VARIATIONAL METHODS FOR ACOUSTIC RADIATION IN A DUCT WITH A SHEAR FLOW AND AN ABSORBING BOUNDARY

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Abstract. The well-posedness of the acoustic radiation in a 2D duct in presence of both a shear flow and an absorbing wall described by the Myers boundary condition is studied thanks to variational methods. Without flow the problem is found well-posed for any impedance value. The presence of a flow complicates the results. With a uniform flow the problem is proven to be always of the Fredholm type but is found well-posed only when considering a dissipative radiation problem. With a general shear flow, the Fredholm property is recovered for a weak enough shear and the dissipative radiation problem requires to introduce extra conditions to be well-posed: enough dissipation, a large enough frequency and non-intuitive conditions on the impedance value.

Key words. Acoustics, time-harmonic radiation, shear flow, impedance boundary condition, variational methods, Fredholm alternative

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1. Introduction.

Wave propagation in a steady flow over an acoustically lined wall has been widely studied due to its applications to noise damping. To reduce fan and engine noise, an important feature is the inclusion of sections of acoustic lining at the intake or exhaust of an aeroengine. These acoustic linings are usually made of a honeycomb of small resonators designed to dampen tonal noise. Alternatively, bulk linings (e.g. foam or wool-type materials) can be used, with the characteristic to dampen broadband noise. The first case is called locally reacting liners because it is characterized by a local impedance defined to relate the acoustic pressure to the normal acoustic displacement at the liner surface. On the contrary the second kind of lining is modelled by a non local impedance through Fourier transforms along the interface air/material.

In this paper we consider the first case characterized by a local impedance and we focus more particularly on the influence of a flow. The boundary conditions often used in numerical simulations incorporate both the impedance of the lining and the effect of the slipping mean flow. Ingard [1] proposed a boundary condition accounting for the continuity of the acoustic normal displacement at the liner surface, between the fluid and the acoustic liner and considering the effect of a mean flow parallel to the surface. Myers [2] extended this result to any arbitrary mean flow along a curved wall. This Ingard-Myers impedance boundary condition has been extensively used and has been verified to correctly represent the limit of a vanishingly-thin inviscid boundary layer over the acoustic lining [3, 4] although the boundary layer needs to be very thin in some cases to give a good approximation [5].

The Ingard-Myers boundary condition is known to have drawbacks. In time-domain simulations, there have been several reports on issues of stability when applying the Myers boundary condition to an acoustically-lined duct with flow [6, 7, 8, 9, 10, 11]. The boundary condition leads to numerical simulations becoming unstable at the grid scale, even in the uniform flow case. However due to a lack of alternatives, the Myers boundary condition is still used, with any instability artifacts considered spurious and filtered out using an artificial selective filter [6, 12, 13, 14, 15].

In this paper we focus on the Myers condition in the time-harmonic regime. It has

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been less studied theoretically because numerical stability issues are in practice less problematic when the frequency is fixed, and the well-posedness of the time-harmonic radiation in presence of a Myers condition is still an open question. The aim is to bring a comprehensive mathematical study of the time-harmonic propagation in a waveguide with an absorbing boundary and to study the influence of various complexities of the flow: no flow, uniform flow and shear flow. Our approach is different from the ones developed in the references already cited because they mostly focus on the time domain and on the behavior of the guided modes. Indeed, these studies restrict to a problem without any source which enables them to use a spatial Fourier transform of the linearized Euler equations. It leads to the Pridmore-Brown equation [16], whose solutions are the guided modes. When one mode is found unstable, in the sense exponentially growing in time, the problem is deduced to be ill-posed. On the contrary in the frequency-domain, other tools must be used. Indeed even if a mode is found unstable in the sense exponentially spatially growing, the ill-posedness cannot be deduced since the modes do not form a complete basis [17], even for a uniform flow. To avoid this procedural problem, in this paper we don't study the guided modes individually. To recover a controlled mathematical framework we prefer to consider a radiation problem due to the presence of a source and to use variational arguments and coercivity properties to study the well-posedness.

The paper is organized as follows. Section 2 concerns a fluid at rest. The acoustic radiation problem and the impedance boundary condition are presented. Thanks to the absorbing boundary, the solution is found of finite energy and thus can be directly sought in an unbounded waveguide. The problem is shown to be well-posed: the proof is easy at low frequencies and more involved at larger frequencies. To conclude in this latter case, we draw inspiration from works on the scattering from unbounded rough surfaces [18]. The extension to a fluid in motion is done in section 3. First is treated in subsection 3.2 the case of a uniform flow. Thanks to the introduction of PMLs to select the outgoing solution, the problem is proven to always be of Fredholm type but to prove well-posedness, a dissipative radiation problem must be considered. The problem is proven to be well-posed, including for weak dissipations. Eventually subsection 3.3 deals with the case of a varying flow. Then the problem is found to remain of Fredholm type for a weak shear and is proven to be well-posed under some extra constraints: the dissipation must be large enough and the physical parameters must follow some laws, a small impedance modulus or a high frequency. These constraints are illustrated numerically.

2. Case of a fluid at rest.

2.1. Geometry and equation. We consider a 2D infinite duct $(X, Y) \in \mathbb{R} \times (0, h)$ of height h filled with a compressible fluid. The real acoustic pressure $\tilde{P}(X, Y, t)$ satisfies the wave equation

$$\Delta\tilde{P} - \frac{1}{c_0^2}\frac{\partial^2\tilde{P}}{\partial t^2} = \tilde{F}(X, Y, t),$$

where $\tilde{F}$ is a real source term and c_0 is the sound speed. To use non-dimensional equations, we introduce $x = X/h$, $y = Y/h$ and the new unknowns $P(x, y, t) = \tilde{P}(X, Y, t)$ and $F(x, y, t) = \tilde{F}(X, Y, t)$. The guide is defined by $\Omega = \{(x, y) \in \mathbb{R} \times (0, 1)\}$. We note $\Gamma_0 = \{(x, y) \in \mathbb{R}^2, y = 0\}$ the lower boundary that we suppose rigid. The upper boundary $\Gamma = \{(x, y) \in \mathbb{R}^2, y = 1\}$ is supposed to absorb the sound and is characterized by a complex impedance Z. For a time harmonic source term $F(x,y,t) = f(x,y)e^{-i\omega t}$ with a frequency $\omega > 0$, we look for the pressure $P(x,y,t) = p(x,y)e^{-i\omega t}$ satisfying the following Helmholtz problem: for $f \in L^2(\Omega)$ and $k = \omega/c_0 > 0$, find $p \in H^1(\Omega)$ such that

$$\left\{ \begin{array}{rcll} (\Delta + k^2)p & = & f & \text{in} \quad \Omega, \\ \partial p/\partial y & = & ikYp & \text{on} \quad \Gamma \quad \text{and} \quad \partial p/\partial y = 0 \quad \text{on} \quad \Gamma_0, \end{array} \right. \tag{1}$$

where $Y = 1/Z$ is the admittance, that we consider constant with $\Re e(Y) > 0$ to produce sound absorption.

Remark 1. *The sign of the real part of the admittance is easy to understand when considering the time domain and restricting to a real admittance: in the transient regime, the pressure $P(x,y,t)$ satisfies the time version of (1) (considered without source term but with some initial conditions not precised here):*

$$\left\{ \begin{array}{rcll} c_0^2 \, \Delta P & = & \partial^2 P/\partial t^2 & \text{in} \quad \Omega, \\ c_0 \, \partial P/\partial y & = & -Y \partial P/\partial t & \text{on} \quad \Gamma \quad \text{and} \quad \partial P/\partial y = 0 \quad \text{on} \quad \Gamma_0. \end{array} \right. \tag{2}$$

Multiplying (2) by $\partial P/\partial t$ is easily deduced the energy balance

$$\frac{dE}{dt} = -\frac{Y}{c_0} \int_\Gamma \left(\frac{\partial P}{\partial t} \right)^2 \mathrm{d}x \quad \text{with} \quad E = \frac{1}{2} \int_\Omega \left[\left(\frac{1}{c_0} \frac{\partial P}{\partial t} \right)^2 + |\boldsymbol{\nabla} P|^2 \right] \mathrm{d}x \mathrm{d}y. \tag{3}$$

It is clear that the energy decreases only if $Y > 0$.

Remark 2. *In [19] is considered a Generalized Impedance Boundary Condition (GIBC) of the form*

$$\frac{\partial p}{\partial y} + \frac{\partial p}{\partial x} \left(\mu \frac{\partial p}{\partial x} \right) + \lambda p = 0.$$

It corresponds to our case with $\mu = 0$ and $\lambda = -ikY$, thus with $\Im m(\lambda) < 0$ for our acoustics applications. On the contrary [19] treats the opposite case $\Im m(\lambda) > 0$ (with the same $e^{-i\omega t}$ convention), well-adapted to electromagnetism applications [20]. Note that our case is less favorable to prove the coercivity of the radiation problem.

For the rest of the paper we consider the time-harmonic regime. Due to the absorption and to the energy balance (3), the pressure decays away from the source and looking for a solution of finite energy, thus being in $H^1(\Omega)$, we will prove that the radiation problem is well-posed, first for low frequencies and then for all frequencies. To do so, we will use the Lax-Milgram theorem. We first notice that the problem (1) is equivalent to the following variational formulation: find $p \in H^1(\Omega)$ such that

$$a(k;p,q) = -\left(f,q\right)_{L^2(\Omega)} \quad \forall q \in H^1(\Omega), \tag{4}$$

where we have introduced the sesquilinear form for all $p, q \in H^1(\Omega)$

$$a(k;p,q) = \int_\Omega \left(\boldsymbol{\nabla} p \cdot \boldsymbol{\nabla} \bar{q} - k^2 p \bar{q} \right) \mathrm{d}x \mathrm{d}y - ikY \int_\Gamma p \bar{q} \, \mathrm{d}x. \tag{5}$$

2.2. Well-posedness at low frequencies. To prove the coercivity of $a(k;p,q)$, we start with a Poincaré-like inequality.

Lemma 1. *For all $p \in H^1(\Omega)$ and all $\lambda > 0$,*

$$\int_\Omega |p|^2 \, \mathrm{d}x \mathrm{d}y \leq (1+\lambda) \int_\Gamma |p|^2 \, dy + \frac{1}{2} \left(1 + \frac{1}{\lambda} \right) \int_\Omega \left| \frac{\partial p}{\partial y} \right|^2 \mathrm{d}x \mathrm{d}y. \tag{6}$$

Proof. For all $p \in C^\infty(\overline{\Omega}) \cap H^1(\Omega)$ and for all $y \in [0,1]$ we have

$$|p(x,y)| \leq |p(x,1)| + \int_y^1 \left| \frac{\partial p}{\partial y}(x,t) \right| \mathrm{d}t.$$

We use Young's inequality: for all $\lambda > 0$,

$$|p(x,y)|^2 \leq (1+\lambda)\,|p(x,1)|^2 + \left(1 + \frac{1}{\lambda}\right) \left(\int_y^1 \left| \frac{\partial p}{\partial y}(x,t) \right| \mathrm{d}t \right)^2.$$

Then thanks to Cauchy-Schwarz inequality, the final result is obtained by integrating on Ω and is extended to $p \in H^1(\Omega)$ by density. □

Now we prove the well-posedness at low frequencies:

Lemma 2. *If*

$$k < \frac{\sqrt{1+2|Y|^2} - 1}{|Y|}, \tag{7}$$

then problem (4) is well-posed.

Proof. We use the Lax-Milgram theorem. $a(k;\cdot,\cdot)$ is continuous on $H^1(\Omega) \times H^1(\Omega)$ because

$$|a(k;p,q)| \leq \int_\Omega \left(|\boldsymbol{\nabla} p||\boldsymbol{\nabla} \bar{q}| + k^2 |p||\bar{q}| \right) \mathrm{d}x\mathrm{d}y + k|Y| \int_\Gamma |p||\bar{q}| \mathrm{d}x.$$

We conclude thanks to the continuity of the trace application: $\exists C_{tr} > 0$ such that $\forall p \in H^1(\Omega)$, $\|p\|_{L^2(\Gamma)} \leq C_{tr} \|p\|_{H^1(\Omega)}$.

