ROUGH PATHS AND SYMMETRIC-STRATONOVICH INTEGRALS DRIVEN BY SINGULAR COVARIANCE GAUSSIAN PROCESSES

BY ALBERTO OHASHI[1,a], FRANCESCO RUSSO[2,b]

[1] *Department of Mathematics, Universidade de Brasília,* [a] *ohashi@mat.unb.br*

[2] *Unité de Mathématiques appliquées, Institut Polytechnique de Paris ,* [b] *francesco.russo@ensta-paris.fr*

We examine the relation between a stochastic version of the rough path integral with the symmetric-Stratonovich integral in the sense of regularization. Under mild regularity conditions in the sense of Malliavin calculus, we establish equality between stochastic rough path and symmetric-Stratonovich integrals driven by a class of Gaussian processes. As a by-product, we show that solutions of multi-dimensional rough differential equations driven by a large class of Gaussian rough paths they are actually solutions to Stratonovich stochastic differential equations. We obtain almost sure convergence rates of the first-order Stratonovich scheme to rough paths integrals in the sense of Gubinelli. In case the time-increment of the Malliavin derivative of the integrands is regular enough, the rates are essentially sharp. The framework applies to a large class of Gaussian processes whose the second-order derivative of the covariance function is a sigma-finite non-positive measure on $\mathbb{R}^2_+$ off diagonal.

1. Introduction. Let X be a d-dimensional continuous Gaussian process over a bounded time interval $[0,T]$ and equipped with a second-order process $\mathbb{X}$ so that $\mathbf{X} = (X, \mathbb{X})$ is a γ-geometric rough path for $\frac{1}{3} < \gamma < \frac{1}{2}$ (see e.g Lyons [23] and Gubinelli [19]). Let $\mathcal{D}_X(\mathbb{R}^d)$ be the space of d-dimensional processes Y controlled by X in the sense that there exists a $\mathbb{R}^{d\times d}$-valued process Y' such that

$$Y_t - Y_s = Y'_s(X_t - X_s) + A_{s,t}, \tag{1}$$

where a two-parameter process A implicitly defined by (1) satisfies

$$\lim_{\epsilon\to 0^+} \frac{1}{\epsilon}\int_0^t \langle A_{s,s+\epsilon}, X_{s+\epsilon} - X_s\rangle ds = 0 \quad \text{(in probability)}, \tag{2}$$

for each $t > 0$. A typical example of a pair $(Y, Y') \in \mathcal{D}_X(\mathbb{R}^d)$ is a controlled rough path in the sense of Gubinelli as described in (5). The goal of this paper is to establish equality of a suitable stochastic version of the rough path integral (in the sense of Gubinelli [19]) with the symmetric-Stratonovich integral (in the sense of stochastic calculus via regularizations, see e.g. [29]) for a given pair $(Y, Y') \in \mathcal{D}_X(\mathbb{R}^d)$. In particular, we study the problem of the (almost sure) convergence rate of the first-order Stratonovich approximation scheme

$$I^0(\epsilon, Y, dX)(t) := \frac{1}{2\epsilon}\int_0^t \Big\langle Y_s, X_{(s+\epsilon)\wedge t} - X_{(s-\epsilon)\vee 0}\Big\rangle ds \tag{3}$$

to rough path integrals driven by $\mathbf{X}$ of controlled rough paths (Y, Y') in the sense of Gubinelli [19].

Stratonovich integrals play a prominent role in stochastic analysis. Since Wong and Zakai's pioneering work [32], we know that the Stratonovich formulation of stochastic differential equations (SDEs)

MSC2020 subject classifications: Primary 60L20; 60G15; 60H05.Secondary 60H07.
Keywords and phrases: Rough paths, Stratonovich integrals.

$$dY_t = f(Y_t)dX_t \tag{4}$$

has the important interpretation of being approximated by a sequence of ordinary differential equations driven by smooth approximations X^n for a continuous semimartingale driving noise X. Lyons establishes the rough path theory in his seminal work [23] and uses a Wong-Zakai-type argument to establish well-posedness of SDEs (4) driven by rather general noises. Rough path theory provides a robust pathwise solution which is continuous with respect to the driving path X. Lyons's deep insight was to realize that what really controls the dynamics in (4) is not the path of X but rather a "natural" lift of X to a random rough path $\mathbf{X}$. Gubinelli [19] observes that, in the regime $\frac{1}{3} < \gamma < \frac{1}{2}$, a consistent integration theory can be formulated by fixing $\mathbf{X}$ and considering integrands of the form (Y, Y') satisfying

$$Y_t - Y_s = Y'_s(X_t - X_s) + R^Y_{s,t}, \tag{5}$$

where $R^Y_{s,t} = O(|t-s|^r)$ a.s for $r = \gamma + \beta$, Y' is β-Hölder continuous with $2\gamma + \beta > 1$. In his approach, the celebrated Sewing lemma (see e.g [12, 19]) plays a fundamental role on the construction of the rough path integral $(Y, Y') \mapsto \big(\int Y d\mathbf{X}, Y\big)$ which is described by

$$\int_0^T Y_s d\mathbf{X}_s = \lim_{\|\Pi\| \to 0} \sum_{t_i \in \Pi} \big\{ \langle Y_{t_i}, X_{t_{i+1}} - X_{t_i} \rangle + Y'_{t_i} \mathbb{X}_{t_i, t_{i+1}} \big\}, \tag{6}$$

almost surely, as the mesh of partitions $\|\Pi\| \to 0$. The role of the underlying probability measure is totally restricted to the construction of the rough path $\mathbf{X}$ and the Sewing lemma is applied pathwisely.

Another approach in dealing with integrals driven by irregular noises is via regularization ([29]) which is based on integral-type approximations of the form

$$\int_0^t Y d^* X_s = \lim_{\epsilon \to 0^+} \frac{1}{\epsilon} \int_0^t \langle Y_s, X(\epsilon, s) \rangle ds, \quad * = +, -, 0, \tag{7}$$

where $\frac{1}{\epsilon} X(\epsilon, s)$ encodes a sort of "derivative approximation" of X and convergence (7) should be interpreted in probability. This gives rise to three different types of stochastic integrals called backward $(+)$, forward $(-)$ and symmetric-Stratonovich (0) integrals. In this approach, none higher-order approximation scheme is employed. The connection with semimartingale theory, Young and Skorohod integrals has been studied over the years by many authors (see e.g [30], [1], [7], [16]). When the driving noise has very low regularity, it turns out that symmetric-Stratonovich integral is the correct choice (see e.g [7], [17], [18]). A one-dimensional theory of symmetric-Stratonovich SDEs is constructed by [10], where the driving noise is a general finite-cubic variation process (in the sense of [11]) and a semimartingale.

