

Weak Dirichlet processes and generalized martingale problems

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Abstract

In this paper we explain how the notion of *weak Dirichlet process* is the suitable generalization of the one of semimartingale with jumps. For such a process we provide a unique decomposition which is new also for semimartingales: in particular we introduce *characteristics* for weak Dirichlet processes. We also introduce a weak concept (in law) of finite quadratic variation. We investigate a set of new useful chain rules and we discuss a general framework of (possibly path-dependent with jumps) martingale problems with a set of examples of SDEs with jumps driven by a distributional drift.

Key words: Weak Dirichlet processes; càdlàg semimartingales; jump processes; martingale problem; singular drift; random measure.

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1 Introduction

The central notion of this work is the one of *weak Dirichlet process* with jumps and the related *martingale problem*. In this work we want in particular to convince the reader that the concept of weak Dirichlet process plays a similar central role as the one of semimartingale. An (\mathcal{F}_t) -weak Dirichlet process X is the sum of an (\mathcal{F}_t) -local martingale M and an (\mathcal{F}_t) -martingale orthogonal process Γ , generally fixed to vanish at zero. When self-explanatory, the filtration will be omitted. A *martingale orthogonal process* A has the property that $[A, N] = 0$ for every continuous martingale N . In particular a purely discontinuous martingale is a martingale orthogonal process. A substitutes the usual bounded variation component V when X is a semimartingale. As a matter of fact, any bounded variation process is (\mathcal{F}_t) -martingale orthogonal, see Proposition 2.14 in [4].

When X is a continuous process, the notion of weak Dirichlet process was introduced in [14] and largely investigated in [24]. In [4], we extended the concept of weak Dirichlet jump process with related calculus. In particular, generalizing the notion of special semimartingale, we introduced the one of special weak Dirichlet process $X = M + \Gamma$, where M is a possibly (càdlàg) local martingale and Γ is a predictable; in that case the decomposition $X = M + \Gamma$ is unique. In fact this was earlier introduced by [10] and partially studied, omitting the mention "special".

An important feature of the calculus beyond semimartingales is the one of stability, which often constitutes a generalization of Itô formula. If X is a càdlàg semimartingale and $f \in C^2(\mathbb{R})$, we recall that $f(X)$ is again a semimartingale by a direct application of Itô formula. However, if f is only of class $C^1(\mathbb{R})$, $f(X)$ is generally no more a semimartingale, nevertheless it remains a finite quadratic variation process, see [35] when X is a continuous process. For instance, if X is a Brownian motion and f is not the difference of convex functions, then it is well-known that $f(X)$

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is not a semimartingale. On the other hand, if f is bounded and of class C^1 and X is a Poisson process, then $f(X)$ is a special semimartingale.

A càdlàg process X will be called *finite quadratic variation process* (see [34, 4]) whenever the u.c.p. limit (which will be denoted by $[X, X]$) of $[X, X]_\varepsilon^{ucp}$ exists, where

$$[X, Y]_\varepsilon^{ucp}(t) := \int_{]0, t]} \frac{(X((s + \varepsilon) \wedge t) - X(s))(Y((s + \varepsilon) \wedge t) - Y(s))}{\varepsilon} ds. \quad (1.1)$$

By Lemma 2.10 in [4], we know that

$$[X, X] = [X, X]^c + \sum_{s \leq \cdot} |\Delta X_s|^2, \quad (1.2)$$

where $[X, X]^c$ is the continuous component. The covariation of two càdlàg processes X and Y was defined in [34] as the u.c.p. limit of $[X, Y]_\varepsilon^{ucp}$, whenever it exists.

The notion of quadratic variation for a non-semimartingale X was introduced in [19] by means of discretizations (instead of regularizations, that we denote here by $[X]$). One also proved that if $f \in C^1(\mathbb{R})$ and $[X]$ exists, then $[f(X)]$ also exists. A natural extension of the notion of the semimartingale is the one of (\mathcal{F}_t) -Dirichlet process introduced in [20] still in the discretization approach, which is the (unique) sum of an (\mathcal{F}_t) -local martingale and a zero quadratic variation process (vanishing at zero). For us, an (\mathcal{F}_t) -Dirichlet process will be the analogous concept in the regularization approach. Let X be such a Dirichlet process, which is obviously in particular an (\mathcal{F}_t) - (even special) weak Dirichlet process. Let $X = M + \Gamma$ its unique decomposition. Observe that the notion of Dirichlet process does not naturally fit the jump case; indeed $[\Gamma, \Gamma] = 0$ implies that Γ is continuous by (1.2), therefore predictable. On the other hand, since X is a finite quadratic variation process, if $f \in C^1(\mathbb{R})$ then $f(X)$ is also a finite quadratic variation process but it is not necessarily a Dirichlet process, see [3].

For applications (for instance to control problems and BSDEs theory, see e.g. [23, 5, 22]), it is useful to investigate stability for functions $f \in C^{0,1}([0, T] \times \mathbb{R}; \mathbb{R})$. Let $f \in C^{0,1}([0, T] \times \mathbb{R}; \mathbb{R})$ and X be a finite quadratic variation process. In general we cannot expect that $f(t, X_t)$ to be a finite quadratic variation process: for instance a very irregular function f not depending on x may not be of finite quadratic variation.

If X is a continuous weak Dirichlet process with finite quadratic variation, it is known that $f(t, X_t)$ is a weak Dirichlet process, see Proposition 3.10 in [24]. This stability result was extended to the case where X is a discontinuous process, provided a specific relation between the jumps of X and f , see [4]. Under these conditions, $f(t, X_t)$ is a special weak Dirichlet process. Recently, an interesting generalization in the continuous framework is provided in [8], where f is a $C^{0,1}$ -path-dependent functional in the sense of horizontal-vertical Dupire derivative.

As mentioned earlier, the second central notion of the present paper is the one of martingale problem, whose classical notion is due to Stroock and Varadhan, see e.g. [36]. In general, one says that a process X is a solution to the martingale problem with respect to a probability \mathbb{P} (on some probability space), to some domain \mathcal{D} and to a time-indexed family of operators L_t if

$$f(X_t) - f(X_0) - \int_0^t L_s f(X_s) ds,$$

is a \mathbb{P} -local martingale. One also says that (X, \mathbb{P}) is a solution to the martingale problem related to \mathcal{D} and L_t . In the classical martingale problem in [36] one takes $\mathcal{D} = C^2(\mathbb{R}^d)$ and L_t is a second order PDE operator. One also knows that the solution of Stroock-Varadhan martingale problem is equivalent to the one of an SDE in law (or weak). In more singular situations, the notion of SDE seems difficult to exploit and define, and in that case it is substituted by the more flexible

notion of martingale problem. This is the case for instance when one investigates the notion of *SDEs with distributional drift*, i.e. of the type

$$X_t = X_0 + \int_0^t \sigma(X_s) dW_s + \int_0^t b(X_s) ds,$$

and b is a Schwartz distribution. In this case it is more comfortable to express those processes as solutions to a martingale problem with respect to some domain \mathcal{D} which is a suitable subset of $C^1(\mathbb{R})$, where the formal map $L_t f(x) := \frac{1}{2} \sigma^2(x) f''(x) + b(x) f'(x)$ is well-defined (independently of t). We remark that this is not the case in general for $f \in C_0^\infty(\mathbb{R})$. This was the object of [17, 18, 33]. Later extensions to the multi-dimensional case were performed, see [16, 12, 9]. Generalizations to the path-dependent case were done by [30, 31]. In all these cases the solutions are constituted by continuous processes which are not necessarily semimartingales. In the literature of jump processes the notion of martingale problem has followed two different routes. The first one is a new formulation of martingale problem given in [27] where the formulation makes essentially use of the notion of characteristics. This approach is particularly natural in the purely discontinuous framework, see e.g. [2]. The second one continues the Stroock-Varadhan approach, see e.g. [29, 6].

We describe now the main contributions of the paper.

1. First we formulate a unique decomposition for a weak Dirichlet process $X = X^c + A$, where X^c is a continuous local martingale and A a martingale orthogonal process vanishing at zero, see Proposition 3.2. Until now unique decompositions of weak Dirichlet processes were established when X has the special weak Dirichlet property, in particular if X is a special semimartingale. Even when $X = M + V$ is a general càdlàg semimartingale, no unique decomposition was available because the property of V to have bounded variation was not enough to determine it. We recall that often purely discontinuous martingales have bounded variation.
2. Let X be a càdlàg process satisfying

$$\sum_{s \leq \cdot} |\Delta X_s|^2 < \infty \quad \text{a.s.} \tag{1.3}$$

Notice that condition (1.3) is equivalent to ask that $(1 \wedge |x|^2) \star \mu^X \in \mathcal{A}_{\text{loc}}^+$ (see Proposition C.1), where μ^X is the jump measure related to X defined in (2.1).

In Corollary 3.18 we prove that if X is a weak Dirichlet process then it is a special weak Dirichlet process if and only if

$$x \mathbb{1}_{\{|x|>1\}} \star \mu^X \in \mathcal{A}_{\text{loc}}. \tag{1.4}$$

This result in particular extends the classical characterization of a special semimartingale, see Proposition 2.29, Chapter II, in [27]. We recall that if X is a special weak Dirichlet process and a semimartingale, then it is a special semimartingale, see Proposition 5.14 in [4].

More generally, let $v : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$ continuous. In Theorem 3.16 we give a necessary and sufficient condition on the weak Dirichlet process $Y_t = v(t, X_t)$ to be a special weak Dirichlet process, namely

$$(v(s, X_{s-} + x) - v(s, X_{s-})) \mathbb{1}_{\{|x|>1\}} \star \mu^X \in \mathcal{A}_{\text{loc}}.$$

Notice that in the literature only sufficient conditions were available for Y to be a special weak Dirichlet process, see for instance Theorem 5.31 in [4].

3. In Theorem 3.10 we provide a first chain rule expanding a process $v(s, X_s)$, where v has no regularity at all, that extends a similar chain rule established in [4] but only for purely jump processes. Indeed, under the conditions (3.7) and (3.8), if $v(t, X_t)$ is an \mathbb{F} -weak Dirichlet process with unique continuous martingale component Y^c , then we have

$$Y = Y^c + (v(s, X_{s-} + x) - v(s, X_{s-})) \frac{k(x)}{x} \star (\mu^X - \nu^X) + \Gamma^k(v) + (v(s, X_{s-} + x) - v(s, X_{s-})) \frac{x - k(x)}{x} \star \mu^X, \quad (1.5)$$

with $\Gamma^k(v)$ a predictable and martingale orthogonal process. Our result constitutes an important tool to solve the identification problem for a BSDE driven by a random measure and a Brownian motion, see Remark 3.11.

4. We relax the notion of finite quadratic variation process, by giving the notion of weakly finite quadratic variation, see Definition 3.29. This is a notion which is more related to the convergence in law of subsequences. The u.c.p convergence of (1.1) is replaced by the fact that, for every $T > 0$, $[X, X]_{0 < \varepsilon \leq \varepsilon_0}^{ucp}(T)$ are tight for $\varepsilon_0 > 0$ small enough. A classical example of weakly finite quadratic variation process comes up when

$$\sup_{0 < \varepsilon \leq \varepsilon_0} [X, X]_{\varepsilon}^{ucp}(T) < \infty \text{ a.s.,}$$

see Remark 3.32-(i). Another example is given when X has finite energy, see Remark 3.32-(ii). It is not difficult to exhibit a process with finite energy which has no quadratic variation, see Example 3.33. Notice that condition (1.3) holds if X is a finite quadratic variation process, and it is also valid under the weaker condition of X being a weak finite quadratic variation process, see Proposition 3.34.

5. Let $v \in C^{0,1}(\mathbb{R}_+ \times \mathbb{R})$. If X is a weakly finite quadratic variation process, then Theorem 3.36 states that the process $Y = v(\cdot, X)$ is a weak Dirichlet process, and identifies its unique continuous local martingale component Y^c of Y , as

$$Y^c = Y_0 + \int_0^\cdot \partial_x v(s, X_s) dX_s^c.$$

The combined results in Theorems 3.10 and 3.36 constitute indeed a $C^{0,1}$ -type chain rule for weak Dirichlet processes which generalizes Theorem 5.15 in [4]. As far as special weak Dirichlet processes are concerned, the corresponding $C^{0,1}$ -type chain rule is given in Corollary 3.37 generalizing Theorem 5.31 in [4], see also Remark 3.38. Notice that in [4] X was also supposed to be a finite quadratic variation process.

6. We introduce the notion of *characteristics* (B^k, C, ν) for weak Dirichlet processes (see Definition 3.25), that extends the corresponding one for semimartingales. We remark that, if X is a weak Dirichlet process, then $\Gamma^k(Id) = B^k \circ X$ with $\Gamma^k(v)$ defined in (1.5), see Corollary 3.22 and Remark 3.23. Given the characteristics (B^k, C, ν) of a weak Dirichlet process X , a natural question is to determine the characteristics of a process $h(\cdot, X)$, where $h \in C^{0,1}$. In fact, it is possible to provide the second and third characteristic of $h(\cdot, X)$ in terms of C and ν , see Remark 3.42, while it is a challenging problem to evaluate the first characteristic. Nevertheless, we are able to solve this problem in the case when h is bijective and time-homogeneous, see Remark 3.43.

7. We introduce a notion of martingale problem, which applies in a general framework including possibly non-Markovian jumps processes and non semimartingales, by generalizing the classical Stroock-Varadhan martingale problem with respect to some domain $\mathcal{D}_{\mathcal{A}} \subseteq C^{0,1}$ (replacing \mathcal{D}) and operator \mathcal{A} (replacing $\partial_t + L_t$), see Definition 4.12. Moreover, here the Lebesgue measure dt can be substituted by some random kernel.

Let X be a càdlàg weakly finite quadratic variation process. The fact that X is a solution to some martingale problem in the sense of Definition 4.12 does not imply that it is a semimartingale (indeed, we are in particular interested in the case where X is not a semimartingale). Among others, if X is a solution of a martingale problem with respect to $\mathcal{D}_{\mathcal{A}}$ and \mathcal{A} , with $\mathcal{D}_{\mathcal{A}}$ dense in $C^{0,1}$, we even do not know if X is a weak Dirichlet process. Corollary 4.21 provides some necessary and sufficient condition conditions under which X is weak Dirichlet. To get those conditions, Theorem 4.3 together with Proposition 4.5 give some crucial preparatory stochastic calculus tools.

8. Section 4.3 relates our inhomogeneous formulation of martingale problem with the (more classical) time-homogeneous expression. Knowing that a process X solves some martingale problem (in the time-homogeneous sense), the fact that it also solves a non-homogeneous martingale problem corresponds to some general chain rule.
9. In Section 4.5 we discuss five classes of examples of martingale problems. The first two are respectively the case of general semimartingales and the case where there is a bijective function h in $C^{0,1}$ such that $h(t, X_t)$ is a semimartingale. The third one concerns discontinuous processes solving martingale problems with distributional drift. For this, existence and uniqueness is discussed systematically in the companion paper [3]. The fourth one is about continuous path-dependent problems involving distributional drifts. The latter one is about the martingale problem solved by a piecewise deterministic Markov process.

2 Preliminaries and notations

In the sequel we will consider the space of functions $u : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$, $(t, x) \mapsto u(t, x)$, which are of class $C^{0,1}$ or $C^{1,2}$. $C_b^{0,1}$ (resp. $C_b^{1,2}$) stands for the class of bounded functions which belong to $C^{0,1}$ (resp. $C^{1,2}$). $C^{0,1}$ is equipped with the topology of uniform convergence on each compact of u and $\partial_x u$. C^0 (resp. C_b^0) will denote the space of continuous functions (resp. continuous and bounded functions) on \mathbb{R} equipped with the topology of uniform convergence on each compact (resp. equipped with the topology of uniform convergence). C^1 (resp. C^2) will be the space of continuously differentiable (twice continuously differentiable) functions $u : \mathbb{R} \rightarrow \mathbb{R}$. C_b^1 (resp. C_b^2) stands for the class of bounded functions which belong to C^1 (resp. C^2). $D(\mathbb{R}_+)$ will denote the space of real càdlàg functions on \mathbb{R}_+ .

Let $T > 0$ be a finite horizon. $C^{0,1}([0, T] \times \mathbb{R})$ will denote the space of functions in $C^{0,1}$ restricted to $[0, T] \times \mathbb{R}$. In the following, $D(0, T)$ (resp. $D_-(0, T)$, $C(0, T)$, $C^1(0, T)$) will indicate the space of real càdlàg (resp. càglàd, continuous, continuously differentiable) functions on $[0, T]$. These spaces are equipped with the uniform norm. We will also indicate by $\|\cdot\|_\infty$ the essential supremum norm and by $\|\cdot\|_{var}$ the total variation norm. Given a topological space E , in the sequel $\mathcal{B}(E)$ will denote the Borel σ -field associated with E .

A stochastic basis $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ is fixed throughout the section. We will suppose that $\mathbb{F} = (\mathcal{F}_t)$ satisfies the usual conditions. By convention, any càdlàg process (or function) defined on $[0, T]$ is extended to \mathbb{R} by continuity. A similar convention is made for random fields (or functions) on $[0, T] \times \mathbb{R}$. Related to \mathbb{F} , the symbol \mathbb{D}^{ucp} will denote the space of all adapted càdlàg processes endowed with the u.c.p. (uniform convergence in probability) topology on each compact interval.

\mathcal{P} (resp. $\tilde{\mathcal{P}} := \mathcal{P} \otimes \mathcal{B}(\mathbb{R})$) will denote the predictable σ -field on $\Omega \times \mathbb{R}_+$ (resp. on $\tilde{\Omega} := \Omega \times \mathbb{R}_+ \times \mathbb{R}$). For a random field W , the simplified notation $W \in \tilde{\mathcal{P}}$ means that W is $\tilde{\mathcal{P}}$ -measurable.

A process X indexed by \mathbb{R}_+ will be said to be with integrable variation if the expectation of its total variation is finite. \mathcal{A} (resp. \mathcal{A}_{loc}) will denote the collection of all adapted processes with integrable variation (resp. with locally integrable variation), and \mathcal{A}^+ (resp. $\mathcal{A}_{\text{loc}}^+$) the collection of all adapted integrable increasing (resp. adapted locally integrable) processes. The significance of locally is the usual one which refers to localization by stopping times, see e.g. (0.39) of [26].

The concept of random measure will be extensively used throughout the paper. For a detailed discussion on this topic and the unexplained notations, we refer to Chapter I and Chapter II, Section 1, in [27], Chapter III in [26], and Chapter XI, Section 1, in [25]. In particular, if μ is a random measure on $[0, T] \times \mathbb{R}$, for any measurable real function H defined on $\Omega \times [0, T]$, one denotes $H \star \mu_t := \int_{]0, t] \times \mathbb{R}} H(\cdot, s, x) \mu(\cdot, ds dx)$, when the stochastic integral in the right-hand side is defined (with possible infinite values).

We recall that a transition kernel $Q(e, dx)$ of a measurable space (E, \mathcal{E}) into another measurable space (G, \mathcal{G}) is a family $\{Q(e, \cdot) : e \in E\}$ of positive measures on (G, \mathcal{G}) , such that $Q(\cdot, C)$ is \mathcal{E} -measurable for each $C \in \mathcal{G}$, see for instance in Section 1.1, Chapter I of [27].

Let X be an adapted càdlàg process. We set the corresponding jump measure μ^X by

$$\mu^X(dt dx) = \sum_{s>0} \mathbb{1}_{\{\Delta X_s \neq 0\}} \delta_{(s, \Delta X_s)}(dt dx). \quad (2.1)$$

We denote by $\nu^X = \nu^{X, \mathbb{P}}$ the compensator of μ^X , see [27] (Theorem 1.8, Chapter II). The dependence on \mathbb{P} will be omitted when self-explanatory. For any random field W , we set

$$\hat{W}_t = \int_{\mathbb{R}} W_t(x) \nu^X(\{t\} \times dx), \quad \tilde{W}_t = \int_{\mathbb{R}} W_t(x) \mu^X(\{t\} \times dx) - \hat{W}_t,$$

whenever they are well-defined. We also define

$$C(W) := |W - \hat{W}|^2 \star \nu^X + \sum_{s \leq \cdot} |\hat{W}_s|^2 (1 - \nu^X(\{s\} \times \mathbb{R})),$$

and, for every $q \in [1, \infty[$, the linear spaces

$$\mathcal{G}^q(\mu^X) = \left\{ W \in \tilde{\mathcal{P}} : \forall s \geq 0 \int_{\mathbb{R}} |W(s, x)| \nu^X(\{s\} \times dx) < \infty, \left[\sum_{s \leq \cdot} |\tilde{W}_s|^2 \right]^{q/2} \in \mathcal{A}^+ \right\},$$

$$\mathcal{G}_{\text{loc}}^q(\mu^X) = \left\{ W \in \tilde{\mathcal{P}} : \forall s \geq 0 \int_{\mathbb{R}} |W(s, x)| \nu^X(\{s\} \times dx) < \infty, \left[\sum_{s \leq \cdot} |\tilde{W}_s|^2 \right]^{q/2} \in \mathcal{A}_{\text{loc}}^+ \right\}.$$

For a random field W on $[0, T] \times \mathbb{R}$ we set the norms $\|W\|_{\mathcal{G}^2(\mu^X)}^2 := \mathbb{E}[C(W)_T]$, $\|W\|_{\mathcal{L}^2(\mu^X)} := \mathbb{E}[|W|^2 \star \nu_T]$, and the space $\mathcal{L}^2(\mu^X) := \{W \in \tilde{\mathcal{P}} : \|W\|_{\mathcal{L}^2(\mu^X)} < \infty\}$.

If $W \in \mathcal{G}_{\text{loc}}^1(\mu^X)$, we call stochastic integral with respect to $\mu^X - \nu^X$ and we denote it by $W \star (\mu^X - \nu^X)$, any purely discontinuous local martingale X such that ΔX and \tilde{W} are indistinguishable, see Definition 1.27, Chapter II, in [27]. We recall that, if $W \in \tilde{\mathcal{P}}$ such that $|W| \star \mu^X \in \mathcal{A}_{\text{loc}}^+$, then $W \in \mathcal{G}_{\text{loc}}^1(\mu^X)$ and $W \star (\mu^X - \nu) = W \star \mu^X - W \star \nu^X$, see Theorem 1.28, Chapter II, in [27]. Moreover, by Theorem 11.21, point 3) in [25], the following statements are equivalent:

1. $W \in \mathcal{G}_{\text{loc}}^2(\mu^X)$;
2. $C(W) \in \mathcal{A}_{\text{loc}}^+$;

3. $W \star (\mu^X - \nu^X)$ is a square integrable local martingale.

In this case $\langle W \star (\mu^X - \nu^X), W \star (\mu^X - \nu^X) \rangle = C(W)$. Finally, if $W \in \mathcal{L}^2(\mu^X)$ then $W \in \mathcal{G}^2(\mu^X)$, and $C(W) = |W|^2 \star \nu^X - \sum_{s \leq \cdot} |\dot{W}_s|^2$. In this case $\|W\|_{\mathcal{G}^2(\mu^X)}^2 \leq \|W\|_{\mathcal{L}^2(\mu^X)}^2$.

3 Weak Dirichlet processes: the suitable generalization of semi-martingales with jumps

3.1 A new unique decomposition

A stochastic basis $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ is fixed throughout the section. Sometimes the dependence on \mathbb{F} will be omitted. Given an adapted (càdlàg) process X on it, we will denote by μ^X its jump measure given in (2.1) and by ν^X the corresponding compensator.

We recall that an \mathbb{F} -weak Dirichlet process is a process of the type

$$X = M + \Gamma, \quad (3.1)$$

where M is an \mathbb{F} -local martingale and Γ is an \mathbb{F} -orthogonal process vanishing at zero, while a special weak Dirichlet process is a weak Dirichlet process $X = M + \Gamma$, where Γ is in addition predictable, see Definitions 5.5. and 5.6 in [4]. For complementary results, the reader can consult Section 5 in [4].

Remark 3.1. Any local martingale M can be uniquely decomposed as the sum of a continuous local martingale M^c and a purely discontinuous local martingale M^d such that $M_0^d = 0$, see Theorem 4.18, Chapter I, in [27]

The decomposition (3.1) is not unique, but the result below proposes a particularly natural one, which is unique.

Proposition 3.2. *Let X be a càdlàg \mathbb{F} -weak Dirichlet process. Then there is a unique continuous \mathbb{F} -local martingale X^c and a unique \mathbb{F} -martingale orthogonal process A vanishing at zero, such that*

$$X = X^c + A. \quad (3.2)$$

Proof. Existence. Since X is an \mathbb{F} -weak Dirichlet process, by (3.1) it is a process of the type $X = M + \Gamma$, with M an \mathbb{F} -local martingale and Γ an \mathbb{F} -martingale orthogonal process vanishing at zero. Recalling Remark 3.1, it follows that X admits the decomposition

$$X = M^c + M^d + \Gamma, \quad (3.3)$$

that provides (3.2) by setting $A := M^d + \Gamma$ and $X^c := M^c$.

Uniqueness. Assume that X admits the two decompositions

$$X = M^1 + A^1, \quad X = M^2 + A^2,$$

with M^1, M^2 continuous \mathbb{F} -local martingales and A^1, A^2 \mathbb{F} -martingale orthogonal processes vanishing at zero. So we have $0 = M^1 - M^2 + A^1 - A^2$. Taking the covariation of previous equality with $M^1 - M^2$, we get $[M^1 - M^2, M^1 - M^2] \equiv 0$. Since $M^1 - M^2$ is a continuous martingale vanishing at zero we finally obtain $M^1 = M^2$ and so $A^1 = A^2$. \square

Remark 3.3. Notice that decomposition (3.3) of the weak Dirichlet X is not unique.