Now we prove that $a(k;\cdot,\cdot)$ is coercive on $H^1(\Omega)$. Let us note $iY = |Y|\, e^{i\zeta}$ where $\zeta = \arg(Y) + \pi/2 \in]0,\pi[$, since $\Re e Y > 0$. We introduce the decomposition $a(k;p,p) = \alpha(p) - e^{i\zeta}\beta(p)$, where we have introduced the forms:

$$\alpha(p) = \int_\Omega \left(|\boldsymbol{\nabla} p|^2 - k^2 |p|^2 \right) \mathrm{d}x\mathrm{d}y \ \text{ and } \ \beta(p) = k|Y| \int_\Gamma |p|^2 \,\mathrm{d}x.$$

The lower bound: $|\alpha - e^{i\zeta}\beta| = |e^{-i\zeta/2}(\alpha - e^{i\zeta}\beta)| \geq |\Im m(e^{-i\zeta/2}\alpha - e^{i\zeta/2}\beta)| \geq -\Im m(e^{-i\zeta/2}\alpha - e^{i\zeta/2}\beta) = \sin(\zeta/2)\,(\alpha+\beta)$, leads to the estimation:

$$|a(k;p,p)| \geq \sin\left(\frac{\zeta}{2}\right) \left[\int_\Omega \left(|\boldsymbol{\nabla} p|^2 - k^2 |p|^2 \right) \mathrm{d}x\mathrm{d}y + k|Y| \int_\Gamma |p|^2 \,\mathrm{d}x \right].$$

Note that $\sin(\zeta/2) \neq 0$ because $\Re e(Y) > 0$. Using (6) leads to

$$|a(k;p,p)| \geq \sin\left(\frac{\zeta}{2}\right) \left\{ \left[1 - \frac{k^2}{2}\left(1 + \frac{1}{\lambda}\right) \right] \int_\Omega |\boldsymbol{\nabla} p|^2 \,\mathrm{d}x\mathrm{d}y + k\left[|Y| - k\,(1+\lambda)\right] \int_\Gamma |p|^2 \,\mathrm{d}x \right\}.$$

To recover the H^1-norm on the right-hand side, we use again (6) leading to

$$\|p\|^2_{H^1(\Omega)} \leq (1+\lambda) \int_\Gamma |p|^2 \,\mathrm{d}x + \frac{1}{2}\left(3 + \frac{1}{\lambda}\right) \int_\Omega |\boldsymbol{\nabla} p|^2 \,\mathrm{d}x\mathrm{d}y.$$

This leads to the lower bound

$$|a(k;p,p)| \geq \sin\left(\frac{\zeta}{2}\right) \min \left[\frac{1 - \frac{k^2}{2}\left(1+\frac{1}{\lambda}\right)}{\frac{1}{2}\left(3 + \frac{1}{\lambda}\right)}, \frac{k\left[|Y| - k\,(1+\lambda)\right]}{(1+\lambda)} \right] \|p\|^2_{H^1(\Omega)}.$$

Therefore a is coercive if λ is chosen such that $[(2/k^2) - 1]^{-1} < \lambda < (|Y|/k) - 1$. This is possible if k is such that $|Y|k^2 + 2k - 2|Y| < 0$, and also if $k^2 < 2$, ensuring that $\lambda > 0$. Since $k > 0$, both conditions are fulfilled if (7) applies. □

2.3. Well-posedness at all frequencies. The previous result can not be easily extended to large frequencies. However it remains true and we have the following

THEOREM 3. *For all* $k > 0$*, the problem* (4) *is well-posed.*

To prove this theorem, we follow the procedure in [18], indicating that we just have to find an a-priori estimate for a solution of (4). It corresponds to prove that exists $C > 0$ such that for all $p \in H^1(\Omega)$ and $f \in L^2(\Omega)$ satisfying (4), it holds that

$$\|p\|_{H^1(\Omega)} \leq C \|f\|_{L^2(\Omega)} . \tag{8}$$

Then as in [18], we invoke Lemmas 4.4 and 4.5 in [21], which show that the a-priori estimate implies an inf-sup condition for the sesquilinear form $a(k; p, q)$ and also the transposed inf-sup condition. Then [[22], Theorem 2.15] yields existence, uniqueness, and boundedness of a solution of (4).

We now focus on establishing the a priori bound (8), which is a rather technical task. First we prove the following lemmas:

LEMMA 4. *If* $p \in H^1(\Omega)$ *is solution of the variational problem* (4) *then*

$$\|\boldsymbol{\nabla} p\|^2_{L^2(\Omega)} - k^2 \|p\|^2_{L^2(\Omega)} \leq (1 + |\Im m(Y)|/\Re e(Y)) \|f\|_{L^2(\Omega)} \|p\|_{L^2(\Omega)} , \tag{9}$$

and

$$k \Re e(Y) \|p\|^2_{L^2(\Gamma)} \leq \|f\|_{L^2(\Omega)} \|p\|_{L^2(\Omega)} . \tag{10}$$

Proof. From (4) and (5) written for $q = p$ is found

$$a(k; p, p) = \int_\Omega \left(|\boldsymbol{\nabla} p|^2 - k^2 |p|^2\right) \mathrm{d}x\mathrm{d}y - k \left[i \Re e(Y) - \Im m(Y)\right] \int_\Gamma |p|^2 \,\mathrm{d}x = -\left(f, p\right)_{L^2(\Omega)} .$$

The real and imaginary parts read

$$\left\{ \begin{array}{rcl} k \Re e(Y) \displaystyle\int_\Gamma |p|^2 \,\mathrm{d}x &=& \Im m \left(f, p\right)_{L^2(\Omega)} , \\ \displaystyle\int_\Omega \left(|\boldsymbol{\nabla} p|^2 - k^2 |p|^2\right) \mathrm{d}x\mathrm{d}y + k \Im m(Y) \int_\Gamma |p|^2 \,\mathrm{d}x &=& -\Re e \left(f, p\right)_{L^2(\Omega)} , \end{array} \right.$$

which gives the two inequalities. □

To go further, we need first to prove that $p \in H^2(\Omega)$.

LEMMA 5. *If* $p \in H^1(\Omega)$ *is solution of the variational problem* (4) *then* $p \in H^2(\Omega)$.

Proof. $p \in H^1(\Omega)$ satisfies (1) which can be written

$$\left\{ \begin{array}{rclcl} -\Delta p + p &=& g & \text{in} & \Omega \ \text{ with } \ g \equiv -f + (k^2 + 1)p, \\ \partial p / \partial y &=& ikYp & \text{on} & \Gamma \ \text{ and } \ \partial p / \partial y = 0 \ \text{ on } \ \Gamma_0. \end{array} \right.$$

The key point is to use the Fourier transform $\hat{p}(\xi, y)$ of $p(x, y)$ and a convenient definition of the H^2-norm. In [23] is indicated that $H^2(\Omega)$ can be equipped with the following norm

$$\|p\|^2_{H^2(\Omega)} = \int_{\mathbb{R}} \left(\|\hat{p}(\xi, \cdot)\|^2_{H^2(0,1)} + \xi^4 \|\hat{p}(\xi, \cdot)\|^2_{L^2(0,1)}\right) d\xi, \tag{11}$$

equivalent to the usual norm in $H^2(\Omega)$. Taking the Fourier transform along the x axis, $\hat{p}(\xi, y)$, noted for simplicity $\varphi(y)$ since ξ is just a parameter, is found to satisfy

$$\begin{cases} -d^2\varphi/dy^2 + (1+\xi^2)\varphi &= h \quad \text{in} \quad (0,1), \\ d\varphi/dy &= ikY\varphi \quad \text{at} \quad y=1 \ \text{and} \ d\varphi/dy = 0 \ \text{at} \ y=0, \end{cases}$$

where $h(y) = \hat{g}(\xi, y)$. Deriving the corresponding variational form and choosing the test field $\psi = \varphi$, we deduce

$$\|d\varphi/dy\|^2_{L^2(0,1)} + (1+\xi^2)\|\varphi\|^2_{L^2(0,1)} - ikY|\varphi(1)|^2 = (h, \varphi)_{L^2(0,1)}.$$

Proceeding as in the proof of lemma 2, we get the coercivity of the left-hand side leading to

$$\sin(\zeta/2)\left(\|d\varphi/dy\|^2_{L^2(0,1)} + (1+\xi^2)\|\varphi\|^2_{L^2(0,1)} + k|Y||\varphi(1)|^2\right) \le |(h,\varphi)_{L^2(0,1)}|.$$

Therefore by the Lax-Milgram theorem we have that for all $\xi \in \mathbb{R}$, $\hat{p}(\xi, \cdot) = \varphi(\cdot) \in H^1(0,1)$ exists and is unique and we derive also two upper bounds:

$$\sin(\zeta/2)\,(1+\xi^2)\|\varphi\|_{L^2(0,1)} \le \|h\|_{L^2(0,1)},$$

that we note

$$(1+\xi^2)\|\varphi\|_{L^2(0,1)} \le C\|h\|_{L^2(0,1)}, \tag{12}$$

where C designs a generic constant (same convention in the following). The other upper bound is

$$\sin(\zeta/2)\left(\|d\varphi/dy\|^2_{L^2(0,1)} + \|\varphi\|^2_{L^2(0,1)}\right) \le \|h\|_{L^2(0,1)}\|\varphi\|_{L^2(0,1)},$$

from which we deduce a control of the H^1-norm of φ: $\|\varphi\|_{H^1(0,1)} \le C\|h\|_{L^2(0,1)}$. To go further and to control the L^2-norm of $d^2\varphi/dy^2$, we come back to the equation satisfied by φ from which is deduced

$$\left\|d^2\varphi/dy^2\right\|_{L^2(0,1)} \le (1+\xi^2)\|\varphi\|_{L^2(0,1)} + \|h\|_{L^2(0,1)} \le C\|h\|_{L^2(0,1)},$$

where (12) has been used. Therefore we deduce the control of the H^2-norm

$$\|\varphi\|^2_{H^2(0,1)} = \left\|d^2\varphi/dy^2\right\|^2_{L^2(0,1)} + \|\varphi\|^2_{H^1(0,1)} \le C\|h\|^2_{L^2(0,1)}.$$

Finally to prove that $p \in H^2(\Omega)$, we just need to show that

$$\|\hat{p}(\xi,\cdot)\|^2_{H^2(0,1)} + \xi^4\|\hat{p}(\xi,\cdot)\|^2_{L^2(0,1)} \le C\|\hat{g}(\xi,\cdot)\|^2_{L^2(0,1)},$$

from which (11) leads, with Plancherel relation, to

$$\|p\|^2_{H^2(\Omega)} \le C\|g\|^2_{L^2(\Omega)}.$$

This is achieved by noting that

$$\|\varphi\|_{H^2(0,1)} + \xi^2\|\varphi\|_{L^2(0,1)} \le \|\varphi\|_{H^2(0,1)} + (1+\xi^2)\|\varphi\|_{L^2(0,1)} \le C\|h\|_{L^2(0,1)},$$

from which is deduced

$$\|\varphi\|^2_{H^2(0,1)} + \xi^4\|\varphi\|^2_{L^2(0,1)} \le \left(\|\varphi\|_{H^2(0,1)} + \xi^2\|\varphi\|_{L^2(0,1)}\right)^2 \le C\|h\|^2_{L^2(0,1)},$$

with $\varphi(y) = \hat{p}(\xi, y)$, $h(y) = \hat{g}(\xi, y)$. □

Now we prove the following technical lemma:

LEMMA 6. *If $p \in H^1(\Omega)$ is solution of the variational problem (4) then*

$$2\left\|\frac{\partial p}{\partial y}\right\|^2_{L^2(\Omega)} \leq \left(1 + \frac{|\Im m(Y)|}{\Re e(Y)} + \frac{k}{\Re e(Y)}(1+|Y|^2) + 2\right)\|f\|_{L^2(\Omega)}\|p\|_{H^1(\Omega)}. \tag{13}$$

Proof. We follow closely the approach in [18], using in particular some Green's like identity [[24], Theorem 4.4, Theorem 3.34]. For any $p \in H^2(\Omega)$, using two times the Green formula, one gets:

$$2\Re e\,(y\partial p/\partial y, \Delta p)_{L^2(\Omega)} = -2\|\partial p/\partial y\|^2_{L^2(\Omega)} + \|\nabla p\|^2_{L^2(\Omega)} + 2\|\partial p/\partial y\|^2_{L^2(\Gamma)} - \|\nabla p\|^2_{L^2(\Gamma)}. \tag{14}$$

On the other side, $p \in H^1(\Omega)$ solution of the variational problem (4) belongs to $H^2(\Omega)$ thanks to lemma 5 and satisfies (1)

$$\begin{cases} (\Delta + k^2)p &= f \quad \text{in} \quad \Omega, \\ \partial p/\partial y &= ikYp \quad \text{on} \quad \Gamma \text{ and } \partial p/\partial y = 0 \text{ on } \Gamma_0. \end{cases}$$

