

ROUGH PATHS AND REGULARIZATION

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ABSTRACT. Calculus via regularizations and rough paths are two methods to approach stochastic integration and calculus close to pathwise calculus. The origin of rough paths theory is purely deterministic, calculus via regularization is based on deterministic techniques but there is still a probability in the background. The goal of this paper is to establish a connection between stochastically controlled-type processes, a concept reminiscent from rough paths theory, and the so-called weak Dirichlet processes. As a by-product, we present the connection between rough and Stratonovich integrals for càdlàg weak Dirichlet processes integrands and continuous semimartingales integrators.

Key words and phrases. Rough paths; calculus via regularization; Gubinelli's derivative.

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1. INTRODUCTION

This paper focuses on two variants of stochastic calculus of pathwise type: calculus via regularization and rough paths. The recent literature on rough paths is very rich and it is impossible to list it here completely. It was started in [37] continued by the monograph [36] which focused on rough differential equations. The corresponding integral was introduced later by M. Gubinelli, see [30]. Later, a great variety of contributions on the subject appeared and it is not possible to list all of them. We refer however to the monograph [22] to a fairly rich list of references and for a complete development of the subject. In spite of some recent work mixing probability and deterministic theory, see e.g. [34, 6, 21], the theory of rough paths is essentially deterministic.

Stochastic calculus via regularization was started first by F. Russo and P. Vallois in [40]. The calculus was later continued in [41, 44, 45] in the framework of continuous integrators, essentially with finite quadratic variation. The case of processes with higher variation was first introduced in [17, 18] and continued in [11, 29, 28, 27, 47,

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4], especially in relation with fractional Brownian motion and related processes. A not very recent survey paper in the framework of finite dimensional processes is [46]. Stochastic calculus via regularization for processes taking values in Banach spaces, with applications to the path-dependent case, was realized in [12, 13] and in [8]. The case of real-valued jump integrators was first introduced in [42] and then deeply investigated in [1] and later by [2]. Applications to mathematical finance (resp. to fluidodynamics modeling) were published in [10] (resp. [19]).

An important notion which emerged in calculus via regularization is the notion of weak Dirichlet processes, started in [17, 26]. Such a process X is the sum of a local martingale M and an orthogonal process A such that $[A, N] = 0$ for any continuous martingale. This constitutes a natural generalization of the notion of semimartingale and of Dirichlet process (in the sense of Föllmer), see [20]. In particular, [26] allowed to establish chain rule type decomposition extending Itô formulae with applications to control theory, see [25]. That concept was extended to the jump case by [7] and its related calculus was performed by [2] with applications to BSDEs, see [3]. In [15, 14] one has performed weak Dirichlet decomposition of real functional of Banach space-valued processes. In [8, 16] one has investigated strict solutions of path-dependent PDEs.

In this paper we wish first to give a key to revisit the theory of rough paths under the perspective of stochastic calculus via regularizations. The idea here is not to summarize the theory of rough paths integrals, but to propose a variant version which is directly probabilistic. In particular, we emphasize the strong link between the notion of weak Dirichlet process and one of stochastically controlled process, which is a stochastic version of the one proposed by Gubinelli [30]. According to Definition 3.2 such a process fulfills

$$(1.1) \quad Y_t - Y_s = Y'_s(X_t - X_s) + R_{s,t}^Y, \quad s < t,$$

where

$$(1.2) \quad \lim_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon} \int_0^t R_{s,s+\varepsilon}^Y (X_{s+\varepsilon} - X_s) ds = 0,$$

in probability for each $t \in [0, T]$.

Here, X is the reference driving noise, Y' is a process (not necessarily admitting γ -Hölder continuous paths). The orthogonality condition (1.2) resembles the 2γ -Hölder-regularity condition reminiscent from [30].

Propositions 3.7 and 3.9 present the connection between weak Dirichlet processes and stochastically controlled processes. In particular, when the reference driving noise is a martingale, then both concepts coincide. As a side effect, Theorem 5.6 shows Stratonovich integration as a stochastic rough-type integration for weak Dirichlet integrands and continuous semimartingale integrators. The connection between rough paths theory with semimartingales has been investigated by some authors. [9] shows pathwise Wong-Zakai-type theorems for Stratonovich SDEs driven by continuous semimartingales. In particular, the integral defined by rough paths theory agrees with Stratonovich integrals for real-valued functions $f(X)$ of the driving noise X , see also Proposition 17.1 in [23]. Recently, [21] introduces a concept of rough semimartingales and develops the corresponding stochastic integration having a deterministic rough path in the background and mixing with p -variation regularity. Beyond semimartingale driving noises, we draw attention to the recent work of [35]. The authors have established the connection between rough integrals and trapezoidal Riemann sum approximations for controlled processes integrands (in the pathwise sense of [30]) and a general class of Gaussian driving noises.

In this article, we take full advantage of the probability measure and the stochastic controllability (1.1) to establish consistency between stochastic rough-type and Stratonovich integrals for more general integrands. In the companion paper in preparation [38], a detailed analysis on stochastic rough-type integrals driven by Gaussian rough paths and their connection with Stratonovich and Skorohod integrals is presented.

The paper is organized as follows. After this introduction, in Section 2.2 we introduce some notations about matrix-valued calculus via regularization. In Section 3 we introduce the notion of stochastically controlled paths and the one of stochastic Gubinelli derivative, under the inspiration of the classical rough paths theory. We link this with the notion of Dirichlet process. In Section 4 we introduce the **second order process** (connected with the Lévy area) and finally in Section 5 discuss the notion of rough stochastic integrals via regularization, examining carefully the case when the integrator is a semimartingale.

2. PRELIMINARY NOTIONS

2.1. Basic notations. We introduce here some basic notations intervening in the paper. $T > 0$ will be a finite fixed horizon. Regarding linear algebra, vectors or

elements of \mathbb{R}^d will be assimilated to column vectors, so that if x is a vector in \mathbb{R}^d , then x^\top is a row vector.

We continue fixing some notations. In the sequel, finite-dimensional Banach spaces E will be equipped with a norm $|\cdot|$, typically $E = \mathbb{R}^d$. Let $T > 0$ be a fixed maturity. For $\alpha \in]0, 1]$, the notation $C^{[\alpha]}([0, T]; E)$ is reserved for E -valued paths defined on $[0, T]$, Hölder continuous of index $\alpha \in]0, 1]$. For $X \in C^{[\alpha]}([0, T]; E)$, the usual seminorm is given by

$$\|X\|_\alpha := \sup_{s, t \in [0, T], s \neq t} \frac{|X_{s,t}|}{|t - s|^\alpha},$$

where we set

$$(2.1) \quad X_{s,t} := X_t - X_s, \quad 0 \leq s, t \leq T.$$

When $E = \mathbb{R}$ we simply write $C^{[\alpha]}([0, T])$

For a two-parameter function $R : [0, T]^2 \rightarrow \mathbb{R}$, vanishing on the diagonal $\{(s, t) | 0 \leq s = t \leq T\}$, we write $R(s, t) := R_{s,t}$. We say that $R \in C^{[\alpha]}([0, T]^2)$ if

$$(2.2) \quad \|R\|_\alpha := \sup_{s, t \in [0, T]^2} \frac{|R_{s,t}|}{|t - s|^\alpha} < \infty.$$

By convention the quotient $\frac{0}{0}$ will set to zero. In the sequel, if $n \in \mathbb{N}^*$, we will extend a function $R \in C([0, T]^n)$ to \mathbb{R}^n by continuity, setting

$$(2.3) \quad R_{t_1, \dots, t_n} := R_{(t_1 \wedge T), \dots, (t_n \wedge T)}.$$

(Ω, \mathcal{F}, P) will be a fixed probability space. Let X^1, X^2 be two stochastic processes, continuous for simplicity.