Remark 3.4. The unique decomposition (3.2) holds (and it is new) in particular for semimartingales. On the other hand, when X is a semimartingale, the notion of X^c (as unique continuous martingale component) was introduced in Definition 2.6, Chapter II, in [27] (in that case it was fixed $X_0^c = 0$). There, a unique decomposition was provided only after having fixed a truncation.

Proposition 3.5. *Let X be an \mathbb{F} -semimartingale. Then $[X, X]^c = \langle X^c, X^c \rangle$.*

Proof. By Proposition 3.2 we have the unique decomposition $X = X^c + A$. On the other hand, being X a semimartingale, we also have that $X = M + V$ with V a bounded variation process, and M a local martingale. Since the unique decomposition $M = M^c + M^d$ holds (see Remark 3.1), we get that $X^c = M^c$ and

$$A = M^d + V. \quad (3.4)$$

Indeed M^d and V are \mathbb{F} -martingale orthogonal processes. Now, by (1.2) and (3.2)

$$[X, X] = [X, X]^c + \sum_{s \leq \cdot} |\Delta A_s|^2. \quad (3.5)$$

On the other hand, the bilinearity of the covariation gives

$$[X, X] = [X^c, X^c] + [A, A] \quad (3.6)$$

also taking into account that A is an \mathbb{F} -martingale orthogonal process. To conclude, comparing (3.5) and (3.6), we have to show that $[A, A] = \sum_{s \leq \cdot} |\Delta A_s|^2$. Now, by (3.4), Proposition 5.3 and Proposition 2.14 in [4],

$$\begin{aligned} [A, A] &= [M^d, M^d] + [V, V] + 2[M^d, V] = \sum_{s \leq \cdot} (|\Delta M_s^d|^2 + |\Delta V_s|^2 + 2\Delta M_s^d \Delta V_s) \\ &= \sum_{s \leq \cdot} |\Delta M_s^d + \Delta V_s|^2 = \sum_{s \leq \cdot} |\Delta A_s|^2. \end{aligned}$$

□

Remark 3.6. The equality $[X, X]^c = \langle X^c, X^c \rangle$ valid for semimartingales (see Proposition 3.5) may be not true for weak Dirichlet processes, even if they are of finite quadratic variation. Indeed, let W, B be two canonical Brownian motions, and \mathbb{F} be the canonical filtration associated with W and B . We set

$$X_t = \int_0^t B_{t-s} dW_s.$$

By Proposition 2.10 of [15], X is an \mathbb{F} -weak Dirichlet and \mathbb{F} -martingale orthogonal. By Proposition 3.2 it follows that $X^c \equiv 0$. On the other hand, by Remark 2.16-(2) in [15], $[X, X]_t = \frac{t^2}{2}$ which is different from zero.

3.2 Fundamental chain rules

From here on we will denote by \mathcal{K} the set of truncation functions, namely

$$\mathcal{K} := \{k : \mathbb{R} \rightarrow \mathbb{R} \text{ bounded with compact support: } k(x) = x \text{ in a neighborhood of } 0\}.$$

A typical choice of $k(x)$ will be $k(x) = x \mathbb{1}_{\{|x| \leq 1\}}$. In this case, $\frac{x-k(x)}{x} = \mathbb{1}_{\{|x| > 1\}}$. We will make use the following assumption on a pair (v, X) , with $v : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$ locally bounded and X a càdlàg process.

Hypothesis 3.7.

$$v(t, X_t) \text{ is a càdlàg process, and for every } t \in \mathbb{R}_+, \Delta v(t, X_t) = v(t, X_t) - v(t, X_{t-}); \quad (3.7)$$

$$\exists k \in \mathcal{K} \text{ such that } (v(s, X_{s-} + x) - v(s, X_{s-})) \frac{k(x)}{x} \in \mathcal{G}_{\text{loc}}^1(\mu^X). \quad (3.8)$$

Remark 3.8. (i) If v is continuous, then the pair (v, X) obviously fulfills (3.7). For a more refined condition on (v, X) to guarantee the validity of (3.7) we refer to Hypothesis 5.34 in [4].

(ii) Assume the validity of (3.7). If there is $a \in \mathbb{R}_+$ such that $\sum_{s \leq \cdot} |\Delta v(s, X_s)| \mathbb{1}_{\{|\Delta X_s| \leq a\}} < \infty$ a.s., then (3.8) is verified. This is trivially verified if $v(\cdot, X)$ is a bounded variation process.

Proposition 3.9. *Let X be an adapted càdlàg process satisfying (1.3). Let $v : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$ be a function of class $C^{0,1}$. Then Hypothesis 3.7 holds true.*

Proof. Condition (3.7) holds true being v continuous, see Remark 3.8-(i). On the other hand, by Proposition C.4,

$$|v(s, X_{s-} + x) - v(s, X_{s-})|^2 \frac{k^2(x)}{x^2} \star \mu^X \in \mathcal{A}_{\text{loc}}^+, \quad \forall k \in \mathcal{K}.$$

In particular, $(v(s, X_{s-} + x) - v(s, X_{s-})) \frac{k(x)}{x} \in \mathcal{G}_{\text{loc}}^2(\mu^X)$, that in turn implies condition (3.8), being $\mathcal{G}_{\text{loc}}^2(\mu^X) \subseteq \mathcal{G}_{\text{loc}}^1(\mu^X)$. \square

Theorem 3.10. *Let X be a càdlàg and \mathbb{F} -adapted process satisfying (1.3). Let $v : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$ be a locally bounded function such that (v, X) satisfies Hypothesis 3.7. Let $Y_t = v(t, X_t)$ be an \mathbb{F} -weak Dirichlet process with continuous martingale component Y^c . Then, for every $k \in \mathcal{K}$, one can write the decomposition*

$$Y = Y^c + M^{k,d} + \Gamma^k(v) + (v(s, X_{s-} + x) - v(s, X_{s-})) \frac{x - k(x)}{x} \star \mu^X, \quad (3.9)$$

with

$$M^{k,d} := (v(s, X_{s-} + x) - v(s, X_{s-})) \frac{k(x)}{x} \star (\mu^X - \nu^X) \quad (3.10)$$

and $\Gamma^k(v)$ a predictable and \mathbb{F} -martingale orthogonal process.

Remark 3.11. (i) Sufficient conditions for Y to be weak Dirichlet are given in Theorem 3.36.

(ii) Theorem 3.10 drastically generalizes the results given in Proposition 5.37 in [4], where we considered the case of an \mathbb{F} -martingale orthogonal process $Y_t = v(t, X_t)$ and such that $\sum_{s \leq \cdot} |\Delta Y_s| \in \mathcal{A}_{\text{loc}}^+$; in particular, $\sum_{s \leq T} |\Delta Y_s| < \infty$ a.s. Notice that, by Remark 3.8-(ii), Hypothesis 3.7 is verified. In that case $Y^c = 0$ by Proposition 3.2.

Remark 3.12. Let \mathcal{D} be the set of functions $v \in \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$ locally bounded such that $v(\cdot, X)$ is a weak Dirichlet process and (v, X) satisfies Hypothesis 3.7. We observe that $v \mapsto \Gamma^k(v)$ in Theorem 3.10 can be seen as a linear map from \mathcal{D} to the space of càdlàg adapted processes. In the sequel \mathcal{D} will also denote a similar set of functions defined on $[0, T] \times \mathbb{R}$ instead of $\mathbb{R}_+ \times \mathbb{R}$, and related to a process $X = (X(t))_{t \in [0, T]}$.

Proof of Theorem 3.10. Let $k \in \mathcal{K}$. We claim that

$$Y^k := Y - (v(s, X_{s-} + x) - v(s, X_{s-})) \frac{x - k(x)}{x} \star \mu^X \quad (3.11)$$

is an \mathbb{F} -special weak Dirichlet process. Indeed, we notice that,

$$Y^k = Y^c + A^Y - (v(s, X_{s-} + x) - v(s, X_{s-})) \frac{x - k(x)}{x} \star \mu^X =: Y^c + \tilde{A}^k, \quad (3.12)$$

where \tilde{A}^k is an \mathbb{F} -martingale orthogonal process. By condition (3.8) and Lemma C.7, $(v(s, X_{s-} + x) - v(s, X_{s-})) \frac{k(x)}{x} \in \mathcal{G}_{\text{loc}}^1(\mu^X)$. Then the process $M^{k,d}$ in (3.10), is a purely discontinuous \mathbb{F} -local martingale, see Section 2. We rewrite (3.12) as

$$Y^k = Y^c + M^{k,d} + \Gamma^k(v), \quad (3.13)$$

with

$$\Gamma^k(v) := \tilde{A}^k - M^{k,d}. \quad (3.14)$$

This gives $\Delta Y_t^k = \Delta M_t^{k,d} + \Delta \Gamma_t^k(v)$. Now, by (3.11) and (3.10) we get

$$\begin{aligned} \Delta Y_t^k &= \int_{\mathbb{R}} (v(s, X_{s-} + x) - v(s, X_{s-})) \frac{k(x)}{x} \mu^X(\{t\} \times dx), \\ \Delta M_t^{k,d} &= \int_{\mathbb{R}} (v(s, X_{s-} + x) - v(s, X_{s-})) \frac{k(x)}{x} \mu^X(\{t\} \times dx) \\ &\quad - \int_{\mathbb{R}} (v(s, X_{s-} + x) - v(s, X_{s-})) \frac{k(x)}{x} \nu^X(\{t\} \times dx), \end{aligned}$$

so that, by (3.13),

$$\Delta \Gamma_t^k(v) = \int_{\mathbb{R}} (v(s, X_{s-} + x) - v(s, X_{s-})) \frac{k(x)}{x} \nu^X(\{t\} \times dx).$$

Recall that an adapted càdlàg process is predictable if and only if its jump process is predictable, see Remark 3.13-2. below. We conclude that $\Gamma^k(v)$ is an \mathbb{F} -predictable process; moreover, by (3.14), $\Gamma^k(v)$ is also an \mathbb{F} -martingale orthogonal process. This yields decomposition (3.9). \square

Remark 3.13. 1. Any càglàd process is locally bounded, see the lines above Theorem 15, Chapter IV, in [32].

2. Let X be a càdlàg adapted process. Recalling that $\Delta X_s = X_s - X_{s-}$, we have the following.

- X is locally bounded if and only if ΔX is locally bounded. As a matter of fact, (X_{s-}) is a càglàd process and therefore locally integrable.
- X is predictable if and only if ΔX is predictable. Indeed, (X_{s-}) is a predictable process being adapted and left-continuous.

Taking $k(x) = x \mathbb{1}_{\{|x| \leq a\}}$ in Theorem 3.10 we get the following result.

Corollary 3.14. *Let X be a càdlàg \mathbb{F} -adapted process satisfying (1.3). Let $v : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$ be a locally bounded function, such that (v, X) satisfies Hypothesis 3.7. Assume moreover that, for some $a \in \mathbb{R}_+$,*

$$|\Delta X_t| \leq a, \quad \forall t \in \mathbb{R}_+. \quad (3.15)$$

Then, if $Y_t = v(t, X_t)$ is an \mathbb{F} -weak Dirichlet process, then it is an \mathbb{F} -special weak Dirichlet process.

The particular case of Corollary 3.14 with $v \equiv \text{Id}$ is stated below.

Corollary 3.15. *Let X be a càdlàg and \mathbb{F} -adapted process satisfying (1.3) and (3.15). Then, if X is an \mathbb{F} -weak Dirichlet process, it is an \mathbb{F} -special weak Dirichlet process.*

Theorem 3.16. *Let X be càdlàg and \mathbb{F} -adapted process satisfying (1.3). Let $v : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$ be a locally bounded function such that (v, X) satisfies Hypothesis 3.7. Set $Y_t = v(t, X_t)$, and assume that Y is an \mathbb{F} -weak Dirichlet process. Then Y is an \mathbb{F} -special weak Dirichlet process if and only if*

$$\exists a \in \mathbb{R}_+ \text{ s.t. } (v(s, X_{s-} + x) - v(s, X_{s-})) \mathbb{1}_{\{|x| > a\}} \star \mu^X \in \mathcal{A}_{\text{loc}}. \quad (3.16)$$

Remark 3.17. If v is bounded and X is a càdlàg and \mathbb{F} -adapted, then condition (3.16) is satisfied because of Lemma C.2.

Proof of Theorem 3.16. Let $a \in \mathbb{R}_+$ and set

$$\tilde{Y} := \sum_{s \leq \cdot} \Delta Y_s \mathbb{1}_{\{|\Delta X_s| > a\}} = (v(s, X_{s-} + x) - v(s, X_{s-})) \mathbb{1}_{\{|x| > a\}} \star \mu^X.$$

By Theorem 3.10 with $k(x) = \mathbb{1}_{\{|x| \leq a\}}$, $Y - \tilde{Y}$ is an \mathbb{F} -special weak Dirichlet process. It follows that Y is an \mathbb{F} -special weak Dirichlet process if and only if \tilde{Y} is an \mathbb{F} -special weak Dirichlet process. \tilde{Y} has bounded variation, so it is a semimartingale, therefore it is a special semimartingale, see Proposition 5.14 in [4]. This can be shown to be equivalent to condition (3.16) by making use of the first three equivalent items of Proposition 4.23, Chapter I, in [27]. \square

Corollary 3.18 below follows from Theorem 3.16 taking $v \equiv \text{Id}$. It extends a characterization stated in Proposition 5.24 in [4]: thereby, X was supposed to belong to a particular class of weak Dirichlet processes.

Corollary 3.18. *Let X be an \mathbb{F} -weak Dirichlet process satisfying (1.3). Then X is an \mathbb{F} -special weak Dirichlet process if and only if*

$$\exists a \in \mathbb{R}_+ \text{ s.t. } x \mathbb{1}_{\{|x| > a\}} \star \mu^X \in \mathcal{A}_{\text{loc}}. \quad (3.17)$$

The following result is the analogous of Theorem 3.10 for special weak Dirichlet processes.

Theorem 3.19. *Let X be a càdlàg and \mathbb{F} -adapted process satisfying (1.3). Let $v : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$ be a locally bounded function such that (v, X) satisfies Hypothesis 3.7. Let $Y_t = v(t, X_t)$ be an \mathbb{F} -weak Dirichlet process with continuous martingale component Y^c . Assume moreover that the pair (v, X) satisfies condition (3.16). Then $Y_t = v(t, X_t)$ is an \mathbb{F} -special weak Dirichlet process, admitting the unique decomposition*

$$Y = Y^c + (v(s, X_{s-} + x) - v(s, X_{s-})) \star (\mu^X - \nu^X) + \Gamma(v), \quad (3.18)$$

with $\Gamma(v)$ a predictable and \mathbb{F} -martingale orthogonal process. Moreover,

$$\Gamma(v) = \Gamma^k(v) + (v(s, X_{s-} + x) - v(s, X_{s-})) \frac{x - k(x)}{x} \star \nu^X, \quad (3.19)$$

with $\Gamma^k(v)$ the predictable and \mathbb{F} -martingale orthogonal process appearing in (3.9).

Proof. Thanks to condition (3.16), by Theorem 3.16 Y is an \mathbb{F} -special weak Dirichlet process. By Theorem 3.10,

$$Y = Y^c + (v(s, X_{s-} + x) - v(s, X_{s-})) \frac{k(x)}{x} \star (\mu^X - \nu^X) + \Gamma^k(v)$$

$$+ (v(s, X_{s-} + x) - v(s, X_{s-})) \frac{x - k(x)}{x} \star \mu^X. \quad (3.20)$$

We add and subtract in (3.20) the term $(v(s, X_{s-} + x) - v(s, X_{s-})) \frac{x - k(x)}{x} \star \nu^X$. Recalling that, for every random field W , $|W| \star \mu^X \in \mathcal{A}_{\text{loc}}^+$ implies $W \in \mathcal{G}_{\text{loc}}^1(\mu^X)$ and $W \star (\mu^X - \nu^X) = W \star \mu^X - W \star \nu^X$ (see Section 2), we get decomposition (3.18) with $\Gamma(v)$ provided by (3.19). \square

Remark 3.20. 1. It directly follows from (3.18) that

$$\Delta \Gamma_s(v) = \int_{\mathbb{R}} (v(s, X_{s-} + x) - v(s, X_{s-})) \star \nu^X(\{s\} \times dx).$$

2. Let X be a càdlàg and \mathbb{F} -adapted process satisfying (1.3). Let $v : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$ be such that (v, X) satisfies Hypothesis 3.7. If v is moreover a bounded function, by Remark 3.17 condition (3.16) is satisfied as well, and all the assumptions of Theorem 3.19 are verified.

Remark 3.21. Theorem 3.19 is in particular useful to solve the so-called identification problem for BSDEs. Proposition 5.37 in [4] allowed to solve the identification problem when the BSDE was exclusively driven by the random measure, see Theorem 3.14 in [5].

Let ζ be a non-decreasing, adapted and continuous process, and λ be a predictable random measure on $\Omega \times [0, T] \times \mathbb{R}$. We consider a BSDE driven by a random measure $\mu - \nu$ and a continuous martingale M of the type

$$\begin{aligned} Y_t &= \xi + \int_{]t, T]} \tilde{g}(s, Y_{s-}, Z_s) d\zeta_s + \int_{]t, T] \times \mathbb{R}} \tilde{f}(s, Y_{s-}, U_s(e)) \lambda(ds de) \\ &\quad - \int_{]t, T]} Z_s dM_s - \int_{]t, T] \times \mathbb{R}} U_s(e) (\mu - \nu)(ds de), \end{aligned} \quad (3.21)$$

whose solution is a triple of processes $(Y, Z, U(\cdot))$. If $Y_t = v(t, X_t)$ for some function v and some adapted càdlàg process X , the identification problem consists in expressing Z and $U(\cdot)$ in terms of v .

- (i) Being $Y_t = v(t, X_t)$ a solution to a BSDE, it is a special weak Dirichlet process (even a special semimartingale), and therefore (v, X) satisfies condition (3.16). Therefore, if Hypothesis 3.7 holds for (v, X) , then Theorem 3.19 allows to identify $U(\cdot)$. More precisely, this provides

$$U(e) \star (\mu - \nu) = (v(s, X_{s-} + x) - v(s, X_{s-})) \star (\mu^X - \nu^X), \quad \text{a.s.}$$

From now on suppose $\mu = \mu^X$, even though this can be generalized, see Hypothesis 2.9 in [5]. This yields

$$H(x) \star (\mu^X - \nu^X) = 0, \quad \text{a.s.,}$$

with $H(x) := U(x) - (v(s, X_{s-} + x) - v(s, X_{s-}))$. If $H \in \mathcal{G}_{\text{loc}}^2(\mu^X)$, then the predictable bracket of $H(x) \star (\mu^X - \nu^X)$ is well-defined and equals $C(H)$. Since $C(H)_T = 0$, we get (see Proposition 2.8 in [5]) that there is a predictable process (l_s) such that

$$H_s(x) = l_s \mathbb{1}_K(s) \quad d\mathbb{P} \nu^X(ds dx) \text{ a.e.,}$$

where $K := \{(\omega, t) : \nu^X(\omega, \{t\} \times \mathbb{R}) = 1\}$.

- (ii) In order to identify the process Z we need that v belongs to $C^{0,1}([0, T] \times \mathbb{R})$ and that X is a weakly finite quadratic variation process, and this will be discussed in Remark 3.39.

Theorems 3.10 and 3.19 with $v \equiv \text{Id}$ give in particular the following result.

Corollary 3.22. *Let X be an \mathbb{F} -weak Dirichlet process satisfying condition (1.3). Let X^c be the continuous martingale part of X . Then, the following holds.*

(i) *Let $k \in \mathcal{K}$. Then X can be decomposed as*

$$X = X^c + k(x) \star (\mu^X - \nu^X) + \Gamma^k(\text{Id}) + (x - k(x)) \star \mu^X, \quad (3.22)$$

with Γ^k the operator introduced in Theorem 3.10.

(ii) *If (3.17) holds, then the \mathbb{F} -special weak Dirichlet process X admits the decomposition*

$$X = X^c + x \star (\mu^X - \nu^X) + \Gamma \quad (3.23)$$

with $\Gamma := \Gamma^k(\text{Id}) + x \mathbb{1}_{\{|x|>1\}} \star \nu^X$.

3.3 The notion of characteristics for weak Dirichlet processes

In the present section we provide a generalization of the concept of characteristics for semimartingales (see Appendix D) in the case of weak Dirichlet processes.

Remark 3.23. Let X be an \mathbb{F} -weak Dirichlet process with jump measure μ^X satisfying condition (1.3). Given $k \in \mathcal{K}$, by Corollary 3.22-(i), we have that $X^k = X - \sum_{s \leq \cdot} [\Delta X_s - k(\Delta X_s)]$ is an \mathbb{F} -special weak Dirichlet process with unique decomposition

$$X^k = X^c + k(x) \star (\mu^X - \nu^X) + B^{k,X}, \quad (3.24)$$

where

- X^c is the unique continuous \mathbb{F} -local martingale part of X introduced in Proposition 3.2;
- $B^{k,X} := \Gamma^k(\text{Id})$, which is in particular a predictable and \mathbb{F} -martingale orthogonal process.

Remark 3.24. When X is a semimartingale, $B^{k,X}$ is a bounded variation process, so in particular \mathbb{F} -martingale orthogonal.

From here on we denote by $\check{\Omega}$ the canonical space of all càdlàg functions $\check{\omega} : \mathbb{R}_+ \rightarrow \mathbb{R}$, namely $\check{\Omega} = D(\mathbb{R}_+)$, and by \check{X} the canonical process defined by $\check{X}_t(\omega) = \check{\omega}(t)$. We also set $\check{\mathcal{F}} = \sigma(\check{X})$, and $\check{\mathbb{F}} = (\check{\mathcal{F}}_t)_{t \geq 0}$. Let μ be the jump measure of \check{X} and ν the compensator of μ under the law $\mathcal{L}(X)$ of X .

Definition 3.25. *We call characteristics of X , associated with $k \in \mathcal{K}$, the triplet $(B^k, C = \langle \check{X}^c, \check{X}^c \rangle, \nu)$ on $(\check{\Omega}, \check{\mathcal{F}}, \check{\mathbb{F}})$ obtained from the unique decomposition (3.24) for \check{X} under $\mathcal{L}(X)$. In particular,*

- (i) B^k is a predictable and $\check{\mathbb{F}}$ -martingale orthogonal process, with $B_0^k = 0$;
- (ii) C is an $\check{\mathbb{F}}$ -predictable and increasing process, with $C_0 = 0$;
- (iii) ν is an $\check{\mathbb{F}}$ -predictable random measure on $\mathbb{R}_+ \times \mathbb{R}$.

Remark 3.26. a) $\mu \circ X$ is the jump measure of X , and its compensator under \mathbb{P} is $\nu \circ X$, see Proposition 10.39-b) in [26].

b) It is not difficult to show that $\check{X}^c \circ X$ is a continuous local martingale under \mathbb{P} .

c) Let Y and Z be two processes on $(\check{\Omega}, \check{\mathcal{F}}, \check{\mathbb{F}})$ such that $[Y, Z]$ exists under $\mathcal{L}(X)$. Then $[Y, Z] \circ X = [Y \circ X, Z \circ X]$ under \mathbb{P} . In particular, $B^k \circ X$ is an \mathbb{F} -martingale orthogonal process.

d) By previous items we have a new decomposition of the process X^k :

$$X^k = \check{X}^c \circ X + k \star (\mu \circ X - \nu \circ X) + B^k \circ X.$$

e) By the uniqueness of decomposition (3.24) we have $X^c = \check{X}^c \circ X$ and $B^{k,X} = B^k \circ X$.

f) $C^X := C \circ X = \langle X^c, X^c \rangle$ by c) and e).

Remark 3.27. Assume that X admits characteristics (B^{k_0}, C, ν) depending to a given truncation function k_0 . Then, if we choose another truncation function k , the corresponding characteristics will be (B^k, C, ν) with

$$B^k \circ X = B^{k_0} \circ X + (k - k_0) \star (\nu \circ X),$$

where $(k - k_0) \star (\nu \circ X) \in \mathcal{A}_{\text{loc}}$, see Lemma C.6.

Remark 3.28. a) Given a process $X \in \mathcal{A}_{\text{loc}}^+$, Theorem 3.17, Chapter I, in [27] shows that there is a unique predictable process $X^p \in \mathcal{A}_{\text{loc}}^+$, called *compensator* of X , such that $X - X^p$ is a local martingale. In particular, X is a special semimartingale.

b) The notion of compensator can be naturally extended. Given a process X , we denote by X^p a process verifying the following conditions:

- (i) X^p is predictable;
- (ii) X^p is martingale orthogonal;
- (iii) $X - X^p$ is a local martingale.

Obviously, such X^p exists if and only if X is a special weak Dirichlet process, and $X^p = \Gamma$, with Γ the process in Corollary 3.22-(ii), so it is uniquely determined.

c) The two notions of X^p in a) and b) coincide when $X \in \mathcal{A}_{\text{loc}}^+$. Indeed, a special semimartingale is a special weak Dirichlet process.

3.4 A weak notion of finite quadratic variation and a stability theorem

In the following we will use the notation (for $t \geq 0$)

$$[X, X]_{\varepsilon}^{\text{ucp}}(t) := \int_0^t \frac{(X_{(s+\varepsilon)\wedge t} - X_s)^2}{\varepsilon} ds, \quad \varepsilon > 0, \quad (3.25)$$

$$C_{\varepsilon}(X, X)(t) := \int_0^t \frac{(X_{s+\varepsilon} - X_s)^2}{\varepsilon} ds, \quad \varepsilon > 0. \quad (3.26)$$

From now on $T > 0$ will denote a fixed maturity time.

Definition 3.29. A càdlàg process $X = (X(t))_{t \in [0, T]}$ is said to be a *weakly finite quadratic variation process* if there is $\varepsilon_0 > 0$ such that the laws of the random variables $[X, X]_{\varepsilon}^{\text{ucp}}(T)$, $0 < \varepsilon \leq \varepsilon_0$, are tight.

