are in F, then a_2b_4 is substituted for $a_2b_3a_3b_4$ and d_2c_4 for $d_2c_3e_3g_3h_3f_3d_3c_4$ to obtain a 2-factor in $\mathcal{H}'(4)$. □

5.3 Meg(n) for $n \neq 0 \ mod(8)$, $n \geq 14$

Now, we give minimal expandable graphs when $n \geq 14$ is not a multiple of 8. The graphs $\mathcal{H}^{-1}(p), \mathcal{H}^{-2}(p), p \geq 2$, are obtained from $\mathcal{H}(p)$ by contracting one edge g_ih_i, respectively two edges $g_ih_i, g_jh_j, i \neq j$. The vertex resulting of the contraction of gh is denoted by gh.

Property 5.4 *$\mathcal{H}^{-1}(p)$ is a $meg(8p-1), p \geq 2$ and $\mathcal{H}^{-2}(p)$ is a $meg(8p-2), p \geq 2$.*

Proof: $\mathcal{H}^{-1}(p)$ contains $m = 11p - 1 = \lceil \frac{11}{8} \times (8p-1) \rceil$ edges. $\mathcal{H}^{-2}(p)$ contains $m = 11p - 2 = \lceil \frac{11}{8} \times (8p-2) \rceil$ edges.

From Properties 5.1 and 5.3 we have $\mathcal{H}(p)$ is a $meg(8p), p \geq 2$. The 2-factor of $\mathcal{H}^{-1}(p)$, resp. $\mathcal{H}^{-2}(p)$, corresponds to the 2-factor of $\mathcal{H}(p)$ where the subsequence of two consecutive vertices g, h in $\mathcal{H}(p)$ is replaced by the contracted vertex gh in $\mathcal{H}^{-1}(p)$, resp. $\mathcal{H}^{-2}(p)$. □

The graph $\mathcal{H}^{+1}(p), p \geq 2$, is obtained from $\mathcal{H}(p)$ by adding a 2-vertex i and the two edges ig, ih to $\mathcal{H}(p)$.

Property 5.5 *$\mathcal{H}^{+1}(p)$ is a $meg(8p+1), p \geq 2$.*

Proof: $\mathcal{H}^{+1}(p)$ contains $m = 11p + 2 = \lceil \frac{11}{8} \times (8p+1) \rceil$ edges.

Observe that all 2-factors of $\mathcal{H}(p)$ contain gh. Let $xy \notin E$. If $x, y \neq i_1$, then the 2-factor of $\mathcal{H}^{+1}(p)$ corresponds to the 2-factor of $\mathcal{H}(p)$ where g_1h_1 in $\mathcal{H}(p)$ is replaced by $g_1i_1h_1$ in $\mathcal{H}^{+1}(p)$. If $x = i_1$, then $gy \notin E$ or $hy \notin E$. W.l.o.g. assume that $gy \notin E$. Then the 2-factor of $\mathcal{H}^{+1}(p)$ corresponds to the 2-factor of $\mathcal{H}(p)$ where h_1g_1 in $\mathcal{H}(p)$ is replaced by $h_1i_1g_1$ in $\mathcal{H}^{+1}(p)$. □

The graph $\mathcal{H}^{+2}(p)$ is obtained from $\mathcal{H}(p)$ as follows: the two edges a_1e_1, c_1e_1 are subdivided into the paths $(a_1, i, e_1), (c_1, j, e_1)$, respectively; the edge ij is added.

Property 5.6 *$\mathcal{H}^{+2}(p)$ is a $meg(8p+2), p \geq 2$.*

Proof: $\mathcal{H}^{+2}(p)$ contains $m = 11p + 3 = \lceil \frac{11}{8} \times (8p+2) \rceil$ edges.