Multiplying $\bar{f}$ by $y\partial p/\partial y$ is obtained

$$2\Re e\,(y\partial p/\partial y, \Delta p)_{L^2(\Omega)} = k^2\|p\|^2_{L^2(\Omega)} - k^2\|p\|^2_{L^2(\Gamma)} + 2\Re e\,(y\partial p/\partial y, f)_{L^2(\Omega)},$$

while (14) becomes for the solution of problem (4):

$$2\Re e\,(y\partial p/\partial y, \Delta p)_{L^2(\Omega)} = -2\|\partial p/\partial y\|^2_{L^2(\Omega)} + \|\nabla p\|^2_{L^2(\Omega)} + k^2|Y|^2\|p\|^2_{L^2(\Gamma)} - \|\partial p/\partial x\|^2_{L^2(\Gamma)}.$$

Combining these two equalities together gives:

$$2\|\partial p/\partial y\|^2_{L^2(\Omega)} = \|\nabla p\|^2_{L^2(\Omega)} - k^2\|p\|^2_{L^2(\Omega)} + T - 2\Re e\,(y\partial p/\partial y, f)_{L^2(\Omega)},$$

where

$$T = k^2|Y|^2\|p\|^2_{L^2(\Gamma)} - \|\partial p/\partial x\|^2_{L^2(\Gamma)} + k^2\|p\|^2_{L^2(\Gamma)}.$$

Using (10) we get an upper bound for T

$$T \leq [k/\Re e(Y)](1+|Y|^2)\|f\|_{L^2(\Omega)}\|p\|_{L^2(\Omega)},$$

and combined with (9), it gives (13). □

Now we can prove the required a priori estimation (8) from which theorem 3 is directly deduced:

LEMMA 7. *If $p \in H^1(\Omega)$ is solution of the variational problem (4) then (8) holds with*

$$C = (1+k^2)\left[\frac{2}{k\Re e(Y)} + \frac{1}{2}\left(1 + \frac{|\Im m(Y)|}{\Re e(Y)} + \frac{k}{\Re e(Y)}(1+|Y|^2) + 2\right)\right] + 1 + \frac{|\Im m(Y)|}{\Re e(Y)}.$$

Proof. We start from (6) with $\lambda = 1$. Using (10) combined with (13) leads to

$$\|p\|^2_{L^2(\Omega)} \leq \left[\frac{2}{k\Re e(Y)} + \frac{1}{2}\left(1 + \frac{|\Im m(Y)|}{\Re e(Y)} + \frac{k}{\Re e(Y)}(1+|Y|^2) + 2\right)\right]\|f\|_{L^2(\Omega)}\|p\|_{H^1(\Omega)}.$$

Then with the control of $\|\nabla p\|^2_{L^2(\Omega)}$ with (9), we get

$$\|p\|^2_{H^1(\Omega)} \leq (1+k^2)\|p\|^2_{L^2(\Omega)} + (1 + |\Im m(Y)|/\Re e(Y))\|f\|_{L^2(\Omega)}\|p\|_{L^2(\Omega)} \leq C\|f\|_{L^2(\Omega)}\|p\|_{H^1(\Omega)}.$$

□

3. The case of a fluid in motion. Now we extend the previous study by adding a motion of the fluid. Restricting to a bounded domain, we will prove that the radiation problem is always of Fredholm type, for no shear or at most for a weak shear (see paragraphs 3.2.5 and 3.3.4) but to prove the well-posedness in an unbounded guide, a main difference with the no-flow case is that we need to introduce some extra dissipation (limiting absorption principle). We consider in the duct Ω an horizontal subsonic shear flow of velocity $U(y)\boldsymbol{e}_x$ with $|U| < c_0$, which in non-dimensional form becomes $M(y)\boldsymbol{e}_x$ with $M(y) = U(y)/c_0$ the Mach number. The acoustics equations are more complicated than in the no-flow case and are detailed now.

3.1. Impedance boundary condition in presence of a flow. In the flow case, the impedance boundary condition reads

$$\partial u_y/\partial t = Y c_0 p \text{ at } y = 1,$$

where $\boldsymbol{u}$ is the acoustic displacement linked to the velocity by $\boldsymbol{v} = D_t\boldsymbol{u}$ with the convective derivative

$$D_t = (1/c_0)\partial/\partial t + M\partial/\partial x. \tag{15}$$

Without flow the condition expressed versus the velocity and the pressure is simply $v_y = c_0 Y p$ but in presence of a uniform flow it becomes $\partial v_y/\partial t = Y c_0 D_t p$ at $y = 1$. In the time-harmonic regime it reads

$$v_y = (iY/k)D_k p,$$

where D_k is the convective operator

$$D_k = M(y)\partial/\partial x - ik. \tag{16}$$

Note that in the no flow case, D_k reduces to $-ik$ and since the Linearized Euler Equations give $\boldsymbol{\nabla} p = ik\boldsymbol{v}$, we recover the no-flow condition in Eq. (1).

For a fluid in motion, the difficulty of the study is weaker when the Mach number is constant. Therefore we present first the case of a uniform flow and then we consider the most difficult case of a varying flow.

3.2. Uniform flow case.

3.2.1. Equations of the problem. We consider a uniform flow M =cst$\neq 0$. Then the Linearized Euler Equations read

$$\left\{\begin{array}{rcl} D_k\boldsymbol{v} + \boldsymbol{\nabla} p &=& 0, \\ D_k p + \operatorname{div}\boldsymbol{v} &=& f, \end{array}\right. \tag{17}$$

with D_k defined in (16). The first relation of (17) implies that $\operatorname{curl}(D_k\boldsymbol{v} + \boldsymbol{\nabla} p) = 0$ where has been used the scalar curl operator defined by $\operatorname{curl}\boldsymbol{v} = \partial_x v_y - \partial_y v_x$. Thus it implies that exists a velocity potential φ such that $\boldsymbol{v} = \boldsymbol{\nabla}\varphi$ with $p = -D_k\varphi$. Indeed the solution of $D_k(\operatorname{curl}\boldsymbol{v}) = 0$ is $\operatorname{curl}\boldsymbol{v} = A(y)\exp(ikx/M)$ and the only causal solution ($\operatorname{curl}\boldsymbol{v} = 0$ when $x \to -\infty$) is $\operatorname{curl}\boldsymbol{v} = 0$. Expressing the impedance boundary condition versus the velocity potential leads to the new equations replacing (1):

$$\left\{\begin{array}{rcll} \Delta\varphi - D_k^2\varphi &=& f & \text{in } \Omega, \\ ik\partial\varphi/\partial y &=& Y D_k^2\varphi & \text{on } \Gamma \text{ and } \partial\varphi/\partial y = 0 \text{ on } \Gamma_0. \end{array}\right. \tag{18}$$

For a fluid at rest $M = 0$, since $p = ik\varphi$ we recover (1). Note that even when $M \neq 0$, the equations (18) can be expressed versus the pressure (in fact, since the admittance Y is constant, p satisfies the same equations). But it will no longer be the case when M is not constant. Since the velocity potential will be a natural unknown in the shear case, we prefer to formulate the problem with the velocity potential in the uniform flow case.

We keep on considering $\Re e(Y) > 0$ although contrary to the case of a fluid at rest, we don't know how to establish an energy balance for the problem (18) as explained in the two following remarks.

Remark 3. *The transient version of* (18)*, without any source term and restricted to a real admittance is:*

$$\begin{cases} \Delta\Phi - \left(\dfrac{1}{c_0}\dfrac{\partial}{\partial t} + M\dfrac{\partial}{\partial x}\right)^2 \Phi = 0 \;\; in \;\; \Omega, \\ \dfrac{1}{c_0}\dfrac{\partial^2\Phi}{\partial y \partial t} = -Y\left(\dfrac{1}{c_0}\dfrac{\partial}{\partial t} + M\dfrac{\partial}{\partial x}\right)^2 \Phi \;\; on \;\; \Gamma \;\; and \;\; \dfrac{\partial\Phi}{\partial y} = 0 \;\; on \;\; \Gamma_0, \end{cases}$$

with $\Phi(x, y, t) = \varphi(x, y)e^{-i\omega t}$*. Multiplying the volume equation by* $\partial\Phi/\partial t$*, we did not succeed in deriving an energy balance. This is due to the term* $\partial^2\Phi/\partial y\partial t$ *which does not appear naturally when applying the Green formula. In fact we suspect that it is not possible to establish an energy balance because in the following we will be able to prove the well-posedness of the time-harmonic problem only when introducing some extra dissipation.*

Remark 4. *Eliminating all the unknowns to work with the velocity only, it is possible to derive an equality close to an energy balance. We start from* (17) *without any source term and expressed in the time domain:*

$$\begin{cases} D_t\boldsymbol{v} + \boldsymbol{\nabla} p = 0 \;\; and \;\; D_t p + \operatorname{div}\boldsymbol{v} = 0 \;\; in \;\; \Omega, \\ \partial v_y/\partial t = Y c_0 D_t p \;\; on \;\; \Gamma \;\; and \;\; v_y = 0 \;\; on \;\; \Gamma_0, \end{cases}$$

with D_t *defined in* (15)*. Eliminating the pressure leads to*

$$\begin{cases} D_t^2\boldsymbol{v} - \boldsymbol{\nabla}\operatorname{div}\boldsymbol{v} = 0 & in \quad \Omega, \\ \partial v_y/\partial t = -Y c_0 \operatorname{div}\boldsymbol{v} & on \quad \Gamma \;\; and \;\; v_y = 0 \;\; on \;\; \Gamma_0. \end{cases}$$

Multiplying by $\partial\boldsymbol{v}/\partial t$ *is easilly deduced the equation*

$$\frac{dE}{dt} = -Yc_0\int_\Gamma (\operatorname{div}\boldsymbol{v})^2\,\mathrm{d}x \;\; with \;\; E = \frac{1}{2}\int_\Omega \left[\left(\frac{1}{c_0}\frac{\partial\boldsymbol{v}}{\partial t}\right)^2 - \left(M\frac{\partial\boldsymbol{v}}{\partial x}\right)^2 + (\operatorname{div}\boldsymbol{v})^2\right]\mathrm{d}x\mathrm{d}y.$$

Thus E *decreases only if* $Y > 0$ *and as in the no flow case, we recover that an admittance with a positive real part corresponds to an absorbing boundary condition. Unfortunately the sign of* E *is not known as soon as* $M \neq 0$*, which prevents from assuring that it is an energy.*

The consequence of these remarks is that we are not allowed to look for a solution of problem (18) in $H^1(\Omega)$. In fact we think that such solution doesn't exist, only a solution in $H^1_{\mathrm{loc}}(\Omega)$ should exist. To characterized this solution, we introduce some extra dissipation, as detailed now.

3.2.2. The dissipative problem. We consider a dissipative radiation problem by extending the frequency k to the upper complex plane. Indeed, to define uniquely the good physical solution of a radiation problem, a usual approach (see [25] for instance) is to use the limiting absorption principle [26]: the frequency k is extended to the complex plane by $k + i\varepsilon$ with $\varepsilon > 0$, which defines the so-called dissipative problem. Then the physical solution is defined as the limit, as ε goes to 0, of the unique H^1solution of the dissipative problem. In [27] is given an interpretation of the limiting absorption principle: it is shown that to extend the frequency to the complex plane in the Helmholtz equation corresponds to add a slight dissipation in the medium in the wave equation (for the problem in time). To apply the limiting absorption principle, we use the following transformation to extend the frequency to the complex plane with $\Im m(k) > 0$

$$k \to k_\theta = ke^{i\theta}, \quad \text{with} \quad 0 < \theta < \pi/2, \tag{19}$$

more convenient than the usual transformation $k \to k + i\varepsilon$. In the following we will determine how θ must be chosen to get well-posedness.

The variational formulation of (18) for $k = k_\theta$ is to find $\varphi \in V$ such that $\forall \psi \in V$

$$a_M(k_\theta; \varphi, \psi) = -(f, \psi)_{L^2(\Omega)}, \tag{20}$$

where the sesquilinear form reads

$$\begin{aligned} a_M(k_\theta;\varphi,\psi) &= \int_\Omega \left[\boldsymbol{\nabla}\varphi\cdot\boldsymbol{\nabla}\overline{\psi} - \left(M\frac{\partial\varphi}{\partial x} - ik_\theta\varphi\right)\left(M\frac{\partial\overline{\psi}}{\partial x} + ik_\theta\overline{\psi}\right)\right]\mathrm{d}x\mathrm{d}y, \\ &+ \frac{Y}{ik_\theta}\int_\Gamma \left(M\frac{\partial\varphi}{\partial x} - ik_\theta\varphi\right)\left(M\frac{\partial\overline{\psi}}{\partial x} + ik_\theta\overline{\psi}\right)\mathrm{d}x. \end{aligned} \tag{21}$$

The boundary term on Γ implies that $H^1(\Omega)$ is no longer the good framework and we must choose the Hilbert space

$$V = \left\{\varphi \in H^1(\Omega), \partial\varphi/\partial x \in L^2(\Gamma)\right\}, \tag{22}$$

equipped with the norm $\|\varphi\|_V^2 = \|\boldsymbol{\nabla}\varphi\|_\Omega^2 + \|\varphi\|_\Omega^2 + \|\partial\varphi/\partial x\|_\Gamma^2$.