Below, $\varepsilon > 0$ will mean $0 < \varepsilon \leq \varepsilon_0$ for some ε_0 small enough. For instance, a family $(Z_{\varepsilon})_{\varepsilon > 0}$ of random variables will indicate a sequence $(Z_{\varepsilon})_{0 < \varepsilon \leq \varepsilon_0}$ for some ε_0 small enough.

Remark 3.30. A finite quadratic variation process is a weakly finite quadratic variation process. Indeed, if $\int_0^t \frac{(X_{(s+\varepsilon)\wedge t} - X_s)^2}{\varepsilon} ds$ converges u.c.p., the random variable $[X, X]_{\varepsilon}^{\text{ucp}}(T)$ converges in probability, and so it also converges in law.

Proposition 3.31. Let $(Z_{\varepsilon})_{\varepsilon > 0}$ be a nonnegative sequence of random variables. Suppose that one of the two items below hold.

(i) $\sup_{\varepsilon>0} Z_\varepsilon < \infty$ a.s.

(ii) $\sup_{\varepsilon>0} \mathbb{E}[Z_\varepsilon] < \infty$.

Then the family of distribution of $(Z_\varepsilon)_{\varepsilon>0}$ is tight.

Proof. Let $\delta > 0$. In order to prove the result, we need to find $M > 0$ such that

$$\mathbb{P}(Z_\varepsilon \notin [0, M]) \leq \delta, \quad \forall \varepsilon > 0.$$

(i) We choose $M = M(\delta) > 0$ such that $\mathbb{P}(\sup_{\varepsilon>0} Z_\varepsilon > M) \leq \delta$.

(ii) For every M , by Markov-Chebichev inequality,

$$\mathbb{P}(Z_\varepsilon > M) \leq \frac{\mathbb{E}[Z_\varepsilon]}{M} \leq \frac{1}{M} \sup_{\varepsilon>0} \mathbb{E}[Z_\varepsilon].$$

We choose M so that the previous upper bound is smaller or equal then δ . \square

Remark 3.32. It follows from Proposition 3.31 that a càdlàg process X is a weakly finite quadratic variation process when one of the following conditions holds:

(i) $\sup_{\varepsilon>0} \int_0^T \frac{(X_{s+\varepsilon} - X_s)^2}{\varepsilon} ds < \infty$ a.s.

(ii) $\sup_{\varepsilon>0} \mathbb{E} \left[\int_0^T \frac{(X_{s+\varepsilon} - X_s)^2}{\varepsilon} ds \right] < \infty$. A process X fulfilling this condition was called a finite energy process, see [10] (in the framework of Föllmer's discretizations) and [35].

Example 3.33. Let X be a square integrable process with weakly stationary increments. Set $V(a) := \mathbb{E}[(X_a - X_0)^2]$, $a \in \mathbb{R}_+$. Let $V(\varepsilon) = O(\varepsilon)$, $\varepsilon > 0$, i.e., there is a constant $C > 0$ such that $V(\varepsilon) \leq C\varepsilon$ for a small $\varepsilon > 0$. Then condition (ii) of Remark 3.32 is verified. A classical example of a process that satisfies that condition is a weak Brownian motion of order $\kappa = 2$, see [21]. In this case $V(a) = a$. Indeed the bivariate distributions of such a process are the same as those of Brownian motion. In general, such a process is not a semimartingale and not even a finite quadratic variation process.

Proposition 3.34. *Suppose that X is a weakly finite quadratic variation process. Then condition (1.3) holds true.*

Remark 3.35. By Proposition C.1, condition (1.3) is equivalent to $(1 \wedge |x|^2) \star \nu^X \in \mathcal{A}_{\text{loc}}^+$. The validity of latter condition was known for semimartingales, see Theorem 2.42, Chapter II, in [27].

Proof. Let γ be a constant. For each fixed ω , let $\tau_0 = \tau_{0,\gamma} := 0$ and set

$$\tau_i(\omega) = \tau_{i,\gamma}(\omega) := \inf_{t>\tau_{i-1,\gamma}(\omega)} \{t : |\Delta X_t| > \gamma\}, \quad i \in \mathbb{N},$$

with the convention that it is $+\infty$ if the set is empty. Notice that, almost surely, there exists a finite number of jumps of X greater than γ . Let thus $N = N(\omega)$ be such that $\tau_{N,\gamma}$ is the maximum of those jump times. Let $\varepsilon > 0$, and define $\Omega_{\varepsilon,\gamma}^0 = \{\omega \in \Omega : \tau_i(\omega) - \tau_{i-1}(\omega) > \varepsilon, i \in \mathbb{N}\}$, with the convention that $\infty - \infty = \infty$. We have that

$$\cup_{\varepsilon} \Omega_{\varepsilon,\gamma}^0 = \Omega, \tag{3.27}$$

up to a null set. We set

$$J_A(\varepsilon) := \sum_{i=1}^N \frac{1}{\varepsilon} \int_{\tau_{i-1}}^{\tau_i} (X_{(s+\varepsilon) \wedge T} - X_s)^2 ds.$$

On $\Omega_{\varepsilon, \gamma}^0$ we have

$$[X, X]_{\varepsilon}^{ucp}(T) \geq J_A(\varepsilon). \quad (3.28)$$

We also know, by Lemma 2.11 in [4], that

$$J_A(\varepsilon) \xrightarrow{\varepsilon \rightarrow 0} \sum_{i=1}^N \mathbb{1}_{]0, T]}(\tau_i) |\Delta X_{\tau_i}|^2 = \sum_{t \leq T} |\Delta X_t|^2 \mathbb{1}_{\{|\Delta X_t| > \gamma\}} \quad \text{a.s.} \quad (3.29)$$

Let $\kappa > 0$. Since X is a weakly finite quadratic variation process, there exists $\ell = \ell(\kappa) > 0$ such that, for every ε (small enough), $\mathbb{P}(\Omega_{\varepsilon, \ell}^c) < \kappa$ with $\Omega_{\varepsilon, \ell} := \{\omega \in \Omega : [X, X]_{\varepsilon}^{ucp}(T) \leq \ell\}$. We get

$$\begin{aligned} & \mathbb{P}\left(\sum_{t \leq T} |\Delta X_t|^2 \mathbb{1}_{\{|\Delta X_t| > \gamma\}} > K\right) \leq \kappa + \mathbb{P}\left(\sum_{t \leq T} |\Delta X_t|^2 \mathbb{1}_{\{|\Delta X_t| > \gamma\}} > K; \Omega_{\varepsilon, \ell}\right) \\ & \leq \kappa + \mathbb{P}\left(|J_A(\varepsilon)| > \frac{K}{2}; \Omega_{\varepsilon, \ell}\right) + \mathbb{P}\left(\left|\sum_{t \leq T} |\Delta X_t|^2 \mathbb{1}_{\{|\Delta X_t| > \gamma\}} - J_A(\varepsilon)\right| > \frac{K}{2}; \Omega_{\varepsilon, \ell}\right) \\ & \leq \kappa + \mathbb{P}\left(|J_A(\varepsilon)| > \frac{K}{2}; \Omega_{\varepsilon, \ell} \cap \Omega_{\varepsilon, \gamma}^0\right) + \mathbb{P}\left(|J_A(\varepsilon)| > \frac{K}{2}; \Omega_{\varepsilon, \ell} \setminus \Omega_{\varepsilon, \gamma}^0\right) \\ & + \mathbb{P}\left(\left|\sum_{t \leq T} |\Delta X_t|^2 \mathbb{1}_{\{|\Delta X_t| > \gamma\}} - J_A(\varepsilon)\right| > \frac{K}{2}\right) \\ & \leq \kappa + \mathbb{P}\left([X, X]_{\varepsilon}^{ucp}(T) > \frac{K}{2}; \Omega_{\varepsilon, \ell} \cap \Omega_{\varepsilon, \gamma}^0\right) + \mathbb{P}\left((\Omega_{\varepsilon, \gamma}^0)^c\right) \\ & + \mathbb{P}\left(\left|\sum_{t \leq T} |\Delta X_t|^2 \mathbb{1}_{\{|\Delta X_t| > \gamma\}} - J_A(\varepsilon)\right| > \frac{K}{2}\right), \end{aligned} \quad (3.30)$$

where we have used (3.28) in the latter inequality. The first probability in the right-hand side of latter inequality equals

$$\mathbb{P}\left([X, X]_{\varepsilon}^{ucp}(T) \wedge \ell > \frac{K}{2}; \Omega_{\varepsilon, \ell} \cap \Omega_{\varepsilon, \gamma}^0\right).$$

Choosing $K = K(\ell, \kappa)$ so that $\frac{K}{2} \leq \ell$, this probability is zero. Therefore, applying the $\limsup_{\varepsilon \rightarrow 0}$ in (3.30), and taking into account (3.27) and (3.29), we get

$$\mathbb{P}\left(\sum_{t \leq T} |\Delta X_t|^2 \mathbb{1}_{\{|\Delta X_t| > \gamma\}} > K\right) \leq \kappa. \quad (3.31)$$

Notice that $\sum_{t \leq T} |\Delta X_t|^2 \mathbb{1}_{\{|\Delta X_t| > \gamma\}}$ converges increasingly to $\sum_{t \leq T} |\Delta X_t|^2$, a.s. when γ converges to zero. Consequently, letting $\gamma \rightarrow 0$ in (3.31), we obtain

$$\mathbb{P}\left(\sum_{t \leq T} |\Delta X_t|^2 > K\right) \leq \kappa.$$

Finally,

$$\mathbb{P}\left(\sum_{t \leq T} |\Delta X_t|^2 = \infty\right) \leq \mathbb{P}\left(\sum_{t \leq T} |\Delta X_t|^2 > K\right) \leq \kappa,$$

so the conclusion follows. \square

Below we give a significant generalization of Proposition 3.10 in [24], where the result was proven when X is continuous and of finite quadratic variation. When X is càdlàg, even in the case when X is a finite quadratic variation process, the result is new. Crucial tools to prove the result are the canonical decomposition stated in Proposition 3.2 and Proposition A.3.

Theorem 3.36. *Let X be an \mathbb{F} -weak Dirichlet process with weakly finite quadratic variation. Let $v \in C^{0,1}([0, T] \times \mathbb{R})$. Then $Y_t = v(t, X_t)$ is an \mathbb{F} -weak Dirichlet with continuous martingale component*

$$Y^c = Y_0 + \int_0^\cdot \partial_x v(s, X_s) dX_s^c. \quad (3.32)$$

Theorem 3.36, together with Theorems 3.16 and 3.19 (recall Proposition 3.9), provides the following result.

Corollary 3.37. *Let X be an \mathbb{F} -weak Dirichlet process with weakly finite quadratic variation. Let $v \in C^{0,1}([0, T] \times \mathbb{R})$ such that (v, X) satisfies condition (3.16). Then $Y_t = v(t, X_t)$ is an \mathbb{F} -special weak Dirichlet process, admitting the unique decomposition*

$$Y = Y_0 + \int_0^\cdot \partial_x v(s, X_s) dX_s^c + (v(s, X_{s-} + x) - v(s, X_{s-})) \star (\mu^X - \nu^X) + \Gamma(v), \quad (3.33)$$

with $\Gamma(v)$ a predictable and \mathbb{F} -martingale orthogonal process.

Remark 3.38. Corollary 3.37 extends the chain rules previously given in this framework in Theorems 5.15 and 5.31 in [4]. More precisely, we have the following.

- In Theorem 5.15 in [4] we already proved that, if X is a \mathbb{F} -weak Dirichlet process of finite quadratic variation, and v is of class $C^{0,1}$, then $Y = v(\cdot, X)$ is again a weak Dirichlet process. However the decomposition established therein was not unique. Here instead in Theorem 3.10, together with Theorem 3.36, we are able to provide an explicit form of its unique decomposition.
- Theorem 5.31 in [4] focused on sufficient conditions on (v, X) so that Y is special weak Dirichlet. Here, by Corollary 3.37, we are able to give the necessary and sufficient condition (3.16) on (v, X) such that Y is a special weak Dirichlet, and we provide its unique decomposition.

Remark 3.39. Let us consider the BSDE (3.21) introduced in Remark 3.21. If $Y_t = v(t, X_t)$ is a solution to (3.21), it is a special weak Dirichlet process, and therefore (v, X) satisfies condition (3.16). Then, if $v \in C^{0,1}([0, T] \times \mathbb{R})$, then Corollary 3.37 allows also to identify Z . More precisely, we get

$$Z_t = \partial_x v(t, X_t) \frac{d\langle X^c, M \rangle_t}{\langle M \rangle_t}, \quad d\mathbb{P} d\langle M \rangle_t\text{-a.e.}$$

Proof of Theorem 3.36. We aim at proving that, for every \mathbb{F} -continuous local martingale N ,

$$[v(\cdot, X), N]_t = \int_0^t \partial_x v(s, X_s) d[X^c, N]_s, \quad t \in [0, T]. \quad (3.34)$$

Indeed, if (3.34) were true, it would imply that $A(v) := v(\cdot, X) - Y^c$ is martingale orthogonal, and therefore by additivity $v(\cdot, X)$ would be a weak Dirichlet process. Then (3.32) would follow by the uniqueness of the continuous martingale part of Y .

Let us thus prove (3.34). The approximating sequence of the left-hand side of (3.34) is

$$\int_0^t [v((s + \varepsilon) \wedge t, X_{(s + \varepsilon) \wedge t}) - v(s, X_s)] \frac{N_{(s + \varepsilon) \wedge t} - N_s}{\varepsilon} ds = I_1(t, \varepsilon) + I_2(t, \varepsilon),$$

with

$$\begin{aligned} I_1(t, \varepsilon) &:= \int_0^t [v((s + \varepsilon) \wedge t, X_{(s+\varepsilon)\wedge t}) - v((s + \varepsilon) \wedge t, X_s)] \frac{N_{(s+\varepsilon)\wedge t} - N_s}{\varepsilon} ds, \\ I_2(t, \varepsilon) &:= \int_0^t [v((s + \varepsilon) \wedge t, X_s) - v(s, X_s)] \frac{N_{(s+\varepsilon)\wedge t} - N_s}{\varepsilon} ds. \end{aligned}$$

Concerning $I_2(t, \varepsilon)$, by stochastic Fubini theorem we get

$$\begin{aligned} I_2(t, \varepsilon) &= \frac{1}{\varepsilon} \int_0^t [v((s + \varepsilon) \wedge t, X_s) - v(s, X_s)] \int_s^{(s+\varepsilon)\wedge t} dN_u \\ &= \int_0^t dN_u \int_{(u-\varepsilon)^+}^u [v((s + \varepsilon) \wedge t, X_s) - v(s, X_s)] \frac{ds}{\varepsilon}. \end{aligned} \quad (3.35)$$

Since

$$\int_0^T d[N, N]_u \left(\int_{(u-\varepsilon)^+}^u [v(s + \varepsilon, X_s) - v(s, X_s)] \frac{ds}{\varepsilon} \right)^2 \xrightarrow{\varepsilon \rightarrow 0} 0 \quad \text{in probability,}$$

by Problem 2.27, Chapter 3, in [28], this is enough to conclude that $I_2(\cdot, \varepsilon) \xrightarrow{\varepsilon \rightarrow 0} 0$ u.c.p. (since N is continuous, it is clear that we can neglect the “ $\wedge t$ ” in (3.35)). Concerning $I_1(t, \varepsilon)$, we have

$$\begin{aligned} I_1(t, \varepsilon) &= \int_0^t [v((s + \varepsilon) \wedge t, X_{(s+\varepsilon)\wedge t}) - v((s + \varepsilon) \wedge t, X_s)] \frac{N_{(s+\varepsilon)\wedge t} - N_s}{\varepsilon} ds \\ &= \int_0^t \int_0^1 \partial_x v((s + \varepsilon) \wedge t, X_s + a(X_{(s+\varepsilon)\wedge t} - X_s)) da (X_{(s+\varepsilon)\wedge t} - X_s) (N_{(s+\varepsilon)\wedge t} - N_s) \frac{ds}{\varepsilon} \\ &= \int_0^t \partial_x v(s, X_s) (X_{(s+\varepsilon)\wedge t} - X_s) (N_{(s+\varepsilon)\wedge t} - N_s) \frac{ds}{\varepsilon} \\ &+ \int_0^t \int_0^1 [\partial_x v((s + \varepsilon) \wedge t, X_s + a(X_{(s+\varepsilon)\wedge t} - X_s)) - \partial_x v(s, X_s)] da (X_{(s+\varepsilon)\wedge t} - X_s) (N_{(s+\varepsilon)\wedge t} - N_s) \frac{ds}{\varepsilon} \\ &=: I_{1,a}(t, \varepsilon) + I_{1,b}(t, \varepsilon). \end{aligned} \quad (3.36)$$

We set $\tilde{I}_{1,a}(t, \varepsilon) := \int_0^t \partial_x v(s, X_s) (X_{s+\varepsilon} - X_s) (N_{s+\varepsilon} - N_s) \frac{ds}{\varepsilon}$. By Proposition A.3 with $g(s) = \partial_x v(s, X_{s-})$, we get

$$\tilde{I}_{1,a}(\cdot, \varepsilon) \xrightarrow{\varepsilon \rightarrow 0} \int_0^\cdot \partial_x v(s, X_s) d[X^c, N]_s, \quad \text{u.c.p.}$$

This also shows the convergence of $I_{1,a}(\cdot, \varepsilon)$ to the same limit, since $I_{1,a}(\cdot, \varepsilon) - \tilde{I}_{1,a}(\cdot, \varepsilon) \xrightarrow{\varepsilon \rightarrow 0} 0$ u.c.p., being N continuous.

It remains to prove that $I_{1,b}(\cdot, \varepsilon) \xrightarrow{\varepsilon \rightarrow 0} 0$ u.c.p. Let $(\varepsilon_n)_n$ be a sequence converging to zero as n goes to infinity. We fix $\kappa > 0$. Recall that X and N have weakly finite quadratic variation. Then, there exists $\ell = \ell(\kappa) > 0$ such that, for every n (big enough), there is an event $\Omega_{n,\ell}$ such that $\mathbb{P}(\Omega_{n,\ell}^c) \leq \kappa$, and for $\omega \in \Omega_{n,\ell}$,

$$\begin{aligned} &\left(\sup_{s \in [0, T]} |X_s(\omega)|^2 + \int_0^T (X_{(s+\varepsilon_n)\wedge T}(\omega) - X_s(\omega))^2 \frac{ds}{\varepsilon_n} \right) \\ &+ \left(\sup_{s \in [0, T]} |N_s(\omega)|^2 + \int_0^T (N_{(s+\varepsilon_n)\wedge T}(\omega) - N_s(\omega))^2 \frac{ds}{\varepsilon_n} \right) \leq \ell. \end{aligned} \quad (3.37)$$

We provide now some estimates which are valid only for $\omega \in \Omega_{n,\ell}$. For this, we proceed as in the proof of Proposition 5.18 in [4] (estimate of the term I_{13}). We enumerate the jumps of $X(\omega)$ on $[0, T]$ by $(t_i)_{i \geq 0}$, and

$$\mathbb{K}^X = \mathbb{K}^{X(\omega)} \text{ is smallest convex compact set including } \{X_t(\omega) : t \in [0, T]\}. \quad (3.38)$$

We fix $\gamma > 0$, and we choose $M = M(\gamma, \omega)$ such that $\sum_{i=M+1}^{\infty} |\Delta X_{t_i}|^2 \leq \gamma^2$. For $\varepsilon > 0$ small enough and depending on ω , we introduce $B(\varepsilon, M) = \bigcup_{i=1}^M]t_{i-1}, t_i - \varepsilon]$ and we decompose $I_{1,b}(t, \varepsilon_n) = J^A(t, \varepsilon_n) + J^B(t, \varepsilon_n)$, with

$$\begin{aligned} J^A(t, \varepsilon_n) &= \sum_{i=1}^M \int_{t_{i-\varepsilon_n}}^{t_i} \frac{ds}{\varepsilon_n} \mathbb{1}_{]0, t]}(s) (X_{(s+\varepsilon_n)\wedge t} - X_s)(N_{(s+\varepsilon_n)\wedge t} - N_s) \\ &\quad \cdot \int_0^1 (\partial_x v((s+\varepsilon_n) \wedge t, X_s + a(X_{(s+\varepsilon_n)\wedge t} - X_s)) - \partial_x v((s+\varepsilon_n) \wedge t, X_s)) da, \\ J^B(t, \varepsilon_n) &= \frac{1}{\varepsilon_n} \int_{]0, t]} (X_{(s+\varepsilon_n)\wedge t} - X_s)(N_{(s+\varepsilon_n)\wedge t} - N_s) R^B(\varepsilon_n, s, t, M) ds, \end{aligned}$$

where

$$R^B(\varepsilon_n, s, t, M) = \mathbb{1}_{B(\varepsilon_n, M)}(s) \int_0^1 [\partial_x v((s+\varepsilon_n) \wedge t, X_s + a(X_{(s+\varepsilon_n)\wedge t} - X_s)) - \partial_x v((s+\varepsilon_n) \wedge t, X_s)] da.$$

Let δ denote the modulus of continuity. By Remark 3.12 in [4] we have for every $s, t \in [0, T]$,

$$R^B(\varepsilon_n, s, t, M) \leq \delta \left(\partial_x v \Big|_{[0, T] \times \mathbb{K}^X}, \sup_l \sup_{\substack{r, a \in [t_{i-1}, t_i] \\ |r-a| \leq \varepsilon_n}} |X_a - X_r| \right),$$

so that Lemma 2.12 in [4] applied successively to the intervals $[t_{i-1}, t_i]$ implies

$$R^B(\varepsilon_n, s, t, M) \leq \delta(\partial_x v|_{[0, T] \times \mathbb{K}^X}, 3\gamma). \quad (3.39)$$

This concludes the estimates restricted to $\omega \in \Omega_{n,\ell}$.

Since N is continuous, by (3.37) (we remember that ℓ is fixed),

$$\sup_{t \leq T} |J^A(t, \varepsilon_n)| \mathbb{1}_{\Omega_{n,\ell}} \leq \sqrt{\ell} \delta(N(\omega), \varepsilon_n) M(\gamma, \omega) \sup_{(t,x) \in [0, T] \times \mathbb{K}^X(\omega)} |\partial_x v| \xrightarrow{n \rightarrow \infty} 0, \quad \text{a.s.} \quad (3.40)$$

On the other hand, we remark that

$$\begin{aligned} \int_0^t (X_{(s+\varepsilon_n)\wedge t}(\omega) - X_s(\omega))^2 \frac{ds}{\varepsilon_n} &= \int_0^{t-\varepsilon_n} (X_{s+\varepsilon_n}(\omega) - X_s(\omega))^2 \frac{ds}{\varepsilon_n} + \int_{t-\varepsilon_n}^t (X_t(\omega) - X_s(\omega))^2 \frac{ds}{\varepsilon_n} \\ &\leq \int_0^T (X_{(s+\varepsilon_n)\wedge T}(\omega) - X_s(\omega))^2 \frac{ds}{\varepsilon_n} + \sup_{s \in [0, T]} |X_s(\omega)|^2. \end{aligned} \quad (3.41)$$

A similar estimate holds replacing X with for N . Consequently, recalling (3.39), we have

$$\begin{aligned} &\sup_{t \in [0, T]} |J^B(t, \varepsilon_n)| \mathbb{1}_{\Omega_{n,\ell}} \\ &\leq \delta(\partial_x v|_{[0, T] \times \mathbb{K}^X(\omega)}, 3\gamma) \sup_{t \in [0, T]} \sqrt{\int_0^t (X_{(s+\varepsilon)\wedge t}(\omega) - X_s(\omega))^2 \frac{ds}{\varepsilon_n} \int_0^t (N_{(s+\varepsilon)\wedge t}(\omega) - N_s(\omega))^2 \frac{ds}{\varepsilon_n}} \end{aligned}$$

$$\begin{aligned}
&\leq \delta(\partial_x v|_{[0, T] \times \mathbb{K}^X(\omega)}, 3\gamma) \cdot \\
&\cdot \sqrt{\left(\sup_{s \in [0, T]} |X_s(\omega)|^2 + \int_0^T (X_{(s+\varepsilon_n) \wedge T}(\omega) - X_s(\omega))^2 \frac{ds}{\varepsilon_n} \right) \left(\sup_{s \in [0, T]} |N_s(\omega)|^2 + \int_0^T (N_{(s+\varepsilon_n) \wedge T}(\omega) - N_s(\omega))^2 \frac{ds}{\varepsilon_n} \right)} \\
&\leq \ell \delta(\partial_x v|_{[0, T] \times \mathbb{K}^X(\omega)}, 3\gamma), \tag{3.42}
\end{aligned}$$

where in the second inequality we have used (3.41).