We show that $\mathcal{H}^{+2}(p)$ is expandable. Let $xy \notin E$. In the case where $x, y \notin \{a_1, c_1, e_1\}$ we proceed as follows: we know that xy can be extended in $\mathcal{H}(p)$; the corresponding 2-factor contains either the subsequence $a_1e_1c_1$, or $a_1e_1g_1$ or $c_1e_1g_1$. In the first case the subsequence is substituted for $a_1ie_1jc_1$, in the second for $a_1ije_1g_1$, in the third for $c_1jie_1g_1$. So xy can be extended in $\mathcal{H}^{+2}(p)$.

Now let $x = i$ (the case where $x = j$ is similar). If $y \neq c_1, g_1$, we know that e_1y can be extended in $\mathcal{H}(p)$; the corresponding 2-factor contains e_1g_1; we replace it by ije_1g_1. If $y = c_1$ (resp. $y = g_1$): in $\mathcal{H}(p)$, the 2-factor corresponding to the non-edge g_1c_1 (resp. g_1a_1) contains $c_1e_1a_1$; we replace it by $c_1je_1ia_1$.

If $y = g_1$: in $\mathcal{H}(p)$, the 2-factor corresponding to the non-edge g_1a_1 contains $c_1e_1a_1$; we replace it by $c_1je_1ia_1$.

So iy can be extended in $\mathcal{H}^{+2}(p)$.

Now let $x = e_1$. We know that g_1y can be extended in $\mathcal{H}(p)$; the corresponding 2-factor contains both g_1h_1 and $a_1e_1c_1$ (or $c_1e_1a_1$, but by symmetry this case is the same). We substitute the first subsequence for $e_1g_1h_1$ and the second for a_1ijc_1 and e_1y can be extended in $\mathcal{H}^{+2}(p)$. □

The graph $\mathcal{H}^{+3}(p), p \geq 2$, is obtained from $\mathcal{H}(p)$ by adding a triangle (i, j, k) and the two edges ia_1, ja_1 to $\mathcal{H}(p)$.

Property 5.7 *$\mathcal{H}^{+3}(p)$ is a meg$(8p + 3), p \geq 2$.*

Proof: $\mathcal{H}^{+3}(p)$ contains $m = 11p + 5 = \lceil \frac{11}{8} \times (8p + 3) \rceil$ edges.

Let $xy \notin E$. If $x, y \notin \{i, j, k\}$, then the 2-factor of $\mathcal{H}^{+3}(p)$ corresponds to the 2-factor of $\mathcal{H}(p)$ plus the cycle (i, j, k, i).

Let $x = i$ (the case $x = j$ is the same): note that $y \notin \{i, j, k, a_1\}$. Suppose first that y is not a neighbor of a_1 in $\mathcal{H}(p)$: ya_1 can be extended to a 2-factor and substituting ya_1 for $yikja_1, (i = x)$, we have a 2-factor for $\mathcal{H}^{+3}(p)$. Now let $y = e_1$. We know that e_1f_1 can be extended in $\mathcal{H}(p)$. The corresponding 2-factor contains $(e_1g_1h_1f_1e_1)$ and a cycle C with the edge a_1b_1. Substituting a_1b_1 for $a_1jkie_1g_1h_1f_1b_1$ we have a 2-factor of $\mathcal{H}^{+3}(p)$. Let $y \in \{b_1, b_2\}$; e_1f_1 can be extended in $\mathcal{H}(p)$, and the corresponding 2-factor contains $(e_1, g_1, h_1, f_1, e_1)$, a cycle C with the edges a_1b_1 and a_1b_2, and a cycle C' with the edge c_1d_1. So replacing a_1y by a_1jkiy and c_1d_1 by $c_1e_1g_1h_1f_1d_1$ we have a 2-factor of $\mathcal{H}^{+3}(p)$.