3.2.3. Well-posedness conditions. As in the no flow case, to prove the well-posedness of the problem (20) we will use the Lax-Milgram theorem. In this aim it is sufficient to show that $\exists C > 0$ such that $\forall \varphi \in V$,

$$|a_M(k_\theta; \varphi, \varphi)| \geq C \|\varphi\|_V^2.$$

To simplify the notations, we introduce the admittance argument $-\pi/2 < \gamma < \pi/2$ (let us recall that $\Re e(Y) > 0$) such that

$$Y = |Y|e^{i\gamma}. \tag{23}$$

THEOREM 8. *For all Mach number $0 < M < 1$ and for all admittance $Y = |Y|e^{i\gamma}$ defined in (23), a critical angle $0 < \theta_c < \theta_{max} \equiv (2\gamma+\pi)/4$ exists such that the problem (20) is well posed for all dissipations associated to an angle $0 < \theta < \theta_c$ where θ is defined in (19) and θ_c is defined in (31).*

REMARK 5. *Since the lower bound for θ is zero, the dissipation is allowed to take very small values. It will not be the case when $M' \neq 0$ for which we will get $\theta > \theta_{min} > 0$.*

To prove latter this theorem, we introduce

$$A_M(\varphi) = a\,\|\partial\varphi/\partial x\|_\Omega^2 + b\,\|\partial\varphi/\partial y\|_\Omega^2 + c\,\|\varphi\|_\Omega^2 + d\,\|\partial\varphi/\partial x\|_\Gamma^2 + 2e\,\Im m\,(\varphi, \partial\varphi/\partial x)_\Gamma + f\,\|\varphi\|_\Gamma^2,$$

with the parameters defined by

$$\begin{array}{lll} a = (1-M^2)\sin(\theta)/k, & b = \sin(\theta)/k, & c = k\sin(\theta), \\ d = |Y|M^2\cos(\gamma-2\theta)/k^2, & e = |Y|M\cos(\gamma-\theta)/k, & f = |Y|\cos(\gamma). \end{array} \tag{24}$$

The quantity $A_M(\varphi)$ is useful since we have the

LEMMA 9. *If $\exists C > 0$ such that $\forall \varphi \in V$ defined in (22),*

$$A_M(\varphi) \geq C\left(\|\nabla\varphi\|_\Omega^2 + \|\varphi\|_\Omega^2 + \|\partial\varphi/\partial x\|_\Gamma^2\right),$$

then $a_M(k_\theta;\cdot,\cdot)$ is coercive and consequently problem (20) is well posed.

Proof. As we will show hereafter, $A_M(\varphi) = \Im m\left(\overline{a_M(k_\theta;\varphi,\varphi)/k_\theta}\right)$. Thus the coercivity comes from

$$\left|\frac{a_M(k_\theta;\varphi,\varphi)}{k_\theta}\right| \geq \Im m\left[\overline{\left(\frac{a_M(k_\theta;\varphi,\varphi)}{k_\theta}\right)}\right] \quad \text{and} \quad |a_M(k_\theta;\varphi,\varphi)| \geq |k_\theta|\left|\frac{a_M(k_\theta;\varphi,\varphi)}{k_\theta}\right|.$$

Now we evaluate $\Im m\left(\overline{a_M(k_\theta;\varphi,\varphi)/k_\theta}\right)$. Noting $\|\cdot\|_\Omega = \|\cdot\|_{L^2(\Omega)}$ and $\|\cdot\|_\Gamma = \|\cdot\|_{L^2(\Gamma)}$, from (21) we get

$$\begin{aligned} \overline{\left(\frac{a_M(k_\theta;\varphi,\varphi)}{k_\theta}\right)} &= \frac{(1-M^2)}{\overline{k_\theta}}\left\|\frac{\partial\varphi}{\partial x}\right\|_\Omega^2 + \frac{1}{\overline{k_\theta}}\left\|\frac{\partial\varphi}{\partial y}\right\|_\Omega^2 - \overline{k_\theta}\,\|\varphi\|_\Omega^2 - 2M\,\Im m\left(\varphi, \frac{\partial\varphi}{\partial x}\right)_\Omega, \\ &\quad + i\frac{\overline{Y}M^2}{\overline{k_\theta}^2}\left\|\frac{\partial\varphi}{\partial x}\right\|_\Gamma^2 + i\frac{\overline{Y}M}{\overline{k_\theta}}\,2\,\Im m\left(\varphi, \frac{\partial\varphi}{\partial x}\right)_\Gamma + i\overline{Y}\,\|\varphi\|_\Gamma^2. \end{aligned}$$

Taking the imaginary part, we get $A_M(\varphi)$. □

Now we look for conditions on θ under which lemma 9 applies. If $M = 0$, we get from (24) the simplifications $d = 0 = e$, $a = \sin(\theta)/k = b$ and thus $A_M(\varphi) = a\,\|\nabla\varphi\|_\Omega^2 + c\,\|\varphi\|_\Omega^2 + f\,\|\varphi\|_\Gamma^2$ has all its coefficients positive. Therefore $A_M(\varphi) \geq a\,\|\nabla\varphi\|_\Omega^2 + c\,\|\varphi\|_\Omega^2$ and the dissipative radiation problem is well-posed in $H^1(\Omega)$ for all $\theta > 0$ ($V \neq H^1(\Omega)$ has to be introduced only when $M > 0$). Note that we have proven in the previous section that it is also true without dissipation ($\theta = 0$).

Now we focus on the case $M > 0$ and to go further we need to establish some lower bounds. First we show the

LEMMA 10. *For all $\mu > 0$,*

$$\|\varphi\|_\Gamma^2 \leq (1+\mu)\,\|\varphi\|_\Omega^2 + (1/\mu)\,\|\partial\varphi/\partial y\|_\Omega^2. \tag{25}$$

Proof. For all $\varphi \in C^\infty(\overline{\Omega}) \cap H^1(\Omega)$, $\forall y \in [0,1]$,

$$|\varphi(x,1)|^2 - |\varphi(x,y)|^2 = \int_y^1 \frac{\partial}{\partial t}\left(|\varphi(x,t)|^2\right)\mathrm{d}t,$$

which is developed in

$$|\varphi(x,1)|^2 = |\varphi(x,y)|^2 + 2\Re e\left(\int_y^1 \overline{\varphi}(x,t)\frac{\partial\varphi}{\partial t}(x,t)\,\mathrm{d}t\right).$$

We use Young's inequality: for all $\mu > 0$ and for $y = 0$ in the integral,

$$|\varphi(x,1)|^2 \leq |\varphi(x,y)|^2 + \mu\int_0^1 |\varphi(x,y)|^2 \mathrm{d}y + (1/\mu)\int_0^1 |\partial\varphi/\partial y(x,y)|^2\,\mathrm{d}y,$$

The result is obtained by integrating for y between 0 and 1 and then for x on $\mathbb{R}$ and is finally extended to $\varphi \in H^1(\Omega)$ by density. □

Then we show the

LEMMA 11. *For all $\lambda > 0$ and $\mu > 0$,*

$$A_M(\varphi) \geq a\,\|\partial\varphi/\partial x\|_\Omega^2 + C_1\,\|\partial\varphi/\partial y\|_\Omega^2 + C_2\,\|\varphi\|_\Omega^2 + C_3\,\|\partial\varphi/\partial x\|_\Gamma^2,$$

with $g = e\lambda - f$, $C_1 = b - (g/\mu)$, $C_2 = c - g(1+\mu)$, $C_3 = d - (e/\lambda)$ and other constants defined in (24).

Proof. We use the Young inequality:

$$\forall\lambda > 0,\ |2\,\Im m\,(\varphi, \partial\varphi/\partial x)_\Gamma| \leq \lambda\,\|\varphi\|_\Gamma^2 + (1/\lambda)\,\|\partial\varphi/\partial x\|_\Gamma^2,$$

to deduce:

$$A_M(\varphi) \geq a\,\|\partial\varphi/\partial x\|_\Omega^2 + b\,\|\partial\varphi/\partial y\|_\Omega^2 + c\,\|\varphi\|_\Omega^2 + (d - e/\lambda)\,\|\partial\varphi/\partial x\|_\Gamma^2 - g\,\|\varphi\|_\Gamma^2.$$

Then using (25) leads to the relation in lemma 11. □

Now we can prove theorem 8 by finding conditions on θ under which lemma 9 applies, thanks to lemma 11.

Proof of theorem 8. We want all the coefficients a, C_1, C_2 and C_3 in lemma 11 to be strictly positive. First we consider the case $C_3 = 0$ thus

$$\lambda = \lambda_0(\theta) \equiv \frac{e}{d} = \frac{k}{M}\frac{\cos(\gamma-\theta)}{\cos(\gamma-2\theta)}. \tag{26}$$

From lemma 11, λ must be strictly positive which implies $2\theta < \gamma + \pi/2$ and which defines the maximum angle

$$\theta_{\max} = (2\gamma + \pi)/4. \tag{27}$$

This upper bound for θ becomes a strong constraint only when $\gamma \to -\pi/2$. For $\lambda = \lambda_0$, $g = g_0 \equiv e\lambda_0 - f$ is found to be equal to

$$g_0(\theta) = |Y|\sin^2(\theta)/\cos(\gamma - 2\theta), \tag{28}$$

and thus $g_0 > 0$. To get $C_1 > 0$ and $C_2 > 0$ we must satisfy $(g_0/b) < \mu < (c/g_0) - 1$ with $\mu > 0$. Thus $\theta \in]0, \theta_{max}[$ must be chosen such that $(g_0/b) < (c/g_0) - 1$ and this is obtained for θ below a critical value. Indeed $(g_0/b)(g_0/c) + (g_0/c) < 1$ can be written as $\tilde{P}_0(u) < 0$ with the polynomial

$$\tilde{P}_0(u) = u^2 + (u/k) - 1, \tag{29}$$

where

$$u(\theta) = g_0(\theta)/\sin(\theta) = |Y|\sin(\theta)/\cos(\gamma - 2\theta). \tag{30}$$

Since $\tilde{P}_0(0) < 0$, this is achieved for $u < u_c$ where $\tilde{P}_0(u_c) = 0$ with the explicit critical value $2u_c = -(1/k) + \sqrt{(1/k)^2 + 4}$. Finally, $\theta \in (0, \theta_{\max}) \to u(\theta)$ is found to be an increasing function of range $(0, \infty)$. $u < u_c$ corresponds to $\theta < \theta_c$, where the critical angle θ_c is defined as the unique solution $\theta \in (0, \theta_{\max})$ of $u(\theta_c) = u_c$ which reads more explicitly

$$2\frac{|Y|\sin(\theta_c)}{\cos(\gamma - 2\theta_c)} = 2u_c \equiv -\frac{1}{k} + \sqrt{\frac{1}{k^2} + 4}. \tag{31}$$

Note that θ_c is surprisingly independent of M. Eventually to satisfy lemma 9 we need conditions under which C_3 is strictly positive. This is achieved by slightly perturbing the condition under which $C_3 = 0$. We take $\lambda_\varepsilon = (e/d) + \varepsilon$ for any $\varepsilon > 0$. Then C_3 becomes C_3^ε such that $C_3^\varepsilon = \varepsilon d/[(e/d) + \varepsilon] > 0$. g becomes $g_\varepsilon \equiv e\lambda_\varepsilon - f = g_0 + e\varepsilon > 0$. The conditions $C_1^\varepsilon \equiv b - (g_\varepsilon/\mu) > 0$ and $C_2^\varepsilon \equiv c - g_\varepsilon(1+\mu) > 0$ lead to $(g_\varepsilon/b) < \mu < (c/g_\varepsilon) - 1$, which implies the condition $\tilde{P}_\varepsilon(u) < 0$ with

$$\tilde{P}_\varepsilon(u) = \tilde{P}_0(u) + \frac{2e\varepsilon g_0}{\sin(\theta)^2} + \left(\frac{e\varepsilon}{\sin(\theta)}\right)^2 + \frac{e\varepsilon}{k\sin(\theta)}.$$

Since $\tilde{P}_\varepsilon(u) \geq \tilde{P}_0(u)$, $\tilde{P}_\varepsilon(u_c(\varepsilon)) = 0$ for $u_c(\varepsilon) \leq u_c(0) = u_c$. Therefore the condition $\tilde{P}_\varepsilon(u) < 0$ is satisfied for $\theta < \theta_c(\varepsilon) \leq \theta_c(0)$, with $\theta_c(0)$ noted previously θ_c. $\theta_c(\varepsilon)$ can be as close as we want to θ_c by taking ε small enough which ensures that the problem is well posed for any $\theta < \theta_c$, as stated in theorem 8.