We continue now the proof of the u.c.p. convergence of $I_{1,b}(\cdot, \varepsilon_n)$. Let $K > 0$. By (3.36) we have

$$\begin{aligned}
\mathbb{P}\left(\sup_{t \in [0, T]} |I_{1,b}(t, \varepsilon_n)| > K \right) &\leq \mathbb{P}\left(\Omega_{n,\ell}^c \right) + \mathbb{P}\left(\sup_{t \in [0, T]} |I_{1,b}(t, \varepsilon_n)| > K, \Omega_{n,\ell} \right) \\
&\leq \kappa + \mathbb{P}\left(\sup_{t \in [0, T]} |J^A(\varepsilon_n, t)| > \frac{K}{2}, \Omega_{n,\ell} \right) + \mathbb{P}\left(\sup_{t \in [0, T]} |J^B(\varepsilon_n, t)| > \frac{K}{2}, \Omega_{n,\ell} \right) \\
&\leq \kappa + \mathbb{P}\left(\sup_{t \in [0, T]} |J^A(\varepsilon_n, t)| > \frac{K}{2}, \Omega_{n,\ell} \right) + \mathbb{P}\left(\frac{K}{2} < \ell \delta(\partial_x v|_{[0, T] \times \mathbb{K}^X(\omega)}, 3\gamma) \right),
\end{aligned}$$

where in the latter inequality we have used (3.42). This holds true for fixed $\kappa, \ell(\kappa), \gamma, K$. So, taking into account (3.40),

$$\limsup_{n \rightarrow \infty} \mathbb{P}\left(\sup_{t \in [0, T]} |I_{1,b}(t, \varepsilon_n)| > K \right) \leq \kappa + \mathbb{P}\left(\frac{K}{2} < \ell \delta(\partial_x v|_{[0, T] \times \mathbb{K}^X(\omega)}, 3\gamma) \right).$$

We let now $\gamma \rightarrow 0$. We get $\delta(\partial_x v|_{[0, T] \times \mathbb{K}^X(\omega)}, 3\gamma) \xrightarrow{\gamma \rightarrow 0} 0$ a.s., so that

$$\mathbb{P}\left(\frac{K}{2} < \ell \delta(\partial_x v|_{[0, T] \times \mathbb{K}^X(\omega)}, 3\gamma) \right) \xrightarrow{\gamma \rightarrow 0} 0.$$

Consequently,

$$\limsup_{n \rightarrow \infty} \mathbb{P}\left(\sup_{t \in [0, T]} |I_{1,b}(t, \varepsilon_n)| > K \right) \leq \kappa,$$

and since κ is arbitrary, this concludes the proof. \square

The result below follows from Theorem 3.36, together with Remark 3.20-2. and Propositions 3.34 and 3.9.

Corollary 3.40. *Let X be an \mathbb{F} -weak Dirichlet with weakly finite quadratic variation. Let $v \in C^{0,1}([0, T] \times \mathbb{R})$ and bounded, and set $Y_t = v(t, X_t)$. Then Y is a \mathbb{F} -special weak Dirichlet process.*

Remark 3.41. Let X be a càdlàg process, $v : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ a continuous function, and set $Y = v(\cdot, X)$. It is well-known that, for fixed $\omega \in \Omega$, $\mu^Y(\omega, \cdot)$ is the push forward of $\mu^X(\omega, \cdot)$ through $\mathcal{H}_\omega : (s, x) \mapsto v(s, X_{s-}(\omega) + x) - v(s, X_{s-}(\omega))$:

$$\mu^Y(\cdot]0, t] \times A) = \int_{\cdot]0, t] \times \mathbb{R}} \mathbb{1}_{A \setminus 0}(v(s, X_{s-} + x) - v(s, X_{s-})) \mu^X(ds dx), \tag{3.43}$$

for all $A \in \mathcal{B}(\mathbb{R})$. In particular,

$$\Delta Y_t = \int_{\mathbb{R}} y \mu^Y(\{t\} \times dy) = \int_{\mathbb{R}} (v(t, X_{t-} + x) - v(t, X_{t-})) \mu^X(\{t\} \times dx).$$

Taking the predictable projection in identity (3.43), we get that

$$\nu^Y(\cdot]0, t] \times A) = \int_{\cdot]0, t] \times \mathbb{R}} \mathbb{1}_{A \setminus 0}(v(s, X_{s-} + x) - v(s, X_{s-})) \nu^X(ds dx), \tag{3.44}$$

for all $A \in \mathcal{B}(\mathbb{R})$.

Remark 3.42. Let X be a weak Dirichlet process of weakly finite quadratic variation with given characteristics (B^k, ν, C) , and $v \in C^{0,1}([0, T] \times \mathbb{R})$.

- (i) By Theorem 3.36, $Y_t = v(t, X_t)$ is a weak Dirichlet process, so it admits characteristics $(\bar{B}^k, \bar{C}, \bar{\nu})$. Moreover, again by Theorem 3.36, recalling Remark 3.26-f),

$$\bar{C} \circ Y = \langle Y^c, Y^c \rangle = \int_{]0, \cdot]} |\partial_x v(s, X_s)|^2 d\langle X^c, X^c \rangle_s = \int_{]0, \cdot]} |\partial_x v(s, X_s)|^2 d(C \circ X)_s. \quad (3.45)$$

If moreover $v(s, \cdot)$ is bijective for every s , from (3.45)-(3.44) we get the explicit form of \bar{C} and $\bar{\nu}$:

$$(\bar{C} \circ Y)_t = \int_{]0, t]} |\partial_x v(s, v^{-1}(s, Y_s))|^2 d(C \circ v^{-1}(\cdot, Y))_s, \quad (3.46)$$

$$\begin{aligned} & (\bar{\nu} \circ Y)(]0, t] \times A) \\ &= \int_{]0, t] \times \mathbb{R}} \mathbb{1}_{A \setminus \{0\}} (v(s, v^{-1}(s, Y_{s-}) + x) - v(s, v^{-1}(s, Y_{s-}))) (\nu \circ v^{-1}(\cdot, Y))(ds dx), \end{aligned} \quad (3.47)$$

for all $A \in \mathcal{B}(\mathbb{R})$.

- (ii) In general, instead, it does not look possible to give explicitly the characteristic \bar{B}^k of Y in terms of the corresponding characteristic of X , even when \bar{B}^k is a bounded variation process. This can however be done for instance when v is a bijective homogeneous function of class C^2 , see Remark 3.43-(ii).

Remark 3.43. Let X be a weak Dirichlet process with weakly finite quadratic variation. Let $h : \mathbb{R} \rightarrow \mathbb{R}$ in C^1 and bijective. By Theorem 3.36, $Y_t := h(X_t)$ is a weak Dirichlet process.

- (i) By Theorem 3.10. we have

$$\begin{aligned} Y &= Y^c + \Gamma^k(h) + (h(X_{s-} + x) - h(X_{s-})) \frac{x - k(x)}{x} \star \mu^X \\ &\quad + (h(X_{s-} + x) - h(X_{s-})) \frac{k(x)}{x} \star (\mu^X - \nu^X), \end{aligned} \quad (3.48)$$

with $\Gamma^k(h)$ a predictable and \mathbb{F} -martingale orthogonal process. The characteristic \bar{B}^k of Y can be determined in terms of ν^X and of the map $\Gamma^k(h)$.

As a matter of fact, by Corollary 3.22-(i) together with Remark 3.23, recalling (3.43)-(3.44), we have

$$\begin{aligned} Y &= Y^c + \bar{B}^{k,Y} + (y - k(y)) \star \mu^Y + k(y) \star (\mu^Y - \nu^Y) \\ &= Y^c + \bar{B}^{k,Y} + (h(X_{s-} + x) - h(X_{s-}) - k(h(X_{s-} + x) - h(X_{s-}))) \star \mu^X \\ &\quad + k(h(X_{s-} + x) - h(X_{s-})) \star (\mu^X - \nu^X) \\ &= Y^c + \bar{B}^{k,Y} + \left[(h(X_{s-} + x) - h(X_{s-})) \frac{k(x)}{x} - k(h(X_{s-} + x) - h(X_{s-})) \right] \star \mu^X \\ &\quad + (h(X_{s-} + x) - h(X_{s-})) \frac{x - k(x)}{x} \star \mu^X + k(h(X_{s-} + x) - h(X_{s-})) \star (\mu^X - \nu^X). \end{aligned} \quad (3.49)$$

Subtracting (3.48) from (3.49), we get

$$0 = \bar{B}^{k,Y} - \Gamma^k(h) + \left[(h(X_{s-} + x) - h(X_{s-})) \frac{k(x)}{x} - k(h(X_{s-} + x) - h(X_{s-})) \right] \star \mu^X$$

$$\begin{aligned}
& + \left[k(h(X_{s-} + x) - h(X_{s-})) - (h(X_{s-} + x) - h(X_{s-})) \frac{k(x)}{x} \right] \star (\mu^X - \nu^X) \\
& = \bar{B}^{k,Y} - \Gamma^k(h) + \left[k(h(X_{s-} + x) - h(X_{s-})) - (h(X_{s-} + x) - h(X_{s-})) \frac{k(x)}{x} \right] \star \nu^X,
\end{aligned}$$

that provides

$$\bar{B}^{k,Y} = \Gamma^k(h) - \left[k(h(X_{s-} + x) - h(X_{s-})) - \frac{(h(X_{s-} + x) - h(X_{s-}))}{x} k(x) \right] \star \nu^X.$$

- (ii) Assume moreover that $h \in C^2$, and that X is a semimartingale with characteristics (B^k, C, ν) . Then it is possible to express the characteristic \bar{B}^k of Y explicitly in terms of the characteristics of X . In particular, this involves a Lebesgue integral with respect to the bounded variation process B^k . As a matter of fact, for every $f \in C^2 \cap C_b^0$, $f(Y)$ is a special semimartingale. By Theorem D.1 applied to $(f \circ h)(X)$, the predictable bounded variation part of $f(Y)$ is given by

$$\begin{aligned}
& (f \circ h)(X_0) + \frac{1}{2} \int_0^\cdot (f \circ h)''(X_s) dC_s^X + \int_0^\cdot (f \circ h)'(X_s) dB_s^{k,X} \\
& + \int_{]0, \cdot] \times \mathbb{R}} ((f \circ h)(X_{s-} + x) - (f \circ h)(X_{s-}) - k(x) (f \circ h)'(X_{s-})) \nu^X(ds dx) \\
& = f(Y_0) + \frac{1}{2} \int_0^\cdot [f''(h(X_s))(h'(X_s))^2 + f'(h(X_s))h''(X_s)] dC_s^X + \int_0^\cdot f'(h(X_s))h'(X_s) dB_s^{k,X} \\
& + \int_{]0, \cdot] \times \mathbb{R}} [(f \circ h)(X_{s-} + x) - (f \circ h)(X_{s-}) - k(x) f'(h(X_{s-}))h'(X_{s-})] \nu^X(ds dx). \quad (3.50)
\end{aligned}$$

On the other hand, again by Theorem D.1 applied to $f(Y)$, and recalling (3.45) and (3.44), the process above is equal to

$$\begin{aligned}
& f(Y_0) + \frac{1}{2} \int_0^\cdot f''(Y_s) dC_s^Y + \int_0^\cdot f'(Y_s) d\bar{B}_s^{k,Y} \\
& + \int_{]0, \cdot] \times \mathbb{R}} (f(Y_{s-} + y) - f(Y_{s-}) - k(y) f'(Y_{s-})) \nu^Y(ds dy) \\
& = f(Y_0) + \frac{1}{2} \int_0^\cdot f''(h(X_s)) (h'(X_s))^2 dC_s^X + \int_0^\cdot f'(h(X_s)) d\bar{B}_s^{k,Y} \\
& + \int_{]0, \cdot] \times \mathbb{R}} [k(x) h'(X_{s-}) - k(h(X_{s-} + x) - h(X_{s-}))] f'(h(X_{s-})) \nu^X(ds dy) \\
& + \int_{]0, \cdot] \times \mathbb{R}} [(f \circ h)(X_{s-} + x) - (f \circ h)(X_{s-}) - k(x) f'(h(X_{s-}))h'(X_{s-})] \nu^X(ds dx). \quad (3.51)
\end{aligned}$$

Subtracting (3.51) from (3.50) we get

$$\begin{aligned}
& \frac{1}{2} \int_0^\cdot f'(h(X_s))h''(X_s) dC_s^X + \int_0^\cdot f'(h(X_s))[h'(X_s)dB_s^{k,X} - d\bar{B}_s^{k,Y}] \\
& - \int_{]0, \cdot] \times \mathbb{R}} [k(x) h'(X_{s-}) - k(h(X_{s-} + x) - h(X_{s-}))] f'(h(X_{s-})) \nu^X(ds dy) = 0. \quad (3.52)
\end{aligned}$$

Define now the unit partition $\chi : \mathbb{R} \rightarrow \mathbb{R}$ as the smooth function $\chi(a)$ equal to 1 if $a \leq -1$, equal to 0 if $a \geq 0$, and such that $\chi(a) \in [0, 1]$ for $a \in (-1, 0)$. Set

$$\chi_N(x) := \chi(|x| - (N + 1)) = \begin{cases} 1 & \text{if } |x| \leq N \\ 0 & \text{if } |x| \geq N + 1 \\ \in [0, 1] & \text{otherwise.} \end{cases} \quad (3.53)$$

Notice that $\chi_N(x)$ is a smooth function. We apply (3.52) with $f = f_N$, where $f_N(0) = 0$ and $f'_N(x) = \chi_N(x)$. Letting $N \rightarrow \infty$ in (3.52), we get

$$\begin{aligned} \bar{B}^{k,Y} &= \frac{1}{2} \int_0^\cdot h''(X_s) dC_s^X + \int_0^\cdot h'(X_s) dB_s^{k,X} \\ &\quad - \int_{]0,\cdot] \times \mathbb{R}} [k(x) h'(X_{s-}) - k(h(X_{s-} + x) - h(X_{s-}))] \nu^X(ds dx). \end{aligned}$$

4 Generalized martingale problems

4.1 Stochastic calculus related to martingale problems

Let (Ω, \mathbb{F}) be a measurable space and \mathbb{P} a probability measure. Suppose that X is a weakly finite quadratic variation process, with canonical filtration $\mathbb{F}^X = (\mathcal{F}_t^X)$, and that, for every v belonging to some linear dense subspace \mathcal{D}^S of $C^{0,1}([0, T] \times \mathbb{R})$, $v(\cdot, X)$ is a weak Dirichlet process. The theorem below provides necessary and sufficient conditions under which $v(\cdot, X)$ is weak Dirichlet for every $v \in C^{0,1}([0, T] \times \mathbb{R})$.

In the sequel we will make use of the following property for the couple (\mathcal{D}^S, X) .

Hypothesis 4.1. *For every $v \in \mathcal{D}^S$, $Y^v := v(\cdot, X)$ is a weak Dirichlet process, with unique continuous local martingale component $Y^{v,c}$.*

Remark 4.2. Let $\mathcal{D}^S \subseteq C^{0,1}([0, T] \times \mathbb{R})$, and X be a càdlàg process satisfying (1.3). If Hypothesis 4.1 holds true for (\mathcal{D}^S, X) , then \mathcal{D}^S is contained in the set \mathcal{D} introduced in Remark 3.12. As a matter of fact, since \mathcal{D}^S is contained in $C^{0,1}$, then Hypothesis 3.7 holds true, see Proposition 3.9.

Theorem 4.3. *Let \mathcal{D}^S be a dense subspace of $C^{0,1}([0, T] \times \mathbb{R})$, and X be a weakly finite quadratic variation process. The following are equivalent.*

1. X is a weak Dirichlet process.
2. (i) $v \mapsto Y^{v,c}$, $\mathcal{D}^S \rightarrow \mathbb{D}^{ucp}$, is continuous in zero.
(ii) Hypothesis 4.1 holds true for (\mathcal{D}^S, X) .
3. $v(t, X_t)$ is a weak Dirichlet process for every $v \in C^{0,1}([0, T] \times \mathbb{R})$.

Proof. 3. \Rightarrow 1. This follows taking $v \equiv \text{Id}$.

1. \Rightarrow 2. By Theorem 3.36, for every $v \in C^{0,1}([0, T] \times \mathbb{R})$, $v(\cdot, X)$ is a weak Dirichlet process, and

$$Y^{v,c} = Y_0 + \int_0^\cdot \partial_x v(s, X_s) dX_s^c.$$

By Problem 2.27, Chapter 3, in [28], this implies the continuity stated in item (i). Moreover, item (ii) trivially holds.

2. \Rightarrow 3. By item (ii), for every $v \in \mathcal{D}^S$, $v(\cdot, X)$ is a weak Dirichlet process, with unique continuous martingale component $Y^{v,c}$. Since by item (i) $v \mapsto Y^{v,c}$, $\mathcal{D}^S \rightarrow \mathbb{D}^{ucp}$, is continuous, it extends continuously to $C^{0,1}$. We will denote in the same way the extended operator. Since the space of continuous local martingales is closed under the u.c.p. convergence, $Y^{v,c}$ is a continuous local martingale for every $v \in C^{0,1}([0, T] \times \mathbb{R})$.

We denote $A^v := v(\cdot, X) - Y^{v,c}$, for every $v \in C^{0,1}([0, T] \times \mathbb{R})$. It remains to prove that A^v is a martingale orthogonal process, namely that, for every continuous local martingale N ,

$$[v(\cdot, X), N] = [Y^{v,c}, N]. \quad (4.1)$$

Step a). Equality (4.1) holds true for every $v \in \mathcal{D}^S$, since $v(\cdot, X_\cdot)$ is a weak Dirichlet process, and therefore $v(\cdot, X_\cdot) - Y^{v,c}$ is a martingale orthogonal process, see Proposition 3.2.

Step b). Let (ε_n) be a sequence converging to zero. We need to show that (recall Proposition A.3 in [4])

$$\int_0^\cdot [v(s + \varepsilon_n, X_{s+\varepsilon_n}) - v(s, X_s)](N_{s+\varepsilon_n} - N_s) \frac{ds}{\varepsilon_n} \rightarrow [Y^{v,c}, N] \quad \text{u.c.p. as } n \rightarrow \infty. \quad (4.2)$$

Indeed, for this, it is enough to show the existence of a subsequence, still denoted by (ε_n) , such that (4.2) holds. Since N is a martingale, $[N, N]$ exists, so that (again by Proposition A.3 in [4])

$$[N, N]_{\varepsilon_n} := \int_0^\cdot (N_{s+\varepsilon_n} - N_s)^2 \frac{ds}{\varepsilon_n} \rightarrow [N, N] \quad \text{u.c.p. as } n \rightarrow \infty.$$

By extraction of subsequence, we can assume that previous convergence holds uniformly almost surely.

Let us then prove (4.2). To this end, we introduce the maps

$$\begin{aligned} T_n : C^{0,1}([0, T] \times \mathbb{R}) &\rightarrow \mathbb{D}^{ucp} \\ v &\mapsto \int_0^\cdot [v(s + \varepsilon_n, X_{s+\varepsilon_n}) - v(s, X_s)](N_{s+\varepsilon_n} - N_s) \frac{ds}{\varepsilon_n}. \end{aligned}$$

They are linear and continuous, \mathbb{D}^{ucp} is an F -space in the sense of [13], Chapter 2.1.

Suppose that we exhibit a metric d_{ucp} related to \mathbb{D}^{ucp} such that,

$$\text{for every fixed } v \in C^{0,1}([0, T] \times \mathbb{R}), \quad T_n(v) \text{ is bounded in } \mathbb{D}^{ucp}. \quad (4.3)$$

By Step a), for every $v \in \mathcal{D}^S$ (dense subset of $C^{0,1}([0, T] \times \mathbb{R})$) we already know that

$$T_n(v) \rightarrow [Y^{v,c}, N] \quad \text{u.c.p. as } n \rightarrow \infty. \quad (4.4)$$

Then Banach-Steinhaus theorem would imply the existence of a linear and continuous map $T : C^{0,1}([0, T] \times \mathbb{R}) \rightarrow \mathbb{D}^{ucp}$ such that $T_n(v) \rightarrow T(v)$ u.c.p. for all $v \in C^{0,1}([0, T] \times \mathbb{R})$. The map $v \mapsto [Y^{v,c}, N]$ is continuous by Proposition A.4. Then by (4.4), $T(v) \equiv [Y^{v,c}, N]$.

Step c). It remains to show (4.3). Let $v \in C^{0,1}([0, T] \times \mathbb{R})$. We have $T_n(v) = T_n^1(v) + T_n^2(v)$, where

$$\begin{aligned} T_n^1(v) &:= \frac{1}{\varepsilon_n} \int_0^\cdot [v(s + \varepsilon_n, X_{s+\varepsilon_n}) - v(s + \varepsilon_n, X_s)](N_{s+\varepsilon_n} - N_s) ds, \\ T_n^2(v) &:= \frac{1}{\varepsilon_n} \int_0^\cdot [v(s + \varepsilon_n, X_s) - v(s, X_s)](N_{s+\varepsilon_n} - N_s) ds. \end{aligned}$$

By similar arguments as in the proof of Proposition 3.10 in [24], $T_n^2(v) \rightarrow 0$ u.c.p. as $n \rightarrow \infty$. Indeed, since v is continuous,

$$\frac{1}{\varepsilon_n^2} \int_0^T \left| \int_{r-\varepsilon_n}^r [v(s + \varepsilon_n, X_s) - v(s, X_s)] ds \right|^2 d[N, N]_r$$

converges to zero, so that

$$\int_0^\cdot \frac{dN_r}{\varepsilon_n} \int_{r-\varepsilon_n}^r [v(s + \varepsilon_n, X_s) - v(s, X_s)] ds \quad (4.5)$$

converges u.c.p. to zero by Problem 2.27, Chapter 3, in [28]. By stochastic Fubini theorem, we observe that the processes in (4.5) are equal to $(T_n^2(v))$ up to a sequence of processes converging u.c.p. to zero. Concerning $(T_n^1(v))$, we have

$$T_n^1(v)(t) := \frac{1}{\varepsilon_n} \int_0^t ds \int_0^1 da \partial_x v(s + \varepsilon_n, X_s + a(X_{s+\varepsilon_n} - X_s))(X_{s+\varepsilon_n} - X_s)(N_{s+\varepsilon_n} - N_s).$$

Step d). We choose $d_{ucp}(X^1, X^2) = \mathbb{E} \left[\sup_{t \leq T} |X_t^1 - X_t^2| \wedge 1 \right]$ if $X^1, X^2 \in \mathbb{D}^{ucp}$. Since $T_n^2(v)$ is a converging sequence, then it is necessarily bounded. To prove that $T_n(v)$ is bounded it remains to prove the same for $T_n^1(v)$. To this end, let $\kappa > 0$. We need to show the existence of δ such that

$$d_{ucp}(\delta T_n^1(v), 0) < \kappa \quad \forall n. \quad (4.6)$$

Let $\mathbb{K}^X(\omega)$ be the random set introduced in (3.38). Now, introducing the (finite) random variables

$$\begin{aligned} \tilde{\Lambda}(\omega) &:= \sup_{s \in [0, T], x \in \mathbb{K}^X(\omega)} |\partial_x v|(s, x), \\ \Lambda(\omega) &:= \tilde{\Lambda}(\omega) \sup_n \left[\int_0^T (N_{s+\varepsilon_n}(\omega) - N_s(\omega))^2 ds \right]^{1/2}, \end{aligned}$$

we get

$$\begin{aligned} \sup_{t \leq T} |T_n^1(v)| &\leq \left[\frac{1}{\varepsilon_n} \int_0^T (X_{s+\varepsilon_n} - X_s)^2 ds \frac{1}{\varepsilon_n} \int_0^T (N_{s+\varepsilon_n} - N_s)^2 ds \right]^{1/2} \tilde{\Lambda} \\ &\leq \left[\frac{1}{\varepsilon_n^2} \int_0^T (X_{s+\varepsilon_n} - X_s)^2 ds \right]^{1/2} \Lambda. \end{aligned}$$

Since X is of weakly finite quadratic variation, we can introduce $M > 0$ such that

$$\mathbb{P} \left(\frac{1}{\varepsilon_n^2} \int_0^T (X_{s+\varepsilon_n} - X_s)^2 ds > M^2 \right) \leq \frac{\kappa}{4}, \quad \mathbb{P}(|\Lambda| > M) \leq \frac{\kappa}{4}, \quad \forall n.$$

Now, setting

$$\Omega_{M,n} := \left(\left\{ \frac{1}{\varepsilon_n^2} \int_0^T (X_{s+\varepsilon_n} - X_s)^2 ds > M^2 \right\} \cup \{|\Lambda| > M\} \right)^c,$$

we notice that $\mathbb{P}(\Omega_{M,n}^c) \leq \frac{\kappa}{2}$. We have

$$\mathbb{E} \left[\sup_{t \leq T} \delta |T_n^1(v)| \wedge 1 \right] = \mathbb{E} \left[1_{\Omega_{M,n}^c} \sup_{t \leq T} \delta |T_n^1(v)| \wedge 1 \right] + \mathbb{E} \left[1_{\Omega_{M,n}} \sup_{t \leq T} \delta |T_n^1(v)| \wedge 1 \right]$$

is bounded by

$$\frac{\kappa}{2} + \mathbb{E} \left[1_{\Omega_{M,n}} \sup_{t \leq T} \delta |T_n^1(v)| \wedge 1 \right] \leq \frac{\kappa}{2} + \mathbb{E}[\delta M^2 \wedge 1] \leq \frac{\kappa}{2} + \delta M^2.$$

Formula (4.6) follows by choosing δ so that $\delta M^2 < \frac{\kappa}{2}$. \square

Remark 4.4. In Section 4.2 we will introduce a suitable notion of path-dependent martingale problem with respect to some operator \mathcal{A} and domain $\mathcal{D}_{\mathcal{A}}$. In particular, if X is a solution to the aforementioned martingale problem, $v(t, X_t)$ is a special semimartingale for every $v \in \mathcal{D}_{\mathcal{A}}$. In that case, if $\mathcal{D}_{\mathcal{A}}$ is a dense subspace in $C^{0,1}$, Theorem 4.3 (with $\mathcal{D}^S = \mathcal{D}_{\mathcal{A}}$) will contribute to obtain a (weak Dirichlet) decomposition for $v(t, X_t)$ to every $v \in C^{0,1}$. In many irregular situations, the identity function does not belong to $\mathcal{D}_{\mathcal{A}}$ but only to $C^{0,1}$; this allows among others to get a sort of Doob-Meyer type decomposition for the process X itself.

The result below reformulates point 2.(i) of Theorem 4.3 in terms of the map Γ^k in Theorem 3.10.

Proposition 4.5. *Let \mathcal{D}^S be a dense subspace of $C^{0,1}([0, T] \times \mathbb{R})$. Let X be a weakly finite quadratic variation process. Assume that Hypothesis 4.1 holds true for (\mathcal{D}^S, X) .*

Then, $v \mapsto Y^{v,c}$, $\mathcal{D}^S \subseteq C^{0,1}([0, T] \times \mathbb{R}) \rightarrow \mathbb{D}^{ucp}$, is continuous in zero if and only if the map Γ^k in Theorem 3.10 restricted to \mathcal{D}^S is continuous in zero with respect to the $C^{0,1}$ -topology.