For $x = k$, if $y \neq a_1$ we proceed as above by replacing the sequence ikj by kij. If $y = a_1$, we do as follows: we know that g_1a_1 can be extended in $\mathcal{H}(p)$; the corresponding 2-factor contains $e_1a_1g_1$. Then substituting $e_1a_1g_1$ for e_1g_1 and adding (a_1, i, j, k, a_1) we obtain a 2-factor of $\mathcal{H}^{+3}(p)$. □

The graph $\mathcal{H}^{+4}(p), p \geq 2$, is obtained from $\mathcal{H}(p)$ by adding a diamond $(i, j, k, l), ij \notin E$ and the edge ia_1 to $\mathcal{H}(p)$.

Property 5.8 *$\mathcal{H}^{+4}(p)$ is a meg$(8p + 4), p \geq 2$.*

Proof: $\mathcal{H}^{+4}(p)$ contains $m = 11p + 6 = \lceil \frac{11}{8} \times (8p + 4) \rceil$ edges.

Let $xy \notin E$. If $x, y \notin \{i, j, k, l\}$, then to obtain a 2-factor of $\mathcal{H}^{+4}(p)$ we add the cycle (i, k, j, l, i) to the 2-factor of $\mathcal{H}(p)$ obtained for the extension of xy.

For $xy = ij$ we take a 2-factor covering $\mathcal{H}(p)$ (see Remark 5.1) and the cycle (i, j, l, k, i) to obtain a 2-factor of $\mathcal{H}^{+4}(p)$.

Let $x = i$: we have $y \notin \{i, j, k, l, a_1\}$. First we suppose that y is not a neighbor of a_1. We know that ya_1 can be extended in $\mathcal{H}(p)$, so in the corresponding 2-factor there is a cycle with the sequence ya_1. Substituting ya_1 for yia_1 and adding the cycle (j, k, l, j) we have a 2-factor for $\mathcal{H}^{+4}(p)$. Now let $y = e_1$: We know that $e_1 f_1$ can be extended in $\mathcal{H}(p)$. The corresponding 2-factor contains $(e_1, g_1, h_1, f_1, e_1)$ and a cycle C with the edge $a_1 b_1$. Substituting the sequence $a_1 b_1$ for $a_1 i e_1 g_1 h_1 f_1 b_1$ and adding (j, k, l, j) we have a 2-factor of $\mathcal{H}^{+4}(p)$. When $y \in \{b_1, b_2\}$ we proceed as follows: $e_1 f_1$ can be extended in $\mathcal{H}(p)$, the corresponding 2-factor contains $(e_1, g_1, h_1, f_1, e_1)$, a cycle C with $a_1 y$, and a cycle C' with $c_1 d_1$. So substituting the sequence $a_1 y$ for $a_1 i y$, the sequence $c_1 d_1$ for $c_1 e_1 g_1 h_1 f_1 d_1$ and adding (j, k, l, j) we have a 2-factor of $\mathcal{H}^{+4}(p)$.

Let $x = k$ (resp.$x = j$): We proceed as for $x = i$ but instead of $a_1 i$ we take $a_1 i l j k$ (resp. $a_1 i l k j$). □

The graph $\mathcal{H}^{+5}(p)$, is obtained from $\mathcal{H}^{+2}(p)$ by subdividing the edge $a_1 b_1$ into the path $a_1 l k b_1$, adding a 2-vertex r together with two edges rl, rk. See Figure 10.

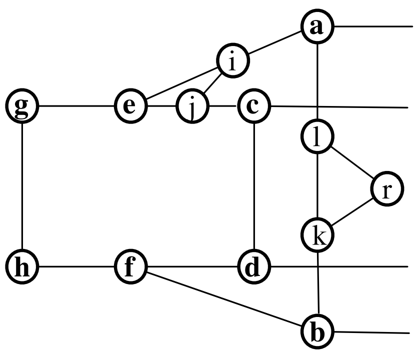

Figure 10: The graph $\mathcal{H}^{+5}(p)$ is a $meg(11p + 7)$.

Property 5.9 *$\mathcal{H}^{+5}(p)$ is a $meg(8p + 5)$.*

Proof: $\mathcal{H}^{+5}(p)$ contains $m = 11p + 7 = \lceil \frac{11}{8} \times (8p + 5) \rceil$ edges.