□

3.2.4. Numerical illustration. It is not possible to get θ_c more explicitly than the solution of the fixed point equation (31) but we can characterize it numerically. Some dependences of θ_c versus several physical parameters are explicit: since u is an increasing function of θ, from (31) is deduced that $\theta_c(k, Y)$ increases when k increases or $|Y|$ decreases. Moreover $\theta_c \to 0$ when $k \to 0$ or when $|Y| \to \infty$. However the variations of $\theta_c(\gamma)$ are not easy to guess and it is why we plot them now numerically.

For $k = 2$, $\theta_c(\gamma)$ solution of (31) is plotted in Fig. 1 for two modulus of the admittance: $|Y| = 1.4$ and $|Y| = 1.6$. The maximum of θ_c is located at $\gamma_{\max} = 2\arcsin(u_c/|Y|)$ with u_c defined in (31). For $|Y| = 1.4$ we get $\gamma_{\max} = 1.18$ and $\theta_c(\gamma_{\max}) = 0.59$ whereas $|Y| = 1.6$ leads to $\gamma_{\max} = 1.02$ and thus $\theta_c(\gamma_{\max}) = 0.51$. Moreover the values of $\theta_c(\gamma)$ at $\gamma = \pm\pi/2$ are expected. When $\gamma \to -\pi/2$, since $0 < \theta_c < \theta_{\max}$ with $\theta_{\max}(\gamma) \to 0$ when $\gamma \to -\pi/2$, naturally $\theta_c(\gamma) \to 0$. For $\gamma \to \pi/2$, we get directly from (31) that $\theta_c(\pi/2) = \arccos(|Y|/2u_c)$ if $|Y|/2u_c \leq 1$ and $\theta_c(\pi/2) = 0$ for $|Y|/2u_c \geq 1$. For $|Y| = 1.4$ we get $|Y|/2u_c = 0.90$ and $\theta_c(\pi/2) = 0.46$ whereas $|Y| = 1.6$ leads to $|Y|/2u_c = 1.03$ and thus $\theta_c(\pi/2) = 0$.

3.2.5. Case without dissipation. Without dissipation ($\theta = 0$ and $k_\theta = k$), we are not able to prove that the problem (20) is well-posed (and we suspect it is not true), as we did in the no-flow case, but we can at least prove that the problem is of Fredholm-type. To do so, we restrict the problem to a bounded domain and we close it with appropriate radiation conditions. The outgoing solution is selected thanks to the introduction of PMLs: the problem is set in a bounded domain $\Omega_\alpha = \overline{\Omega_d} \cup \Omega_\pm^L$ composed of the central domain $\Omega_d = \{(x,y); |x| < d, 0 < y < 1\}$

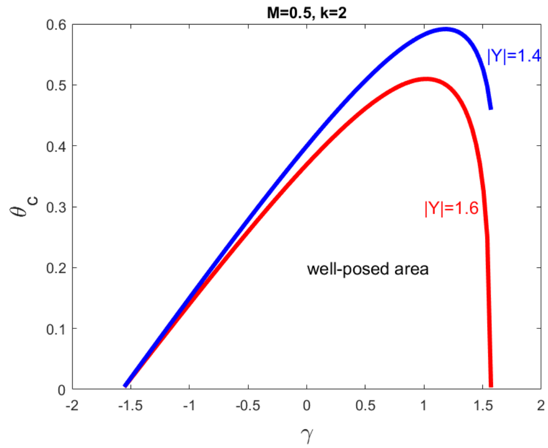


FIG. 1. *For $k = 2$, θ_c in (31) versus $\gamma = \arg Y$ for $|Y| = 1.4$ in blue and $|Y| = 1.6$ in red*

containing the source (for d large enough) and of surrounding absorbing layers $\Omega_\pm^L$ of length L: $\Omega_\pm^L = \{(x, y); d < \pm x < d + L, 0 < y < 1\}$. The introduction of PMLs amounts to the transformation of the differential operator $\partial/\partial x \longrightarrow \alpha(x)\,\partial/\partial x$ in the governing equations of the problem. The complex function α is assumed to be unity in Ω_d and constant and equal to the complex scalar α^*, satisfying the following hypotheses $\mathrm{Re}(\alpha^*) > 0$, $\mathrm{Im}(\alpha^*) < 0$ to produce absorption (see [28] for a more thorough description and justification). For a source $f \in L^2(\Omega)$, the radiation problem in presence of PMLs reads

(32)
$$\left\{\begin{array}{rclll} \alpha\dfrac{\partial}{\partial x}\left(\alpha\dfrac{\partial\varphi}{\partial x}\right) + \dfrac{\partial^2\varphi}{\partial y^2} - D_\alpha^2\varphi & = & f & \text{in} & \Omega_\alpha, \\ \partial\varphi/\partial y & = & (Y/ik)D_\alpha^2\varphi & \text{on} & \Gamma_\alpha = \{(x,1); |x| < d+L\}, \\ \partial\varphi/\partial y & = & 0 & \text{on} & \Gamma_\alpha^0 = \{(x,0); |x| < d+L\}, \\ \varphi & = & 0 & \text{on} & \Sigma_\pm, \end{array}\right.$$

where $\mathrm{D}_\alpha = M\alpha\partial/\partial x - ik$ and where the purpose of the Dirichlet condition on $\Sigma_\pm = \{(x, y); \pm x = d + L, 0 < y < 1\}$ is to select the outgoing solution. This problem has the equivalent variational form:

$$(33) \qquad \left\{\begin{array}{l} \text{Find } \varphi \in U = \{\varphi \in H^1_{\Sigma,0}(\Omega_\alpha) \text{ with } \partial\varphi/\partial x \in L^2(\Gamma_\alpha)\} \\ \text{ such that } \; a_\alpha(\varphi,\psi) = -\,(f/\alpha,\psi)_{L^2(\Omega_\alpha)} \; \text{ for all } \psi \in U, \end{array}\right.$$

where $H^1_{\Sigma,0}(\Omega_\alpha) = \{\varphi \in H^1(\Omega_\alpha),\ \varphi|_{\Sigma_\pm} = 0\}$ and where the sesquilinear form $a_\alpha(\varphi, \psi)$ is defined as:

$$\int_{\Omega_\alpha} \frac{1}{\alpha}\left(\boldsymbol{\nabla}_\alpha\varphi\cdot\boldsymbol{\nabla}_\alpha\bar{\psi} - \mathrm{D}_\alpha\varphi\,\overline{\mathrm{D}}_\alpha\bar{\psi}\right)\mathrm{d}x\mathrm{d}y + \frac{Y}{ik}\int_{\Gamma_\alpha}\frac{1}{\alpha}\left(\mathrm{D}_\alpha\varphi\,\overline{\mathrm{D}}_\alpha\bar{\psi}\right)\mathrm{d}x,$$

where $\overline{\mathrm{D}}_\alpha = M\alpha\partial/\partial x + ik$ and $\boldsymbol{\nabla}_\alpha = (\alpha\partial/\partial x, \partial/\partial y)$.

LEMMA 12. *Problem (33) is of Fredholm type.*

Proof. We show that $a_\alpha(\varphi, \psi)$ is the sum of a compact part and a coercive part. The proof of compactness is classic (remember that $\varphi|_{\Gamma_\alpha} \in H^1(\Gamma_\alpha)$) and the coerciveness is obtained by proving that $\exists C > 0$ such that $\forall \varphi \in U$ defined in (33):

$$|b_\alpha(\varphi)| \geq C \left(\int_{\Omega_\alpha} |\nabla \varphi|^2 \, \mathrm{d}x\mathrm{d}y + \int_{\Gamma_\alpha} |\partial\varphi/\partial x|^2 \, \mathrm{d}x \right),$$

with the non-compact part of $a_\alpha(\varphi, \psi)$ defined as

$$b_\alpha(\varphi) = \int_{\Omega_\alpha} \left[(1 - M^2)\alpha \left| \frac{\partial \varphi}{\partial x} \right|^2 + \frac{1}{\alpha} \left| \frac{\partial \varphi}{\partial y} \right|^2 \right] \mathrm{d}x\mathrm{d}y + \frac{Y}{ik} \int_{\Gamma_\alpha} M^2 \alpha \left| \frac{\partial \varphi}{\partial x} \right|^2 \mathrm{d}x. \tag{34}$$

It is true for any value of the admittance as soon as the numerical parameter α^* is chosen such that $-2\arg(\alpha^*) < \pi/2 + \arg Y$ (remember that $\arg(\alpha^*) < 0$). □

Note that this condition on the PML parameter α^* becomes hard to fulfill in the limit $\gamma = \arg Y \to -\pi/2$. It is consistent with the results for the dissipative problem set in an unbounded domain of the previous paragraphs: then it was hard to find a good dissipation to get a well-posed problem in the same limit.

To go further, we are not able to prove uniqueness of problem (33), which would imply well-posedness from Fredholm alternative. A classic approach to prove uniqueness in a waveguide is to perform a Fourier transform along x and then to use the completeness of the transverse modes of the guide. Here the transverse modes are easy to determine but the associated theoretical framework is not well suited to prove completeness. The difficulty is that the transverse modes satisfy a quadratic and not self-adjoint eigenvalue problem. Completeness is proven only in the no flow case: then the problem reduces to a linear eigenvalue problem, still not self-adjoint but at least symmetric. Then we recover the same transverse modes than when studying water waves propagation and, for a fixed k, excepting for a countable sequence of values of Y, the transverse modes have been proven [29, 30] to form a basis of $H^{1/2}(0, 1)$.

Although we don't know how to prove it, we postulate that problem (33) is well-posed outside a countable sequence of frequencies tending to infinity. This is typical in acoustic radiation problems [31] and it would explain why the problem (20) is well-posed only for a dissipation $\theta > 0$, preventing to consider the limit $\theta \to 0$: it is because the limit would not exist on a set of frequencies (even though this set is small, of zero measure).

3.3. Shear flow case. Now we study the general case of a varying flow. The effect of a mean shear flow on the acoustic perturbations has already been studied [32, 33] but with other tools. It has been done in the absence of source, thanks to a Fourier transform of the linearised Euler equations. Then compared to the uniform flow case, the novelty is that among the numerical solutions of the Pridmore-Brown [16] equation, unstable hydrodynamic modes (spatially exponentially growing) can appear [34]. Thanks to our variational approach, we can consider a radiation problem and thus study a realistic solution combining all the Pridmore-Brown modes together. The main novelty compared to the uniform flow case is that it will not be always possible to find a dissipation value for which the acoustic problem is well-posed. This is expected since enough dissipation must be introduced to attenuate a possible unstable mode.