Proof. The equivalence property follows by Lemma C.5 and formula (3.9) in Theorem 3.10. \square

We end this section relating the brackets of some martingales associated with a semimartingale of the type $v(\cdot, X)$, where X is a càdlàg process, to the map $\Gamma(v)$ in Theorem 3.19.

Proposition 4.6. *Let X be a càdlàg process satisfying (1.3). Let $v \in C^{0,1}([0, T] \times \mathbb{R})$ and bounded such that $Y := v(\cdot, X)$ is a semimartingale, with unique local martingale component Y^c . Then, Y is a special semimartingale with unique martingale part N satisfying*

$$\langle N, N \rangle = \Gamma(v^2) - 2 \int_0^\cdot v(s, X_{s-}) d\Gamma_s(v) - \sum_{0 \leq s \leq \cdot} |\Delta \Gamma_s(v)|^2. \quad (4.7)$$

Moreover,

$$\langle Y^c, Y^c \rangle = \Gamma(v^2) - 2 \int_0^\cdot v(s, X_{s-}) d\Gamma_s(v) - [v(s, X_{s-} + x) - v(s, X_{s-})]^2 \star \nu^{X, \mathbb{P}} \quad (4.8)$$

$$= \Gamma^c(v^2) - 2 \int_0^\cdot v(s, X_{s-}) d\Gamma_s^c(v) - [v(s, X_{s-} + x) - v(s, X_{s-})]^2 \star \nu^{X, \mathbb{P}, c}, \quad (4.9)$$

where $\nu^{X, \mathbb{P}, c}$ (resp. $\Gamma^c(v)$) is the continuous part of $\nu^{X, \mathbb{P}}$ (resp. of $\Gamma(v)$), where $\Gamma(v)$ is the process appearing in Theorem 3.19.

Remark 4.7. Formula (4.7) implies in particular that if $\Gamma(v)$ and $\Gamma(v^2)$ are continuous, then $\langle N, N \rangle$ is continuous as well.

Remark 4.8. By Theorem D.1, Y^{v^2} is a semimartingale. On the other hand, $Y^v := v(\cdot, X)$ and $Y^{v^2} := v^2(\cdot, X)$ are special weak Dirichlet processes by Theorem 3.16 and Remark 3.17, being $v, v^2 \in C^{0,1}$ and bounded. This implies that Y^v and Y^{v^2} are special semimartingales.

Proof of Proposition 4.6. We first prove identity (4.9). By Theorem 3.19, we get that

$$N_t := Y_t - Y_0 - \Gamma_t(v), \quad (4.10)$$

$$\bar{N}_t := Y_t^2 - Y_0^2 - \Gamma_t(v^2) \quad (4.11)$$

are local martingales under \mathbb{P} . Applying the integration by parts formula, and taking into account (4.10) and (4.11), we get

$$\begin{aligned} [Y, Y]_t &= Y_t^2 - Y_0^2 - 2 \int_0^t Y_{s-} dY_s \\ &= \Gamma_t(v^2) - 2 \int_0^t v(s, X_{s-}) d\Gamma_s(v) + \bar{N}_t - 2 \int_0^t v(s, X_{s-}) dN_s. \end{aligned} \quad (4.12)$$

We now show that

$$[v(s, X_{s-} + x) - v(s, X_{s-})]^2 \star \nu^{X, \mathbb{P}} \in \mathcal{A}_{\text{loc}}^+. \quad (4.13)$$

As a matter of fact,

$$\begin{aligned} & [v(s, X_{s-} + x) - v(s, X_{s-})]^2 \star \nu^{X, \mathbb{P}} \\ &= [v(s, X_{s-} + x) - v(s, X_{s-})]^2 \frac{k^2(x)}{x^2} \star \nu^{X, \mathbb{P}} + [v(s, X_{s-} + x) - v(s, X_{s-})]^2 \frac{x^2 - k^2(x)}{x^2} \star \nu^{X, \mathbb{P}}. \end{aligned} \quad (4.14)$$

The first term in the right-hand side of (4.14) belongs to $\mathcal{A}_{\text{loc}}^+$ by Proposition C.4. On the other hand, let $c > 0$ such that $k(x) = x$ on $[-c, c]$. Then the second term in the right-hand side of (4.14) belongs to $\mathcal{A}_{\text{loc}}^+$ by Lemma C.2 and the fact that v is bounded.

At this point, by (1.2) and Proposition 3.5,

$$\begin{aligned} [Y, Y] &= \langle Y^c, Y^c \rangle + \sum_{s \leq \cdot} |\Delta Y_s|^2 = \langle Y^c, Y^c \rangle + [v(s, X_{s-} + x) - v(s, X_{s-})]^2 \star \mu^X \\ &= \langle Y^c, Y^c \rangle + [v(s, X_{s-} + x) - v(s, X_{s-})]^2 \star \nu^{X, \mathbb{P}} + [v(s, X_{s-} + x) - v(s, X_{s-})]^2 \star (\mu^X - \nu^{X, \mathbb{P}}). \end{aligned} \quad (4.15)$$

Since (4.12) and (4.15) provide two decompositions of the same special semimartingale $[Y, Y]$, we get (4.8). Denoting by C the right hand-side of (4.8), we remark that C is a continuous bounded variation process. Therefore, taking into account Remark 3.20-1., (4.8) implies (4.9).

On the other hand, Theorem 3.19 implies the unique decomposition (4.10) with

$$N_t := Y_t^c - Y_0 + [v(s, X_{s-} + x) - v(s, X_{s-})] \star (\mu^X - \nu^{X, \mathbb{P}})_t. \quad (4.16)$$

Now we notice that, by formula (4.13), the stochastic integral appearing in the right-hand side of (4.16) is a locally square integrable martingale, and its predictable bracket yields

$$[v(s, X_{s-} + x) - v(s, X_{s-})]^2 \star \nu^{X, \mathbb{P}} - \sum_{0 \leq s \leq \cdot} \left| \int_{\mathbb{R}} [v(s, X_{s-} + x) - v(s, X_{s-})] \nu^{X, \mathbb{P}}(\{s\} \times dx) \right|^2,$$

see the end of Section 2. Since that purely discontinuous martingale is martingale orthogonal,

$$\begin{aligned} \langle N, N \rangle &= \langle Y^c, Y^c \rangle + [v(s, X_{s-} + x) - v(s, X_{s-})]^2 \star \nu^{X, \mathbb{P}} \\ &\quad - \sum_{0 \leq s \leq \cdot} \left| \int_{\mathbb{R}} [v(s, X_{s-} + x) - v(s, X_{s-})] \nu^{X, \mathbb{P}}(\{s\} \times dx) \right|^2. \end{aligned} \quad (4.17)$$

Identity (4.7) follows by plugging (4.8) in (4.17). \square

4.2 Definition and main properties of the martingale problem

Given $\eta \in D_-(0, T)$ (resp. $\zeta \in D(0, T)$), we will use the notation

$$\eta^t(s) := \begin{cases} \eta(s) & \text{if } s < t, \\ \eta(t) & \text{if } s \geq t \end{cases} \quad \left(\text{resp. } \zeta^{t-}(s) := \begin{cases} \zeta(s) & \text{if } s < t, \\ \zeta(t-) & \text{if } s \geq t \end{cases} \right).$$

Let $\mathcal{D} \subseteq C^{0,1}([0, T] \times \mathbb{R})$, $\Lambda : \mathcal{D} \rightarrow C([0, T] \times D_-(0, T))$ be a measurable map and $\gamma : [0, T] \times D_-(0, T) \rightarrow \mathbb{R}$. We consider the following hypothesis for the triplet $(\mathcal{D}, \Lambda, \gamma)$.

Hypothesis 4.9. *For every $\eta \in D_-(0, T)$, $\gamma(\cdot, \eta)$ is of bounded variation. We also suppose the non-anticipating property holds for Λ and γ , i.e., for every $v \in \mathcal{D}$, $\eta \in D_-(0, T)$, $(\Lambda v)(t, \eta) = (\Lambda v)(t, \eta^t)$ and $\gamma(t, \eta) = \gamma(t, \eta^t)$.*

Definition 4.10. Fix $N \in \mathbb{N}$. Let $\mathcal{D}_{\mathcal{A}} \subseteq C^{0,1}([0, T] \times \mathbb{R})$, $\Lambda_i : \mathcal{D}_{\mathcal{A}} \rightarrow C([0, T] \times D_-(0, T))$ and $\gamma_i : [0, T] \times D_-(0, T) \rightarrow \mathbb{R}$, such that $(\mathcal{D}_{\mathcal{A}}, \Lambda_i, \gamma_i)$ fulfill Hypothesis 4.9 for all $i = 1, \dots, N$. For every $v \in \mathcal{D}_{\mathcal{A}}$, $\eta \in D_-(0, T)$, we set

$$(\mathcal{A}v)(ds, \eta) := \sum_{i=1}^N \Lambda_i v(s, \eta) \gamma_i(ds, \eta). \quad (4.18)$$

Remark 4.11. We have the decomposition $(\mathcal{A}v)(ds, \cdot) = (\mathcal{A}v)((ds)^c, \cdot) + (\mathcal{A}v)(\Delta s, \cdot)$, with

$$(\mathcal{A}v)(\Delta s, \cdot) := \sum_{i=1}^N \Lambda_i v(s, \cdot) \gamma_i(\Delta s, \cdot), \quad (4.19)$$

$$(\mathcal{A}v)((ds)^c, \cdot) := \sum_{i=1}^N \Lambda_i v(s, \cdot) \gamma_i^c(ds, \cdot). \quad (4.20)$$

Definition 4.12. Let \mathcal{A} and $\mathcal{D}_{\mathcal{A}}$ as in Definition 4.10. A process X is said to solve the martingale problem (under a probability \mathbb{P}) with respect to \mathcal{A} , $\mathcal{D}_{\mathcal{A}}$ and $x_0 \in \mathbb{R}$, if

- (i) condition (1.3) holds under \mathbb{P} ;
- (ii) for any $v \in \mathcal{D}_{\mathcal{A}}$ and bounded,

$$M_t^v := v(t, X_t) - v(0, x_0) - \int_0^t (\mathcal{A}v)(ds, X^{s-}) \quad (4.21)$$

is an (\mathcal{F}_t^X) -local martingale.

Remark 4.13. Let \mathcal{A} and $\mathcal{D}_{\mathcal{A}}$ as in Definition 4.10, and set

$$\mathcal{X} = \{M^v : v \in \mathcal{D}_{\mathcal{A}}\}, \quad (4.22)$$

with M^v defined in (4.21). Then (X, \mathbb{P}) solves the martingale problem with respect to \mathcal{A} , $\mathcal{D}_{\mathcal{A}}$ and x_0 if and only if \mathbb{P} is solution of the martingale problem in Definition 1.3, Chapter III, in [27] associated to $(\Omega, \mathcal{F}, \mathbb{F})$, \mathcal{X} in (4.22), and \mathbb{P}_H corresponding to δ_{x_0} (for any $B \in \mathcal{B}(\mathbb{R})$, $\mathbb{P}_H(B) = \delta_{x_0}(B_0)$ with $B_0 = \{\omega(0) \in \mathbb{R} : \omega \in B\}$).

Definition 4.14 (Existence). We say that the martingale problem related to \mathcal{A} , $\mathcal{D}_{\mathcal{A}}$ and x_0 meets existence if there exists a couple (X, \mathbb{P}) on some (Ω, \mathcal{F}) , solution to the martingale problem in the sense of Definition 4.12.

Definition 4.15 (Uniqueness). We say that the martingale problem related to \mathcal{A} , $\mathcal{D}_{\mathcal{A}}$ and x_0 admits uniqueness if, given two spaces $(\Omega_1, \mathcal{F}_1)$ and $(\Omega_2, \mathcal{F}_2)$, and two solutions (X_1, \mathbb{P}_1) and (X_2, \mathbb{P}_2) to the martingale problem in the sense of Definition 4.12, then $\mathcal{L}(X_1)|_{\mathbb{P}_1} = \mathcal{L}(X_2)|_{\mathbb{P}_2}$.

Remark 4.16. Let $(\check{\Omega}, \check{\mathcal{F}})$ be the canonical space, and \mathbb{F} be the canonical filtration of the canonical process \check{X} . (X, \mathbb{P}) is a solution the the martingale problem related to \mathcal{A} , $\mathcal{D}_{\mathcal{A}}$ and x_0 , if and only if $(\check{X}, \mathcal{L}(X))$ is a solution to the martingale problem related to \mathcal{A} , $\mathcal{D}_{\mathcal{A}}$ and x_0 .

4.3 Time-homogeneous towards time-inhomogeneous martingale problem

Definition 4.17. Let $\mathcal{D}_{\mathcal{L}} \subseteq C_b^0$ and $\mathcal{L} : \mathcal{D}_{\mathcal{L}} \rightarrow C([0, T] \times D_-(0, T))$. We say that (X, \mathbb{P}) fulfills the time-homogeneous martingale problem with respect to $\mathcal{D}_{\mathcal{L}}$, \mathcal{L} and x_0 , if for any $f \in \mathcal{D}_{\mathcal{L}}$ and bounded, the process

$$M_t^f := f(X_t) - f(x_0) - \int_0^t (\mathcal{L}f)(s, X^{s-}) ds \quad (4.23)$$

is an (\mathcal{F}_t^X) -local martingale under \mathbb{P} .

Theorem 4.18. Assume that $C([0, T] \times D_-(0, T))$ is equipped with the metric topology of the uniform convergence on closed balls. Let $\mathcal{D}_{\mathcal{L}}$ be a Fréchet space, topologically included in C_b^0 , and equipped with some metric $d_{\mathcal{L}}$. Let $\mathcal{L} : \mathcal{D}_{\mathcal{L}} \rightarrow C([0, T] \times D_-(0, T))$ be a continuous map. (X, \mathbb{P}) fulfills the time-homogeneous martingale problem in Definition 4.17 with respect to $\mathcal{D}_{\mathcal{L}}$, \mathcal{L} and x_0 , if and only if (X, \mathbb{P}) fulfills the time-inhomogeneous martingale problem in Definition 4.12 with respect to x_0 ,

$$\mathcal{D}_{\mathcal{A}} := C^1([0, T]; \mathcal{D}_{\mathcal{L}}), \quad (4.24)$$

and

$$(\mathcal{A}v)(dt, \eta) := (\partial_t v(t, \eta) + (\mathcal{L}v)(t, \eta))dt, \quad v \in \mathcal{D}_{\mathcal{A}}, \quad \eta \in D_-(0, T). \quad (4.25)$$

Remark 4.19. (i) A typical example of metric $d_{\mathcal{L}}$ comes from the graph topology. Let $\mathcal{D}_{\mathcal{L}} \subseteq C_b^0$, and $\mathcal{L} : \mathcal{D}_{\mathcal{L}} \rightarrow C([0, T] \times D_-(0, T))$ be a measurable map. Assume that $\mathcal{D}_{\mathcal{L}}$ is equipped with the graph topology of \mathcal{L} : $v_n \rightarrow 0$ in $\mathcal{D}_{\mathcal{L}}$ if and only if $v_n \rightarrow 0$ in C_b^0 and $\mathcal{L}v_n \rightarrow 0$ in $C([0, T] \times D_-(0, T))$. Then \mathcal{L} is obviously a continuous map.

(ii) $\mathcal{D}_{\mathcal{A}}$ in (4.24) is constituted by bounded functions whose derivative in time is also bounded.

Proof. The *if* implication is trivial choosing v not depending on time.

Let us now prove the *only if* implication. We need to show that, for every $v \in \mathcal{D}_{\mathcal{A}}$,

$$M_t^v := v(t, X_t) - v(0, x_0) - \int_0^t (\partial_s v(s, X_s) + (\mathcal{L}v)(s, X^{s-})) ds \quad (4.26)$$

is a local martingale. If $v = f \in \mathcal{D}_{\mathcal{L}}$ then (4.26) is obviously a local martingale. Suppose that $v(t, x) = a(t)f(x)$, with $a \in C^1(0, T)$, and $f \in \mathcal{D}_{\mathcal{L}}$. By integrating by parts, we get

$$\begin{aligned} v(t, X_t) &= a(0)f(x_0) + \int_0^t a'(s)f(X_s)ds + \int_0^t a(s)df(X_s) \\ &= a(0)f(x_0) + \int_0^t a'(s)f(X_s)ds + \int_0^t a(s)(\mathcal{L}f)(X^{s-})ds + \int_0^t a(s)dM_s^f \\ &= v(0, x_0) + \int_0^t (\mathcal{A}v)(ds, X^{s-}) + M_t^v, \end{aligned} \quad (4.27)$$

where M^v is a local martingale. So (4.26) is a local martingale when $u \in \hat{\mathcal{D}}_{\mathcal{A}}$, with $\hat{\mathcal{D}}_{\mathcal{A}}$ the set in Lemma B.8.

Let now $v \in \mathcal{D}_{\mathcal{A}}$. By Lemma B.8, there is a sequence $v_n \in \hat{\mathcal{D}}_{\mathcal{A}}$, and such that $v_n \xrightarrow{n \rightarrow \infty} v$ with respect to $C^1([0, T]; \mathcal{D}_{\mathcal{L}})$. By (4.27) we have that

$$v_n(t, X_t) = v_n(0, x_0) + \int_0^t (\partial_s v_n(s, X^{s-}) + (\mathcal{L}v_n)(s, X^{s-}))ds + M_t^{v_n}, \quad (4.28)$$

where M^{v_n} is a local martingale. Since X_{s-} is a càglàd process, it is locally bounded, see Remark 3.13-1. So, for every $\ell > 0$, we define $\tau^\ell := \inf\{t \in [0, T] : |X_{t-}| \geq \ell\}$, with the usual convention that $\inf \emptyset = +\infty$. Clearly, $\tau^\ell \uparrow +\infty$ a.s. Then, on $\Omega_\ell := \{\tau^\ell \leq T\}$, $\sup_{s \leq T} |X_{(s \wedge \tau^\ell)-}| \leq \ell$ a.s., and $\sup_{s \leq T} |X^{(s \wedge \tau^\ell)-}| \leq \ell$ a.s. Since $\mathcal{L}v_n \xrightarrow{n \rightarrow \infty} \mathcal{L}v$, $\partial_s v_n \xrightarrow{n \rightarrow \infty} \partial_s v$ uniformly on compacts, it follows that $M^{v_n} \xrightarrow{n \rightarrow \infty} M^v$ u.c.p.

It remains to show that M^v is a local martingale. To this end, it is enough to show that, for every ℓ , $(M^v)^{\tau^\ell}$ is a martingale. Now, by (4.28),

$$(M^{v_n})_t^{\tau^\ell} = v_n(t \wedge \tau^\ell, X_{t \wedge \tau^\ell}) - v_n(0, x_0) - \int_0^{t \wedge \tau^\ell} (\partial_s v_n(s, X_{s-}) + (\mathcal{L}v_n)(s, X^{s-})) ds,$$

so

$$\begin{aligned} \sup_{t \leq T} |(M^{v_n})_t^{\tau^\ell}| &\leq 2 \sup_{t \in [0, T], |x| \leq \ell} |v_n(t, x)| + T \sup_{t \in [0, T], |x| \leq \ell} |(\partial_t v_n)(t, x)| + T \sup_{t \in [0, T], \|\eta\|_\infty \leq \ell} |(\mathcal{L}v_n)(t, \eta)| \\ &=: R, \end{aligned} \quad (4.29)$$

being $C([0, T] \times D_-(0, T))$ equipped with the metric topology of the uniform convergence on closed balls. On the other hand, recalling that $M^{v_n} \rightarrow M^v$ u.c.p. and therefore $(M^{v_n})^{\tau^\ell} \rightarrow (M^v)^{\tau^\ell}$ u.c.p., by (4.29) we have in particular that $\sup_{t \leq T} |(M^v)_t^{\tau^\ell}| \leq R$. It follows that $(M^v)^{\tau^\ell}$ is a martingale. This concludes the proof. \square

The following is a consequence of Theorem 4.18 in the Markovian context.

Theorem 4.20. *Let $\mathcal{D}_{\mathcal{L}_M}$ be a Fréchet space, topologically included in C_b^0 , and equipped with some metric $d_{\mathcal{L}_M}$. Let $\mathcal{L}_M : \mathcal{D}_{\mathcal{L}_M} \rightarrow C^0$ be a continuous map. Set $\mathcal{D}_{\mathcal{L}} := \mathcal{D}_{\mathcal{L}_M}$ equipped with the same topology of $\mathcal{D}_{\mathcal{L}_M}$,*

$$\mathcal{D}_{\mathcal{A}} := C^1([0, T]; \mathcal{D}_{\mathcal{L}_M}) \quad (4.30)$$

and, for every $f \in \mathcal{D}_{\mathcal{L}_M}$, $v \in \mathcal{D}_{\mathcal{A}}$, $\eta \in D_-(0, T)$,

$$(\mathcal{L}f)(t, \eta) := (\mathcal{L}_M f)(\eta_t), \quad (\mathcal{A}v)(dt, \eta) := (\partial_t v(t, \eta_t) + (\mathcal{L}_M v(t, \cdot))(\eta_t)) dt. \quad (4.31)$$

Then the following are equivalent.

(i) (X, \mathbb{P}) fulfills the time-homogeneous martingale problem in Definition 4.17 with respect to $\mathcal{D}_{\mathcal{L}}$, \mathcal{L} and x_0 .

(ii) For any $f \in \mathcal{D}_{\mathcal{L}_M}$,

$$M^f := f(X_\cdot) - f(x_0) - \int_0^\cdot \mathcal{L}_M f(X_s) ds \quad (4.32)$$

is an (\mathcal{F}_t^X) -local martingale under \mathbb{P} .

(iii) (X, \mathbb{P}) fulfills the time-inhomogeneous martingale problem in Definition 4.12 with respect to $\mathcal{D}_{\mathcal{A}}$, \mathcal{A} and x_0 .

Proof. By (4.31), it is easy to see that $\mathcal{L} : \mathcal{D}_{\mathcal{L}} \rightarrow C([0, T] \times D_-(0, T))$ is continuous, where $C([0, T] \times D_-(0, T))$ is equipped with the topology of uniform convergence on closed balls. Items (i) and (ii) are equivalent by construction. Items (i) and (iii) equivalent by Theorem 4.18. \square

4.4 Weak Dirichlet characterization of the solutions to the martingale problem

In the sequel we prove that stochastic calculus framework in Section 4.1 fits in the concrete examples of martingale problems.

Suppose that (X, \mathbb{P}) is a solution to the martingale problem in Definition 4.12 related to \mathcal{A} , $\mathcal{D}_{\mathcal{A}}$, and x_0 . First of all, we set $\mathcal{D}^S = \mathcal{D}_{\mathcal{A}}$. By definition of the martingale problem, for every $v \in \mathcal{D}^S$ and bounded, $v(\cdot, X)$ is a special semimartingale. In addition, by Proposition 3.9, Hypothesis 3.7 holds true being $v \in C^{0,1}([0, T] \times \mathbb{R})$. We can therefore apply Theorem 3.19, that provides decomposition (3.18) and $\Gamma(v)$ in (3.19). By the uniqueness of the decomposition of the special semimartingale, we get that

$$\Gamma(v) = \int_0^\cdot (\mathcal{A}v)(ds, X^{s-}) \quad (4.33)$$

and

$$\Gamma^k(v) = \int_0^\cdot (\mathcal{A}v)(ds, X^{s-}) - (v(s, X_{s-} + x) - v(s, X_{s-})) \frac{x - k(x)}{x} \star \nu^{X, \mathbb{P}}. \quad (4.34)$$

Taking into account Remark 3.20-1., we get in particular that, for every $v \in \mathcal{D}_{\mathcal{A}}$,

$$(\mathcal{A}v)(\Delta t, X^{t-}) = \int_{\mathbb{R}} (v(t, X_{t-} + x) - v(t, X_{t-})) \nu^{X, \mathbb{P}}(\{t\} \times dx), \quad t \in [0, T]. \quad (4.35)$$

Corollary 4.21. *Let \mathbb{P} be a probability on (Ω, \mathcal{F}) , and X be a càdlàg process with weakly finite quadratic variation. Let $\mathcal{D}_{\mathcal{A}}$ be a dense subset of $C^{0,1}([0, T] \times \mathbb{R})$, and \mathcal{A} be the operator in Definition 4.10. For every $v \in \mathcal{D}_{\mathcal{A}}$, set $Y_t^v = v(t, X_t)$, and denote by $Y^{v,c}$ its unique continuous martingale component. If (X, \mathbb{P}) is a solution to the martingale problem in Definition 4.12 related to $\mathcal{D}_{\mathcal{A}}$, \mathcal{A} and x_0 , then the following are equivalent.*

(i) $v \mapsto Y^{v,c}$, $\mathcal{D}_{\mathcal{A}} \subseteq C^{0,1}([0, T] \times \mathbb{R}) \rightarrow \mathbb{D}^{ucp}$, is continuous in zero.

(ii) The map

$$v \mapsto \int_0^\cdot (\mathcal{A}v)(ds, X^{s-}) - (v(s, X_{s-} + x) - v(s, X_{s-})) \frac{x - k(x)}{x} \star \nu^{X, \mathbb{P}}$$

is continuous in zero with respect to the $C^{0,1}$ -topology.

(iii) X is a weak Dirichlet process.