From Property 5.6 we know that $\mathcal{H}^{+2}(p)$ is expandable.

Let $xy \notin E$. If xy is also an non-edge of $\mathcal{H}^{+2}(p)$ and the corresponding 2-factor contains $a_1 b_1$ then we substitute the sequence $a_1 b_1$ for $a_1 l r k b_1$; else if $a_1 b_1$ is not in the 2-factor, so we add the cycle (k, l, r, k) to the 2-factor. In both cases we obtain a 2-factor for $\mathcal{H}^{+5}(p)$.

Now let $xy = a_1 b_1$: $(a_1, b_2, a_2, b_3, a_3, \ldots, b_p, a_p, b_1), (c_1, j, i, e_1, g_1, h_1, f_1, d_1, c_1)$, $(c_i, e_i, g_i, h_i, f_i, d_i, c_i), 2 \leq i \leq p$, and (k, l, r, k) is a 2-factor.

Let $x = k$ (the case $x = l$ is symmetric). If $y \neq a_1$: $a_1 y$ can be extended in $\mathcal{H}^{+2}(p)$; the corresponding 2-factor contains the path $a_1 \cdots y$; we substitute it for $krla_1 \cdots y$ and we have a 2-factor for $\mathcal{H}^{+5}(p)$. If $y = a_1$: $(k, r, l, a_1), (c_1, d_1, d_p, c_p, d_{p-1}, c_{p-1}, \ldots, d_2, c_2, e_2, g_2, h_2, f_2, b_2, a_2, b_3, f_3, h_3, g_3, e_3, a_3, b_4, f_4, h_4, g_4, e_4, a_4, \ldots, b_p, f_p, h_p, g_p, e_p, a_p, b_1, f_1, h_1, g_1, e_1, i, j, c_1)$ is a 2-factor.

Let $x = r$. Remark that all 2-factors containing the non-edge ky contain also kr; to extend ry we replace $krla_1 \cdots y$ (resp.(k, r, l, a_1)) by $rkla_1 \cdots y$ (resp.(r, k, l, a_1)) in the 2-factors above. □

5.4 Meg(n) for $10 \leq n \leq 13$

The constructions given in the previous sections do not cover the cases $10 \leq n \leq 13$. We give constructions for all the situations.

We start from the graph G_{10} shown in Figure 11. Note that G_{10} and $G_{10} - a$ are hamiltonian.

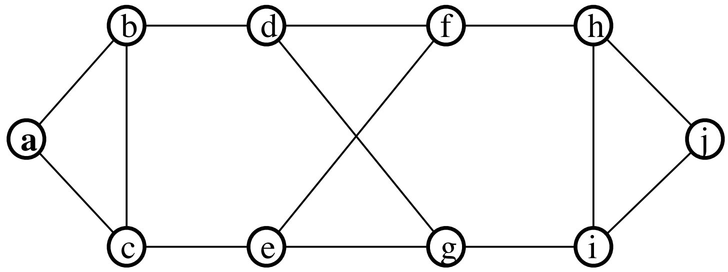

Figure 11: The graph G_{10} is a $meg(10)$.

Property 5.10 *G_{10} is a $meg(10)$.*

Proof: G_{10} contains $m = 14 = \lceil \frac{11}{8} \times 10 \rceil$ edges.

Let $xy \notin E$.

- $xy = ad$: $(a, b, c, e, f, h, j, i, g, d)$ yields a 2-factor; by symmetry ae, jf, jg can be extended;
- $xy = af$: $(a, b, c, e, g, d, f), (h, i, j, h)$ yields a 2-factor; by symmetry ag, jd, je can be extended;
- $xy = ah$: $(a, b, c, e, f, d, g, i, j, h)$ yields a 2-factor; by symmetry ai, jb, jc can be extended;
- $xy = aj$: $(a, b, c, e, f, d, g, i, h, j)$ yields a 2-factor;
- $xy = be$: $(b, a, c, e), (d, g, i, j, h, f, d)$ yields a 2-factor; by symmetry cd, hg, if can be extended;
- $xy = bf$: $(b, a, c, e, g, d, f), (h, i, j, h)$ yields a 2-factor; by symmetry $cg, hd, ie, bg, cf, he, id$ can be extended;
- $xy = bh$: $(b, a, c, e, f, d, g, i, j, h)$ yields a 2-factor; by symmetry ci, bi, ch can be extended;