Coming back to the rough path theory approach, in one hand, in general, one cannot avoid the inclusion of the second-order process $\mathbb{X}$ in (6). On the other hand, in the case the driving noise X is an $\mathbb{F}$-continuous semimartingale, the classical Stratonovich stochastic integral coincides with the stochastic rough path integral (9) driven by a Stratonovich second-order process $\mathbb{X}$ and $\mathcal{D}_X(\mathbb{R}^d)$ coincides with the space of $\mathbb{F}$-weak Dirichlet processes. In this direction, we refer to the recent paper [15] and also Friz and Victoir's book [section 17.2, [14]]. We also drive attention to [22] where the authors produce first-order trapezoidal approximations for (6) in case X belongs to a rather general class of Gaussian processes and Y' is also controllable in the sense of [19]. In [26] one shows that the presence of $\mathbb{X}$ in (6) can be neglected in case X is a "typical price path" with finite quadratic variation, which confirms earlier considerations in [9]. In the case Y is a gradient system or a solution of a rough differential equation driven by a class of Gaussian geometric rough paths, then it is known that Skorohod correction terms can be derived. In this direction, see e.g [20] and [4, 5], respectively.

The above results suggest that the rough path integral of a pair $(Y, Y') \in \mathcal{D}_X(\mathbb{R}^d)$ driven by Gaussian geometric rough paths can be recast as a purely first-order symmetric-Stratonovich stochastic integral in the sense of regularization [29]. The main result of this paper demonstrates that this is almost the case, at least for a large class of Gaussian driving noises and derivatives Y' satisfying weak regularity conditions in the sense of Malliavin calculus. More precisely, recall that the symmetric-Stratonovich integral, in the sense of regularization ([29]), it is defined by

$$\int_0^t Y_s d^0 X_s := \lim_{\epsilon \downarrow 0} I^0(\epsilon, Y, dX)(t) \quad \text{(in probability)}. \tag{8}$$

In this article, the rough path integral will be interpreted in the sense of regularization and the convergence topology will be given in probability rather than almost sure:

$$\int_0^t Y_s d\mathbf{X}_s = \lim_{\epsilon \to 0^+} \frac{1}{\epsilon} \int_0^t \Big(\langle Y_s, X_{s+\epsilon} - X_s \rangle + Y'_s \mathbb{X}_{s,s+\epsilon} \Big) ds \quad \text{(in probability)}, \tag{9}$$

where the area process $\mathbb{X}$ is given by

$$\mathbb{X}_{s,t} = \int_s^t (X_r - X_s) d^0 X_r, \tag{10}$$

and by convention we set $X_u = X_T; u > T$. In this article, we show *equivalence* between the two integrals in the following sense. Let $\mathbb{D}^{1,p}$ be the Malliavin-Watanabe space associated with the Malliavin derivative $\mathbf{D}$ supported by a probability measure $\mathbb{P}$. The equivalence is stated below in an informal way. The reader is referred to Theorem 5.1 for the precise statement of Theorem 1.1 described below.

THEOREM 1.1. *Let X be a d-dimensional continuous Gaussian process with covariance kernel R whose the Schwartz second-order derivative of R is a non-positive sigma-finite measure $d\mu = \partial^2 R dx$ which is absolutely continuous w.r.t Lebesgue on $[0,T]^2$ off diagonal. Assume $(Y, Y') \in \mathcal{D}_X(\mathbb{R}^d)$ and there exist $p, q > 2$ such that $t \mapsto Y'_t$ is a $\mathbb{D}^{1,p}$-valued continuous function and*

$$\int_0^T \int_{v_2}^T \sup_{s \geq v_1 \text{ or } s < v_2} \|\mathbf{D}_{v_1} Y'_s - \mathbf{D}_{v_2} Y'_s\|^q_{L^q(\mathbb{P})} |\partial^2 R(v_1, v_2)|^{\frac{q}{2}} dv_1 dv_2 < \infty. \tag{11}$$

Then, (8) exists, if and only if, (9) exists. Moreover, when $(Y, Y') \in \mathcal{D}_X(\mathbb{R}^d)$ is integrable, we have

$$\begin{aligned} \int_0^t Y_s d\mathbf{X}_s &= \int_0^t Y_s d^0 X_s \\ &= \lim_{\epsilon \to 0^+} \frac{1}{\epsilon} \int_0^t \Big(\langle Y_s, X_{s+\epsilon} - X_s \rangle + Y'_s Sym(\mathbb{X}_{s,s+\epsilon}) \Big) ds, \end{aligned} \tag{12}$$

in probability, where $Sym(\mathbb{X})$ is the symmetric part of $\mathbb{X}$ given by (10).

Theorem 1.1 almost immediately implies that solutions of rough differential equations driven by $\mathbf{X} = (X, \mathbb{X})$ (see Example 7) are also solutions to multi-dimensional Stratonovich SDEs of the form

$$Y_t = Y_0 + \int_0^t V(Y_s) d^0 X_s, \tag{13}$$

for smooth coefficients V. See Corollary 5.1.

The equivalence presented in Theorem 1.1 yields the investigation of the ($L^2(\mathbb{P})$ and almost sure) rate of convergence of the first-order Stratonovich approximation scheme $I^0(\epsilon, Y, dX)(T)$ to a rough path integral. For sake of conciseness, we approach this problem in the case of the fractional Brownian motion driving noise. In the sequel, we summarize two major consequences of Theorem 1.1 applied to rough path integrals driven by a fractional Brownian motion with parameter $\frac{1}{3} < H < \frac{1}{2}$.

COROLLARY 1.1. *Let $X = (X_1, \ldots, X_d)$ be a d-dimensional fractional Brownian motion with parameter $\frac{1}{3} < H < \frac{1}{2}$. Let $\mathbf{X} = (X, \mathbb{X})$ be the geometric rough path given by (10). Assume that $f : \mathbb{R}^d \to \mathbb{R}^d \in C_b^2$ and fix $\rho > 0$ such that $0 < \rho < 2H - \frac{1}{2}$. Then,*

$$\left| \int_0^T f(X_s) d\mathbf{X}_s - I^0(2^{-n}, f(X), dX)(T) \right| \lesssim 2^{-n\rho} \to 0, \tag{14}$$

almost surely, as $n \to +\infty$.

COROLLARY 1.2. *Let $X = (X_1, \ldots, X_d)$ be a d-dimensional fractional Brownian motion with parameter $\frac{1}{3} < H < \frac{1}{2}$. Let $\mathbf{X} = (X, \mathbb{X})$ be the geometric rough path given by (10) and $V \in C_b^3\big(\mathbb{R}^d, \mathcal{L}(\mathbb{R}^d, \mathbb{R}^d)\big)$. Let Y be the solution of the rough differential equation*

$$Y_t = Y_0 + \int_0^t V(Y_s) d\mathbf{X}_s; 0 \leq t \leq T.$$

Fix $\frac{1}{3} < \gamma < H$ and ρ such that $0 < \rho < \gamma + 2H - 1$. Then

$$\left| \int_0^T Y_s d\mathbf{X}_s - I^0(2^{-n}, Y, dX)(T) \right| \lesssim 2^{-n\rho} \to 0, \tag{15}$$

almost surely, as $n \to +\infty$.

1.1. *Discussion of results and idea of the proofs.* In the sequel, we fix a Gaussian process X satisfying Assumptions A, B, C, D, E and F with a given exponent $-\frac{4}{3} < \alpha < -1$ realizing

$$|\partial^2 R(s,t)| \lesssim |t - s|^{\alpha} + \phi(s,t), \tag{16}$$

on $[0, T]^2 \setminus D$, where D is the diagonal of $[0, T]^2$. The typical example is the bifractional Brownian motion with exponents $\frac{1}{3} < HK < \frac{1}{2}$, $H \in (0, 1)$ and $K \in (0, 1]$. In this section, we discuss the main results of this paper, namely, Theorems 5.1, 6.2 and their Corollaries 1.1 and 1.2.