Proof. The equivalence of items (i) and (iii) comes from Theorem 4.3. The equivalence of items (i) and (ii) follows from Proposition 4.5 together with (4.34). \square

Corollary 4.22. *Let $\mathcal{D}_{\mathcal{A}}$ be a dense subset of $C^{0,1}([0, T] \times \mathbb{R})$, and \mathcal{A} be the operator in Definition 4.10. Let (X, \mathbb{P}) is a solution to the martingale problem in Definition 4.12 with respect to $\mathcal{D}_{\mathcal{A}}$, \mathcal{A} , and $x_0 \in \mathbb{R}$. Then, for every $v \in \mathcal{D}_{\mathcal{A}}$ and bounded, the unique martingale part N of the special semimartingale $Y_t := v(t, X_t)$ is given by*

$$\langle N, N \rangle = \int_0^\cdot (\mathcal{A}v^2)(ds, X^{s-}) - 2 \int_0^\cdot v(s, X_{s-}) (\mathcal{A}v)(ds, X^{s-}) - \sum_{0 \leq s \leq \cdot} |(\mathcal{A}v)(\Delta s, X^{s-})|^2. \quad (4.36)$$

Moreover,

$$\begin{aligned} \langle Y^{v,c}, Y^{v,c} \rangle &= \int_0^\cdot (\mathcal{A}v^2)(ds, X^{s-}) - 2 \int_0^\cdot v(s, X_{s-}) (\mathcal{A}v)(ds, X^{s-}) \\ &\quad - [v(s, X_{s-} + x) - v(s, X_{s-})]^2 \star \nu^{X, \mathbb{P}}, \end{aligned} \quad (4.37)$$

where $Y^{v,c}$ is the unique continuous martingale part of $Y^v = v(\cdot, X)$, and $(\mathcal{A}v)(\Delta s, \cdot)$ is introduced in (4.19).

Proof. The result follows from Proposition 4.6 together with (4.33). \square

4.5 Examples of martingale problems

4.5.1 Semimartingales

Let X be an adapted càdlàg real semimartingale with characteristics (B^k, C, ν) , with decomposition $\nu(\omega, ds dx) = \phi_s(\omega, dx)d\chi_s(\omega)$. For further notations we refer to Section D. X verifies condition (1.3), since it is a semimartingale, see Remark 3.35.

Then (X, \mathbb{P}) is a solution of the martingale problem in Definition 4.12 with respect to \mathcal{A} , $\mathcal{D}_{\mathcal{A}}$ and x_0 , with $\mathcal{D}_{\mathcal{A}}$ the set of functions in $C_b^{1,2}$ restricted to $[0, T] \times \mathbb{R}$ and, for every $v \in \mathcal{D}_{\mathcal{A}}$,

$$\begin{aligned} (\mathcal{A}v)(ds, \eta) &:= \partial_s v(s, \eta_s) ds + \frac{1}{2} \partial_{xx}^2 v(s, \eta_s) d(C \circ \eta)_s + \partial_x v(s, \eta_s) d(B^k \circ \eta)_s \\ &\quad + \int_{\mathbb{R}} (v(s, \eta_s + x) - v(s, \eta_s) - k(x) \partial_x v(s, \eta_s)) \phi_s(\eta, dx) d\chi_s(\eta), \quad \eta \in D_-(0, T). \end{aligned}$$

This follows by Remark D.2.

4.5.2 Weak Dirichlet processes derived from semimartingales

Let X be a weak Dirichlet process with characteristics (B^k, C, ν) , with $\nu(\omega, ds dx) = \phi_s(\omega, dx)d\chi_s(\omega)$. For further notations we refer to Section 3.3. Assume that there exists $h \in C^{0,1}$, $h(t, \cdot)$ bijective and $h(t, X_t)$ is a semimartingale with characteristics $(\bar{B}^k, \bar{C}, \bar{\nu})$. Being $h^{-1} \in C^{0,1}$, we have

$$\Delta X_s = h^{-1}(s, Y_{s-} + \Delta Y_s) - h^{-1}(s, Y_{s-}) = \Delta Y_s \int_0^1 \partial_x h^{-1}(s, Y_{s-} + a\Delta Y_s) da,$$

so condition (1.3) for Y (see previous subsection) implies the same for X .

Then by Example 4.5.1, (X, \mathbb{P}) is a solution of the martingale problem in Definition 4.12 with respect to \mathcal{A} , $\mathcal{D}_{\mathcal{A}}$ and x_0 , with $\mathcal{D}_{\mathcal{A}}$ the set of functions $v \in C_b^{0,1}$ such that $v \circ h^{-1} \in C^{1,2}$, restricted to $[0, T] \times \mathbb{R}$, and, for every $v \in \mathcal{D}_{\mathcal{A}}$,

$$\begin{aligned} (\mathcal{A}v)(ds, \gamma) &= \partial_s (v \circ h^{-1})(s, h(s, \gamma_s)) ds + \frac{1}{2} \partial_{xx}^2 (v \circ h^{-1})(s, h(s, \gamma_s)) (\partial_x h(s, \gamma_s))^2 d(C \circ \gamma)_s \\ &\quad + \partial_x (v \circ h^{-1})(s, h(s, \gamma_s)) d(\bar{B}^k \circ h(\cdot, \gamma))_s \\ &\quad + (v(s, \gamma_s + x) - v(s, \gamma_s) - k(h(s, \gamma_s + x) - h(s, \gamma_s)) \partial_x h^{-1}(s, h(s, \gamma_s)) \partial_x v(s, \gamma_s)) \phi_s(\gamma, dx) d\chi_s(\gamma) \end{aligned}$$

for every $\gamma \in D_-(0, T)$.

4.5.3 Discontinuous Markov processes with distributional drift

In [3] we study existence and uniqueness for a time-homogeneous martingale problem with distributional drift in a discontinuous Markovian framework.

In this section, we will consider a fixed $\alpha \in [0, 1]$. If $\alpha \in]0, 1[$, C_{loc}^α denotes the space of functions locally α -Hölder continuous. By $C_{\text{loc}}^{1+\alpha}$ we will denote the functions in C^1 whose derivative is α -Hölder continuous. For convenience, we set $C_{\text{loc}}^0 := C^0$, $C_{\text{loc}}^1 := C^1$, $C_{\text{loc}}^2 := C^2$.

Let $k \in \mathcal{K}$ be a continuous function. Let $\beta = \beta^k : \mathbb{R} \rightarrow \mathbb{R}$ and $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ be continuous functions, with σ not vanishing at zero. We consider formally the PDE operator of the type

$$L\psi = \frac{1}{2} \sigma^2 \psi'' + \beta' \psi' \tag{4.38}$$

in the sense introduced by [17, 18]. For a mollifier $\phi \in \mathcal{S}(\mathbb{R})$ with $\int_{\mathbb{R}} \phi(x) dx = 1$, we set

$$\phi_n(x) := n \phi(nx), \quad \beta'_n := \beta' * \phi_n, \quad \sigma_n := \sigma * \phi_n.$$

Hypothesis 4.23. 1. We assume the existence of the function

$$\Sigma(x) := \lim_{n \rightarrow \infty} 2 \int_0^x \frac{\beta'_n}{\sigma_n^2}(y) dy \quad (4.39)$$

in C^0 , independently from the mollifier.

2. The function Σ in (4.39) is upper and lower bounded, and belongs to C_{loc}^α .

The following proposition and definition are given in [17], see respectively Proposition 2.3 and the definition at page 497.

Proposition 4.24. Hypothesis 4.23-1 is equivalent to ask that there is a solution $h \in C^1$ to $Lh = 0$ such that $h(0) = 0$ and

$$h'(x) := e^{-\Sigma(x)}, \quad x \in \mathbb{R}. \quad (4.40)$$

In particular, $h'(0) = 1$, and h' is strictly positive so the inverse function by $h^{-1} : \mathbb{R} \rightarrow \mathbb{R}$ is well defined.

Let \mathcal{D}_L be the set of functions $f \in C^1$ such that there is $\phi \in C^1$ with

$$f' = e^{-\Sigma} \phi. \quad (4.41)$$

For any $f \in \mathcal{D}_L$, we set

$$Lf = \frac{\sigma^2}{2} (e^\Sigma f')' e^{-\Sigma}. \quad (4.42)$$

This defines without ambiguity $L : \mathcal{D}_L \subset C^1 \rightarrow C^0$. We also define

$$\mathcal{D}_{\mathcal{L}_M} := \mathcal{D}_L \cap C_{\text{loc}}^{1+\alpha} \cap C_b^0, \quad (4.43)$$

equipped with the graph topology of L , the natural topology of $C_{\text{loc}}^{1+\alpha}$ and the uniform convergence topology.

Then we consider a transition kernel $Q(\cdot, dx)$ from $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ into $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, with $Q(y, \{0\}) = 0$, satisfying the following condition.

Hypothesis 4.25. For all $B \in \mathcal{B}(\mathbb{R})$, the map $y \mapsto \int_B (1 \wedge |x|^{1+\alpha}) Q(y, dx)$ is bounded and the measure-valued map $y \mapsto (1 \wedge |x|^{1+\alpha}) Q(y, dx) =: \tilde{Q}(y, dx)$ is continuous in the total variation topology.

For every $f \in \mathcal{D}_{\mathcal{L}_M}$, we finally introduce the operator

$$\mathcal{L}_M f(y) := Lf(y) + \int_{\mathbb{R} \setminus \{0\}} (f(y+x) - f(y) - k(x) f'(y)) Q(y, dx). \quad (4.44)$$

Under Hypothesis 4.25, the operator above takes values in C^0 , see [3].

In [3] we study the following Markovian martingale problem.

Definition 4.26. We say that (X, \mathbb{P}) fulfills the time-homogeneous Markovian martingale problem with respect to $\mathcal{D}_{\mathcal{L}_M}$ in (4.43), \mathcal{L}_M in (4.44) and $x_0 \in \mathbb{R}$, if for any $f \in \mathcal{D}_{\mathcal{L}_M}$, the process

$$f(X_\cdot) - f(x_0) - \int_0^\cdot \mathcal{L}_M f(X_s) ds$$

is an (\mathcal{F}_t^X) -local martingale under \mathbb{P} .

Remark 4.27. Under Hypotheses 4.23 and 4.25, in [3] we provide existence and uniqueness for the Markovian martingale problem in Definition 4.26. Moreover, the solution (X, \mathbb{P}) is a finite quadratic variation process, with $\nu^{X, \mathbb{P}}(ds dx) = Q(X_{s-}, dx)ds$.

We can state the following theorem.

Theorem 4.28. *Assume that Hypotheses 4.23 and 4.25 hold true. If (X, \mathbb{P}) is a solution to the martingale problem in Definition 4.26, then it is a solution to the martingale problem in Definition 4.12 with respect to x_0 ,*

$$\mathcal{D}_A := C^1([0, T]; \mathcal{D}_{\mathcal{L}_M}), \quad (4.45)$$

and

$$\begin{aligned} (\mathcal{A}v)(dt, \eta) &:= \partial_t v(t, \eta_t) dt \\ &+ Lv(t, \eta_t) dt + \int_{\mathbb{R}} (v(t, \eta_t + x) - v(t, \eta_t) - k(x) \partial_y v(t, \eta_t)) Q(\eta_t, dx) dt, \quad v \in \mathcal{D}_A, \eta \in D_-(0, T). \end{aligned} \quad (4.46)$$

Moreover, that martingale problem meets uniqueness.

Remark 4.29. By [3], $\mathcal{D}_{\mathcal{L}_M}$ in (4.43) is dense in C^1 . Then, by Lemma B.9, \mathcal{D}_A in (4.45) is dense in $C^{0,1}([0, T] \times \mathbb{R})$.

Proof of Theorem 4.28. A solution (X, \mathbb{P}) to the martingale problem in Definition 4.26 exists by Remark 4.27. By the equivalence (ii)-(iii) in Theorem 4.20, (X, \mathbb{P}) fulfills the time-inhomogeneous martingale problem in Definition 4.12 with respect to x_0 , \mathcal{D}_A in (4.45) and \mathcal{A} in (4.46).

Concerning uniqueness, given a solution (X, \mathbb{P}) to the martingale problem by Definition 4.12, we know that, for every $v \in \mathcal{D}_A$, M^v is a local martingale. In particular this holds for v not depending on time, which implies the validity of the martingale problem not depending on time in the sense of Definition 4.26, for which we have uniqueness, see Remark 4.27. \square

Below we discuss some other properties of the solution to our martingale problem. We evaluate first the quadratic variation of the martingale component of a function of X belonging to \mathcal{D}_A . We can in particular apply Corollary 4.22 to the case of the operator \mathcal{A} in (4.46).

Proposition 4.30. *Let (X, \mathbb{P}) be a solution to the martingale problem in Definition 4.26. Then, for every $v \in \mathcal{D}_A$ in (4.45) we have*

$$\langle Y^{v,c}, Y^{v,c} \rangle_t = \int_0^t \sigma^2(X_s) (\partial_x v(s, X_s))^2 ds, \quad (4.47)$$

where $Y_t^{v,c}$ denotes the unique continuous martingale part of $v(t, X_t)$. In particular, $v \mapsto Y^{v,c}$, $\mathcal{D}_A \subseteq C^{0,1}([0, T] \times \mathbb{R}) \rightarrow \mathbb{D}^{ucp}$, is continuous in zero.

Proof of Proposition 4.30. By Theorem 4.28, (X, \mathbb{P}) is a solution to the martingale problem in Definition 4.12 with respect to x_0 , \mathcal{D}_A in (4.45) and \mathcal{A} in (4.46).

Assume that (4.47) holds. Let $v_n \in \mathcal{D}_A$ such that $v_n \rightarrow 0$ in $C^{0,1}([0, T] \times \mathbb{R})$. By (4.47), $\langle Y^{v_n,c}, Y^{v_n,c} \rangle_T$ converges to zero in probability. Then, by Problem 5.25 in [28], Chapter 1, it follows that,

$$Y^{v_n,c} \rightarrow 0 \quad \text{u.c.p. as } n \rightarrow \infty. \quad (4.48)$$

It remains to prove (4.47). Take $v \in \mathcal{D}_A$. Then formula (4.37) in Corollary 4.22 applied to \mathcal{A} in (4.46) (taking into account Remark 4.27) yields

$$\langle Y^{v,c}, Y^{v,c} \rangle_t = \int_0^t (\mathcal{A}v^2 - 2v\mathcal{A}v)(ds, X^{s-}) - \int_{]0,t] \times \mathbb{R}} [v(s, X_{s-} + x) - v(s, X_{s-})]^2 Q(X_{s-}, dx) ds$$

$$\begin{aligned}
&= \int_0^t (\partial_s v^2(s, X_s) - 2v(s, X_s) \partial_s v(s, X_s)) ds + \int_0^t (Lv^2 - 2vLv)(s, X_s) ds \\
&+ \int_{]0, t] \times \mathbb{R}} (v^2(s, X_{s-} + x) - v^2(s, X_{s-}) - 2k(x)v(s, X_{s-}) \partial_x v(s, X_{s-})) Q(X_{s-}, dx) ds \\
&- 2 \int_{]0, t] \times \mathbb{R}} v(s, X_s) (v(s, X_{s-} + x) - v(s, X_{s-}) - k(x) \partial_x v(s, X_{s-})) Q(X_{s-}, dx) ds \\
&- \int_{]0, t] \times \mathbb{R}} [v(s, X_{s-} + x) - v(s, X_{s-})]^2 Q(X_{s-}, dx) ds \\
&= \int_0^t (Lv^2 - 2vLv)(s, X_s) ds = \int_0^t \sigma^2(X_s) (\partial_x v(s, X_s))^2 ds,
\end{aligned}$$

where the latter equality follows from the fact that $Lv^2 = 2vLv + (\sigma \partial_x v)^2$, see Propositions 2.10 in [17]. \square

Corollary 4.31. *Let (X, \mathbb{P}) be a solution to the martingale problem in Definition 4.26.*

(i) *For every $v \in C^{0,1}$, $Y^v := v(\cdot, X)$ is a weak Dirichlet process. In particular, X is a weak Dirichlet process.*

(ii) *(4.47) holds for every $v \in C^{0,1}$.*

Proof. By Theorem 4.28, (X, \mathbb{P}) is a solution to the martingale problem in Definition 4.12 with respect to x_0 , $\mathcal{D}_{\mathcal{A}}$ in (4.45) and \mathcal{A} in (4.46).

(i) By Remark 4.29, $\mathcal{D}_{\mathcal{A}}$ in (4.45) is dense in $C^{0,1}([0, T] \times \mathbb{R})$. Thanks to Proposition 4.30, $v \mapsto Y^{v,c}$ is a continuous map. Since X is a finite quadratic variation process (see Remark 4.27), we can apply Corollary 4.21, which states that X is a weak Dirichlet process. Theorem 3.36 concludes the proof of item (i).

(ii) Let $v \in C^{0,1}$. Since $\mathcal{D}_{\mathcal{A}}$ is dense in $C^{0,1}$ (see Remark 4.29), there exists a sequence $(v_n) \in \mathcal{D}_{\mathcal{A}}$ converging to v in $C^{0,1}$. Since $v \mapsto Y^{v,c}$ is continuous, $Y^{v_n,c} \rightarrow Y^{v,c}$, u.c.p. By Proposition A.4, $\langle Y^{v_n,c}, Y^{v_n,c} \rangle \rightarrow \langle Y^{v,c}, Y^{v,c} \rangle$. The result follows from Proposition 4.30 since $v \mapsto \int_0^t \sigma^2(X_s) (\partial_x v(s, X_s))^2 ds$ is continuous. \square

Theorem 4.32. *Let (X, \mathbb{P}) be a solution to the martingale problem in Definition 4.26. Then there exists an (\mathcal{F}_t) -Brownian motion W^X such that*

$$\begin{aligned}
X &= x_0 + \int_0^\cdot \sigma(X_s) dW_s^X + \int_{]0, \cdot] \times \mathbb{R}} k(x) (\mu^X(ds dx) - Q(X_{s-}, dx) ds) + \lim_{n \rightarrow \infty} \int_0^\cdot Lf_n(X_s) ds \\
&+ \int_{]0, \cdot] \times \mathbb{R}} (x - k(x)) \mu^X(ds dx),
\end{aligned} \tag{4.49}$$

for every sequence $(f_n)_n \subseteq \mathcal{D}_{\mathcal{L}_M}$ such that $f_n \xrightarrow[n \rightarrow \infty]{} Id$ in C^1 . The limit appearing in (4.49) holds in the u.c.p. sense.

Proof. By Theorem 4.28, (X, \mathbb{P}) is a solution to the martingale problem in Definition 4.12 with respect to x_0 , $\mathcal{D}_{\mathcal{A}}$ in (4.45) and \mathcal{A} in (4.46). By Corollary 3.22 we have

$$X = X^c + k(x) \star (\mu^X - \nu^{X, \mathbb{P}}) + \Gamma^k(Id) + (x - k(x)) \star \mu^X,$$

where Γ^k is the operator defined in Theorem 3.10. By Proposition 3.9 and Remark 3.12, Γ^k is well defined in particular on $C^{0,1}$. By Proposition 4.30, we can apply Proposition 4.5 with $\mathcal{D}^S = \mathcal{D}_{\mathcal{A}}$,

which yields that $v \mapsto \Gamma^k(v)$ restricted to $\mathcal{D}_{\mathcal{A}}$ is continuous. Since $\mathcal{D}_{\mathcal{A}}$ is dense in $C^{0,1}$, $\Gamma^k(v)$ is the continuous extension of the map defined in (4.34).

At this point we evaluate $\Gamma^k(Id)$. Take $(f_n)_n \subseteq \mathcal{D}_{\mathcal{L}_M}$ such that $f_n \rightarrow Id$ in C^1 . By (4.34) together with (4.46), we get

$$\begin{aligned} \Gamma^k(Id) &= \lim_{n \rightarrow \infty} \Gamma^k(f_n) \\ &= \lim_{n \rightarrow \infty} \left(\int_0^\cdot (\mathcal{A}f_n)(ds, X^{s-}) - \int_{]0, \cdot] \times \mathbb{R}} (f_n(X_{s-} + x) - f_n(X_{s-})) \frac{x - k(x)}{x} Q(X_{s-}, dx) ds \right) \\ &= \lim_{n \rightarrow \infty} \left(\int_0^\cdot Lf_n(X_{s-}) ds + \int_{]0, \cdot] \times \mathbb{R}} (f_n(X_{s-} + x) - f_n(X_{s-}) - k(x) f'_n(X_{s-})) Q(X_{s-}, dx) ds \right. \\ &\quad \left. - (f_n(X_{s-} + x) - f_n(X_{s-})) \frac{x - k(x)}{x} \star \nu^{X, \mathbb{P}} \right) \\ &= \lim_{n \rightarrow \infty} \int_0^\cdot Lf_n(X_s) ds. \end{aligned}$$

In order to get (4.49) it remains to identify X^c . Setting $W_t^X := \int_0^t \frac{1}{\sigma(X_s)} dX_s^c$, we have

$$\langle W^X, W^X \rangle_t = \int_0^t \frac{1}{\sigma(X_s)} d\langle X^c, X^c \rangle_s.$$

On the other hand, Corollary 4.31-(ii) with $v \equiv Id$ yields $\langle X^c, X^c \rangle = \int_0^\cdot \sigma^2(X_s) ds$, which implies that $\langle W^X, W^X \rangle_t = t$, and by Lévy representation theorem, W^X is a Brownian motion. Finally, by construction we have $X^c = \int_0^\cdot \sigma(X_s) dW_s$. \square

4.5.4 Continuous path-dependent SDEs with distributional drift

Let σ, β be continuous real functions and $L : \mathcal{D}_L \rightarrow C^0$ in (4.42), with \mathcal{D}_L the subset of C^1 introduced just before. We only suppose Hypothesis 4.23-1. Let $G^d : [0, T] \times D_-(0, T) \rightarrow \mathbb{R}$ be a Borel bounded functional, uniformly continuous on closed balls. We set $G : [0, T] \times C(0, T) \rightarrow \mathbb{R}$ as the restriction of G^d . In [30] one investigates the martingale problem related to a path-dependent SDE of the type

$$dX_t = \sigma(X_t) dW_t + (\beta'(X_t) + G(t, X^t)) dt. \quad (4.50)$$

We set $\mathcal{D}_{L,b} := \mathcal{D}_L \cap C_b^0$.

Definition 4.33. (X, \mathbb{P}) is solution to the (non-Markovian) path-dependent martingale problem related to (4.50) an initial condition x_0 if, for every $f \in \mathcal{D}_{L,b}$,

$$M^f := f(X_t) - f(x_0) - \int_0^t ((Lf)(X_s) + f'(X_s)G(s, X^s)) ds \quad (4.51)$$

is an (\mathcal{F}^X) -local martingale under \mathbb{P} .

Proposition 4.34. Let (X, \mathbb{P}) be a solution to the martingale problem in the sense of Definition 4.33. Then X is necessarily a continuous process.

Proof. Let h be the function introduced in Proposition 4.24, and set $Y = h(X)$. For every $\phi \in C^2$, we denote by L^0 the classical PDE operator $L^0\phi(y) = \frac{(\sigma h')^2 (h^{-1}(y))}{2} \phi''(y)$. By Propositions 2.13 in [17], $\phi \in \mathcal{D}_{L^0}(= C^2)$ if and only if $\phi \circ h \in \mathcal{D}_L$. This implies that $\phi \in C_b^2$ if and only if $\phi \circ h \in \mathcal{D}_{L,b}$. Moreover, $L(\phi \circ h) = (L^0\phi) \circ h$ for every $\phi \in C^2$.

Since (X, \mathbb{P}) fulfills the time-homogeneous martingale problem in Definition 4.33, for every $f \in \mathcal{D}_{L,b}$,

$$f(X_\cdot) - f(x_0) - \int_0^\cdot ((Lf)(X_s) + f'(X_s)G(s, X^s))ds$$

is an (\mathcal{F}_t^X) -local martingale under \mathbb{P} . Setting $y_0 = h^{-1}(x_0)$, this yields that, for every $\tilde{f} \in C_b^2$,

$$\tilde{f}(Y_\cdot) - \tilde{f}(y_0) - \frac{1}{2} \int_0^\cdot (\sigma h')^2(h^{-1}(Y_s)) \tilde{f}''(Y_s) ds - \int_0^\cdot h'(h^{-1}(Y_s)) G(s, h^{-1}(Y^s)) \tilde{f}'(Y_s) ds$$

is an (\mathcal{F}_t^Y) -local martingale under \mathbb{P} . It follows from Theorem D.1 that Y is a semimartingale with characteristics $B = \int_0^\cdot b(s, \check{Y}) ds$, $C = \int_0^\cdot c(s, \check{Y}) ds$, $\nu(ds dz) = 0$, where $b(s, \eta) := h'(h^{-1}(\eta(s)))G(s, h^{-1}(\eta^s))$ and $c(s, \eta) := (\sigma h')^2(h^{-1}(\eta(s)))$. Consequently $\mu^Y(ds dy) = 0$, so $Y = h(X)$ is necessarily a continuous process, and the same holds for X . \square

Lemma 4.35. $\mathcal{D}_{L,b}$ is dense in \mathcal{D}_L equipped with its graph topology. In particular, $\mathcal{D}_{L,b}$ is dense in C^1 .

Proof. We consider the sequence (χ_N) introduced in (3.53). Let $f \in \mathcal{D}_L$. We define a sequence $(f_N) \subset C^1$ such that $f_N(0) = f(0)$ and $f'_N = \chi_N f'$. $f_N \in \mathcal{D}_L$ since $f'_N = (\phi \chi_N) e^{-\Sigma}$ and $\phi \chi_N \in C^1$, where ϕ has been defined in (4.41). Now, each f_N is a bounded function since f'_N has compact support. Clearly, $f_N \rightarrow f$ in C^1 . Moreover, making use of (4.42) we get

$$Lf_N = \frac{\sigma^2}{2} (e^\Sigma \chi_N f')' e^{-\Sigma} = \frac{\sigma^2}{2} (\phi \chi_N)' e^{-\Sigma} \rightarrow \frac{\sigma^2}{2} \phi' e^{-\Sigma} = Lf \quad \text{in } C^0.$$

This concludes the proof. \square

Corollary 4.36. *Existence and uniqueness of a solution to the martingale problem in Definition 4.33 holds true.*

Proof. By Theorem 4.23 in [30], there is a solution (X, \mathbb{P}) in the sense of Definition 4.33 which is even continuous. This shows existence.