We introduce

$$(2.4) \quad C(\varepsilon, X^1, X^2)(t) = \int_0^t \frac{(X_{s+\varepsilon}^1 - X_s^1)(X_{s+\varepsilon}^2 - X_s^2)}{\varepsilon} ds, \quad t \geq 0.$$

In the sequel (\mathcal{F}_t) will be a filtration fulfilling the usual condition.

Definition 2.1. (1) The **covariation** of X^1 and X^2 is the continuous process (whenever it exists) $[X^1, X^2]$ such that, for $t \geq 0$,

$$C(\varepsilon, X^1, X^2)(t) \text{ converges in probability to } [X^1, X^2]_t.$$

We say that the covariation $[X^1, X^2]$ exists in the strong sense if moreover

$$(2.5) \quad \sup_{0 < \varepsilon \leq 1} \int_0^T \frac{|(X_{s+\varepsilon}^1 - X_s^1)(X_{s+\varepsilon}^2 - X_s^2)|}{\varepsilon} ds < \infty.$$

- (2) A vector of processes $(X^1, \dots, X^d)^\top$ is said to have all its **mutual covariations** if $[X^i, X^j]$ exists for every $1 \leq i, j \leq d$.
- (3) A real process X is said to be **strong finite cubic variation process**, see [18], if there is a process ξ such that, for every $t \in [0, T]$,

$$\int_0^t \frac{|X_{s+\varepsilon}^1 - X_s^1|^3}{\varepsilon} ds \rightarrow \xi,$$

in probability. If $\xi = 0$ then X is said to have zero cubic variation.

- (4) A real-valued (continuous) (\mathcal{F}_t) -**martingale orthogonal process** A is a continuous adapted process such that $[A, N] = 0$ for every (\mathcal{F}_t) -local martingale N . A real-valued (continuous) (\mathcal{F}) -**weak Dirichlet process** is the sum of a continuous (\mathcal{F}_t) -local martingale M and an (\mathcal{F}_t) -martingale orthogonal process.

Remark 2.2. (1) If X^1, X^2 are two semimartingales then $(X^1, X^2)^\top$ has all its mutual covariations, see Proposition 1.1 of [43] and $[X^1, X^2]$ is the classical covariation of semimartingales.

- (2) It may happen that $[X^1, X^2]$ exists but $(X^1, X^2)^\top$ does not have all its mutual covariations, see Remark 22 of [46].
- (3) If X^1 (resp. X^2) has α -Hölder (resp. β -Hölder) paths with $\alpha + \beta > 1$, then $[X^1, X^2] = 0$, see Propositions and 1 of [46].

Suppose that $M = (M^1, \dots, M^d)$, and M^1, \dots, M^d are real-valued local martingales. In particular M^\top is an \mathbb{R}^d -valued local martingale. We denote by $\mathcal{L}^2(d[M, M])$ the space of processes $H = (H^1, \dots, H^d)$ where H^1, \dots, H^d are real progressively measurable processes and

$$(2.6) \quad \sum_{i,j} \int_0^T H_s^i H_s^j d[M^i, M^j]_s < \infty \quad \text{a.s.}$$

$\mathcal{L}^2(d[M, M])$ is an F -space with respect to the metrizable topology d_2 defined as follows: (H^n) converges to H when $n \rightarrow \infty$ if

$$\sum_{i,j} \int_0^T ((H^n)_s^i - H_s^i)((H^n)_s^j - H_s^j) d[M^i, M^j]_s \rightarrow 0,$$

in probability, when $n \rightarrow \infty$.

Similarly as in (27), in Section 4.1 of [46], one can prove the following.

Proposition 2.3. *Let X^1, X^2 be two processes such that $(X^1, X^2)^\top$ has all its mutual covariations, and H be a continuous (excepted eventually on a countable number of points) real-valued process, then*

$$\frac{1}{\varepsilon} \int_0^\cdot H_s (X_{s+\varepsilon}^1 - X_s^1)(X_{s+\varepsilon}^2 - X_s^2) ds \rightarrow \int_0^\cdot H_s d[X^1, X^2]_s$$

in the ucp sense, when $\varepsilon \rightarrow 0$.

2.2. Matrix-valued integrals via regularization. Here we will shortly discuss about matrix-valued stochastic integrals via regularizations. Let $\mathbb{M}^{n \times d}$ be the linear space of the real $n \times d$ matrices, which in the rough paths literature are often associated with tensors.

For every $(s, t) \in \Delta := \{(s, t) | 0 \leq s \leq t \leq T\}$, we introduce two $\mathbb{M}^{n \times d}$ -valued stochastic integrals via regularizations. Let X (resp. Y) be an \mathbb{R}^d -valued (resp. \mathbb{R}^n -valued) continuous process (resp. locally integrable process) indexed by $[0, T]$.

So $X = (X^1, \dots, X^d)^\top$ (resp. $Y = (Y^1, \dots, Y^n)^\top$).

$$(2.7) \quad \int_s^t Y \otimes d^- X := \lim_{\varepsilon \rightarrow 0^+} \int_s^t Y_r \frac{(X_{r+\varepsilon} - X_r)^\top}{\varepsilon} dr,$$

$$(2.8) \quad \left(\text{resp. } \int_s^t Y \otimes d^o X := \lim_{\varepsilon \rightarrow 0^+} \int_s^t \frac{Y_r + Y_{r+\varepsilon}}{2} \frac{(X_{r+\varepsilon} - X_r)^\top}{\varepsilon} dr \right),$$

provided that previous limit holds in probability and the random function $t \mapsto \int_0^t Y \otimes d^- X$, (resp. $t \mapsto \int_0^t Y \otimes d^o X$), admits a continuous version. In particular

$$\left(\int_s^t Y \otimes d^- X \right) (i, j) = \int_s^t Y^i \otimes d^- X^j.$$

We remark that $\int_s^t Y \otimes d^- X$ exists if and only if $\int_s^t Y^i \otimes d^- X^j$ exist for every $1 \leq i \leq n, 1 \leq j \leq d$.