Under the regularity condition (11), Theorem 5.1 implies that if one relax almost sure convergence to convergence in probability, the anti-symmetric part $\text{Anti}(\mathbb{X})$ plays no role in the convergence of the integral in (12). Moreover, one can compute the stochastic rough integral (9) through a first-order symmetric-Stratonovich scheme $I^0(\epsilon, Y, dX)$ without involving the higher-order term $\mathbb{X}$. Let $\mathbf{X} = (X, \mathbb{X})$ be the geometric process defined by (10) and let $\text{Anti}(\mathbb{X})$ be the antisymmetric part of $\mathbb{X}$. The main argument in the proof of Theorem 5.1 is the verification that the convergence

$$\lim_{\epsilon\to 0^+} \frac{1}{\epsilon}\int_0^t \left\langle Y'_s, \text{Anti}(\mathbb{X}_{s,s+\epsilon})\right\rangle_{\mathbf{F}} ds = 0 \quad \text{(in probability)} \tag{17}$$

holds true in typical situations for $(Y, Y') \in \mathcal{D}_X(\mathbb{R}^d)$, where $\langle\cdot,\cdot\rangle_{\mathbf{F}}$ denotes the Frobenius inner product on the space of $d \times d$-matrices. The analysis of (17) starts with the representation of $\text{Anti}(\mathbb{X})$ in terms of the divergence operator. In a second step, we provide delicate estimates on Skorohod integrals involving Y' and components of $\text{Anti}(\mathbb{X})$. Convergence (17) (in the sense of Riemann sum) is analyzed by [22], where the authors assume a pathwise second-order additional decomposition for Y, where Y' follows (5) equipped with a second Gubinelli's derivative Y''. On the one hand, in contrast to [22], none second-order pathwise expansion of Y is employed in our framework. See Example 8 and Remark 5.2. On the other hand, we assume Malliavin-type regularity on Y'. We believe (11) is the natural stochastic regularity condition to insure (17) and, indeed, (11) it is fulfilled for a large class of examples.

We stress the simplest possible case takes place when $(Y, Y') \in \mathcal{D}_X(\mathbb{R}^d)$ is a controlled rough path in the Gubinelli's sense and Y' is *symmetric*. This case is examined by [13] and one can reduce the relevant information to the *reduced rough path* $\mathbf{X} = (X, \text{Sym}(\mathbb{X}))$. In this work, we show that this phenomenon takes place in typical situations much beyond the symmetric case. For instance, when Y' is deterministic (see e.g Example 8), condition (11) requires only continuity of $t \mapsto Y'_t$. We stress the equality (12) provided by Theorem 5.1 can fail outside the class of integrands $(Y, Y') \in \mathcal{D}_X(\mathbb{R}^d)$. Indeed, in general, $\lim_{\epsilon\downarrow 0} I^0(\epsilon, Y, dX)(t)$ does not require the structure (1) and (2) imposed on the set $\mathcal{D}_X(\mathbb{R}^d)$. See Lemma 6.3 for a simple case. We emphasize Theorem 5.1 can be extended to the less regular case $-\frac{3}{2} < \alpha \leq -\frac{4}{3}$ by working with a corresponding level-3 Stratonovich geometric process. We postpone this analysis to a future work.

In Theorem 6.2, we further study the precise limiting behavior $I^0(\epsilon, Y, dX)(T)$ to a symmetric-Stratonovich integral in a broader regime $-\frac{3}{2} < \alpha < -1$. The rate of convergence to a rough path integral is then obtained by Theorem 5.1 and restricting to the case $-\frac{4}{3} < \alpha < -1$. Theorem 6.2 presents a sufficient condition for the existence of the symmetric-Stratonovich integral under Malliavin regularity conditions and we explore an $L^2(\mathbb{P})$-rate of convergence. Although other classes of Gaussian process can also be presented in Theorem 6.2, for convenience of exposition, we only present the case of the fractional Brownian motion $\frac{1}{4} < H < \frac{1}{2}$ so that $\alpha = 2H - 2$ and $\phi = 0$ in (16). This is the content of Theorem 6.2. For this purpose, we explore a decomposition of (3) in terms of "Skorohod component plus a trace term". This type of approximate decomposition has already appeared in the seminal work of [25] in the Brownian motion context and also in the fractional Brownian motion context in the works [2, 1]. They both explored undirect density-type arguments of simple processes which do not allow the obtention of convergence rates. Recently, in the particular case of rough differential equations, [4, 5] also explore such type of decomposition without convergence rates. The recent work [31] mixes the Malliavin derivative trace with Gubinelli's derivative without the obtention of convergence rates.

In the present work, we obtain the rates based on the regularity of the shifted process $Y_{\cdot+r}$ when $r \to 0$ (see Proposition 4.1 and Lemma 4.4) and a detailed analysis on $Tr(\mathbf{D}Y)_\epsilon$ (see (64)) in terms of two assumptions. There exist $\gamma > 0$ and $\eta > 0$ such that $2\gamma + 2H - 1 > 0$ and $\eta + 2H - 1 > 0$ which realize (40) and (76), respectively. Corollaries 1.1 and 1.2 are consequences of Theorems 6.2 and 5.1. The $L^2(\mathbb{P})$-convergence rate is given by

$$\mathbb{E}\left|\int_0^T Y_s d\mathbf{X}_s - I^0(\epsilon, Y, dX)(T)\right|^2 \lesssim \epsilon^{2\gamma+2H-1} + \epsilon^{2(\eta+2H-1)}, \tag{18}$$

for every $\epsilon > 0$ sufficiently small. At this point, it is important to discuss the sharpness of the convergence rates. Typically, in non-trivial situations, we expect $0 < \gamma \leq H$. If the time increment of the Malliavin derivative is regular enough in the sense that $\eta \geq \gamma + \frac{1}{2} - H$ in (76), then the leading term in the right-hand side of (18) is $\epsilon^{2\gamma+2H-1}$ and the rate becomes $\epsilon^{(4H-1)-}$ as long as $\gamma \uparrow H$. We observe in case $\gamma = H$ and $\eta \geq \frac{1}{2}$, we get the exact rate ϵ^{4H-1}. In these cases, the rate is essentially sharp considering that the Lévy area diverges when $H = \frac{1}{4}$, see e.g [8]. This can also be viewed in Corollary 1.1. Unfortunately, in case $\gamma + \frac{1}{2} - H > \eta$, the leading term in the right-hand side of (18) is $\epsilon^{2(\eta+2H-1)}$ and then the $L^2(\mathbb{P})$-rate becomes $\epsilon^{(6H-2)-}$ as long as $\eta \uparrow H$. In this case, it is not sharp. This happens in the case of the rough differential equation as described in Corollary 1.2. This phenomenon appears due to high singularity found in the Radon-Nikodym derivative $\partial^2 R(r_1, r_2) = H(2H-1)|r_1 - r_2|^{2H-2}$ on $[0,T]^2 \setminus D$ which, in our strategy, it requires the existence of

$$\int_{0\leq r_1 < r_2 \leq T} \mathrm{tr}[\mathbf{D}_{r_1} Y_{r_2} - \mathbf{D}_{r_2-} Y_{r_2}] |r_1 - r_2|^{2H-2} dr_1 dr_2.$$

In order to improve the rate $\epsilon^{(6H-2)-}$ in the case $\eta \uparrow H$ and $\gamma + \frac{1}{2} - H > \eta$, one needs to work with a distinct decomposition of $I^0(\epsilon, Y, dX)(T)$ not involving the trace of the Malliavin derivative of Y.