- $xy = de$: $(d, f, h, j, i, g, e), (a, b, c, a)$ yields a 2-factor; by symmetry fg can be extended.

□

The graph G_{11} shown in Figure 12 is obtained from G_{10} by the addition of the vertex k, the removal of dg, ef and the addition of dk, ek, fk, gk to E.

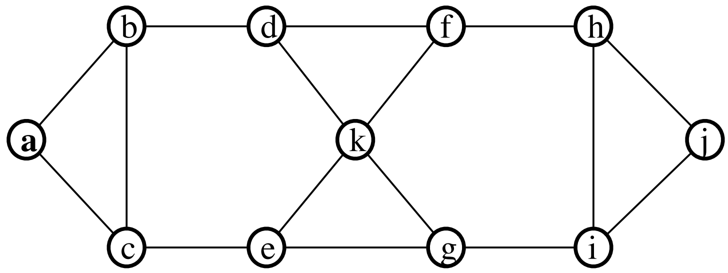

Figure 12: The graph G_{11} is a $meg(11)$.

Property 5.11 *G_{11} is a $meg(11)$.*

Proof: G_{11} contains $m = 16 = \lceil \frac{11}{8} \times 11 \rceil$ edges.

Let $xy \notin E$. If $xy \notin \{dg, ef, kz\ z \in \{a, \ldots, j\}\}$ from Property 5.10 xy can be extended to a 2-factor F in G_{10}. Since at least one of dg, ef, df, eg is an edge of F, F can easily be modified to a 2-factor F' of G_{11} by replacing the edge by the 3-path passing through k.

- $xy = dg$: $(d, f, k, e, g), (a, b, c, a), (h, i, j, h)$ is a 2-factor; by symmetry ef can be extended;
- $xy = ak$: $(a, b, c, e, g, i, j, h, f, d, k)$ is a 2-factor; by symmetry jk can be extended;
- $xy = bk$: $(b, a, c, e, g, i, j, h, f, d, k)$ is a 2-factor; by symmetry ck, hk, ik can be extended.

□

The graph G_{12} shown in Figure 13 is obtained from G_{10} by the addition of the vertices k, l, the removal of dg, ef and the addition of dk, ek, fl, gl, kl to E.

Property 5.12 *G_{12} is a $meg(12)$.*

Proof: G_{12} contains $m = 17 = \lceil \frac{11}{8} \times 12 \rceil$ edges.

Let $xy \notin E$. If $xy \notin \{dg, ef, kz, lz\ z \in \{a, \ldots, j\}\}$ from Property 5.10 xy can be extended to a 2-factor F in G_{10}. Since at least one of dg, ef, df, eg is an edge of F, F can easily be modified to a 2-factor F' of G_{12} by replacing the edge by the 4-path passing through k and l.

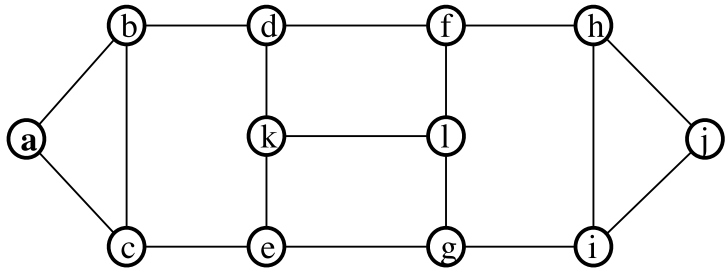

Figure 13: The graph G_{12} is a $meg(12)$.