Suppose now that Y is continuous. We denote by $[X, Y]$ the matrix

$$[X, Y](i, j) = [X^i, Y^j], 1 \leq i \leq d, 1 \leq j \leq n,$$

provided those covariations exist. If $n = d$ and $X = Y$, previous matrix exists for instance if and only if X has all its mutual covariations.

We will denote by $[X, X]^{\mathbb{R}}$ the **scalar quadratic variation** defined as the real continuous process (if it exists) such that

$$[X, X]_t^{\mathbb{R}} := [X^{\top}, X] = \lim_{\varepsilon \rightarrow 0} \int_0^t \frac{|X_{s+\varepsilon} - X_s|^2}{\varepsilon} ds, \quad t \in [0, T],$$

when the limit holds in probability. $[X, X]^{\mathbb{R}}$, when it exists, is an increasing process. When X^i are finite quadratic variation processes for every $1 \leq i \leq d$, then

$$[X, X]^{\mathbb{R}} = \sum_{i=1}^d [X^i, X^i].$$

We recall that \mathbb{R}^d -valued continuous process is called semimartingale with respect to a filtration (\mathcal{F}_t) , if all its components are semimartingales.

3. STOCHASTICALLY CONTROLLED PATHS AND GUBINELLI DERIVATIVE

In [30], the author introduced a class of controlled paths Y by a reference function.

Definition 3.1. (*Gubinelli*). *Let X be a function belonging to $C^{[\gamma]}([0, T]; E)$ with $\frac{1}{3} < \gamma < \frac{1}{2}$. An element Y of $C^{[\gamma]}([0, T])$ is called **weakly controlled** (by X) if there exists a function $Y' \in C^{[\gamma]}([0, T]; E)$ (here by convention, Y' will be a row vector), so that the remainder term R defined by the relation*

$$Y_{s,t} = Y'_s(X_t - X_s) + R_{s,t}, \quad s, t \in [0, T],$$

belongs to $C^{[2\gamma]}([0, T]^2)$.

From now on X will stand for a fixed \mathbb{R}^d -valued reference continuous process. The definition below is inspired by previous one.

Definition 3.2. (1) *We say that an \mathbb{R} -valued stochastic process Y is **stochastically controlled** by X if there exists an \mathbb{R}^d -valued stochastic process Y' (here again indicated by a row vector) so that the remainder term R^Y defined by the relation*

$$(3.1) \quad Y_t - Y_s = Y'_s(X_t - X_s) + R_{s,t}^Y, \quad s < t,$$

satisfies

$$(3.2) \quad \lim_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon} \int_0^t R_{s,s+\varepsilon}^Y(X_{s+\varepsilon} - X_s) ds = 0,$$

*in probability for each $t \in [0, T]$. Y' is called **stochastic Gubinelli derivative**.*

- (2) \mathcal{D}_X will denote the couples of processes (Y, Y') satisfying (3.1) and (3.2).
- (3) If Y is an \mathbb{R}^n -valued process whose components are Y^1, \dots, Y^n , then Y is said to be stochastically controlled by X if every component Y^i is stochastically controlled by X . The matrix Y' whose rows are stochastic Gubinelli derivatives $(Y^i)'$ of Y^i is called (matrix) stochastic Gubinelli derivative of Y . The relations (3.1) and (3.2) also make sense in the vector setting. $\mathcal{D}_X(\mathbb{R}^n)$ will denote the couples (Y, Y') , where Y is a \mathbb{R}^n -valued process, being stochastically controlled by X and Y' is a Gubinelli derivative. We remark that R^Y also depends on the process X .

Similarly to the theory of (deterministic) controlled rough paths, in general, Y can admit different stochastic Gubinelli derivatives. However Proposition 3.7 states sufficient conditions for uniqueness.

Let us now provide some examples of stochastically controlled processes.

Example 3.3. Let X be an \mathbb{R}^d -valued continuous process having all its mutual covariations. Let Y be an \mathbb{R} -valued process such that, for every $1 \leq i \leq d$, $[Y, X^i]$ exists in the strong sense and $[Y, X^i] = 0$. Consider for instance the three following particular cases.

- (Y, X^1, \dots, X^d) has all its mutual covariations and $[Y, X] = 0$. In this case, for every $1 \leq i \leq d$, $[Y, X^i]$ exists in the strong sense.
- Let Y (resp. X) be a γ' -continuous (resp. γ -continuous) process with $\gamma + \gamma' > 1$. Again $[Y, X^i]$ admits its mutual covariations in the strong sense and $[Y, X^i] = 0$, for every $1 \leq i \leq d$ since

$$\int_0^T |Y_{s+\varepsilon} - Y_s| |X_{s+\varepsilon}^i - X_s^i| \frac{ds}{\varepsilon} \leq \text{const } \varepsilon^{\gamma+\gamma'-1} \rightarrow 0,$$

when $\varepsilon \rightarrow 0_+$, for every $1 \leq i \leq d$.

We recall that, under those conditions, the Young integral $\int_0^t Y d^{(y)}X, t \in [0, T]$ exists, see [48, 5].

- If $X^i, 1 \leq i \leq d$, are continuous bounded variation processes and Y is a.s. locally bounded.

- (1) We claim that Y is stochastically controlled by X with $Y' \equiv 0$.
- (2) If moreover $[X, X]^{\mathbb{R}} \equiv 0$, then Y' can be any locally bounded process: therefore the stochastic Gubinelli derivative is not unique.

Indeed, for $0 \leq s \leq t \leq T$, write

$$Y_t - Y_s = Y'_s(X_t - X_s) + R_{s,t}^Y.$$

(1) If $Y' \equiv 0$ we have

$$\frac{1}{\varepsilon} \int_0^t R_{s,s+\varepsilon}(X_{s+\varepsilon} - X_s) ds \rightarrow 0,$$

when $\varepsilon \rightarrow 0_+$, since

$$(3.3) \quad \frac{1}{\varepsilon} \int_0^t (Y_{s+\varepsilon} - Y_s)(X_{s+\varepsilon} - X_s) ds \rightarrow [Y, X] = 0,$$

when $\varepsilon \rightarrow 0$.

(2) If $[X, X]^{\mathbb{R}} \equiv 0$ and Y' is a locally bounded process, then we also have

$$\lim_{\varepsilon \rightarrow 0_+} \frac{1}{\varepsilon} \int_0^t Y'_s(X_{s+\varepsilon} - X_s)(X_{s+\varepsilon} - X_s) ds = 0, \quad t \in [0, T].$$

This follows by

$$\lim_{\varepsilon \rightarrow 0_+} \frac{1}{\varepsilon} \int_0^t |Y'_s| |X_{s+\varepsilon} - X_s|^2 ds = 0, \quad t \in [0, T],$$

$[X, X]^{\mathbb{R}} = 0$ and Kunita-Watanabe inequality, see e.g. Proposition 1.4) of [46].

We leave the detailed proof to the reader. The result follows by (3.3).

In the second example we show that a weakly controlled process in the sense of Gubinelli is a stochastically controlled process.