Concerning uniqueness, let (X, \mathbb{P}) be a solution of the martingale problem in the sense of Definition 4.33. By Proposition 4.34, $f(X)$ is necessarily continuous for every $f \in \mathcal{D}_{L,b}$. Moreover the process in (4.51) is a martingale also for every $f \in \mathcal{D}_L$ not necessarily bounded. Indeed, let $f \in \mathcal{D}_L$. By Lemma 4.35 there is a sequence $f_N \in \mathcal{D}_{L,b}$ converging to f in \mathcal{D}_L . This implies that M^{f_N} converges to M^f u.c.p. We remark that the space of continuous local martingales is closed with respect to the u.c.p. convergence topology so M^f is again a continuous local martingale. The conclusion follows by Proposition 4.24 in [30] which states uniqueness in the framework of continuous processes. \square

Proposition 4.37. *Let (X, \mathbb{P}) be a solution to the martingale problem in Definition 4.33. Then (X, \mathbb{P}) is a solution to the martingale problem in Definition 4.12 with respect to x_0 ,*

$$\mathcal{D}_A := C^1([0, T]; \mathcal{D}_{L,b}) \tag{4.52}$$

and

$$(Av)(ds, \eta) = (\partial_s v(s, \eta(s)) + Lv(s, \eta(s)) + G^d(s, \eta^s) \partial_x v(s, \eta(s))) ds. \tag{4.53}$$

Proof. We apply Theorem 4.18 with $\mathcal{D}_L := \mathcal{D}_{L,b}$ equipped with its graph topology. \square

Remark 4.38. Consider the martingale problem in Definition 4.12 with respect to \mathcal{D}_A in (4.52), \mathcal{A} in (4.53), and $x_0 \in \mathbb{R}$. Replacing G^d in (4.53) with another Borel extension of G one gets the same solution to the martingale problem.

Proceeding analogously as for the proof of Theorem 4.32 we can prove the following result.

Proposition 4.39. *Let (X, \mathbb{P}) be a solution to the martingale problem in Definition 4.33. Then we have the following.*

- (i) X is a weak Dirichlet process.
- (ii) There exists an (\mathcal{F}_t) -Brownian motion W^X such that

$$X = x_0 + \int_0^\cdot \sigma(X_s) dW_s^X + \int_0^\cdot G^d(s, X^s) ds + \lim_{n \rightarrow \infty} \int_0^\cdot Lf_n(X_s) ds, \quad (4.54)$$

for every sequence $(f_n)_n \subseteq \mathcal{D}_L$ such that $f_n \xrightarrow[n \rightarrow \infty]{} Id$ in C^1 . The limit in (4.54) holds in the u.c.p. sense.

4.5.5 The PDMPs case

Let X be a piecewise deterministic Markov process (PDMP) generated by a marked point process (T_n, ζ_n) , where $(T_n)_n$ are increasing random times such that $T_n \in]0, \infty[$, where either there is a finite number of times $(T_n)_n$ or $\lim_{n \rightarrow \infty} T_n = +\infty$, and ζ_n are random variables in $[0, 1]$. We will follow the notations in [11], Chapter 2, Sections 24 and 26. The behavior of the PDMP X is described by a triplet of local characteristics (h, λ, Q) : $h :]0, 1[\rightarrow \mathbb{R}$ is a Lipschitz continuous function, $\lambda :]0, 1[\rightarrow \mathbb{R}$ is a measurable function such that $\sup_{x \in]0, 1[} |\lambda(x)| < \infty$, and Q is a transition probability measure on $[0, 1] \times \mathcal{B}(]0, 1[)$. Some other technical assumptions are specified in the over-mentioned reference, that we do not recall here. Let us denote by $\Phi(s, x)$ the unique solution of $g'(s) = h(g(s))$, $g(0) = x$. Then

$$X(t) = \begin{cases} \Phi(t, x), & t \in [0, T_1[\\ \Phi(t - T_n, \zeta_n), & t \in [T_n, T_{n+1}[, n \in \mathbb{N}, \end{cases} \quad (4.55)$$

and, for any $x_0 \in [0, 1]$, verifies the equation (provided the second integral in the right-hand side is well-defined)

$$X_t = x_0 + \int_0^t h(X_s) ds + \int_{]0, t] \times \mathbb{R}} x \mu^X(ds dx) \quad (4.56)$$

with

$$\mu^X(ds dx) = \sum_{n \geq 1} 1_{\{\zeta_n \in]0, 1[\}} \delta_{(T_n, \zeta_n - \zeta_{n-1})}(ds dx). \quad (4.57)$$

Moreover, we introduce the predictable process counting the number of jumps of X from the boundary of its domain:

$$p_t^* = \sum_{0 < s \leq t} 1_{\{X_{s-} \in \{0, 1\}\}}. \quad (4.58)$$

The knowledge of (h, λ, Q) completely specifies the law of X , see Section 24 in [11], and also Proposition 2.1 in [1]. In particular, let \mathbb{P} be the unique probability measure under which the compensator of μ^X has the form

$$\nu^X(ds dx) = \tilde{Q}(X_{s-}, dx) (\lambda(X_{s-}) ds + dp_s^*), \quad (4.59)$$

where $\tilde{Q}(y, dx) = Q(y, y + dx)$, and λ is trivially extended to $[0, 1]$ by the zero value. Notice that X is a finite variation process, so (1.3) holds true. According to Theorem 31.3 and subsequent Section 31.5 in [11], for every measurable absolutely continuous function $v : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$ such that $(v(t, X_{t-} + x) - v(t, X_{t-})) \star \mu^X \in \mathcal{A}_{\text{loc}}^+$,

$$\begin{aligned} & v(t, X_t) - v(0, x_0) \\ & - \int_{]0, t]} \left(\partial_s v(s, X_{s-}) + h(X_{s-}) \partial_x v(s, X_s) + \lambda(X_{s-}) \int_{\mathbb{R}} (v(s, X_{s-} + x) - v(s, X_{s-})) \tilde{Q}(X_{s-}, dx) \right) ds \\ & - \int_{]0, t] \times \mathbb{R}} (v(s, X_{s-} + x) - v(s, X_{s-})) \tilde{Q}(X_{s-}, dx) dp_s^* \end{aligned}$$

is an (\mathcal{F}^X) -local martingale under \mathbb{P} . Therefore, (X, \mathbb{P}) solves the martingale problem in Definition 4.12 with respect to \mathcal{A} , $\mathcal{D}_A := C^1([0, T] \times \mathbb{R})$ and x_0 , with

$$(\mathcal{A}v)(ds, \eta) := \Lambda_1 v(s, \eta_{s-}) \gamma_1(ds, \eta_{s-}) + \Lambda_2 v(s, \eta_{s-}) \gamma_2(ds, \eta_{s-}), \quad v \in \mathcal{D}_A, \eta \in D(0, T),$$

with, for any $y \in \mathbb{R}$, $\gamma_1(ds, y) = ds$, $\gamma_2(ds, y) = dp_s^*(y)$, and

$$\begin{aligned} \Lambda_1 v(s, y) &= \partial_s v(s, y) + h(y) \partial_y v(s, y) + \lambda(y) \int_{\mathbb{R}} (v(s, y + x) - v(s, y)) \tilde{Q}(y, dx), \quad y \in]0, 1[, \\ \Lambda_2 v(s, y) &= \int_{\mathbb{R}} (v(s, y + x) - v(s, y)) \tilde{Q}(y, dx), \quad y \in \{0, 1\}. \end{aligned}$$

Appendix

A Some technical results on the (weak) finite quadratic variation

Proposition A.1. *Let $Y = (Y(t))_{t \in [0, T]}$ and $X = (X(t))_{t \in [0, T]}$ be respectively a càdlàg and a continuous process. Then*

$$[X, Y]_{\varepsilon}^{ucp}(t) = C_{\varepsilon}(X, Y)(t) + R(\varepsilon, t)$$

with $R(\varepsilon, t) \xrightarrow{\varepsilon \rightarrow 0} 0$ u.c.p.

Proof. See Proposition A.3 in [4]. □

Lemma A.2. *Let $G_n : C(0, T) \rightarrow \mathbb{R}$, $n \in \mathbb{N}$, be a sequence of functions such that*

$$(i) \sup_n \|G_n\|_{var} \leq M \in [0, +\infty[,$$

$$(ii) G_n \xrightarrow{n \rightarrow \infty} 0 \text{ uniformly.}$$

Then, for every $g : [0, T] \rightarrow \mathbb{R}$ càglàd, $\int_0^\cdot g dG_n \xrightarrow{n \rightarrow \infty} 0$, uniformly.

Proof. For every $n \in \mathbb{N}$, let us define the operator $T_n : D_-(0, T) \rightarrow C(0, T)$, $g \mapsto \int_0^\cdot g dG_n$. We denote by $\mathcal{E}_-(0, T)$ the linear space of càglàd step functions of the type $\sum_i c_i \mathbb{1}_{]a_i, b_i]}$. We first notice that, by (ii), if $g \in \mathcal{E}_-(0, T)$, then $T_n(g) \xrightarrow{n \rightarrow \infty} 0$, uniformly.

On the other hand, $\mathcal{E}_-(0, T)$ is dense in $D_-(0, T)$, see Lemma 1, Chapter 3, in [7] (that lemma is written for càdlàg function; however the same follows for càglàd functions since the time reversal of a càdlàg function is càglàd).

Let now $g \in D_-(0, T)$. Since g is bounded, we have $\sup_{s \in [0, T]} |g(s)| \leq m$ for some constant m . We get $\|T_n(g)\|_{\infty} \leq Mm$. The conclusion follows by the Banach-Steinhaus theorem, see e.g. Chapter 1.2 in [13]. □

Proposition A.3. *Let X be an \mathbb{F} -weak Dirichlet with weakly finite quadratic variation. Let g be a càglàd process, and N be a continuous \mathbb{F} -local martingale. Then*

$$\int_0^t g(s)(X_{s+\varepsilon} - X_s)(N_{s+\varepsilon} - N_s) \frac{ds}{\varepsilon} \xrightarrow{\varepsilon \rightarrow 0} \int_0^t g(s) d[X^c, N]_s, \quad t \in [0, T], \quad \text{u.c.p.}$$

Proof. For $\varepsilon > 0$, we set $F_\varepsilon(t) := C_\varepsilon(X, N)(s)$. By Proposition 3.2, $X = X^c + A$ with A a martingale orthogonal process. Therefore, recalling Proposition A.1,

$$F_\varepsilon(t) \xrightarrow{\varepsilon \rightarrow 0} F(t) := [X^c, N]_t, \quad \text{u.c.p.}$$

Let ε_n be a sequence converging to zero as $n \rightarrow \infty$. It is sufficient to show the existence of a subsequence, still denoted by ε_n , such that

$$\int_0^\cdot g(s) dF_{\varepsilon_n}(s) \xrightarrow{n \rightarrow \infty} \int_0^\cdot g(s) dF(s), \quad \text{u.c.p.} \quad (\text{A.1})$$

By extracting a sub-subsequence, there exists a null set \mathcal{N} such that

$$F_{\varepsilon_n}(t) \xrightarrow{n \rightarrow \infty} F(t), \quad \text{uniformly for all } \omega \notin \mathcal{N}. \quad (\text{A.2})$$

We remark that N has also weakly finite quadratic variation. Let $\kappa > 0$. For any $\ell > 0$, we denote by $\Omega_{n,\ell}$ the subset of $\omega \in \Omega$ such that

$$\begin{aligned} \int_0^T (X_{(s+\varepsilon_n) \wedge T}(\omega) - X_s(\omega))^2 \frac{ds}{\varepsilon_n} + \int_0^T (N_{(s+\varepsilon_n) \wedge T}(\omega) - N_s(\omega))^2 \frac{ds}{\varepsilon_n} &\leq \ell, \\ \langle X^c, X^c \rangle_T + \langle N, N \rangle_T &\leq \ell. \end{aligned} \quad (\text{A.3})$$

In particular, we can choose ℓ such that $\mathbb{P}(\Omega_{n,\ell}^c) \leq \kappa$. Moreover, since g is locally bounded, without restriction of generality we can take

$$\sup_{s \in [0, T]} |g(s)| \leq \ell, \quad \forall \omega \in \Omega_{n,\ell}. \quad (\text{A.4})$$

Collecting (A.3) and (A.4), on $\Omega_{n,\ell}$ we have

$$\begin{aligned} \sup_{0 \leq t \leq T} \left| \int_0^t g(s) dF_{\varepsilon_n}(s) \right| &\leq \ell \|F_{\varepsilon_n}\|_{\text{var}} \\ &\leq \ell \sqrt{\int_0^T (X_{s+\varepsilon_n}(\omega) - X_s(\omega))^2 \frac{ds}{\varepsilon_n}} \sqrt{\int_0^T (N_{s+\varepsilon_n}(\omega) - N_s(\omega))^2 \frac{ds}{\varepsilon_n}} \\ &= \ell \sqrt{\int_0^T (X_{(s+\varepsilon_n) \wedge T}(\omega) - X_s(\omega))^2 \frac{ds}{\varepsilon_n}} \sqrt{\int_0^T (N_{(s+\varepsilon_n) \wedge T}(\omega) - N_s(\omega))^2 \frac{ds}{\varepsilon_n}} \leq \ell^2, \\ \sup_{0 \leq t \leq T} \left| \int_0^t g(s) dF(s) \right| &\leq \ell \|F\|_{\text{var}} \leq \ell \sqrt{\langle X^c, X^c \rangle_T + \langle N, N \rangle_T} \leq \ell^2. \end{aligned} \quad (\text{A.5})$$

Let us come back to prove (A.1). We set

$$\chi_n(g) := \sup_{0 \leq t \leq T} \left| \int_0^t g(s) dF_{\varepsilon_n}(s) - \int_0^t g(s) dF(s) \right|.$$

Let $K > 0$. Using (A.5), together with Chebyshev's inequality, we get

$$\begin{aligned}
\mathbb{P}(\chi_n(g) > K) &\leq \mathbb{P}(\Omega_{n,\ell}^c) + \mathbb{P}(\{\chi_n(g) > K\} \cap \Omega_{n,\ell} \cap \mathcal{N}^c) \\
&\leq \kappa + \mathbb{P}(\{\chi_n(g) > K\} \cap \Omega_{n,\ell} \cap \mathcal{N}^c) \\
&= \kappa + \mathbb{P}(\{\chi_n(g) \wedge 2\ell^2 > K\} \cap \Omega_{n,\ell} \cap \mathcal{N}^c) \\
&= \kappa + \mathbb{P}((\chi_n(g) \wedge 2\ell^2) \mathbb{1}_{\Omega_{n,\ell} \cap \mathcal{N}^c} > K) \\
&\leq \kappa + \frac{\mathbb{E}[(\chi_n(g) \wedge 2\ell^2) \mathbb{1}_{\Omega_{n,\ell} \cap \mathcal{N}^c}]}{K^2}.
\end{aligned} \tag{A.6}$$

To prove that previous expectation goes to zero as n goes to infinity, by using Lebesgue's dominated convergence theorem, it remains to show that

$$\chi_n(g) \mathbb{1}_{\Omega_{n,\ell} \cap \mathcal{N}^c} \xrightarrow[n \rightarrow \infty]{} 0 \quad \text{a.s.} \tag{A.7}$$

To this end, let us set $G_n := \mathbb{1}_{\Omega_{n,\ell}}(F_n - F)$. Since by (A.3) we have $\|G_n\|_{\text{var}} \leq \frac{\ell}{2}$, the convergence in (A.7) follows from Lemma A.2. Consequently, by (A.6), $\limsup_{n \rightarrow \infty} \mathbb{P}(\chi_n(g) > K) \leq \kappa$. Since $\kappa > 0$ is arbitrary, the proof is concluded. \square

Proposition A.4. *Let $(M^n(t))_{t \in [0, T]}$ (resp. $(N^n(t))_{t \in [0, T]}$) be a sequence of continuous local martingales, converging u.c.p. to M (resp. to N). Then*

$$[M^n, N^n] \xrightarrow[n \rightarrow \infty]{} [M, N] \quad \text{u.c.p.}$$

In order to prove Proposition A.4 we first give a technical result.

Lemma A.5. *Let $A, \delta > 0$. Let M be a continuous local martingale vanishing at zero. We have*

$$\mathbb{P}([M, M]_T \geq A) \leq \frac{\mathbb{E}[\max_{t \in [0, T]} |M_t|^2 \wedge \delta^2]}{A} + \mathbb{P}\left(\max_{t \in [0, T]} |M_t| \geq \delta\right). \tag{A.8}$$

Proof. We bound the left-hand side of (A.8) by

$$\mathbb{P}\left([M, M]_T \geq A, \max_{t \in [0, T]} |M_t| \leq \delta\right) + \mathbb{P}\left(\max_{t \in [0, T]} |M_t| \geq \delta\right) := I_1 + I_2.$$

Let $\tau := \inf\{s \in [0, T] : |M_s| \geq \delta\}$. We notice that on $\Omega_0 := \{\omega \in \Omega : \max_{t \in [0, T]} |M_t(\omega)| \leq \delta\}$ we have $M = M^\tau$. Therefore, by the definition of covariation, $[M, M] = [M^\tau, M^\tau]$ on Ω_0 , so that

$$I_1 = \mathbb{P}\left([M^\tau, M^\tau]_T \geq A, \max_{t \in [0, T]} |M_t| \leq \delta\right).$$

Using Chebyshev and Burkholder-Davis-Gundy inequalities, we get that there is a constant $c > 0$ such that

$$\begin{aligned}
I_1 &\leq \mathbb{P}([M^\tau, M^\tau]_T \geq A) \leq \frac{\mathbb{E}[[M^\tau, M^\tau]_T]}{A} \leq \frac{c \mathbb{E}[\sup_{t \in [0, T]} |M_t^\tau|^2]}{A} = \frac{c \mathbb{E}[\sup_{t \in [0, T]} |M_t^\tau|^2 \wedge \delta^2]}{A} \\
&\leq \frac{c \mathbb{E}[\sup_{t \in [0, T]} |M_t|^2 \wedge \delta^2]}{A}.
\end{aligned}$$

\square

Proof of Proposition A.4. Obviously, we can take $M \equiv N \equiv 0$. By polarity arguments, it is enough to suppose $(M^n) \equiv (N^n)$ and to prove that $[M^n, M^n]_T \rightarrow 0$ in probability. Let $\varepsilon > 0$, and

let N_0 such that, for every $n \geq N_0$, $\mathbb{P}(\sup_{t \in [0, T]} |M_t^n| \geq 1) \leq \varepsilon$. Let $A > 0$. For $n \geq N_0$, Lemma A.5 gives

$$\mathbb{P}([M^n, M^n]_T \geq A) \leq \frac{\mathbb{E} \left[\max_{t \in [0, T]} |M_t^n|^2 \wedge 1 \right]}{A} + \varepsilon.$$

By taking $n \rightarrow \infty$ we get

$$\limsup_{n \rightarrow \infty} \mathbb{P}([M^n, M^n]_T \geq A) \leq \varepsilon$$

and the result follows from the arbitrariness of ε . \square

B Some results on densities of martingale problems' domains

We have the following density results.

Lemma B.1. *Let $M > 0$ and E be a topological vector F -space with some metric d . We introduce the distance*

$$\bar{d}(e_1, e_2) := \sup_{t \in [0, M]} d(te_1, te_2), \quad e_1, e_2 \in E.$$

Then d and \bar{d} are equivalent in the following sense: if $\varepsilon > 0$, there is $\delta > 0$ such that

$$d(e_1, e_2) \leq \delta \Rightarrow \bar{d}(e_1, e_2) \leq \varepsilon, \quad e_1, e_2 \in E, \quad (\text{B.1})$$

and

$$\bar{d}(e_1, e_2) \leq \delta \Rightarrow d(e_1, e_2) \leq \varepsilon, \quad e_1, e_2 \in E. \quad (\text{B.2})$$

Proof. (B.2) is immediate since $d \leq \bar{d}$.

(B.1) follows since we can easily prove that $e \mapsto \bar{d}(e, 0)$ is continuous. This follows because $(t, e) \mapsto te$ and therefore $(t, e) \mapsto d(te, 0)$ is continuous. \square

Definition B.2. *An F -space E is said to be generated by a countable sequence of seminorms $(\|\cdot\|_{\alpha \in \mathbb{N}})$ if it can be equipped with the distance*

$$d_E(x, y) := \sum_{\alpha} \frac{\|x - y\|_{\alpha}}{1 + \|x - y\|_{\alpha}} 2^{-\alpha}. \quad (\text{B.3})$$

The space $C^0([0, T]; E)$ will be equipped with the distance $d(f, g) := \sup_{t \in [0, T]} d_E(f(t), g(t))$.

Remark B.3. C^1 , C^2 , C^0 , and $\mathcal{D}_L \cap C_{\text{loc}}^{\alpha} \cap C_b^0$ (see (4.43)) are F -spaces generated by a countable sequence of seminorms.

Definition B.4. *Let E be an F -space generated by a countable sequence of seminorms. Let d_E be the distance in (B.3) related to E . A function $f : [0, T] \rightarrow E$ is said to be $C^1([0, T]; E)$ if there exists $f' : [0, T] \rightarrow E$ continuous such that*

$$\lim_{\varepsilon \rightarrow 0} \sup_{t \in [0, T]} d_E \left(\frac{f(t + \varepsilon) - f(t)}{\varepsilon}, f'(t) \right) = 0. \quad (\text{B.4})$$

We remark that by Definition B.4 f' is continuous. We are not aware about Bochner integrability for functions taking values in E . For this we make use of Riemann integrability. The following lemma makes use of classical arguments, which exploits the fact that f is uniformly continuous.

Lemma B.5. Let $f : [0, T] \rightarrow B$ be a continuous function, where B is a seminormed space. We denote by

$$s_n(f)(t) = \sum_{k=0}^{2^n-1} f(kt2^{-n})2^{-n}, \quad t \in [0, T],$$

the Riemann sequence related to the dyadic partition. Then $(s_n(f))$ is Cauchy in $C^0([0, T]; B)$, namely

$$\sup_{t \in [0, T]} \|(s_n(f) - s_m(f))(t)\|_B \rightarrow 0 \quad \text{as } n, m \rightarrow \infty.$$

Remark B.6. Let E be an F -space generated by a countable sequence of seminorms $(\|\cdot\|_{\alpha \in \mathbb{N}})$. Let d_E be the distance in (B.3) related to E . Let f_n, f_m (resp. f) be sequences of functions (resp. a function) from $[0, T]$ to E .

(i) $\sup_{t \in [0, T]} d_E(f_n(t), f(t)) \xrightarrow{n \rightarrow \infty} 0$ is equivalent to $\sup_{t \in [0, T]} \|f_n(t) - f(t)\|_{\alpha} \xrightarrow{n \rightarrow \infty} 0$;

(ii) $\sup_{t \in [0, T]} d_E(f_n(t), f_m(t)) \xrightarrow{n, m \rightarrow \infty} 0$ is equivalent to $\sup_{t \in [0, T]} \|f_n(t) - f_m(t)\|_{\alpha} \xrightarrow{n, m \rightarrow \infty} 0$.

Remark B.7 (Riemann integral). Let E be an F -space generated by a countable sequence of seminorms $(\|\cdot\|_{\alpha \in \mathbb{N}})$. By Remark B.6, $C^0([0, T]; E)$ is an F -space. Let $f \in C^0([0, T]; E)$. By Lemma B.5, $(s_n(f))$ is Cauchy with respect to all seminorms $\|\cdot\|_{\alpha \in \mathbb{N}}$. By Remark B.6, the sequence $(s_n(f))$ is Cauchy with respect to d . Being $C^0([0, T]; E)$ an F -space, it is complete, and $(s_n(f))$ converges to an element in $C^0([0, T]; E)$ that we denote by $\int_0^b f(s)ds$. For $a, b \in [0, T]$ we write $\int_a^b f(s)ds = \int_0^b f(s)ds - \int_0^a f(s)ds$. It is not difficult to show that Riemann integral satisfies the following properties.

1. $\int_a^b f(t)dt = \int_{a-h}^{b-h} f(t+h)dt$, $h \in \mathbb{R}$ (by definition of the integral).
2. $\lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_t^{t+\varepsilon} f(s)ds \rightarrow f(t)$ in $C^0([0, T]; E)$, $\lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_{t-\varepsilon}^t f(s)ds \rightarrow f(t)$ in $C^0([0, T]; E)$ (by Remark B.6-(i) and the fact that f is uniformly continuous).
3. If $f \in C^0([0, T]; E)$, then $f(\cdot) = f(0) + \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_0^\cdot (f(s+\varepsilon) - f(s))ds$ (by items 1. and 2. above).
4. If $f \in C^1([0, T]; E)$, then $f(\cdot) = f(0) + \int_0^\cdot f'(s)ds$ (by item 3. above and Definition B.4).

Lemma B.8. Let $\mathcal{D}_{\mathcal{L}}$ be a topological vector F -space generated by a countable sequence of seminorms, equipped with the metric $d_{\mathcal{L}} (= d_E)$ in (B.3) related to $\mathcal{D}_{\mathcal{L}} (= E)$.