- $xy = dg$: $(d, b, a, c, e, k, l, f, h, j, i, g)$ is a 2-factor; by symmetry ef can be extended;
- $xy = kf$: $(k, l, g, e, c, a, b, d, f), (h, i, j, h)$ is a 2-factor; by symmetry kg, dl, el can be extended;
- $xy = ka$: $(k, d, f, l, g, e, c, b, a), (h, i, j, h)$ is a 2-factor; by symmetry lj can be extended;
- $xy = kb$: $(k, d, f, l, g, e, c, a, b), (h, i, j, h)$ is a 2-factor; by symmetry kc, lh, li can be extended;
- $xy = kj$: $(k, l, g, e, c, a, b, d, f, h, i, j)$ is a 2-factor; by symmetry la can be extended;
- $xy = ki$: $(k, l, g, e, c, a, b, d, f, h, j, i)$ is a 2-factor; by symmetry kh, lb, lc can be extended.

□

The graph G_{13} shown in Figure 14 is obtained from G_{10} by the addition of the vertices k, l, o, the removal of ac. $\{a, k, l, o\}$ induce the diamond with $ao \notin E$.

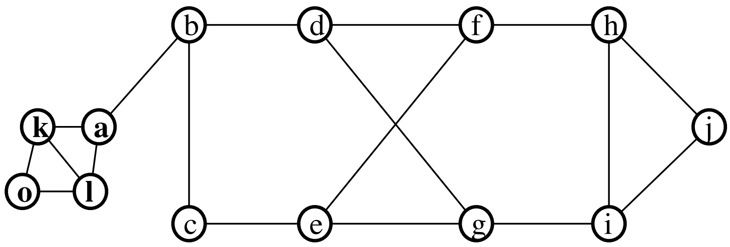

Figure 14: The graph G_{13} is a $meg(13)$.

Property 5.13 *G_{13} is a $meg(13)$.*

Proof: G_{13} contains $m = 18 = \lceil \frac{11}{8} \times 13 \rceil$ edges.

Let $xy \notin E$. If $x, y \notin \{a, k, l, o\}$, then we take F the corresponding 2-factor of G_{10}. F has bac as subsequence. Substituting bac for bc and adding (a, k, o, l, a) we have a 2-factor for G_{13}. If $xy = ao$ then (a, k, l, o) and an hamiltonian cycle of $G_{10} - a$ is a 2-factor for G_{13}.

Let $x = a$. If $y \in \{d, f, h, j\}$ we know from Property 5.10 that there exists a 2-factor F for G_{10} which does not contain ac. So adding (k, l, o, k) we have a 2-factor for G_{13}.

- $y = c$: $(a, b, d, f, h, j, i, g, e, c), (k, l, o, k)$ is a 2-factor;
- $y = e$: $(a, b, c, e), (d, g, i, j, h, f, d), (k, l, o, k)$ is a 2-factor;
- $y = g$: $(a, b, c, e, f, d, g), (h, i, j, h), (k, l, o, k)$ is a 2-factor;
- $y = i$: $(a, b, c, e, g, d, f, h, j, i), (k, l, o, k)$ is a 2-factor.

Now when $x \in \{k, l, o\}$ we use the 2-factors we obtained above for $x = a$: the first sequence begins with $kola$ or $loka$ or $okla$ instead of a. □

6 Conclusion

We have determined the values of $Exp_2(n)$ for all values of n. It could be interesting to characterize the $meg(n)$, for instance by a (finite ?) collection of forbidden induced subgraphs. Furthermore one could consider a generalization of the basic problem: the edges of a complete graph K_n are coloured in blue and in red. We want to color a minimal set of red edges in white so that any red edge uv can be extended to a 2-factor using only white edges. In our case, we had only white and red edges. Furthermore instead of just one red edge we could require that any appropriate subset of d red edges could be extended to a 2-factor by adding only white edges. More generally we could consider k-factors, $k \geq 3$, or even other structured set of edges.

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