Example 3.4. Let X be an \mathbb{R}^d -valued γ -Hölder continuous process, with $\frac{1}{3} < \gamma < \frac{1}{2}$. Let Y be a γ -Hölder continuous real-valued process such that there exists an \mathbb{R}^d -valued process Y' , so that the remainder term R^Y , given through the relation

$$Y_{s,t} = Y'_s(X_t - X_s) + R_{s,t}^Y,$$

belongs to $C^{[2\gamma]}([0, T]^2)$. In particular ω -a.s., Y is weakly controlled by X . Then, Y is stochastically controlled by X . Indeed a.s.

$$(3.4) \quad \sup_{0 \leq t \leq T} \left| \frac{1}{\varepsilon} \int_0^t R_{s,s+\varepsilon}^Y(X_{s+\varepsilon} - X_s) ds \right| \leq T \|R\|_{2\gamma} \|X\|_{\gamma} \varepsilon^{3\gamma-1} \rightarrow 0,$$

as $\varepsilon \rightarrow 0^+$. In particular the result follows because $\gamma > \frac{1}{3}$.

Example 3.5. Let X be an d -dimensional continuous semimartingale. Let $Z = (Z^1, \dots, Z^d)$ where the components Z^1, \dots, Z^d are càglàd progressively measurable processes. We set

$$Y_t = \int_0^t Z_s \cdot dX_s := \sum_{i=1}^d \int_0^t Z_s^i dX_s^i, \quad t \in [0, T].$$

Then, the real-valued process Y is stochastically controlled by X and Z is a Gubinelli stochastic derivative.

Indeed, for $s, t \in [0, T]$ such that $s \leq t$, we define R^Y implicitly by the relation

$$Y_t - Y_s = Z_s^\top (X_t - X_s) + R_{s,t}^Y.$$

We have

$$\frac{1}{\varepsilon} \int_0^t R_{s, s+\varepsilon}^Y (X_{s+\varepsilon} - X_s) ds = I_1(t, \varepsilon) - I_2(t, \varepsilon),$$

with

$$\begin{aligned} I_1(t, \varepsilon) &= \frac{1}{\varepsilon} \int_0^t (Y_{s+\varepsilon} - Y_s)(X_{s+\varepsilon} - X_s) ds \\ (3.5) \quad I_2(t, \varepsilon) &= \frac{1}{\varepsilon} \int_0^t Z_s^\top (X_{s+\varepsilon} - X_s)(X_{s+\varepsilon} - X_s) ds. \end{aligned}$$

$I_1(t, \varepsilon)$ converges in probability to

$$(3.6) \quad [Y, X]_t = \int_0^t Z_s^\top d[X, X]_s, \quad t \in [0, T],$$

by Proposition 9 of [46]. We emphasize that the k -component of the integral on the right-hand side of (3.5) is

$$\frac{1}{\varepsilon} \sum_{j=1}^d \int_0^t Z_s^j (X_{s+\varepsilon}^j - X_s^j)(X_{s+\varepsilon}^k - X_s^k) ds.$$

Reasoning component by component, it can be also shown by Proposition 2.3. that $I_2(t, \varepsilon)$ also converges in probability to the right-hand side of (3.6).

Example 3.6. Let X be an d -dimensional process whose components are finite strong cubic variation processes and at least one component has a zero cubic variation. Let $f \in C^2(\mathbb{R}^d)$. Then $Y = f(X)$ is a stochastically controlled process by X with stochastic Gubinelli derivative $Y' = (\nabla f)^\top(X)$.

We prove the result for $d = 1$, leaving to the reader the general case. Let $\omega \in \Omega$ be fixed, but underlying. Let $0 \leq s \leq t \leq T$. Then, Taylor's formula yields

$$f(X_t) - f(X_s) = f'(X_s)(X_t - X_s) + R_{s,t}^Y,$$

where

$$R_{s,t}^Y = (X_t - X_s)^2 \int_0^1 f''(X_s + a(X_t - X_s))(1-a)da.$$

$$\left| \frac{1}{\varepsilon} \int_0^t R_{s,s+\varepsilon}^Y (X_{s+\varepsilon} - X_s) ds \right| \leq \sup_{\xi \in I(\omega)} |f''(\xi)| \int_0^t |X_{s+\varepsilon} - X_s|^3 \frac{ds}{\varepsilon},$$

where

$$I(\omega) = \left[-\min_{t \in [0,T]} X_t(\omega), \max_{t \in [0,T]} X_t(\omega) \right].$$

Since the integral on the right-hand side converges in probability (even ucp) to zero, R^Y fulfills (3.2).

When X is an (\mathcal{F}_t) -local martingale, Proposition 3.7 below shows that somehow a process Y is stochastically controlled if and only if Y is an (\mathcal{F}_t) -weak Dirichlet process.

Proposition 3.7. *Let $X = M$ be an \mathbb{R}^d -valued continuous (\mathcal{F}_t) -local martingale. Let Y be an \mathbb{R} -valued continuous adapted process.*

- (1) *Suppose that Y is a weak Dirichlet process. Then Y is stochastically controlled by M .*
- (2) *Suppose that Y is stochastically controlled by M and the stochastic Gubinelli derivative Y' is progressively measurable and càglàd. Then Y is a weak Dirichlet process with decomposition $Y = M^Y + A^Y$ where*

$$M_t^Y = \int_0^t Y'_s dM_s, t \in [0, T]$$

and A^Y is an (\mathcal{F}_t) -martingale orthogonal process.

- (3) *(Uniqueness). There is at most one stochastic Gubinelli's derivative Y' in the class of càglàd progressively measurable processes, w.r.t to the Doléans measure $\mu_{[X]}(d\omega, dt) := d[X, X]_t^{\mathbb{R}}(\omega) \otimes dP(\omega)$.*

Proof. For simplicity we suppose that $d = 1$.

- (1) Suppose that Y is a weak Dirichlet process with canonical decomposition

$$Y = M^Y + A^Y,$$

where M^Y is the local martingale and A^Y such that $A_0^Y = 0$, is a predictable process such that $[A^Y, N] = 0$ for every continuous local martingale N . By Galtchouk-Kunita-Watanabe decomposition, see [33, 24], there exist Z and O such that

$$M_t^Y = Y_0 + \int_0^t Z_s dM_s + O_t, \quad t \in [0, T].$$

Moreover O is a continuous local martingale such that $[O, M] = 0$. Then,

$$Y_t = Y_0 + \int_0^t Z_s dM_s + O_t + A_t^Y, t \in [0, T].$$

We set $Y' := Z$. Hence,

$$Y_t - Y_s = Y'_s(M_t - M_s) + R_{s,t}^Y,$$

where we set

$$(3.7) \quad R_{s,t}^Y = \int_s^t (Z_r - Z_s)^\top dM_r + O_t - O_s + A_t^Y - A_s^Y.$$

Condition (3.2) follows by Remark 3.8 and the fact that $[O, M] = [A^Y, M] = 0$.