3.3.1. Equations of the problem. We consider a shear flow $M(y)\boldsymbol{e}_x$ of regularity $M \in C^1([0, 1])$. We suppose also that $0 < M(y) < 1$, the case of a vanishing flow leading to specific difficulties hard to handle ($M \to 0$ is a singular limit, see (37)). When $M \neq$cst, (18) is not valid and we choose to use the Goldstein equations [35] because they are convenient since they are a direct extension of (18). The velocity has no longer a potential but reads $\boldsymbol{v} = \boldsymbol{\nabla}\varphi + \boldsymbol{\xi}$ associated to the pressure $p = -D_{k_\theta}\varphi$ with $D_{k_\theta} = M(y)\partial/\partial x - ik_\theta$. The acoustic unknown $\varphi \in V$ defined in (22) and the hydrodynamic unknown $\boldsymbol{\xi} \in W = \{\boldsymbol{\xi} \in (L^2(\Omega))^2, \partial\boldsymbol{\xi}/\partial x \in (L^2(\Omega))^2\}$ (such that $D_{k_\theta}\boldsymbol{\xi} \in (L^2(\Omega))^2$) satisfy the Goldstein equations: they are made of the following acoustic propagation equation for φ

$$(35) \qquad \left\{ \begin{array}{l} \operatorname{div}(\boldsymbol{\nabla}\varphi + \boldsymbol{\xi}) - D_{k_\theta}^2\varphi = f \quad \text{in} \quad \Omega, \\ \dfrac{\partial\varphi}{\partial y} + \xi_y = \dfrac{Y}{ik_\theta} D_{k_\theta}^2 \varphi \ \text{ on } \ \Gamma \ \text{ and } \ \dfrac{\partial\varphi}{\partial y} + \xi_y = 0 \ \text{ on } \ \Gamma_0, \end{array} \right.$$

(if $\boldsymbol{\xi} = \boldsymbol{0}$ we recover (18)) coupled to the transport equations for $\boldsymbol{\xi}$

$$(36) \qquad \left\{ \begin{array}{lcll} D_{k_\theta}\xi_x & = & -M'(y)\,(\partial\varphi/\partial y + \xi_y) & \text{in} \quad \Omega, \\ D_{k_\theta}\xi_y & = & M'(y)\partial\varphi/\partial x & \text{in} \quad \Omega. \end{array} \right.$$

The transport equations are solved explicitly and we prove the

LEMMA 13. *The $(L^2(\Omega))^2$ solution $\boldsymbol{\xi}(\varphi)$ of (36) is*

$$(37) \qquad \left\{ \begin{array}{lcl} \xi_y(x,y) & = & \dfrac{M'}{M} \displaystyle\int_{-\infty}^{x} e^{i\frac{k_\theta}{M}(x-s)} \dfrac{\partial\varphi}{\partial x}(s,y)\,\mathrm{d}s, \\ \xi_x(x,y) & = & -\dfrac{M'}{M} \displaystyle\int_{-\infty}^{x} e^{i\frac{k_\theta}{M}(x-s)} \left(\dfrac{\partial\varphi}{\partial y} + \xi_y\right)(s,y)\,\mathrm{d}s, \end{array} \right.$$

and satisfies for any $\tau > 0$ and any $\varphi \in H^1(\Omega)$:

$$(38) \qquad \left| \int_\Omega (\boldsymbol{\xi} \cdot \boldsymbol{\nabla}\overline{\varphi})\,\mathrm{d}x\mathrm{d}y \right| \leq \left(\frac{s_1}{c}\right)^2 \left\| \frac{\partial\varphi}{\partial x} \right\|_\Omega^2 + \left(\frac{s_1}{c}\right) \left(\tau \left\| \frac{\partial\varphi}{\partial x} \right\|_\Omega^2 + \frac{1}{\tau} \left\| \frac{\partial\varphi}{\partial y} \right\|_\Omega^2 \right),$$

where $s_1 = \sup_{y\in[0,1]} |M'(y)|$ is the maximum flow shear and $c = -\Im m(k_\theta) = k\sin(\theta)$ defined in (24).

Proof. The $(L^2(\Omega))^2$ solution $\boldsymbol{\xi}$ of (36) is obtained thanks to the causal Green function $G(x,y) = (H(x)/M)\exp(ik_\theta x/M(y))$ with H the Heaviside function. Then, since $\exp(ik_\theta x/M) \notin L^2(\Omega)$, the only L^2 solution is $\xi_y(x,y) = G(\cdot,y) * (M'(y)\,\partial\varphi/\partial x(\cdot,y))$ and using

$$\|G(\cdot,y) * \partial\varphi/\partial x(\cdot,y)\|_{L^2(\mathbb{R})} \leq \|G(\cdot,y)\|_{L^1(\mathbb{R})} \|\partial\varphi/\partial x(\cdot,y)\|_{L^2(\mathbb{R})} \quad \text{with} \quad \|G(\cdot,y)\|_{L^1(\mathbb{R})} = 1/c,$$

we get $\int_\mathbb{R} |\xi_y(s,y)|^2\,\mathrm{d}s \leq |M'/c|^2 \int_\mathbb{R} |\partial\varphi/\partial x(s,y)|^2\,\mathrm{d}s$. Using $s_1 = \sup_{y\in[0,1]}|M'(y)|$ we finally get

$$\left\{ \begin{array}{lcl} \|\xi_y\|_\Omega & \leq & (s_1/c)\,\|\partial\varphi/\partial x\|_\Omega, \\ \|\xi_x\|_\Omega & \leq & (s_1/c)\,\|\partial\varphi/\partial y + \xi_y\|_\Omega \leq (s_1/c)\,\|\partial\varphi/\partial y\|_\Omega + (s_1/c)^2\,\|\partial\varphi/\partial x\|_\Omega. \end{array} \right.$$

Eventually we deduce the upper bound

$$\begin{aligned} \left| \int_\Omega (\boldsymbol{\xi} \cdot \boldsymbol{\nabla}\overline{\varphi})\,\mathrm{d}x\mathrm{d}y \right| &\leq \|\xi_x\|_\Omega\,\|\partial\varphi/\partial x\|_\Omega + \|\xi_y\|_\Omega\,\|\partial\varphi/\partial y\|_\Omega, \\ &\leq (s_1/c)^2\,\|\partial\varphi/\partial x\|_\Omega^2 + 2\,(s_1/c)\,\|\partial\varphi/\partial x\|_\Omega\,\|\partial\varphi/\partial y\|_\Omega, \end{aligned}$$

and thus for any $\tau > 0$, thanks to the Young inequality:

$$\left| \int_\Omega (\boldsymbol{\xi} \cdot \boldsymbol{\nabla}\overline{\varphi})\,\mathrm{d}x\mathrm{d}y \right| \leq (s_1/c)^2\,\|\partial\varphi/\partial x\|_\Omega^2 + (s_1/c)\left(\tau\,\|\partial\varphi/\partial x\|_\Omega^2 + (1/\tau)\,\|\partial\varphi/\partial y\|_\Omega^2\right).$$

□

Thanks to the resolution of the transport equation, the scattering problem for the unknown φ alone can be derived:

LEMMA 14. *The variational formulation of* (35) *is: find* $\varphi \in V$ *defined in* (22) *such that* $\forall \psi \in V$

$$\tilde{a}_M(k_\theta; \varphi, \psi) = -\left(f, \psi\right)_{L^2(\Omega)}, \tag{39}$$

with

$$\tilde{a}_M(k_\theta; \varphi, \psi) = a_{M(y)}(k_\theta; \varphi, \psi) + \int_\Omega \boldsymbol{\xi}(\varphi) \cdot \boldsymbol{\nabla}\overline{\psi}\, \mathrm{d}x\mathrm{d}y.$$

$a_{M(y)}(k_\theta; \cdot, \cdot)$ *is* $a_M(k_\theta; \cdot, \cdot)$ *defined in* (21) *evaluated for a varying Mach profile* $M = M(y)$ *and* $\boldsymbol{\xi}(\varphi)$ *is defined in* (37).

3.3.2. Well-posedness conditions. Now we derive conditions under which (39) is well-posed. The main novelty in the shear case is that $\exists \theta_{\min} > 0$ such that the problem is well-posed only with enough dissipation: for $\theta_{\min} < \theta < \theta_c$ (for a uniform flow, $\theta_{\min} = 0$). The well-posedness conditions are given in the forthcoming theorem 16 which will be given later since several notations must be introduced before. To establish this theorem, we proceed as in the uniform case: we introduce

$$\tilde{A}_M(\varphi) = C_4' \left\|\partial\varphi/\partial x\right\|_\Omega^2 + C_1' \left\|\partial\varphi/\partial y\right\|_\Omega^2 + C_2' \left\|\varphi\right\|_\Omega^2 + C_3' \left\|\partial\varphi/\partial x\right\|_\Gamma^2, \tag{40}$$

and we will show in the proof of lemma 15 that it is a lower bound of $\Im m\left(\overline{\tilde{a}_M(k_\theta;\varphi,\varphi)/k_\theta}\right)$.
The constants are defined by

$$\begin{aligned} C_4' &= a' - \frac{1}{k}\left(\frac{s_1}{c}\right)^2 - \left(\frac{s_1}{c}\right)\frac{\tau}{k}, \quad & C_1' &= b' - \frac{g'}{\mu} \text{ with } b' = b - \left(\frac{s_1}{c}\right)\frac{1}{k\tau}, \\ C_2' &= c - g'(1+\mu), & C_3' &= d' - e'/\lambda, \end{aligned} \tag{41}$$

for all $\lambda, \mu, \tau > 0$, with b, c and f already defined in (24), with the new parameters

$$\begin{aligned} a' &= (1 - s_0^2)\sin(\theta)/k, \quad & d' &= |Y| i_0^2 \cos(\gamma - 2\theta)/k^2, \\ e' &= |Y| s_0 \cos(\gamma - \theta)/k, & g' &= e'\lambda - f, \end{aligned}$$

with the upper and lower bounds of the flow velocity

$$s_0 = \sup_{y\in[0,h]} |M(y)| \quad \text{and} \quad i_0 = \inf_{y\in[0,h]} |M(y)|,$$

and with the shear s_1 defined in lemma 13. $\tilde{A}_M(\varphi)$ is introduced because we have the new lemma similar to lemma 9:

LEMMA 15. *If* $\exists C > 0$ *such that* $\forall \varphi \in V$,

$$\tilde{A}_M(\varphi) \geq C\left(\left\|\boldsymbol{\nabla}\varphi\right\|_\Omega^2 + \left\|\varphi\right\|_\Omega^2 + \left\|\partial\varphi/\partial x\right\|_\Gamma^2\right),$$

then $\tilde{a}_M(k_\theta; \cdot, \cdot)$ *is coercive and consequently problem* (39) *is well posed.*

Proof. Starting from

$$\overline{\left(\frac{\tilde{a}_M(k_\theta;\varphi,\varphi)}{k_\theta}\right)} = \frac{1}{\overline{k_\theta}}\left\|\sqrt{1-M^2}\frac{\partial\varphi}{\partial x}\right\|_\Omega^2 + \frac{1}{\overline{k_\theta}}\left\|\frac{\partial\varphi}{\partial y}\right\|_\Omega^2 - \overline{k_\theta}\left\|\varphi\right\|_\Omega^2 - 2\,\Im m\left(M\varphi, \frac{\partial\varphi}{\partial x}\right)_\Omega,$$

$$+i\frac{\overline{Y}}{\overline{k_\theta}^2}\left\|M\frac{\partial\varphi}{\partial x}\right\|_\Gamma^2+i\frac{\overline{Y}}{\overline{k_\theta}}\,2\,\Im m\left(M\varphi,\frac{\partial\varphi}{\partial x}\right)_\Gamma+i\overline{Y}\,\|\varphi\|_\Gamma^2+\frac{1}{\overline{k_\theta}}\int_\Omega\overline{\boldsymbol{\xi}(\varphi)}\cdot\boldsymbol{\nabla}\varphi\,\mathrm{d}x\mathrm{d}y,$$

proceeding as for lemmas 9 and 11, we get for all $\lambda,\mu>0$ the lower bound

$$\Im m\left[\overline{\left(\frac{\tilde{a}_M(k_\theta;\varphi,\varphi)}{k_\theta}\right)}\right]\geq a'\left\|\frac{\partial\varphi}{\partial x}\right\|_\Omega^2+\left(b-\frac{g'}{\mu}\right)\left\|\frac{\partial\varphi}{\partial y}\right\|_\Omega^2,$$

$$+\left[c-g'(1+\mu)\right]\|\varphi\|_\Omega^2+\left(d'-\frac{e'}{\lambda}\right)\left\|\frac{\partial\varphi}{\partial x}\right\|_\Gamma^2-\frac{1}{k}\left|\int_\Omega\overline{\boldsymbol{\xi}(\varphi)}\cdot\boldsymbol{\nabla}\varphi\,\mathrm{d}x\mathrm{d}y\right|.$$

Note that this is the same inequality than in lemma 11 with two differences: the constants are now written with a prime since they have been extended from a uniform flow to a varying flow (M is replaced by i_0, s_0 or s_1) and the extra hydrodynamic unknown $\boldsymbol{\xi}$ is also involved.