The paper is organized as follows. In Section 2, we fix some notation and we define some basic objects. Section 3 presents the basic elements of the Gaussian space of the driving noise. Section 4 presents some important tools from Malliavin calculus. Section 5 presents Theorem 5.1 and some examples. Section 6 presents the proof of Theorem 6.2, Corollaries 1.1 and 1.2. The proof of Theorem 5.1 is given in the Appendix A.

2. Preliminaries. At first, we introduce some notation. In the sequel, finite-dimensional spaces will be equipped with a norm $|\cdot|$ and T is a finite terminal time. The notation $\mathcal{C}^\alpha$ is reserved for α-Hölder continuous paths defined on $[0,T]$ for $\alpha \in (0,1]$, with values in some finite-dimensional space. For $f \in \mathcal{C}^\alpha$, the usual seminorm is given by

$$\|f\|_\alpha := \sup_{s,t\in[0,T]} \frac{|f_{s,t}|}{|t-s|^\alpha},$$

where, when convenient, we use the notation $f_{s,t} := f_t - f_s$. The sup-norm on the space of continuous functions will be denoted by $\|\cdot\|_\infty$. For a two-parameter function g, we write $g \in \mathcal{C}_2^\beta$ if

$$\|g\|_{\mathcal{C}_2^\beta} := \sup_{s,t\in[0,T]} \frac{|g_{s,t}|}{|t-s|^\beta} < \infty,$$

for $\beta > 0$. We further write $a \lesssim b$ for two positive quantities to express an estimate of the form $a \leq Cb$. By convention, any continuous function f defined on $[0,T]$ will be extended (when necessary) to the real line $\mathbb{R}$ as

$$f(t) := \begin{cases} f(0); \text{ if } t \leq 0 \\ f(T); \text{ if } t \geq T. \end{cases}$$

Throughout this article, we are given a reference continuous $\mathbb{R}^d$-valued stochastic process X equipped with a second-order $\mathbb{R}^{d\times d}$-valued stochastic process $\mathbb{X}$ which satisfies the Chen's relation

$$\mathbb{X}_{s,t} - \mathbb{X}_{s,u} - \mathbb{X}_{u,t} = \big(X^i_{s,u}X^j_{u,t}\big)_{1\le i,j\le d}, \tag{19}$$

for every $(s,u,t)\in[0,T]^3$. We then write $\mathbf{X}=(X,\mathbb{X})$. Let us consider

$$\begin{aligned}\mathbb{X}_{s,t} &= \frac{1}{2}\Big(\mathbb{X}^{i,j}_{s,t}+\mathbb{X}^{j,i}_{s,t}\Big)+\frac{1}{2}\Big(\mathbb{X}^{i,j}_{s,t}-\mathbb{X}^{j,i}_{s,t}\Big); 1\le i,j\le d,\\ &=: \mathrm{Sym}(\mathbb{X}_{s,t})+\mathrm{Anti}(\mathbb{X}_{s,t}).\end{aligned}$$

Throughout this paper, all stochastic processes are defined on a given probability space $\big(\Omega,\mathcal{F},\mathbb{P}\big)$.

DEFINITION 2.1. We say that a pair $\mathbf{X}=(X,\mathbb{X})$ is a **geometric process** if

$$\begin{aligned}\mathrm{Sym}(\mathbb{X}_{s,t}) &= \frac{1}{2}[(X_t-X_s)\otimes(X_t-X_s)]\\ &:= \frac{1}{2}(X^i_t-X^i_s)(X^j_t-X^j_s)\ ; 1\le i,j\le d,\ s,t\in[0,T].\end{aligned} \tag{20}$$

DEFINITION 2.2. Given a reference process X, we say that an $\mathcal{L}(\mathbb{R}^d,\mathbb{R}^n)$-valued stochastic process Y is **stochastically controlled** by X if there exists an $\mathcal{L}(\mathbb{R}^d,\mathcal{L}(\mathbb{R}^d,\mathbb{R}^n))$-valued stochastic process Y' so that the remainder term R^Y given implicitly by the relation

$$Y_t - Y_s = Y'_s X_{s,t} + R^Y_{s,t}, \tag{21}$$

is orthogonal to X, i.e.,

$$\lim_{\epsilon\to0^+}\frac{1}{\epsilon}\int_0^t R^Y_{s,s+\epsilon}X_{s,s+\epsilon}ds = \mathbf{0}, \tag{22}$$

in probability for each $t\in[0,T]$.

This defines the set $\mathcal{D}_X(\mathcal{L}(\mathbb{R}^d,\mathbb{R}^n))$ of all stochastically controlled processes (Y,Y') satisfying (21) and (22). When $n=1$, we write $\mathcal{D}_X(\mathbb{R}^d):=\mathcal{D}_X(\mathcal{L}(\mathbb{R}^d,\mathbb{R}))$.

REMARK 2.1. *Clearly, the concept of stochastically controlled processes does not depend on a Gaussian structure for the driving noise. In fact, if $\mathbb{F}$ is a filtration and X is a continuous $\mathbb{F}$-local martingale, then the class of stochastically controlled processes coincides with the class of continuous $\mathbb{F}$-weak Dirichlet processes, see Prop 3.7 in [15].*

Next, we illustrate the fundamental role played by the notion of cubic (resp. strong cubic) variation introduced by [11], see Section 2.1.

EXAMPLE 1. Let X be a d-dimensional continuous process whose coordinates are finite strong cubic variation processes and at least one component has a zero cubic variation. The typical example is fractional Brownian motion with parameter $\frac{1}{3}\le H<1$. Let $f:\mathbb{R}^d\to\mathbb{R}^d\in C^2$. Then, $f(X)\in\mathcal{D}_X(\mathbb{R}^d)$. For details, see Example 3.6 in [15].

Inspired by Gubinelli [19], let us now give the definition of the integral in the sense of regularization.

DEFINITION 2.3. For a given $\mathbb{R}^d$-valued reference process $\mathbf{X} = (X, \mathbb{X})$, we say that $(Y, Y') \in \mathcal{D}_X(\mathbb{R}^d)$ is **rough stochastically integrable** if

$$\int_0^t Y_s d\mathbf{X}_s := \lim_{\epsilon \to 0^+} \frac{1}{\epsilon} \int_0^t \Big(Y_s X_{s,s+\epsilon} + Y'_s \mathbb{X}_{s,s+\epsilon} \Big) ds \tag{23}$$

exists in probability for each $t \in [0, T]$.

We observe Y' can be viewed as an $\mathcal{L}(\mathbb{R}^{d\times d}, \mathbb{R})$-valued process via the canonical injection $\mathcal{L}(\mathbb{R}^d, \mathcal{L}(\mathbb{R}^d, \mathbb{R})) \hookrightarrow \mathcal{L}(\mathbb{R}^{d\times d}, \mathbb{R})$. Moreover, we make an abuse of notation: we omit the dependence of the integral on Y' which in general affects the limit but it is usually clear from the context.