- (2) Suppose now that Y is stochastically controlled by M with càglàd stochastic Gubinelli derivative Y' . Then, there is R^Y such that (3.1) and (3.2) hold. Setting $t = s + \varepsilon$, we have

$$(3.8) \quad Y_{s+\varepsilon} - Y_s = \int_s^{s+\varepsilon} Y'_r dM_r + \int_s^{s+\varepsilon} (Y'_s - Y'_r) dM_r + R_{s,s+\varepsilon}^Y.$$

where R^Y fulfills (3.2). We have

$$(3.9) \quad Y_{s+\varepsilon} - Y_s = \int_s^{s+\varepsilon} Y'_r dM_r + \tilde{R}_{s,s+\varepsilon}^Y,$$

where

$$\tilde{R}_{s,s+\varepsilon}^Y = \int_s^{s+\varepsilon} (Y'_s - Y'_r) dM_r + R_{s,s+\varepsilon}^Y,$$

fulfills (3.2) by Remark 3.8.

Let N be a continuous local martingale. Multiplying (3.9) by $N_{s+\varepsilon} - N_s$, integrating from 0 to t , dividing by ε , using (3.2) and by Proposition 9 of [46], going to the limit, gives

$$[Y, N]_t = \int_0^t Y'_r d[M, N]_r, \quad t \in [0, T].$$

This obviously implies that Y is a weak Dirichlet process with martingale component $M^Y = Y_0 + \int_0^\cdot Y'_r dM_r$.

- (3) We discuss now the uniqueness of the stochastic Gubinelli derivative. Given two decompositions of Y , taking the difference, we reduce the problem to the following. Let Y' be a càglàd process and R^Y , such that (3.2) holds for $Y = 0$, i.e. for every $0 \leq s < t \leq T$

$$(3.10) \quad 0 = Y'_s(M_t - M_s) + R_{s,t}^Y,$$

satisfies

$$(3.11) \quad \lim_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon} \int_0^t R_{s, s+\varepsilon}^Y (M_{s+\varepsilon} - M_s) ds = 0,$$

in probability for each $t \in [0, T]$. We need to show that Y' vanishes. Setting $t = s + \varepsilon$ in (3.10), multiplying both sides by $M_{s+\varepsilon} - M_s$ integrating, for every $t \in [0, T]$, taking into account (3.11) we get

$$\lim_{\varepsilon \rightarrow 0^+} \int_0^t Y'_s (M_{s+\varepsilon} - M_s)^2 ds = 0,$$

in probability. According to Remark 2.3, the left-hand side of previous expression equals (the limit even holds ucp)

$$\int_0^\cdot Y'_s d[M, M]_s \equiv 0.$$

This concludes the uniqueness result. □

Remark 3.8. *It is not difficult to prove the following. Let X be an \mathbb{R}^d -valued continuous semimartingale with canonical decomposition $X = M + V$. Let Z be a process in $\mathcal{L}^2(d[M, M])$. Then*

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_0^\cdot ds \left(\int_s^{s+\varepsilon} (Z_r - Z_s) dX_r \right) (X_{s+\varepsilon} - X_s) = 0$$

ucp.

The result below partially extends Proposition 3.7.

Proposition 3.9. *Let $X = M + V$ be an \mathbb{R}^d -valued (\mathcal{F}_t) -continuous semimartingale, where M is a continuous local martingale and V is a bounded variation process vanishing at zero. Let Y be a real-valued weak Dirichlet process*

$$Y = M^Y + A^Y,$$

where M^Y is the continuous local martingale component and A^Y is a (\mathcal{F}_t) -martingale orthogonal process vanishing at zero. Then the following holds.

- (1) *Y is stochastically controlled by X .*
- (2) *If Y' is a càglàd stochastic Gubinelli's derivative then*

$$(3.12) \quad [Y, X]_t = \int_0^t Y'_s d[X, X]_s$$

Proof. By Galtchouk-Kunita-Watanabe decomposition, there exist Z and O such that

$$M_t^Y = Y_0 + \int_0^t Z_s dM_s + O_t, t \in [0, T],$$

where $Z \in \mathcal{L}^2(d[M, M])$, O is a continuous local martingale such that $[O, M] = 0$. We recall that the space $\mathcal{L}^2(d[M, M])$ was defined at (2.6). Then,

$$Y_t = Y_0 + \int_0^t Z_s dM_s + O_t + A_t^Y, t \in [0, T].$$

Hence,

$$(3.13) \quad Y_t - Y_s = Y'_s(X_t - X_s) + R_{s,t}^Y,$$

where we set $Y' = Z$, $R_{s,t}^Y = \int_s^t (Z_r - Z_s) dM_r + O_{s,t} + A_{s,t}^Y$.

Now we recall

$$(3.14) \quad [O, M] = [A^Y, M] = 0.$$

Taking into account Remark 3.8, (3.13) and (3.14) show condition (3.11), which implies (1).

Then, by Remark 2.3 we have

$$\lim_{\varepsilon \rightarrow 0} \int_0^t (Y_{s+\varepsilon} - Y_s) \frac{X_{s+\varepsilon} - X_s}{\varepsilon} ds = \int_0^t Y'_s d[X, X]_s,$$

so that (2) is established. \square

An interesting consequence of Proposition 3.9 is given below.

Corollary 3.10. *Every continuous (\mathcal{F}_t) -weak Dirichlet process is stochastically controlled by any (\mathcal{F}_t) -continuous semimartingale.*

4. THE SECOND ORDER PROCESS AND ROUGH INTEGRAL VIA REGULARIZATION

In the rough paths theory, given a driving integrator function X , in order to perform integration, one needs a supplementary ingredient, often called **second order integral** or improperly called **Lévy area**, generally denoted by \mathbb{X} . The couple $\mathbf{X} = (X, \mathbb{X})$ is often called **enhanced rough path**.

In our setup, we are given, an \mathbb{R}^d -valued continuous stochastic process X , which is our reference. We introduce a stochastic analogue of the second order integral in the form of an $\mathbb{M}^{d \times d}$ -valued random field $\mathbb{X} = (\mathbb{X}_{s,t})$, indexed by $[0, T]^2$, vanishing on the diagonal. \mathbb{X} will be called **second-order process**. For $s \leq t$, $\mathbb{X}_{s,t}$ represents formally

a double (stochastic) integral $\int_s^t (X_r - X_s) \otimes dX_r$, which has to be properly defined. By symmetry, \mathbb{X} can be extended to $[0, T]^2$, setting, for $s \geq t$,

$$\mathbb{X}_{s,t} := \mathbb{X}_{t,s}.$$

The pair $\mathbf{X} = (X, \mathbb{X})$ is called **stochastically enhanced process**.