Eventually, using (38) we can eliminate $\boldsymbol{\xi}$ and improve the lower bound: for all $\tau>0$,

$$\Im m\left[\overline{\left(\frac{\tilde{a}_M(k_\theta;\varphi,\varphi)}{k_\theta}\right)}\right]\geq\left[a'-\frac{1}{k}\left(\frac{s_1}{c}\right)^2-\left(\frac{s_1}{c}\right)\frac{\tau}{k}\right]\left\|\frac{\partial\varphi}{\partial x}\right\|_\Omega^2,$$

$$+\left[b-\frac{g'}{\mu}-\left(\frac{s_1}{c}\right)\frac{1}{k\tau}\right]\left\|\frac{\partial\varphi}{\partial y}\right\|_\Omega^2+\left[c-g'(1+\mu)\right]\|\varphi\|_\Omega^2+\left(d'-\frac{e'}{\lambda}\right)\left\|\frac{\partial\varphi}{\partial x}\right\|_\Gamma^2,$$

with the right hand side noted $\tilde{A}_M(\varphi)$ in (40). □

Now our aim is to write for a shear flow a theorem similar to theorem 8, providing the conditions on θ under which lemma 15 applies. Thus we want all the coefficients C_1', C_2', C_3' and C_4' in (40) to be strictly positive. Compared to the no-flow case, we have the extra parameter τ to adjust. First we choose $C_4'=0$ defined in (41) and thus we choose

$$\tau=\tau_0(\theta)\equiv(1-s_0^2)\frac{k\sin^2(\theta)}{s_1}-\frac{s_1}{k\sin(\theta)}=\frac{s_1}{k}\left(\frac{x^2}{\sigma^2}-\frac{1}{x}\right),$$

where we have noted $x=\sin(\theta)\in]0,1[$ and where we have introduced the new parameter

$$\sigma=\frac{s_1}{k\sqrt{1-s_0^2}},\tag{42}$$

which will be important in the following. We call it the instability parameter since it is linked to the possible existence of unstable solutions defined as solutions growing exponentially in time. More precisely we postulate that a velocity profile $M(y)$ can be unstable only if σ is large enough. It is true if $\sigma=0$ since then the velocity is uniform and thus stable. It is also true for a shear flow with a maximum velocity s_0 fixed: then in [36] is proven that a compressible velocity profile of fixed maximum velocity s_0 can allow the development of instabilities only if the profile has an inflexion point (as for an incompressible flow) and if s_1 is above a threshold.

We choose also $C_3'=0$ defined in (41) thus

$$\lambda=\lambda_0'\equiv\frac{e'}{d'}=k\frac{s_0}{i_0^2}\frac{\cos(\gamma-\theta)}{\cos(\gamma-2\theta)},$$

similar to (26), where M is replaced by i_0^2/s_0. To get $\lambda_0' > 0$, we restrict to $\theta < \theta_{\max}$ already defined in (27). The parameter g_0' defined by $g_0' = e'\lambda_0' - f$ is found to be

$$g_0'(\theta) = |Y| \left[\zeta \frac{\sin^2(\theta)}{\cos(\gamma - 2\theta)} + (\zeta - 1)\cos(\gamma) \right],$$

with $\zeta = (s_0/i_0)^2$ and is thus found positive. For $\zeta = 1$ is recovered g_0 defined in (28) for a uniform flow.

Eventually we have to determine conditions under which $C_1' > 0$ and $C_2' > 0$. $C_1' > 0$ requires at least $b' = b - (s_1/ck\tau_0) > 0$. τ_0 is an increasing function which vanishes at x_τ such that $x_\tau^3 = \sigma^2$. For $x_\tau < x < x_{\max}$ with

$$x_{\max} \equiv \sin(\theta_{\max}) = \sin[(2\gamma + \pi)/4],$$

we find

$$b'(\theta) = \frac{1}{k} \left(x - \frac{1}{\frac{x^3}{\sigma^2} - 1} \right).$$

b' is an increasing function vanishing at x_σ defined as the unique positive solution of

(43) $$\frac{x_\sigma^4}{x_\sigma + 1} = \sigma^2,$$

and b' is positive above the threshold $x_\sigma > x_\tau$. Then the remaining conditions to fulfill for $x > x_\sigma$ are $C_1' > 0$ and $C_2' > 0$ and these lead to a condition similar to the one for a uniform flow

(44) $$\frac{g_0'}{b'} < \mu < \frac{c}{g_0'} - 1,$$

with a positive left-hand side for $x_\sigma < x$ (we recall that μ must be positive). The existence of μ satisfying the condition (44) is equivalent to $P_\sigma < 0$ on $(x_\sigma, x_{\max})$ with

(45) $$P_\sigma(x) = P_0(x)Q_\sigma(x) + \frac{\sigma^2}{x^4} v(x)^2,$$

where

$$P_0(x) \equiv P_{\sigma=0}(x) = \tilde{P}_0(v(x)) = v(x)^2 + \frac{v(x)}{k} - 1,$$

with $\tilde{P}_0$ recalled here but already defined in (29) and v defined by

(46) $$v(x) = \frac{g_0'}{x} = |Y| \left[\zeta \frac{x}{\cos(\gamma - 2\theta(x))} + (\zeta - 1) \frac{\cos\gamma}{x} \right],$$

where $\theta(x) = \arcsin(x)$ and with

(47) $$Q_\sigma(x) = 1 - \sigma^2 \frac{x+1}{x^4}.$$

The sign of P_σ has to be determined on $x \in (x_\sigma, x_{\max})$. In the uniform flow case ($\sigma = 0$, $\zeta = 1$), P_σ in (45) reduces to R_0 defined by

(48) $$R_0(x) \equiv \tilde{P}_0(u(x)) = u(x)^2 + \frac{u(x)}{k} - 1,$$

where u has been defined in (30) versus θ, that we can also write versus x

$$u(x) = |Y|x/\cos(\gamma - 2\theta(x)). \quad (49)$$

Then we found that $R_0(x) < 0$ for $0 < x_c^0 < x_{\max}$ with x_c^0 the zero of R_0 which, from (31), is also defined as the unique solution in $(0, x_{\max})$ of

$$2u(x_c^0) = 2u_c \equiv -\frac{1}{k} + \sqrt{\frac{1}{k^2} + 4}. \quad (50)$$

For $\sigma \neq 0$, as for $R_0(x)$ we still have $\lim_{x \to x_{\max}} P_\sigma(x) = \infty$ since $v(x) \to \infty$ but the main difference is that although for $\sigma = 0$, $R_0(0)$ was negative, $P_\sigma(x_\sigma) = v(x_\sigma)^2\sigma^2/x_\sigma^4$ is positive as soon as $\sigma > 0$. Therefore the existence of negative values of $P_\sigma(x)$ is no longer guaranteed when $\sigma \neq 0$. Since $Q_\sigma > 0$ for $x > x_\sigma$ from (47), the existence of x such that $P_\sigma(x) < 0$ requires at least $P_0(x) < 0$ from (45), thus $v(x) < u_c$ defined in (31). Contrary to the behavior of u, $x \to v(x)$ is not an increasing function. Indeed from (46) is obtained that

$$v(x) = \zeta u(x) + (\zeta - 1)\frac{|Y|\cos\gamma}{x}, \quad (51)$$

with $x \to u(x)$ an increasing function but $x \to |Y|\cos(\gamma)/x$ is a decreasing function. Therefore the solutions of the inequality $v(x) < u_c$ are not easy to characterize. The only easy result is that since $u(x) < u_c$ for $x < x_c^0$ defined in (50) and since $v \geq u$ from (51), $v(x) < u_c$ implies that $x < x_c^0$.

Thanks to these notations, we can write the following theorem generalizing theorem 8 to a varying flow:

THEOREM 16. *For all admittance $Y = |Y|e^{i\gamma}$ defined in (23) and all instability parameter σ defined in (42), we define the set I_σ by*

$$I_\sigma = \{x \in (x_\sigma, x_c^0), P_\sigma(x) < 0\}, \quad (52)$$

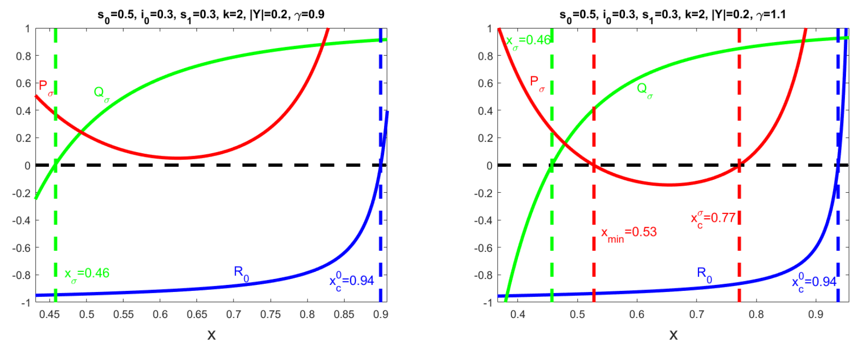
with x_σ defined in (43), x_c^0 defined in (50) and with the convention $I_\sigma = \emptyset$ if $x_c^0 \leq x_\sigma$. If I_σ is not empty, then the problem (39) is well posed for all dissipation associated to the angle θ defined in (19) such that $\sin(\theta) \in I_\sigma$.

REMARK 6. *Note that in the uniform flow case, the problem was well-posed as soon as $\theta > 0$. In the shear flow case, we need to introduce enough dissipation ($\theta > \theta_{min}$ with $\sin\theta_{min} = x_\sigma$) to expect to get the well-posedness of (39).*

Note also that the existence of θ_{min} is not a strong constraint since it is easy to get θ_{min} small: $\theta_{min} \to 0$ if $\sigma \to 0$ from (43), thus for a small shear s_1 and/or k large. As already mentioned, $\sigma \to 0$ is expected to imply the existence of no instability and thus no need to introduce a strong dissipation.

Proof of theorem 16.

I_σ has been defined previously to impose the conditions $C_3' = 0 = C_4'$. But as for the uniform case in the proof of theorem 8, we show now that it is easy to get C_3' and C_4' strictly positive for a set as close as we want to I_σ. We take $\lambda_\varepsilon = (e'/d) + \varepsilon$ such that $C_3' > 0$ for all $\varepsilon > 0$ from (41) and we take $\tau_\eta = \tau - \eta$ such that $C_4' > 0$ for any $\eta > 0$ from (41). The constants C_1' and C_2' depend continuously on ε and η. It is straightforward to check that the conditions $C_1'(\varepsilon, \eta) > 0$ and $C_2'(\varepsilon, \eta) > 0$ lead to a slight perturbation of (44) and thus to a set $I_\sigma^{\varepsilon,\eta} \subset I_\sigma$ with $I_\sigma^{\varepsilon,\eta} \to I_\sigma$ when



(a) (b)

Fig. 2. *$P_\sigma(x)$ defined in (45) in red, $Q_\sigma(x)$ defined in (47) in green and $R_0(x)$ defined in (48) in blue for $s_0 = 0.5$, $i_0 = 0.3$, $s_1 = 0.3$, $k = 2$ and (a): $Y = 0.2\, e^{i0.9}$, (b): $Y = 0.2\, e^{i1.1}$.*

$(\varepsilon, \eta) \to 0$. Then the theorem is a consequence of lemma 15 since all the constants in (40) are strictly positive. □

Let us analyze the theorem 16. We don't have general criteria for the existence of a non-empty I_σ set ensuring the well-posedness of the problem (39) but we have global tendencies given by the

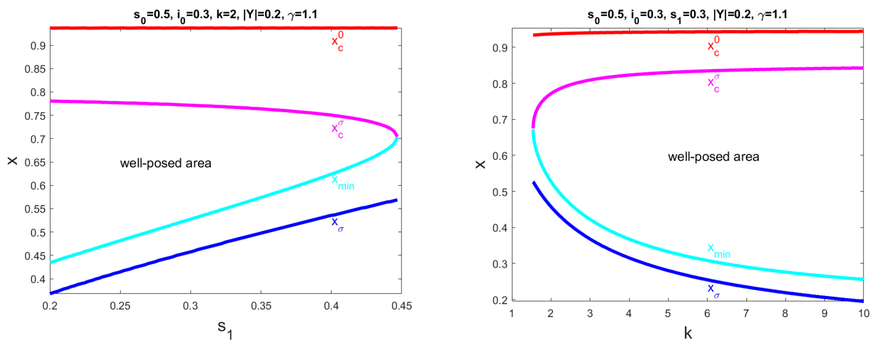
Lemma 17. *The set I_σ defined in (52) is empty if at least one of the following condition is fulfilled: σ is large or k is small or $|Y|$ is large.*

Remark 7. *In other words, to get a non-empty set I_σ, necessary conditions are: σ small enough and k large enough and $|Y|$ small enough. This will be confirmed by the forthcoming numerical illustrations*

Note that the condition on the instability parameter σ was expected since the velocity profile is expected to be stable for σ small enough.