The next result is a simple consequence of the Sewing Lemma in the context of geometric rough paths (see e.g [19, 13, 23, 14]).

LEMMA 2.4. *Let $\mathbf{X} = (X, \mathbb{X})$ be a random γ-geometric rough path in the sense of [19], where $X \in \mathcal{C}^\gamma$ and $\mathbb{X} \in \mathcal{C}_2^{2\gamma}$ a.s with $\frac{1}{3} < \gamma < \frac{1}{2}$. Let (Y, Y') be a controlled rough path in sense of [19], i.e., Y is an $\mathbb{R}^d$-valued process with γ-Hölder continuous paths, Y' is an $\mathcal{L}(\mathbb{R}^d, \mathcal{L}(\mathbb{R}^d, \mathbb{R}))$-valued process with γ-Hölder continuous paths so that the remainder term R^Y given implicitly by relation*

$$Y_t - Y_s = Y'_s X_{s,t} + R^Y_{s,t} \tag{24}$$

satisfies $R^Y \in \mathcal{C}_2^{2\gamma}$ a.s. Then, $(Y, Y') \in \mathcal{D}_X(\mathbb{R}^d)$ and the limit

$$\lim_{\epsilon \to 0^+} \frac{1}{\epsilon} \int_0^{\cdot} \Big(Y_s (X_{s+\epsilon} - X_s) + Y'_s \mathbb{X}_{s,s+\epsilon} \Big) ds$$

exists almost surely and uniformly on $[0, T]$. Moreover, it coincides with the rough path integral as described in [19].

For a proof of Lemma 2.4, see Proposition 6.1 in [15].

3. The structure of the Gaussian space. In this section, we describe the class of the Gaussian driving noises that we will consider in this article. In the sequel, W is a (zero mean) real-valued Gaussian continuous process such that $W_0 = 0$ a.s. Let us denote

$$R(s_1, s_2) := \mathbb{E}[W_{s_1} W_{s_2}]; (s_1, s_2) \in \mathbb{R}^2_+.$$

A priori, R is only continuous on $\mathbb{R}^2_+$ and hence $\partial^2 R := \frac{\partial^2 R}{\partial s_1 \partial s_2}$ will be interpreted in the sense of distributions. We denote

$$D := \{(s_1, s_2) \in \mathbb{R}^2_+; s_1 = s_2\}$$

and to shorten notation, sometimes the elements of $\mathbb{R}^2_+ \setminus D$ will be denoted by $\mathbf{v} = (v_1, v_2)$. A priori, $\partial_u R(u, v), \partial_v R(u, v)$ and $\partial^2 R(u, v)$ are Schwartz distributions. We explore regularity of R outside the diagonal D. Throughout the paper, the following assumptions will be in force.

Assumption A For every $s \in [0,T]$, $R(dx;s)$ is a non-negative finite measure absolutely continuous w.r.t Lebesgue.

Assumption B We suppose the product of the distribution $\partial^2 R$ with the smooth function $(s_1 - s_2)$

$$\partial^2 R(s_1,s_2)(s_1-s_2)$$

is a regular distribution on $\mathbb{R}^2_+$ which is a real Radon measure that we denote by $\bar{\mu}$.

Assumption C

(i) $\partial^2 R$ is a sigma-finite non-positive measure and absolutely continuous w.r.t Lebesgue on $\mathbb{R}^2_+ \setminus D$. With a slight abuse of notation, we denote it by $d\mu = \partial^2 R dx$ on $\mathbb{R}^2_+ \setminus D$. We assume that the Radon-Nikodym derivative satisfies

$$\big|\partial^2 R(s_1,s_2)\big| \lesssim \Big(|s_1-s_2|^\alpha + \phi(s_1,s_2)\Big), \tag{25}$$

for $(s_1,s_2)\in[0,T]^2\setminus D$, where $-\frac{3}{2}<\alpha<-1$ and there exists $L>1$ such that $\phi:[0,T]^2\setminus D\to\mathbb{R}$ is a symmetric p-integrable function over $[0,T]^2\setminus D$ for every $p\in(1,L)$. Of course, the Radon-Nikodym derivative $\partial^2 R$ has support on $[0,T]^2\setminus D$.

(ii) $\mathrm{Var}\big(W_t - W_s\big) \lesssim |t-s|^{\alpha+2}$, where $-\frac{3}{2}<\alpha<-1$ is the exponent given in Assumption C(i).

Of course, C(ii) and the Gaussian property imply that W has γ-Hölder continuous paths for any $\frac{1}{4}<\gamma<\frac{\alpha}{2}+1$. Under Assumption B, one can check the total variation measure $|\mu|$ is absolutely continuous w.r.t the total variation measure $|\bar{\mu}|$ with Radon-Nikodym derivative given by $\frac{1}{|y-x|}$.

Assumption D Let ϕ and α be respectively the symmetric function and the exponent which appear in Assumption C. There exists a non-increasing integrable function $\varphi:[0,T]\to\mathbb{R}_+$ such that

1. $\int_a^b |\phi(r_1,r_2)|dr_1 \lesssim |b-a|^{\frac{\alpha+2}{2}}\varphi(r_2)$
2. $\int_c^d \varphi(y)dy \lesssim |d-c|^{\frac{\alpha+2}{2}}$
3. $s^{\frac{\alpha+2}{2}}\varphi(s)\in L^1[0,T]$, for every a,b,c,d in $[0,T]$.
4. $$\big|R(v_1,T)-R(v_2,T)\big| \lesssim |v_1-v_2|^{\alpha+2}$$
   for every $v_1,v_2\in[0,T]^2\setminus D$.

EXAMPLE 2. Let W be a fractional Brownian motion with exponent $0<H<\frac{1}{2}$. Then,

$$R(s_1,s_2)=\frac{1}{2}\Big(\tilde{s}_1^{2H}+\tilde{s}_2^{2H}-|\tilde{s}_2-\tilde{s}_1|^{2H}\Big);(s_1,s_2)\in\mathbb{R}^2_+,$$

where $\tilde{s}_i = s_i \wedge T$ for $i = 1,2$. Assumptions A, B and C are fulfilled. Indeed, $s_1 \mapsto R(s_1;s)$ is absolutely continuous for each $s \in \overline{\mathbb{R}}_+$, where

$$\partial_{s_1} R(s_1,T) = \begin{cases} H[s_1^{2H-1} + (T-s_1)^{2H-1}] \text{ if } s_1 < T \\ 0 \text{ if } s_1 > T. \end{cases}$$

Moreover,

$$\bar{\mu}(ds_1 ds_2) = H(2H-1)|s_1 - s_2|^{2H-1}\text{sgn}(s_1 - s_2)1\!\!1_{[0,T]^2\setminus D}(s_1,s_2)ds_1 ds_2$$

and

$$\partial^2 R(s_1,s_2) = H(2H-1)|s_1 - s_2|^{2H-2}1\!\!1_{[0,T]^2\setminus D}(s_1,s_2),$$

for $(s_1,s_2) \in \mathbb{R}_+^2 \setminus D$. Assumption D is fulfilled for $\frac{1}{4} < H < \frac{1}{2}$ and $\phi = 0$.