Remark 4.1. (1) *In the classical rough paths framework, if X is a deterministic γ -Hölder continuous path with $\frac{1}{3} < \gamma < \frac{1}{2}$, \mathbb{X} is supposed to belong to $C^{[2\gamma]}([0, T]^2)$ and to fulfill the so called **Chen's relation** below.*

$$(4.1) \quad -\mathbb{X}_{u,t} + \mathbb{X}_{s,t} - \mathbb{X}_{s,u} = (X_u - X_s)(X_t - X_u)^\top, \quad u, s, t \in [0, T].$$

(2) *In the literature one often introduces a decomposition of \mathbb{X} into a symmetric and an antisymmetric component, i.e.*

$$\begin{aligned} \text{sym}(\mathbb{X}_{s,t})(i, j) &:= \frac{1}{2} \left(\mathbb{X}_{s,t}(i, j) + \mathbb{X}_{s,t}(j, i) \right) \\ \text{anti}(\mathbb{X}_{s,t})(i, j) &:= \frac{1}{2} \left(\mathbb{X}_{s,t}(i, j) - \mathbb{X}_{s,t}(j, i) \right), \end{aligned}$$

$1 \leq i, j \leq d$, so that

$$(4.2) \quad \mathbb{X}_{s,t} = \text{sym}(\mathbb{X}_{s,t}) + \text{anti}(\mathbb{X}_{s,t}).$$

(3) *We say that the pair $\mathbf{X} = (X, \mathbb{X})$ is **geometric** if*

$$\text{sym}(\mathbb{X}_{st}) = \frac{1}{2} (X_t - X_s)(X_t - X_s)^\top, \quad s, t \in [0, T].$$

A typical second-order process \mathbb{X} is defined setting

$$(4.3) \quad \mathbb{X}_{s,t} := \int_s^t (X_r - X_s) \otimes d^\circ X_r,$$

provided that previous definite symmetric integral exists, for every $0 \leq s \leq t \leq T$, see (2.8).

We can also consider another \mathbb{X} , replacing the symmetric integral with the forward integral, i.e.

$$(4.4) \quad \mathbb{X}_{s,t} := \int_s^t (X_r - X_s) \otimes d^- X_r,$$

provided that previous definite forward integrals exist, exists, for every $(s, t) \in [0, T]^2, 0 \leq s \leq t \leq T, 0$ see (2.7).

Example 4.2. Let X be an \mathbb{R}^d -valued continuous semimartingale. Then, for $1 \leq i, j \leq d$, one often considers

$$\mathbb{X}_{s,t}^{\text{stra}}(i, j) := \left(\int_s^t (X_r - X_s) \otimes d^\circ X_r \right) (i, j) = \int_s^t (X_r^i - X_s^i) \circ dX_r^j$$

and

$$\mathbb{X}_{s,t}^{\text{ito}}(i, j) := \left(\int_s^t (X_r - X_s) \otimes d^- X_r \right) (i, j) = \int_s^t (X_r^i - X_s^i) dX_r^j,$$

where the integrals in the right-hand side are respectively intended in the Stratonovich and Itô sense.

5. ROUGH STOCHASTIC INTEGRATION VIA REGULARIZATIONS

In this section we still consider our \mathbb{R}^d -valued reference process X , equipped with its second-order process \mathbb{X} . Inspired by [30], we start with the definition of the integral.

Definition 5.1. A couple $(Y, Y') \in \mathcal{D}_X$ is **rough stochastically integrable** if

$$(5.1) \quad \int_0^t Y_s d\mathbf{X}_s := \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_0^t \left(Y_s X_{s,s+\varepsilon}^\top + Y'_s \mathbb{X}_{s,s+\varepsilon} \right) ds$$

exists in probability for each $t \in [0, T]$. Previous integral is called **rough stochastic integral** and it is a row vector.

We remark that if $Y' = 0$ the rough stochastic integral coincides with the forward integral $\int_0^t Y d^- X, t \in [0, T]$. In previous definition, we make an abuse of notation: we omit the dependence of the integral on Y' which in general affects the limit but it is usually clear from the context.

We introduce now a backward version of $\int_0^t Y d\mathbf{X}$, i.e. the **backward rough integral**

$$\int_0^t Y_s d\overleftarrow{\mathbf{X}}_s := \lim_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon} \int_0^t \left(Y_{s+\varepsilon} X_{s,s+\varepsilon}^\top + Y'_{s+\varepsilon} \mathbb{X}_{s,s+\varepsilon} \right) ds,$$

in probability for $(Y, Y') \in \mathcal{D}_X$. Previous expression is again a row vector.

Remark 5.2. Given an \mathbb{R}^n -valued process $(Y_{t \in [0, T]})$, we denote $\hat{Y}_t := Y_{T-t}, t \in [0, T]$.

- (1) The introduction of the backward rough integral is justified by the following observation. By an easy change of variables $s \mapsto T - s$ we easily show that, for every $t \in [0, T]$,

$$(5.2) \quad \int_0^t Y_s d\overleftarrow{\mathbf{X}}_s = - \int_{T-t}^T \hat{Y}_s d\hat{\mathbf{X}}_s.$$

This holds of course with the convention that \hat{Y} is equipped with \hat{Y}' as Gubinelli derivative.

(2) (5.2) is reminiscent of a well-known property which states that

$$\int_0^t Y d^+ X = - \int_{T-t}^T \hat{Y} d^- \hat{X},$$

where the left-hand side is the **backward integral** $\int_0^t Y d^+ X$, see Proposition 1.3), see [46].

Let us give a simple example which connects deterministic regularization approach with rough paths.

Proposition 5.3. *Let $\mathbf{X} = (X, \mathbb{X})$ be an a.s. enhanced rough path, where a.s. $X \in C^{[\gamma]}([0, T])$ with $\frac{1}{3} < \gamma < \frac{1}{2}$. We suppose that a.s. $\mathbb{X} \in C^{[2\gamma]}([0, T]^2)$ and it fulfills the Chen's relation. Let Y be a process such that a.s. its paths are weakly controlled in the sense of Definition 3.1 with Gubinelli derivative Y' . The following properties hold.*

(1) *The limit*

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_0^\cdot \left(Y_s X_{s, s+\varepsilon}^\top + Y'_s \mathbb{X}_{s, s+\varepsilon} \right) ds,$$

*exists uniformly on $[0, T]$ and it coincides a.s. with the **Gubinelli integral**.*

In particular, (5.1) exists.

(2) *The limit*

$$(5.3) \quad \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_0^\cdot \left(Y_{s+\varepsilon} X_{s, s+\varepsilon}^\top + Y'_{s+\varepsilon} \mathbb{X}_{s, s+\varepsilon} \right) ds,$$

exists uniformly on $[0, T]$ a.s. and it coincides a.s. with the rough Gubinelli integral as described in [30].