Proof. The key point is that I_σ is empty if $x_\sigma \geq x_c^0$. From (43), x_σ is found to be an increasing function of σ and tends to infinity when $\sigma \to \infty$. Moreover from (49) and (50), we deduce that $x_c^0(k, Y)$ decreases when k decreases or $|Y|$ increases and $x_c^0(k, Y) \to 0$ when $k \to 0$ or $|Y| \to \infty$. Therefore $x_\sigma \geq x_c^0$ is necessarily satisfied if σ too large or k too small or $|Y|$ too large. □

3.3.3. Numerical illustrations. Now we illustrate numerically on some examples the theoretical derived bounds for the parameters given in lemma 17 for the well-posedness of problem (39). In all the tested situations, when I_σ exists it has been found as a one-piece interval, of the form $I_\sigma = (x_{\min}, x_c^\sigma) \subset (x_\sigma, x_c^0)$, with $x_{\min}$ and x_c^σ the two zeros of P_σ. The upper zero $x_c^\sigma < x_c^0$ is the generalization of x_c^0 in the sense: $x_c^\sigma \to x_c^0$ when $\sigma \to 0$. We illustrate now numerically this empirical relation $I_\sigma = (x_{\min}, x_c^\sigma)$. In Fig. 2(a) and Fig. 2(b) we explain how we determine $x_{\min}$ and x_c^σ and in this aim we represent the variations of $P_\sigma(x)$ defined in (45) for the flow parameters $s_0 = 0.5$, $i_0 = 0.3$, $s_1 = 0.3$, the frequency $k = 2$ and the admittance $|Y| = 0.2$ with two values of the admittance argument γ. For the argument $\gamma = 0.9$ in Fig. 2(a), we look for zeros of P_σ on (x_σ, x_c^0) where the interval boundaries are respectively the zeros of Q_σ defined in (47) and of R_0 defined in (48). $Q_\sigma(x)$ is plotted in green and vanishes at $x_\sigma = 0.46$ represented as a green vertical dashed line. We plot also $R_0(x)$ in blue which vanishes at $x_c^0 = 0.94$ represented as a blue vertical



(a) (b)

FIG. 3. $x_\sigma \leq x_{min} \leq x_c^\sigma \leq x_c^0$ *for* $s_0 = 0.5$, $i_0 = 0.3$, $Y = 0.2\, e^{i1.1}$*: (a) versus* s_1 *for* $k = 2$*, (b) versus* k *for* $s_1 = 0.3$

dashed line. $P_\sigma(x)$ is found to never vanish, leading to an empty I_σ set.

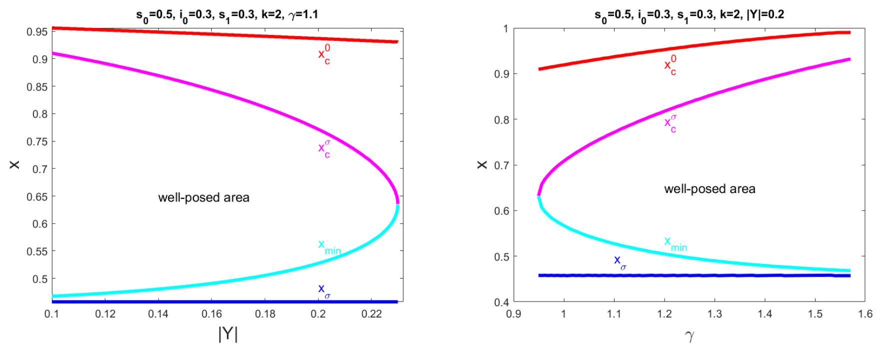
Fig. 2(b) corresponds to the same parameters but for the argument $\gamma = 1.1$. P_σ is found to vanish two times and noting $x_{\min} = 0.53$ and $x_c^\sigma = 0.77$ the lower and upper zeros of P_σ, represented as red vertical dashed lines, we find $I_\sigma = (x_{\min}, x_c^\sigma)$ as a one-piece set.

Now we extend the numerical illustrations and we consider the influence of the flow parameters and of the acoustics parameters. We did not find general laws for the existence of non-empty I_σ but we have checked numerically that the general tendencies given by lemma 17 are relevant. Let us recall that necessary conditions for the existence of a non-empty I_σ are: σ small enough and k large enough and $|Y|$ small enough (the influence of γ is not easy to characterize theoretically). We illustrate now numerically these tendencies and in the following figures, we characterize the influence of the parameters σ, k, $|Y|$ and $\gamma = \arg Y$.

Fig. 3(a) studies the influence of the instability parameter σ for the parameters of the flow $s_0 = 0.5$, $i_0 = 0.3$ and for $k = 2$, $Y = 0.2\, e^{i1.1}$. σ defined in (42) is changed by varying the flow-shear s_1. We plot the four functions $x_\sigma \leq x_{\min} \leq x_c^\sigma \leq x_c^0$ versus s_1. x_c^0 is constant from (50) with $x_c^0 = 0.90$. There are two conclusions. First and as already stated, we find that when I_σ exists, it is a one-piece set of the form $I_\sigma = (x_{\min}, x_c^\sigma)$. Then and as expected, it is found that I_σ exists only for σ small enough, $s_1 < 0.45$, when the flow is more likely to be stable. The problem (39) has been proven to be well-posed if $x_{\min} < x < x_c^\sigma$: this defines a "well-posed area" as indicated on Fig. 3(a) such that if (s_1, x) is chosen in this area, then problem (39) is well-posed. We recall that $x = \sin\theta$ with θ measuring the dissipation.

For the three coming illustrations, the parameters of the flow are fixed: $s_0 = 0.5$, $i_0 = 0.3$ and $s_1 = 0.3$. Fig. 3(b) studies the influence of the frequency k for $Y = 0.2\, e^{i1.1}$. As expected, I_σ exists only for k large enough, $k > 1.54$. Fig. 4(a) studies the influence of $|Y|$ for $k = 2$ and $\gamma = 1.1$ and as expected, I_σ exists only for $|Y|$ small enough, $|Y| < 0.23$.

We finish with the influence of the argument γ of the admittance, for which we don't have general tendencies. It is illustrated in Fig. 4(b) for $k = 2$ and $|Y| = 0.2$. From (42), σ and thus x_σ are constant, $x_\sigma = 0.46$. The set $I_\sigma = (x_{\min}, x_c^\sigma) \subset (x_\sigma, x_c^0)$ is found to exist only for $\gamma > 0.95$.



(a) (b)

FIG. 4. *$x_\sigma \le x_{min} \le x_c^\sigma \le x_c^0$ for $s_0 = 0.5$, $i_0 = 0.3$, $s_1 = 0.3$, $k = 2$, (a) versus $|Y|$ for $\gamma = 1.1$, (b) versus γ for $|Y| = 0.2$*

3.3.4. Case without dissipation. As in the uniform flow case, without dissipation we don't know how to prove well-posedness of (39) but we can prove that the problem is Fredholm. The problem with a shear flow (35) and (36), extended to the presence of PMLs is

(53)
$$\begin{cases} \alpha\dfrac{\partial}{\partial x}\left(\alpha\dfrac{\partial\varphi}{\partial x}+\xi_x\right)+\dfrac{\partial}{\partial y}\left(\dfrac{\partial\varphi}{\partial y}+\xi_y\right) & - \quad D_\alpha^2\varphi = f \qquad \text{in} \quad \Omega_\alpha, \\ \left(M\alpha\dfrac{\partial}{\partial x}-ik\right)\xi_x & = \quad -M'(y)\left(\dfrac{\partial\varphi}{\partial y}+\xi_y\right) \quad \text{in} \quad \Omega_\alpha, \\ \left(M\alpha\dfrac{\partial}{\partial x}-ik_\theta\right)\xi_y & = \quad M'(y)\alpha\dfrac{\partial\varphi}{\partial x} \qquad \text{in} \quad \Omega_\alpha, \\ ik\partial\varphi/\partial y = YD_\alpha^2\varphi \ \text{ on } \Gamma_\alpha & \text{and} \quad \partial\varphi/\partial y = 0 \ \text{ on } \Gamma_\alpha^0, \\ \varphi = 0 \ \text{ on } \Sigma_\pm & \text{and} \quad \boldsymbol{\xi} = 0 \ \text{ on } \Sigma_-, \end{cases}$$

where Γ_α, Γ_α^0 and $\Sigma_\pm$ are defined in (32). As in the absorbing case, $\boldsymbol{\xi}$ is explicitly determined to get a problem depending only on φ. $\boldsymbol{\xi} \in (L^2(\Omega_\alpha))^2$ is given by an expression similar to (37) but extended to the presence of PMLs. As for (37), $\boldsymbol{\xi}$ is found proportional to M' and thus $||\boldsymbol{\xi}||_{L^2}$ is bounded by the flow shear s_1. This will be important in the final estimate of the forthcoming proof.

The variational form of (53) is the same than in the uniform flow case (33) where the sesquilinear form $a_\alpha(\varphi,\psi)$ is replaced by:

$$\tilde{a}_\alpha(\varphi,\psi) = a_\alpha(\varphi,\psi) + \int_{\Omega_\alpha} \frac{1}{\alpha}\boldsymbol{\xi}(\varphi)\cdot\boldsymbol{\nabla}_\alpha\bar{\psi}\,\mathrm{d}x\mathrm{d}y.$$

LEMMA 18. *For a flow shear s_1 small enough, problem (53) is of Fredholm type.*

Proof. We show that $\tilde{a}_\alpha(\varphi,\psi)$ is the sum of a compact part and a coercive part. As in the uniform flow case the proof of compactness is classical and coerciveness is obtained by proving that $\exists C > 0$ such that $\forall\varphi \in U$ defined in (33):

(54) $$\left|b_\alpha(\varphi) + \int_{\Omega_\alpha}\frac{1}{\alpha}\boldsymbol{\xi}(\varphi)\cdot\boldsymbol{\nabla}_\alpha\bar{\varphi}\,\mathrm{d}x\mathrm{d}y\right| \ge C\left(\int_{\Omega_\alpha}|\boldsymbol{\nabla}\varphi|^2\,\mathrm{d}x\mathrm{d}y + \int_{\Gamma_\alpha}\left|\frac{\partial\varphi}{\partial x}\right|^2\mathrm{d}x\right).$$

The term $b_\alpha(\varphi)$ defined in (34) was already involved in the uniform flow case and we already know from the proof of lemma 12 that under the condition $-2\arg(\alpha^*) < \pi/2 + \arg Y$, $\exists C^0 > 0$ such that $\forall \varphi \in U$:

$$|b_\alpha(\varphi)| \geq C^0 \left(\int_{\Omega_\alpha} |\boldsymbol{\nabla}\varphi|^2 \, \mathrm{d}x\mathrm{d}y + \int_{\Gamma_\alpha} \left|\frac{\partial \varphi}{\partial x}\right|^2 \mathrm{d}x \right).$$

Therefore we just need to find an upper bound for $\left|\int_{\Omega_\alpha} (1/\alpha)\, \boldsymbol{\xi}(\varphi) \cdot \boldsymbol{\nabla}_\alpha \bar{\varphi} \, \mathrm{d}x\mathrm{d}y\right|$. From the explicit expression of $\boldsymbol{\xi}(\varphi)$ is deduced a constant $C_\alpha > 0$ (depending on the geometry and on the flow parameters) such that

$$\left| \int_{\Omega_\alpha} \left(\xi_x \frac{\partial \overline{\varphi}}{\partial x} + \xi_y \frac{1}{\alpha} \frac{\partial \overline{\varphi}}{\partial y} \right) \mathrm{d}x\mathrm{d}y \right| \leq C_\alpha \int_{\Omega_\alpha} |\boldsymbol{\nabla}\varphi|^2 \, \mathrm{d}x\mathrm{d}y,$$

and finally is deduced $C = C^0 - C_\alpha$ in (54). Coerciveness is obtained when $C > 0$. As in (38), C_α is proportional to the shear s_1 and thus C_α is small when s_1 is small. Therefore the problem is Fredholm for a flow shear s_1 small enough: this condition of σ small was already involved when considering the problem without PMLs but with dissipation. □

4. Conclusion. Thanks to variational methods, we have studied the well-posedness of the time-harmonic radiation in a waveguide with a Myers absorbing boundary condition on a boundary. The main tendencies are the followings. Without flow, the radiation problem is always well-posed. In presence of a uniform flow, it is proven to be always of Fredholm-type and well-posed as soon as just a little dissipation is introduced. For a varying flow, the problem is Fredholm for a shear weak enough and the well-posedness requires at least the introduction of enough dissipation, still with moderate values of the flow shear.

To go further, let us mention that in the literature some progresses have been made in the time domain to correct the illposedness induced by a uniform flow over an impedance lining. Modifications to the Myers boundary condition have been suggested, by incorporating a thin-but-nonzero thickness boundary layer over the lining, leading to various so called modified Myers boundary conditions [34, 37, 38, 39]. These boundary conditions remove the illposedness while still retaining the simplicity of a uniform flow, with the thin boundary layer being incorporated within the boundary condition. Moreover they match well [38] with solutions to the full linearised Euler equations [40]. The extensions of the modified Myers boundary conditions to the time-harmonic regime and their inclusion in our study would be interesting to increase the domain of well-posedness of the considered radiation problem, but such extensions are not straightforward since these conditions have complicated expression preventing them from fitting naturally into a variational formulation, contrary to the classical Myers condition.

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