EXAMPLE 3. Let $W = B^{H,K}$ be a bifractional Brownian motion with parameters $H \in (0,1), K \in (0,1)$. It is known (see e.g [28])

$$R(s_1,s_2) = 2^{-K}\left[(\tilde{s}_1^{2H} + \tilde{s}_2^{2H})^K - |\tilde{s}_1 - \tilde{s}_2|^{2HK}\right],$$

where $\tilde{s}_i = s_i \wedge T$. One can easily check

$$\partial_{s_1} R(s_1,s_2) = 2HK2^{-K}\Big[(s_1^{2H} + s_2^{2H})^{K-1}s_1^{2H-1} - |s_1 - s_2|^{2HK-1}\text{sign}(s_1 - s_2)\Big]$$

for $s_1, s_2 \in (0,T)$. Then,

$$\partial^2 R(s_1,s_2) = 2^{-K}\Big[(4H^2K(K-1))(s_1^{2H}+s_2^{2H})^{K-2}(s_1s_2)^{2H-1} + 2HK(2HK-1)|s_1-s_2|^{2HK-2}\Big],$$

for $(s_1,s_2) \in [0,T]^2 \setminus D$,

$$\partial_{s_1} R(s_1,\infty) = \begin{cases} 2HK2^{-K}\Big[(s_1^{2H} + T^{2H})^{K-1}s_1^{2H-1} + (T-s_1)^{2HK-1}\Big]; \text{ if } s_1 \in (0,T) \\ 0; \text{ if } s_1 > T, \end{cases}$$

and

$$\bar{\mu}(ds_1ds_2) = 1\!\!1_{[0,T]^2}(s_1,s_2)2^{-K}\Big[4H^2K(K-1)(s_1^{2H} + s_2^{2H})^{K-2}(s_1s_2)^{2H-1}(s_1 - s_2)^2$$
$$+ 2HK(2HK-1)|s_1 - s_2|^{2HK}\Big]ds_1ds_2.$$

Since $2^{K-2}(s_1s_2)^{H(K-2)} \geq (s_1^{2H} + s_2^{2H})^{K-2}$, then, we notice the existence of a positive constant $C(H,K,T)$ such that

$$\partial_{s_1} R(s_1,\infty) \leq C(H,K,T)\Big\{s_1^{2H-1} + (T-s_1)^{2HK-1}\Big\} \tag{26}$$

for every $s_1 > 0$,

$$\big|\partial^2 R(s_1,s_2)\big| \leq C(H,K,T)\Big\{(s_1s_2)^{HK-1} + |s_1 - s_2|^{2HK-2}\Big\}, \tag{27}$$

for every $(s_1, s_2) \in [0,T]^2 \setminus D$. The function, $\phi(s_1,s_2) = (s_1 s_2)^{HK-1}$ is p-integrable over $[0,T]^2 \setminus D$ for every $1 < p < \frac{1}{1-HK}$. Therefore, Assumptions A, B, C and D are fulfilled for $\frac{1}{4} < HK < \frac{1}{2}$. We observe bifractional Brownian motion does not have stationary increments for $K < 1$, it is HK-self similar with γ-Hölder continuous paths for $\gamma < HK$. See e.g [28] for details.

3.1. *Reproducing kernel Hilbert space and related properties.* In this section, we set the basic elements of the reproducing kernel Hilbert space associated with R.

Throughout this section, $X = (X^1, \ldots, X^d)$ is a d-dimensional centered process with iid components satisfying Assumptions A, B, C and D. In the sequel, let $C_0^1(\mathbb{R}_+, \mathbb{R}^d)$ be the space of $\mathbb{R}^d$-valued C^1-functions with compact support in $\mathbb{R}_+$ and $\langle \cdot, \cdot \rangle$ is the standard inner product on $\mathbb{R}^d$. We also denote $e_j; j = 1, \ldots, d$ as the canonical basis of $\mathbb{R}^d$ and $\mathbf{1} = \sum_{\ell=1}^d e_\ell$.

Let $I : C_0^1(\mathbb{R}_+, \mathbb{R}^d) \to L^2(\mathbb{P})$ be the linear mapping defined by

$$I(f) := \int_0^\infty f_s dX_s := \langle f(+\infty), X_\infty \rangle - \int_0^{+\infty} \langle X_s, df(s) \rangle.$$

Let $\tilde{L}_R(\mathbb{R}^d)$ be the linear space of all Borel functions $f : \mathbb{R}_+ \to \mathbb{R}^d$ such that

i $\int_0^\infty |f|^2(s)|R|(ds, \infty) < \infty$,
ii $\int_{\mathbb{R}_+^2 \setminus D} |f(s_1) - f(s_2)|^2 |\mu|(ds_1 ds_2) < \infty$,

where $\mu = \partial^2 R dx$ is a sigma-finite non-positive measure with support on $[0,T]^2 \setminus D$. For $f \in \tilde{L}_R(\mathbb{R}^d)$, we define

$$\|f\|^2_{L_R(\mathbb{R}^d)} := \int_0^\infty |f(s)|^2 R(ds, \infty) - \frac{1}{2} \int_{\mathbb{R}_+^2 \setminus D} |f(s_1) - f(s_2)|^2 \mu(ds_1 ds_2). \tag{28}$$

It is possible to show $\tilde{L}_R(\mathbb{R}^d)$ is a Hilbert space w.r.t the inner-product associated with (28) and

$$\mathbb{E}|I(f)|^2 = \|f\|^2_{L_R(\mathbb{R}^d)}, \tag{29}$$

for every $f \in C_0^1(\mathbb{R}_+, \mathbb{R}^d)$. Let $L_R(\mathbb{R}^d)$ be the closure of $C_0^1(\mathbb{R}_+, \mathbb{R}^d)$ w.r.t $\| \cdot \|_{L_R(\mathbb{R}^d)}$ as a subset of $\tilde{L}_R(\mathbb{R}^d)$. If $d = 1$, we will write $L_R = L_R(\mathbb{R})$. Then, $I : C_0^1(\mathbb{R}_+, \mathbb{R}^d) \to L^2(\mathbb{P})$ can be uniquely extended to a linear isometry

$$I : L_R(\mathbb{R}^d) \to L^2(\mathbb{P}). \tag{30}$$

One can check $L_R(\mathbb{R}^d)$ is a separable Hilbert space, bounded variation functions with compact support belong to $L_R(\mathbb{R}^d)$ and

$$\int_0^\infty \varphi dX = - \int_0^\infty \langle X, d\varphi \rangle, \tag{31}$$

for every bounded variation function φ with compact support. This implies that

$$R(s,t) = \langle \mathbb{1}_{[0,t]}, \mathbb{1}_{[0,s]} \rangle_{L_R}; s, t \in [0,T]. \tag{32}$$

See Propositions 6.18, 6.14, 6.22, 6.32 and 6.33 in [21] for the proof of these results. For obvious reasons, we call I as the Paley - Wiener integral associated with X. We denote

$$I(f) := \int_0^\infty f dX; f \in L_R(\mathbb{R}^d).$$

There is a connection between the Paley-Wiener integral introduced above and integrals via regularization [29].