(3) *The rough stochastic integrals $\int_0^\cdot Y_s d\mathbf{X}_s$ and $\int_0^\cdot Y_s \overleftarrow{d}\mathbf{X}_s$ exist and they are equal a.s. to the Gubinelli integral.*

Remark 5.4. *When Y is γ' -Hölder continuous and X is γ -Hölder continuous, with $\gamma + \gamma' > 1$, Proposition 3. in Section 2.2 of [46] stated that the Young integral $\int_0^t Y d^{(y)} X$, equals both the forward and backward integrals $\int_0^t Y d^\mp X$. Proposition 5.3 states an analogous theorem for the Gubinelli integral, which equals both $\int_0^\cdot Y_s d\mathbf{X}_s$ and $\int_0^\cdot Y_s \overleftarrow{d}\mathbf{X}_s$.*

We introduce now the notion of multi-increments. Let $k \in \{1, 2, 3\}$. We denote by \mathcal{C}_k the space of continuous functions $g : [0, T]^k \rightarrow \mathbb{R}$, denoted by $(t_1, \dots, t_k) \mapsto g_{t_1, \dots, t_k}$ such that $g_{t_1, \dots, t_k} = 0$ whenever $t_i = t_{i+1}$ for some $1 \leq i \leq k-1$.

For $g \in \mathcal{C}_2$, we have defined $\|g\|_\alpha$ at (2.2). For $g \in \mathcal{C}_3$, we set

$$\|g\|_{\alpha,\beta} := \sup_{s,u,t \in [0,T]} \frac{|g_{tus}|}{|u-s|^\alpha |t-s|^\beta},$$

$$\|g\|_\mu := \inf \left\{ \sum_i \|g^i\|_{\rho_i, \mu - \rho_i}; g = \sum_i g^i, 0 < \rho_i < \mu \right\},$$

where the latter infimum is taken over all sequences $\{g^i \in \mathcal{C}_3\}$ such that $g = \sum_i g^i$ and for all choices of $\rho_i \in]0, \mu[$. We say that $g \in C^\mu([0, T]^3)$ if $\|g\|_\mu < \infty$.

We introduce the maps

$$(1) \quad \delta_1 : \mathcal{C}_1 \rightarrow \mathcal{C}_2 \text{ defined by } (\delta_1 f)_{s,t} = f(t) - f(s).$$

$$(2) \quad \delta_2 : \mathcal{C}_2 \rightarrow \mathcal{C}_3 \text{ defined by}$$

$$\delta_2 f_{t_1, t_2, t_3} = -f_{t_2, t_3} + f_{t_1, t_3} - f_{t_1, t_2}.$$

If $k = 1, 2$ and $f \in \mathcal{C}_k$, $\delta_k f$ is called k -**increment** of the function f .

In the proof of Proposition 5.3, as in [31], it is crucial to make use of the so called **Sewing Lemma**. The lemma below follows directly from Proposition 2.3 in [31].

Lemma 5.5. *Let $g \in \mathcal{C}_2$ such that $\delta_2 g \in C^{[\mu]}([0, T]^3)$, for some $\mu > 1$. Then, there exists a unique (up to a constant) $I \in \mathcal{C}_1$ and $\mathcal{R} \in C^{[\mu]}([0, T]^2)$ such that*

$$g = \delta_1 I + \mathcal{R}.$$

Proof (of Proposition 5.3).

(1) We set

$$(5.4) \quad A_{s,t} = Y_s(X_t - X_s)^\top + Y'_s \mathbb{X}_{s,t}, \quad (s, t) \in [0, T]^2.$$

Then the 2-increment of A is given by

$$\begin{aligned} (\delta_2 A)_{t_1, t_2, t_3} &= Y_{t_1}(X_{t_3} - X_{t_1})^\top + Y'_{t_1} \mathbb{X}_{t_1, t_3} \\ &\quad - Y_{t_2}(X_{t_3} - X_{t_2})^\top - Y'_{t_2} \mathbb{X}_{t_2, t_3} - Y_{t_1}(X_{t_2} - X_{t_1})^\top - Y'_{t_1} \mathbb{X}_{t_1, t_2} \\ &= (Y_{t_2} - Y_{t_1})(X_{t_2} - X_{t_3})^\top + Y'_{t_1}(\mathbb{X}_{t_1, t_3} - \mathbb{X}_{t_2, t_3} - \mathbb{X}_{t_1, t_2}) \\ &\quad - (Y'_{t_2} - Y'_{t_1}) \mathbb{X}_{t_2, t_3} \\ &= \left\{ Y_{t_2} - Y_{t_1} - Y'_{t_1}(X_{t_2} - X_{t_1}) \right\} (X_{t_2} - X_{t_3})^\top + (\delta_1 Y')_{t_1 t_2} \mathbb{X}_{t_2, t_3} \\ (5.5) \quad &= R_{t_1, t_2}^Y (\delta_1 X)_{t_2, t_3}^\top + (\delta_1 Y')_{t_1, t_2} \mathbb{X}_{t_2, t_3}, \end{aligned}$$

where the third equality follows by Chen's relation. By Definition 3.1 we have a.s. $Y' \in C^{[\gamma]}([0, T])$, $R^Y \in C^{[2\gamma]}([0, T]^2)$ and we also have $\mathbb{X} \in C^{[2\gamma]}([0, T]^2)$. Consequently $\delta_2 A \in C^{[3\gamma]}([0, T]^3)$.

Then, setting $\mu = 3\gamma$, outside a null set, Lemma 5.5 applied to $g = A$, provides an unique (up to a constant) a continuous process I such that

$$A_{s,s+\varepsilon} = I_{s+\varepsilon} - I_s + \mathcal{R}_{s,s+\varepsilon},$$

where $\mathcal{R} \in C^{[3\gamma]}([0, T]^2)$. For a given $\varepsilon > 0$ and $t \in [0, T]$, we then have

$$\frac{1}{\varepsilon} \int_0^t A_{s,s+\varepsilon} ds = \frac{1}{\varepsilon} \int_0^t I_{s,s+\varepsilon} ds + \frac{1}{\varepsilon} \int_0^t \mathcal{R}_{s,s+\varepsilon} ds$$

and

$$(5.6) \quad \lim_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon} \int_0^t I_{s,s+\varepsilon} ds = I_t - I_0,$$

uniformly in $[0, T]$. By using the fact that $\mathcal{R} \in C^{[3\gamma]}([0, T]^2)$, we have

$$\frac{1}{\varepsilon} \sup_{s \in [0, T]} |\mathcal{R}_{s,s+\varepsilon}| \leq \frac{\varepsilon^{3\gamma}}{\varepsilon} \|\mathcal{R}\|_{3\gamma} \rightarrow 0,$$

as $\varepsilon \downarrow 0$. This completes the proof.

- (2) We fix ω . The quantity (5.3) converges to I_t where I is again the (unique) function appearing in the Sewing Lemma 5.5. The arguments are similar to those of item 1.
- (3) This is a direct consequence of previous points and the fact that a.s. I also coincides with the Gubinelli integral.