We suppose that $\mathcal{D}_{\mathcal{L}}$ is topologically embedded in C^0 . Let $\mathcal{L} : \mathcal{D}_{\mathcal{L}} \rightarrow C^0([0, T] \times D_-(0, T))$ be a continuous map. Let $\hat{\mathcal{D}}_{\mathcal{A}}$ be the subspace of $\mathcal{D}_{\mathcal{A}} := C^1([0, T]; \mathcal{D}_{\mathcal{L}})$ constituted by functions of the type

$$u(t, x) = \sum_k a_k(t)u_k(x), \quad a_k \in C^1(0, T), \quad u_k \in \mathcal{D}_{\mathcal{L}}. \quad (\text{B.5})$$

Then $\hat{\mathcal{D}}_{\mathcal{A}}$ is dense in $\mathcal{D}_{\mathcal{A}}$ equipped with the metric $d_{\mathcal{A}}$ governing the following convergence:
 $u_n \rightarrow 0$ in $\mathcal{D}_{\mathcal{A}}$ if

$$u_n \rightarrow 0 \text{ in } C^0([0, T]; \mathcal{D}_{\mathcal{L}}) \text{ and } \partial_t u_n \rightarrow 0 \text{ in } C^0([0, T]; \mathcal{D}_{\mathcal{L}}).$$

Proof. Let $u \in \mathcal{D}_{\mathcal{A}}$. We have to prove that there is a sequence $u_n \in \hat{\mathcal{D}}_{\mathcal{A}}$ such that $u_n \rightarrow u$ in $\mathcal{D}_{\mathcal{A}}$. We denote $v := u'$, i.e. the time derivative. We divide the proof into two steps.

First step: approximation of v . In this step we only use the fact that $E = \mathcal{D}_{\mathcal{L}}$ is an F -space.

Notice that $v \in C^0([0, T]; \mathcal{D}_{\mathcal{L}})$. Let $\varepsilon > 0$. Since v is uniformly continuous, by Lemma B.1, there exists $\delta > 0$ such that

$$|t - s| \leq \delta \Rightarrow \bar{d}_{\mathcal{L}}(v(t), v(s)) < \frac{\varepsilon}{2}, \quad (\text{B.6})$$

where $\bar{d}_{\mathcal{L}}$ is the metric related to $d_{\mathcal{L}}$ introduced in Lemma B.1 with $M = 1$. We consider a dyadic partition of $[0, T]$ given by $t_k = 2^{-n}kT$, $k \in \{0, \dots, 2^n\}$, $n \in \mathbb{N}$. We define the open covering of $[0, T]$ given by

$$U_k^n = \begin{cases} [t_0, t_1[& k = 0, \\]t_{k-1}, t_{k+1}[\cap[0, T] & k \in \{1, \dots, 2^n\}. \end{cases} \quad (\text{B.7})$$

We also introduce a smooth partition of the unit φ_k^n , $k \in \{0, \dots, 2^n\}$. In particular, $\sum_{k=0}^{2^n} \varphi_k^n = 1$, $\varphi_k^n \geq 0$, and $\text{supp } \varphi_k^n \subset U_k^n$. We define

$$v_n(t) := \sum_{k=0}^{2^n} v(t_k) \varphi_k^n(t).$$

We notice that, if $t \in [t_{k-1}, t_k[$, then $v_n(t) = v(t_{k-1})\varphi_{k-1}^n(t) + v(t_k)\varphi_k^n(t)$ and $v(t) = v(t)\varphi_{k-1}^n(t) + v(t)\varphi_k^n(t)$. Since $d_{\mathcal{L}}$ is a homogeneous distance, using the triangle inequality, we have

$$\begin{aligned} d_{\mathcal{L}}(v_n(t), v(t)) &= d_{\mathcal{L}}((v(t_{k-1}) - v(t))\varphi_{k-1}^n(t) + (v(t_k) - v(t))\varphi_k^n(t), 0) \\ &\leq d_{\mathcal{L}}((v(t_{k-1}) - v(t))\varphi_{k-1}^n(t), 0) + d_{\mathcal{L}}((v(t_k) - v(t))\varphi_k^n(t), 0) \\ &\leq d_{\mathcal{L}}(v(t_{k-1})\varphi_{k-1}^n(t), v(t)\varphi_{k-1}^n(t)) + d_{\mathcal{L}}(v(t_k)\varphi_k^n(t), v(t)\varphi_k^n(t)) \\ &\leq \bar{d}_{\mathcal{L}}(v(t_{k-1}), v(t)) + \bar{d}_{\mathcal{L}}(v(t_k), v(t)). \end{aligned} \quad (\text{B.8})$$

Then we choose N such that $2^{-N}T \leq \delta$. Recalling (B.6), we obtain from (B.8) that

$$n > N \Rightarrow \sup_{t \in [0, T]} d_{\mathcal{L}}(v_n(t), v(t)) \leq \varepsilon.$$

This shows that $v_n \rightarrow v = \partial_t u$ in $C^0([0, T]; \mathcal{D}_{\mathcal{L}})$.

Second step: approximation of u . Now we set

$$u_n(t) := u(0) + \sum_{k=0}^{2^n} v(t_k) \bar{\varphi}_k^n(t),$$

where $\bar{\varphi}_k^n(t) := \int_0^t \varphi_k^n(s) ds$. We remark that $u_n(t) = u(0) + \int_0^t v_n(s) ds$ in the sense of Remark B.7. We have to show that $\sup_{t \in [0, T]} d_{\mathcal{L}}(u_n(t), u(t))$ converges to zero. To this end, by Remark B.6, it is enough to show that, for every fixed α ,

$$\sup_{t \in [0, T]} \|u_n(t) - u(t)\|_{\alpha} \xrightarrow{\varepsilon \rightarrow 0} 0. \quad (\text{B.9})$$

Since u is uniformly continuous with respect to $d_{\mathcal{L}}$, obviously it has the same property with respect to the seminorm α . For every $t \in [0, T]$,

$$\|u_n(t) - u(t)\|_{\alpha} \leq \int_0^t \|v_n(s) - v(s)\|_{\alpha} ds \leq T \sup_{s \in [0, T]} \|v_n(s) - v(s)\|_{\alpha} \leq C \sup_{s \in [0, T]} d_{\mathcal{L}}(v_n(s), v(s))$$

that converges to zero by the first step of the proof. This implies (B.9) and concludes the proof.

□

Lemma B.9. *Let $\mathcal{D}_{\mathcal{L}_M}$ be a topological vector F -space generated by a countable sequence of seminorms, and let $\mathcal{D}_{\mathcal{A}}$ be defined in (4.30). If $\mathcal{D}_{\mathcal{L}_M}$ is dense in C^1 , then $\mathcal{D}_{\mathcal{A}}$ is dense in $C^{0,1}$.*

Proof. We start by noticing that $C^{0,1} = C^0([0, T]; C^1)$. Then we divide the proof into two steps. *First step.* $C^0([0, T]; \mathcal{D}_{\mathcal{L}_M})$ is dense in $C^0([0, T]; C^1)$.

In this step we only use the fact that $E := C^1$ is an F -space. Let d_E be the distance in (B.3) related to E . Let $f \in C^0([0, T]; C^1)$. Let $\delta > 0$. We need to show the existence of $f^\varepsilon \in C^0([0, T]; \mathcal{D}_{\mathcal{L}_M})$ such that

$$d_E(f(t), f^\varepsilon(t)) \leq \delta, \quad \forall t \in [0, T].$$

Let $0 = t_1 < t_2 < \dots < t_n = T$ be a subdivision with mesh ε . Since $\mathcal{D}_{\mathcal{L}_M}$ is dense in C^1 , using Lemma B.1, for every $i = 1, \dots, n$, there is $f_{t_i}^\varepsilon \in \mathcal{D}_{\mathcal{L}_M}$ such that

$$\bar{d}_E(f(t_i), f_{t_i}^\varepsilon) \leq \frac{\delta}{6}, \tag{B.10}$$

where \bar{d}_E is the metric in Lemma B.1 with $M = 1$. The candidate now is

$$f^\varepsilon(t) = f_{t_i}^\varepsilon + \frac{t - t_{i-1}}{t_i - t_{i-1}}(f_{t_i}^\varepsilon - f_{t_{i-1}}^\varepsilon), \quad t \in [t_i, t_{i+1}[.$$

To prove that f^ε is a good approximation of f , we define $f^\pi : [0, T] \rightarrow C^1$ as

$$f^\pi(t) = f(t_{i-1}) + \frac{t - t_{i-1}}{t_i - t_{i-1}}(f(t_i) - f(t_{i-1})), \quad t \in [t_i, t_{i+1}[.$$

We evaluate the difference between f^π and f^ε . Setting $A := \frac{t - t_{i-1}}{t_i - t_{i-1}} \in [0, 1]$, taking into account the homogeneity of d and the triangle inequality, we have

$$\begin{aligned} d_E(f^\pi(t), f^\varepsilon(t)) &= d\left(f(t_{i-1}) + A(f(t_i) - f(t_{i-1})), f_{t_{i-1}}^\varepsilon + A(f_{t_i}^\varepsilon - f_{t_{i-1}}^\varepsilon)\right) \\ &\leq d_E(f(t_{i-1}), f_{t_{i-1}}^\varepsilon) + \bar{d}_E(f(t_i), f_{t_i}^\varepsilon) + \bar{d}_E(f(t_{i-1}), f_{t_{i-1}}^\varepsilon) \leq 3\frac{\delta}{6} = \frac{\delta}{2}, \end{aligned}$$

where in the latter inequality we have used (B.10). This shows that

$$\sup_{t \in [0, T]} d_E(f^\pi(t), f^\varepsilon(t)) \leq \frac{\delta}{2}.$$

Second step. $C^1([0, T]; \mathcal{D}_{\mathcal{L}_M})$ is dense in $C^0([0, T]; \mathcal{D}_{\mathcal{L}_M})$.

In this step we set $E := \mathcal{D}_{\mathcal{L}_M}$. Let $f \in C^0([0, T]; \mathcal{D}_{\mathcal{L}_M})$, and set $f_\varepsilon(t) := \frac{1}{\varepsilon} \int_{t-\varepsilon}^t f(s) ds$, $t \in [0, T]$. This integral is well-defined, and converges to f by Remark B.7-2. \square

C Some further technical results

Proposition C.1. *Condition (1.3) is equivalent to ask that*

$$(1 \wedge |x|^2) \star \mu^X \in \mathcal{A}_{\text{loc}}^+.$$

Proof. We start by noticing that

$$(1 \wedge |x|^2) \star \mu^X = \int_{\mathbb{R}} \mathbb{1}_{|x| > 1} \mu^X(ds dx) + \int_{\mathbb{R}} \mathbb{1}_{|x| \leq 1} x^2 \mu^X(ds dx)$$

$$= \sum_{s \leq \cdot} \mathbb{1}_{\{|\Delta X_s| > 1\}} + \sum_{s \leq \cdot} |\Delta X_s|^2 \mathbb{1}_{\{|\Delta X_s| \leq 1\}}.$$

The process $\sum_{s \leq \cdot} \mathbb{1}_{\{|\Delta X_s| > 1\}}$ is locally bounded having bounded jumps (see Remark 3.13-2.), and therefore it is locally integrable. On the other hand, since there is almost surely a finite number of jumps above one, condition (1.3) is equivalent to

$$S := \sum_{s \leq \cdot} |\Delta X_s|^2 \mathbb{1}_{\{|\Delta X_s| \leq 1\}} < \infty \quad \text{a.s.} \quad (\text{C.1})$$

Being S a process with bounded jumps, (C.1) is equivalent to say that S is locally bounded and therefore locally integrable. This concludes the result. \square

In the sequel of the section we consider a càdlàg process X satisfying condition (1.3).

Lemma C.2. *For all $c > 0$,*

$$\mathbb{1}_{\{|x| > c\}} \star \mu^X \in \mathcal{A}_{\text{loc}}^+.$$

Proof. It is enough to prove the result for $c \leq 1$. We have

$$\mathbb{1}_{\{|x| > c\}} \star \mu^X = \frac{1}{c^2} c^2 \mathbb{1}_{\{|x| > c\}} \star \mu^X \leq \frac{1}{c^2} (|x|^2 \wedge 1) \star \mu^X$$

and the result follows by (1.3) and Proposition C.1. \square

Lemma C.3. *Let $v : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$ be a locally bounded function. Then for all $0 < a_0 \leq a_1$,*

$$|v(s, X_{s-} + x) - v(s, X_{s-})| \mathbb{1}_{\{a_0 < |x| \leq a_1\}} \star \mu^X \in \mathcal{A}_{\text{loc}}^+.$$

Proof. Without restriction of generality, we can take $a_0 < 1 \leq a_1$. Since X_{s-} is càdlàg, it is locally bounded, and therefore we can consider the localizing sequence $(\tau_n)_n$ such that, for every $n \in \mathbb{N}$, $\tau_n = \inf\{s \in \mathbb{R}_+ : |X_{s-}| \leq n\}$. Then, for every $n \in \mathbb{N}$, on $[0, \tau_n]$,

$$\begin{aligned} \mathbb{1}_{[0, \tau_n]}(s) |v(s, X_{s-} + x) - v(s, X_{s-})| \mathbb{1}_{\{a_0 < |x| \leq a_1\}} \star \mu^X &\leq 2 \sup_{y \in (-n-a_1, n+a_1)} |v(s, y)| \mathbb{1}_{\{a_0 < |x| \leq a_1\}} \star \mu^X \\ &\leq 2 \sup_{y \in (-n-a_1, n+a_1)} |v(s, y)| \mathbb{1}_{\{|x| > a_0\}} \star \mu^X \end{aligned}$$

that belongs to \mathcal{A}^+ by (1.3) and Lemma C.2. \square

We now recall the following fact, which constitutes a generalization of Proposition 4.5 (formula (4.4) and considerations below) in [4], obtained replacing $\mathbb{1}_{\{|x| \leq 1\}}$ with $\frac{k(x)}{x}$, with $k \in \mathcal{K}$.

Proposition C.4. *Let $v : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$ be a function of class $C^{0,1}$. Then, for every $k \in \mathcal{K}$,*

$$|v(s, X_{s-} + x) - v(s, X_{s-})|^2 \frac{k^2(x)}{x^2} \star \mu^X \in \mathcal{A}_{\text{loc}}^+.$$

In particular, the process $(v(s, X_{s-} + x) - v(s, X_{s-})) \frac{k(x)}{x} \star (\mu^X - \nu^X)$ is a square integrable purely discontinuous local martingale.

We continue by giving a basic lemma.

Lemma C.5. *Let $k \in \mathcal{K}$. The maps*

$$\begin{aligned} (i) \quad & v \mapsto (v(s, X_{s-} + x) - v(s, X_{s-})) \frac{x - k(x)}{x} \star \mu^X =: D^v, \\ (ii) \quad & v \mapsto (v(s, X_{s-} + x) - v(s, X_{s-})) \frac{k(x)}{x} \star (\mu^X - \nu^X) =: M^{v,d}, \end{aligned}$$

from $C^{0,1}$ with values in \mathbb{D}^{ucp} are well-defined and continuous.

Proof. Let $T > 0$.

(i) For every $v \in C^{0,1}$, the process D^v is defined pathwise. Indeed, let $a_0 > 0$ such that $k(x) = x$ for $x \in [-a_0, a_0]$. Then

$$\begin{aligned} & \sup_{t \in [0, T]} \left| \int_{]0, t] \times \mathbb{R}} (v(s, X_{s-}(\omega) + x) - v(s, X_{s-}(\omega))) \frac{x - k(x)}{x} \mu^X(ds dx) \right| \\ & \leq C \sum_{0 < s \leq T} |\Delta v(s, X_s)| \mathbb{1}_{\{|\Delta X_s| > a_0\}}, \end{aligned} \tag{C.2}$$

for some constant $C = C(k)$. Since the number of jumps of X larger than a_0 is finite, previous quantity is finite a.s.

Let $v_\ell \rightarrow 0$ in $C^{0,1}$ as $\ell \rightarrow \infty$. The continuity follows since replacing v by v_ℓ , (C.2) converges to zero a.s., taking into account that v_ℓ converges uniformly on compact sets.

(ii) For every $v \in C^{0,1}$, $M^{v,d}$ is a square integrable local martingale by Proposition C.4. Moreover, again by Proposition C.4, taking $v = Id$, we have $|k(x)|^2 \star \nu^X \in \mathcal{A}_{loc}^+$.

Let $\tau_n^1 := \inf\{t \geq 0 : |X_{t-}| \geq n\}$ and $\tau_n^2 \uparrow \infty$ be an increasing sequence of stopping times such that $\int_{]0, \tau_n^2] \times \mathbb{R}} |k(x)|^2 \nu^X(ds dx) \in \mathcal{A}^+$ and $M_{\tau_n^2 \wedge \cdot}^{v,d}$ is a square integrable martingale. Take $\tau_n := \tau_n^1 \wedge \tau_n^2$. At this point, let us fix $\varepsilon > 0$. We have

$$\begin{aligned} \mathbb{P}\left(\sup_{t \in [0, T]} |M_t^{v,d}| > \varepsilon\right) &= \mathbb{P}\left(\sup_{t \in [0, T]} |M_t^{v,d}| > \varepsilon, \tau_n \leq T\right) + \mathbb{P}\left(\sup_{t \in [0, T]} |M_t^{v,d}| > \varepsilon, \tau_n > T\right) \\ &\leq \mathbb{P}\left(\tau_n \leq T\right) + \mathbb{P}\left(\sup_{t \in [0, T]} |M_{\tau_n \wedge t}^{v,d}| > \varepsilon\right). \end{aligned}$$

Let us now choose $n_0 \in \mathbb{N}$ in such a way that there exists $\delta > 0$ such that $\mathbb{P}(\tau_n \leq T) \leq \delta$ for all $n \geq n_0$. For $n \geq n_0$, applying the Chebyshev inequality, previous inequality gives

$$\mathbb{P}\left(\sup_{t \in [0, T]} |M_t^{v,d}| > \varepsilon\right) \leq \delta + \mathbb{P}\left(\sup_{t \in [0, T]} |M_{\tau_n \wedge t}^{v,d}| > \varepsilon\right) \leq \delta + \frac{\mathbb{E}\left[\sup_{t \in [0, T]} |M_{\tau_n \wedge t}^{v,d}|^2\right]}{\varepsilon^2}. \tag{C.3}$$

By Doob inequality we have

$$\mathbb{E}\left[\sup_{t \in [0, T]} |M_{\tau_n \wedge t}^{v,d}|^2\right] \leq 4\mathbb{E}[|M_{\tau_n \wedge T}^{v,d}|^2] = 4\mathbb{E}\left[\langle M^{v,d}, M^{v,d} \rangle_{\tau_n \wedge T}\right]. \tag{C.4}$$

Denoting by $[-m, m]$ the compact support of k , we have

$$\begin{aligned} \langle M^{v,d}, M^{v,d} \rangle_{\tau_n \wedge T} &\leq \int_{]0, \tau_n \wedge T] \times \mathbb{R}} \int_0^1 da |\partial_x v(s, X_{s-} + ax)|^2 |k(x)|^2 \nu^X(ds dx) \\ &\leq \sup_{y \in [-(m+n), m+n], s \in [0, T]} |\partial_x v(s, y)|^2 \int_{]0, \tau_n \wedge T] \times \mathbb{R}} |k(x)|^2 \nu^X(ds dx). \end{aligned} \tag{C.5}$$

Collecting (C.4) and (C.5), inequality (C.3) becomes

$$\mathbb{P}\left(\sup_{t \in [0, T]} |M_t^{v, d}| > \varepsilon\right) \leq \delta + \frac{4}{\varepsilon^2} \sup_{y \in [-(m+n), m+n], s \in [0, T]} |\partial_x v(s, y)|^2 \mathbb{E}\left[\int_{]0, \tau_n \wedge t] \times \mathbb{R}} |k(x)|^2 \nu^X(ds dx)\right].$$

Let us now show the continuity with respect to v . Let $v_\ell \rightarrow 0$ in $C^{0,1}$ as $\ell \rightarrow \infty$. Previous estimate shows that

$$\limsup_{\ell \rightarrow \infty} \mathbb{P}\left(\sup_{t \in [0, T]} |M_t^{v_\ell, d}| > \varepsilon\right) \leq \delta.$$

By the arbitrariness of δ this shows that $(M^{v_\ell, d})$ converges to zero u.c.p. \square

Lemma C.6. *Let $k_1, k_2 \in \mathcal{K}$. Then $|k_1(x) - k_2(x)| \star \nu^X \in \mathcal{A}_{\text{loc}}^+$.*

Proof. Let a_0 such that $k_1(x) = k_2(x) = x$ on $[-a_0, a_0]$. Then

$$|k_1(x) - k_2(x)| \star \nu^X = |k_1(x) - k_2(x)| \mathbb{1}_{\{|x| > a_0\}} \star \nu^X \leq (\|k_1\|_\infty + \|k_2\|_\infty) \mathbb{1}_{\{|x| > a_0\}} \star \nu^X$$

that belongs to $\mathcal{A}_{\text{loc}}^+$ by Lemma C.2. \square

Lemma C.7. *Let $v : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$ be a locally bounded function. Condition (3.8) is equivalent to*

$$(v(s, X_{s-} + x) - v(s, X_{s-})) \frac{k(x)}{x} \in \mathcal{G}_{\text{loc}}^1(\mu^X), \quad \forall k \in \mathcal{K}.$$

Proof. Let $k_1, k_2 \in \mathcal{K}$. It is enough to show that $\left|(v(s, X_{s-} + x) - v(s, X_{s-})) \frac{k_1(x) - k_2(x)}{x}\right| \star \mu^X \in \mathcal{A}_{\text{loc}}^+$. Let $a_0, a_1 > 0$ such that $k_1(x) = k_2(x) = x$ on $|x| \leq a_0$ $k_1(x) = k_2(x) = 0$ for $|x| > a_1$. We have

$$\begin{aligned} & \left|(v(s, X_{s-} + x) - v(s, X_{s-})) \frac{k_1(x) - k_2(x)}{x}\right| \star \mu^X \\ &= \left|(v(s, X_{s-} + x) - v(s, X_{s-})) \frac{k_1(x) - k_2(x)}{x}\right| \mathbb{1}_{\{a_0 < |x| \leq a_1\}} \star \mu^X \\ &\leq \frac{\|k_1\|_\infty + \|k_2\|_\infty}{a_0} \mathbb{1}_{\{a_0 < |x| \leq a_1\}} \star \mu^X \end{aligned}$$

that belongs to $\mathcal{A}_{\text{loc}}^+$ by Lemma C.3.

D Recalls on discontinuous semimartingales and related Jacod's martingale problems

We recall that a special semimartingale is a semimartingale X which admits a decomposition $X = M + V$, where M is a local martingale and V is a finite variation and predictable process such that $V_0 = 0$, see Definition 4.21, Chapter I, in [27]. Such a decomposition is unique, and is called canonical decomposition of X , see respectively Proposition 3.16 and Definition 4.22, Chapter I, in [27]. In the following we set $\tilde{\mathcal{K}} := \{k : \mathbb{R} \rightarrow \mathbb{R} \text{ bounded: } k(x) = x \text{ in a neighborhood of } 0\}$.

Assume now that X is a semimartingale with jump measure μ^X . Given $k \in \tilde{\mathcal{K}}$, the process $X^k = X - \sum_{s \leq \cdot} [\Delta X_s - k(\Delta X_s)]$ is a special semimartingale with unique decomposition

$$X^k = X^c + M^{k, d} + B^{k, X}, \tag{D.1}$$

where $M^{k,d}$ is a purely discontinuous local martingale such that $M_0^{k,d} = 0$, X^c is the unique continuous martingale part of X (it coincides with the process X^c introduced in Proposition 3.2), and $B^{k,X}$ is a predictable process of bounded variation vanishing at zero.

Let now $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{F}})$ be the canonical filtered space, and \tilde{X} the canonical process. According to Definition 2.6, Chapter II in [27], the characteristics of X associated with $k \in \tilde{\mathcal{K}}$ is then the triplet (B^k, C, ν) on $(\Omega, \mathcal{F}, \mathbb{F})$ such that the following items hold.

- (i) B^k is $\tilde{\mathbb{F}}$ -predictable, with finite variation on finite intervals, and $B_0^k = 0$, i.e., $B^{k,X} = B^k \circ X$ is the process in (D.1);
- (ii) C is a continuous process of finite variation with $C_0 = 0$, i.e., $C^X := C \circ X = \langle \tilde{X}^c, \tilde{X}^c \rangle$.
- (iii) ν is an $\tilde{\mathbb{F}}$ -predictable random measure on $\mathbb{R}_+ \times \mathbb{R}$, i.e., $\nu^X := \nu \circ X$ is the compensator of μ^X .

Theorem D.1 (Theorem 2.42, Chapter II, in [27]). *Let X be an adapted càdlàg process. Let B^k be an $\tilde{\mathbb{F}}$ -predictable process, with finite variation on finite intervals, and $B_0^k = 0$, C be an $\tilde{\mathbb{F}}$ -adapted continuous process of finite variation with $C_0 = 0$, and ν be an $\tilde{\mathbb{F}}$ -predictable random measure on $\mathbb{R}_+ \times \mathbb{R}$. There is equivalence between the two following statements.*

- (i) X is a real semimartingale with characteristics (B^k, C, ν) .
- (ii) For each bounded function f of class C^2 , the process

$$\begin{aligned} & f(X_t) - f(X_0) - \frac{1}{2} \int_0^t f''(X_s) dC_s^X - \int_0^t f'(X_s) dB_s^{k,X} \\ & - \int_{]0,t] \times \mathbb{R}} (f(X_{s-} + x) - f(X_{s-}) - k(x) f'(X_{s-})) \nu^X(ds dx) \end{aligned} \quad (\text{D.2})$$

is a local martingale.

Remark D.2. Assuming item (i) in Theorem D.1, if f is a bounded function of class $C^{1,2}$, formula (D.2) can be generalized into

$$\begin{aligned} & f(t, X_t) - f(0, X_0) - \int_0^t \frac{\partial}{\partial s} f(s, X_s) ds - \frac{1}{2} \int_0^t \frac{\partial^2}{\partial x^2} f(s, X_s) dC_s^X - \int_0^t \frac{\partial}{\partial x} f(s, X_s) dB_s^{k,X} \\ & - \int_{]0,t] \times \mathbb{R}} \left(f(s, X_{s-} + x) - f(s, X_{s-}) - k(x) \frac{\partial}{\partial x} f(s, X_{s-}) \right) \nu^X(ds dx), \quad t \in [0, T]. \end{aligned}$$

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