DEFINITION 3.1. Let Y be an $\mathbb{R}^d$-valued process with locally integrable paths. Let

$$I^0(\epsilon, Y, dX)(t) := \frac{1}{2\epsilon}\int_0^t \langle Y_s, X_{s+\epsilon} - X_{s-\epsilon}\rangle ds; 0 \le t \le T.$$

We set

$$\int_0^t Y d^0 X := \lim_{\epsilon \downarrow 0} I^0(\epsilon, Y, dX)(t) \quad (\mathbb{P} - \text{probability});\ 0 \le t \le T.$$

The random variable $\int_0^t Y d^0 X$ is called the **symmetric-Stratonovich integral** of Y w.r.t X when exists.

REMARK 3.1. *We observe the symmetric-Stratonovich integral (if it exists) is the limit in probability of*

$$\int_0^t Y d^0 X = \lim_{\epsilon \downarrow 0} \frac{1}{2\epsilon}\int_0^t \langle Y_{u+\epsilon} + Y_u, X_{u+\epsilon} - X_u\rangle du;\ 0 \le t \le T. \tag{33}$$

See Remark 3.2 in [11].

LEMMA 3.2. *Let $f : \mathbb{R}_+ \to \mathbb{R}^d$ be a cadlag and bounded function and suppose the existence of $V_f : \mathbb{R}_+ \to \mathbb{R}_+$ such that $V_f(0) = 0$,*

$$|f(s_1) - f(s_2)| \le V_f(s_1 - s_2); s_1, s_2 \ge 0, \tag{34}$$

and

$$\int_{\mathbb{R}^2_+} V_f^2(s_1 - s_2)|\mu|(ds_1 ds_2) < \infty. \tag{35}$$

Then,

$$\int_0^\infty f_s d^0 X_s = \int_0^\infty f_s dX_s. \tag{36}$$

In particular, $\int_0^\infty f d^0 X$ exists.

For a proof of Lemma 3.2 when $d = 1$, see Prop. 6.34 in [21]. The same arguments apply to the multidimensional case.

EXAMPLE 4. If $f : [0, T] \to \mathbb{R}^d$ is θ-Hölder continuous where θ satisfies $2\theta + \alpha + 1 > 0$, where α is the exponent in Assumption C, then $\int_0^\cdot f d^0 X$ exists.

3.2. *Doubled Paley-Wiener integrals.* In this section, we introduce the doubled Paley-Wiener associated with X. For this purpose, we select the pair (X^1, X^2) from X and we make use the Hilbert tensor product $L_{2,R} := L_R \otimes L_R$ of L_R equipped with the norm

$$\|h\|^2_{L_{2,R}} := \int_0^\infty \|h(\cdot, r)\|^2_{L_R} R(dr, \infty) - \frac{1}{2}\int_{\mathbb{R}^2_+ \setminus D} \|h(\cdot, r_1) - h(\cdot, r_2)\|^2_{L_R} \mu(dr_1 dr_2) \tag{37}$$

for $h : \mathbb{R}^2_+ \to \mathbb{R} \in L_{2,R}$.

For an elementary tensor of the form $g = g^1 \otimes g^2$ for $g^1, g^2 \in L_R$, we define

$$I_2(g) := \int_0^\infty g^1 dX^1 \int_0^\infty g^2 dX^2.$$

A routine computation allows us to state that I_2 can be extended to a linear isometry between $L_{2,R}$ and $L^2(\mathbb{P})$ and we denote

$$I_2(g) := \int_{\mathbb{R}^2_+} g(s_1, s_2) dX^1_{s_1} dX^2_{s_2},$$

for $g \in L_{2,R}$. The operator I_2 will be called the doubled Paley-Wiener integral of $g \in L_{2,R}$ w.r.t (X^1, X^2).

LEMMA 3.3. *Under Assumptions A, B, C, we have:*

1. $X^1 \in L^2(\Omega, L_R)$
2. *If* $h(s_1, s_2) = \mathbb{1}_{[0, s_1 \wedge T]}(s_2)$, *then*

$$\int_{\mathbb{R}^2_+} h(s_1, s_2) dX^1_{s_1} dX^2_{s_2} = \int_0^\infty X^2_s dX^1_s.$$

For a proof of Lemma 3.3 when $d = 1$, see Prop. 6.48, Corollary 6.49 in [21] and notice that Assumption C implies condition (6.35) in [21], namely

$$\int_{\mathbb{R}^2_+} \mathrm{Var}(X^1_{t_1+r} - X^1_{t_2+r}) |\mu|(dt_1 dt_2) < \infty.$$

The same arguments apply to the multidimensional case.

4. Malliavin calculus tools. With the Paley-Wiener integral (30) at hand, it is natural to construct a Malliavin calculus based on $L_R(\mathbb{R}^d)$. Let $\mathcal{S}_{\mathbb{R}^d}$ be the set of cylindrical random variables of the form

$$F = f\left(\int_0^\infty \phi_1 dX, \dots, \int_0^\infty \phi_m dX\right), \tag{38}$$

where $f \in C^\infty_b(\mathbb{R}^m)$($f$ is a smooth real-valued function on $\mathbb{R}^m$ where f and all its partial derivatives are bounded), $\phi_1, \dots, \phi_m \in C^1_0(\mathbb{R}_+, \mathbb{R}^d)$ and $m \geq 1$. One can prove that $\mathcal{S}_{\mathbb{R}^d}$ is a dense subset of $L^2(\mathbb{P})$ (see the proof of Th 7.4 in [21]). For a cylinder random variable of the form (38), we then define

$$\mathbf{D}_t F = \sum_{j=1}^{m} \partial_i f\left(\int_0^\infty \phi_a dX, \ldots, \int_0^\infty \phi_n dX\right)\phi_i(t); t \geq 0.$$

One can check that $\mathbf{D} : \mathcal{S}_{\mathbb{R}^d} \to L^2(\mathbb{P})$ is a densely defined and closable operator satisfying the classical properties of the Gross-Sobolev-Malliavin derivative on the Gaussian space $\Big((\Omega, \mathcal{F}, \mathbb{P}); L_R(\mathbb{R}^d)\Big)$. For details, we refer the reader to Section 8 in [21].

The Malliavin-Watanabe spaces associated with $L_{\mathbb{R}}(\mathbb{R}^d)$ are given by $\mathbb{D}^{1,p}$ for $p > 1$ equipped with the norms

$$\|F\|_{1,p} := \Bigg[\mathbb{E}|F|^p + \mathbb{E}\|\mathbf{D}F\|^p_{L_R(\mathbb{R}^d)}\Bigg]^{\frac{1}{p}}.$$

Let $\mathcal{S}_{L_R(\mathbb{R}^d)}$ be the set of smooth d-dimensional stochastic processes of the form

$$F = \sum_{j=1}^{n} F_j v_j, v_j \in L_R(\mathbb{R}^d), F_j \in \mathcal{S}_{\mathbb{R}^d}.$$

It is a standard procedure in Malliavin calculus to consider $\mathbf{D}$ as a closable operator from $\mathcal{S}_{L_R(\mathbb{R}^d)} \subset L^p(\Omega; L_R(\mathbb{R}^d))$ into $L^p(\Omega; L_{2,R}(\mathbb{R}^{d\times d}))$, where $L_{2,R}(\mathbb{R}^{d\times d}) := L_R(\mathbb{R}^d) \otimes L_R(\mathbb{R}^d)$ is the Hilbert tensor product of $L_R(\mathbb{R}^d)$ equipped with the natural norm associated with (37). Let $\mathbb{D}^{1,p}(L_R(\mathbb{R}^d))$ be the completion of $\mathcal{S}_{L_R(\mathbb{R}^d)}$ w.r.t

$$\|F\|_{1,p,L_R(\mathbb{R}^d)} := \Bigg[\mathbb{E}\|F\|^p_{L_R(\mathbb{R}^d)} + \mathbb{E}\|\mathbf{D}F\|^p_{L_{2,R}(\mathbb{R}^{d\times d})}\Bigg]^{\frac{1}{p}},$$

for $p \geq 1$.