□

Theorem 5.6. *Let $X = (X_t)_{t \in [0, T]}$ be a given continuous (\mathcal{F}_t) -semimartingale with values in \mathbb{R}^d and Y be an (\mathcal{F}_t) -weak Dirichlet process. We set $\mathbb{X} := \mathbb{X}^{\text{stra}}$, see Example 4.2.*

Then the rough stochastic integral of Y (with càglàd progressively measurable, stochastic Gubinelli derivative Y') with respect to $\mathbf{X} = (X, \mathbb{X})$ coincides with the Stratonovich integral i.e.

$$(5.7) \quad \int_0^\cdot Y_s d\mathbf{X}_s = \int_0^\cdot Y_s \circ dX_s.$$

Remark 5.7. (1) *In Proposition 3.9 we have shown the existence of a progressively measurable process Y' such that (Y, Y') belongs to \mathcal{D}_X .*

(2) (5.7) implies that the value of the rough stochastic integral does not depend on Y' .

Proof (of Theorem 5.6).

The rough stochastic integral $\int_0^t Y d\mathbf{X}_s$ defined in (5.1) exists if we prove in particular that the two limits below

$$(5.8) \quad \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_0^t Y_s X_{s,s+\varepsilon}^\top ds \quad \text{and} \quad \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_0^t Y'_s \mathbb{X}_{s,s+\varepsilon} ds,$$

exist in probability. We will even prove the ucp convergence of (5.8). Let us fix $i \in \{1, \dots, d\}$. By Proposition 6. in [46] we have

$$(5.9) \quad \lim_{\varepsilon \rightarrow 0} \int_0^t Y_s \frac{X_{s+\varepsilon}^i - X_s^i}{\varepsilon} ds = \int_0^t Y_s d^- X_s^i = \int_0^t Y_s dX_s^i,$$

ucp, where the second integral in the equality is the usual Itô's stochastic integral.

We show now that

$$(5.10) \quad \frac{1}{\varepsilon} \int_0^t Y'_s \mathbb{X}_{s,s+\varepsilon} ds \rightarrow \frac{1}{2} [Y, X]_t, t \in [0, T],$$

holds ucp as $\varepsilon \rightarrow 0$.

Let $i \in \{1, \dots, d\}$. We write, for every $t \in [0, T]$, an element of the vector $\frac{1}{\varepsilon} \int_0^t Y'_s \mathbb{X}_{s,s+\varepsilon} ds$ as

$$\frac{1}{\varepsilon} \left(\int_0^t Y'_s \mathbb{X}_{s,s+\varepsilon} ds \right)_i = \sum_{k=1}^d \int_0^t (Y'_s)^k \left(\frac{1}{\varepsilon} \int_s^{s+\varepsilon} (X_r^k - X_s^k) \circ dX_r^i \right) ds.$$

The definition of Stratonovich integral yields

$$\begin{aligned} \sum_{k=1}^d \int_0^t (Y'_s)^k \left(\frac{1}{\varepsilon} \int_s^{s+\varepsilon} (X_r^k - X_s^k) \circ dX_r^i \right) ds &= \sum_{k=1}^d \int_0^t (Y'_s)^k \left(\frac{1}{\varepsilon} \int_s^{s+\varepsilon} (X_r^k - X_s^k) dX_r^i \right) ds \\ &\quad + \frac{1}{2\varepsilon} \sum_{k=1}^d \int_0^t (Y'_s)^k [X^k - X_s^k, X^i]_{s,s+\varepsilon} ds. \end{aligned}$$

Obviously $[X^k - X_s^k, X^i] = [X^k, X^i]$. Since the covariations $[X^k, X^i]$ are bounded variation processes, item 7. of Proposition 1. in [46] shows that the second term in the right-hand side of the latter identity converges in ucp as $\varepsilon \rightarrow 0$ to

$$\frac{1}{2} \sum_{k=1}^d \int_0^t (Y'_r)^k d[X^k, X^i]_r = \frac{1}{2} \left(\int_0^t Y'_r d[X]_r \right)_i = \frac{1}{2} [Y, X^i]_t,$$

where the latter equality follows by (3.12) in Proposition 3.9.

We complete the proof if we show that for every $i \in \{1, \dots, d\}$ and $k \in \{1, \dots, d\}$ the ucp limit

$$(5.11) \quad \int_0^t (Y'_s)^k \left(\frac{1}{\varepsilon} \int_s^{s+\varepsilon} (X_r^k - X_s^k) dX_r^i \right) ds \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0,$$

holds. Let $M^i + V^i$ be the canonical decomposition of the semimartingale X^i . By usual localization arguments we can reduce to the case when $[M^i], \|V^i\|(T), X^i, (Y')$ are bounded processes. Using the stochastic Fubini's Theorem (see Theorem 64, Chapter 6 in [39]), we can write

$$\int_0^t (Y'_s)^k \left(\frac{1}{\varepsilon} \int_s^{s+\varepsilon} (X_r^k - X_s^k) dX_r^i \right) ds = \int_0^{t+\varepsilon} \left(\frac{1}{\varepsilon} \int_{(r-\varepsilon)^+}^{r \wedge t} (Y'_s)^k (X_r^k - X_s^k) ds \right) dX_r^i.$$

For $\varepsilon > 0$, and $k \in \{1, \dots, d\}$, let us define the auxiliary process

$$\xi^\varepsilon(t) := \frac{1}{\varepsilon} \int_{(r-\varepsilon)^+}^{r \wedge t} (Y'_s)^k (X_r^k - X_s^k) ds.$$

Controlling the border terms as usual, by Problem 5.25 Chapter 1. of [32] (5.11), it remains to show that the limit in probability

$$(5.12) \quad \int_0^T |\xi^\varepsilon(r)|^2 d[X^i]_r \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0 \quad \text{holds.}$$

Denoting by $\delta(X, \cdot)$ the continuity modulus of X on $[0, T]$,

$$\int_0^T |\xi^\varepsilon(r)|^2 d[X^i]_r \leq \delta(X, \varepsilon)^2 \sup_{s \in [0, T]} |(Y'_s)^k|^2 [X^i]_T,$$

which obviously converges a.s. to zero. This concludes the proof of (5.10).

Combining (5.9) and (5.10) we finish the proof of (5.7). \square

Through a similar but simpler proof (left to the reader) than the one of Theorem 5.6 we have the following.

Theorem 5.8. *Let $X = (X_t)_{t \in [0, T]}$ be a given continuous (\mathcal{F}_t) -semimartingale with values in \mathbb{R}^d and let Y be a.s. bounded and progressively measurable. Suppose moreover that Y has a càglàd progressively measurable Gubinelli derivative Y' . We set $\mathbb{X} := \mathbb{X}^{\text{ito}}$, see Example 4.2. Then the rough stochastic integral of Y with respect to $\mathbf{X} = (X, \mathbb{X})$ coincides with the Itô integral of Y with respect to X , i.e.*

$$(5.13) \quad \int_0^\cdot Y_s d\mathbf{X}_s = \int_0^\cdot Y_s dX_s.$$

Theorems 5.6 and 5.8 somehow extend Proposition 5.1 in [23] and Corollary 5.2 in [22]. In this paper, (Y, Y') does not necessarily have Hölder continuous paths with the classical regularity in the sense of rough paths.

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