Next, we present some key elementary properties.

LEMMA 4.1. *Under Assumptions A, B, C and D, we have* $X \in \mathbb{D}^{1,2}(L_R(\mathbb{R}^d))$ *and* $\mathbf{D}_{t_2} X^i_{t_1} = \mathbb{1}_{[0,T\wedge t_1]}(t_2)$ *for each* $i = 1, \ldots, d$.

LEMMA 4.2. *Let* ρ *be a finite Borel measure on* $\mathbb{R}_+, a : \mathbb{R}^2_+ \to \mathbb{R}$ *be a Borel function and* Y *be a* $\mathbb{R}^d$*-valued stochastic process. We suppose the following.*

1. $a(s, \cdot) \in L_R$ *for a.a* s *w.r.t* ρ.
2. $\int_0^\infty \|a(s, \cdot)\|^2_{L_R} \rho(ds) < \infty$
3. $t \mapsto Y_t \in \mathbb{D}^{1,2}(\mathbb{R}^d)$ *is continuous and bounded on supp* ρ.

Then, the process

$$Z_t = \int_0^\infty a(s,t) Y_s \rho(ds)$$

belongs to $\mathbb{D}^{1,2}(L_R(\mathbb{R}^d))$ *and*

$$\mathbf{D}_\tau Z_t = \int_0^\infty a(t,s) \mathbf{D}_\tau Y_s \rho(ds),\ \tau \geq 0.$$

For proofs of these results when $d=1$, see Prop. 9.7 and 9.14 in [21]. The same arguments apply to the multidimensional case.

The Gross-Sobolev-Malliavin derivative operator $\big(\mathbf{D}, \mathbb{D}^{1,2}\big)$ admits an adjoint which is a densely defined closable linear operator $\big(\boldsymbol{\delta}, \text{dom } \boldsymbol{\delta}\big)$, where $\mathbb{D}^{1,2}(L_R(\mathbb{R}^d)) \subset \text{dom } \boldsymbol{\delta} \subset L^2(\Omega, L_R(\mathbb{R}^d))$.

DEFINITION 4.3. If $u\mathbb{1}_{[0,t]} \in \text{dom } \boldsymbol{\delta}$ for every $t\ge 0$, then we define

$$\int_0^t u_s \boldsymbol{\delta} X_s := \boldsymbol{\delta}(u\mathbb{1}_{[0,t]}); t\ge 0.$$

Of course, if $u\in L_R(\mathbb{R}^d)$, then $\int_0^t u_s\boldsymbol{\delta}X_s = I(u\mathbb{1}_{[0,t]})$ for every $t\ge 0$.

For a given $Y\in\mathbb{D}^{1,2}(L_R(\mathbb{R}^d))$, we denote

$$\bar{Y}^\epsilon_u := \frac{1}{2\epsilon}\int_\epsilon^{T-\epsilon} Y_s \mathbb{1}_{[u-\epsilon,u+\epsilon]}(s)ds,$$

for $0\le u\le T$ and $2\epsilon < T$. We observe if $Y\in\mathbb{D}^{1,2}(L_R(\mathbb{R}^d))$ and $t\mapsto Y_t\in\mathbb{D}^{1,2}(\mathbb{R}^d)$ is continuous, then one can check (see the proof of Proposition 4.1) that $\bar{Y}^\epsilon\in\mathbb{D}^{1,2}(L_R(\mathbb{R}^d))$. Fubini's theorem and the multiplication rule of random variables with Skorohod integrals allow us to write

$$\frac{1}{2\epsilon}\int_\epsilon^{T-\epsilon}\langle Y_s, X_{s+\epsilon}-X_{s-\epsilon}\rangle ds = \int_0^T \bar{Y}^\epsilon_s\boldsymbol{\delta}X_s + \frac{1}{2\epsilon}\int_\epsilon^{T-\epsilon}\sum_{i=1}^d\langle \mathbf{D}Y^i_s, e_i\mathbb{1}_{[s-\epsilon,s+\epsilon]}\rangle_{L_R(\mathbb{R}^d)}ds.$$

If, in addition, there exists $q>2$ such that $\sup_{0\le t\le T}\mathbb{E}|Y_t|^q<\infty$, then Assumption C(ii), Jensen and Hölder's inequality yield

$$\frac{1}{2\epsilon}\int_0^T\langle Y_s, X_{s+\epsilon}-X_{s-\epsilon}\rangle ds = \int_0^T \bar{Y}^\epsilon_u\boldsymbol{\delta}X_u + \frac{1}{2\epsilon}\int_\epsilon^{T-\epsilon}\sum_{i=1}^d\langle \mathbf{D}Y^i_s, e_i\mathbb{1}_{[s-\epsilon,s+\epsilon]}\rangle_{L_R(\mathbb{R}^d)}ds + O_{L^2(\mathbb{P})}(\epsilon^{\alpha+2}). \tag{39}$$

Next, we obtain the convergence rate of $\int_0^T(\bar{Y}^\epsilon_u - Y_u)\boldsymbol{\delta}X_u$. For this purpose, we will impose the following assumption: In the sequel, $\alpha\in(-\frac{3}{2},-1)$ is the exponent of Assumption C.

Assumption SK: There exists $\gamma\in(0,1]$ such that

$$\|Y_t-Y_s\|^2_{\mathbb{D}^{1,2}(\mathbb{R}^d)}\lesssim|t-s|^{2\gamma}, \tag{40}$$

where $2\gamma+\alpha+1>0$.

LEMMA 4.4. *Let $X = (X_1, \ldots, X_d)$ be a d-dimensional Gaussian process satisfying Assumptions A, B and C. Let $\alpha \in (-\frac{3}{2}, -1)$ be the exponent of Assumption C. Assume $Y \in \mathbb{D}^{1,2}(L_R(\mathbb{R}^d))$ satisfies Assumption SK with $2\gamma + \alpha + 1 > 0$. Then, Y satisfies*

$$\|Y_{\cdot+r} - Y\|^2_{\mathbb{D}^{1,2}(L_R(\mathbb{R}^d))} \lesssim |r|^{2\gamma+\alpha+1},$$

for every $|r| \in (0, 1)$.

PROOF. Fix $-\frac{3}{2} < \alpha < -1$. Recall that $\partial R(\cdot, T)$ is a finite non-negative measure whose support is $[0, T]$ and $|\mu|$ is a sigma-finite positive measure whose support is $[0, T]^2 \setminus D$. We observe we can write $\mathbb{D}^{1,2}(L_R(\mathbb{R}^d))$ as the tensor product $\mathbb{D}^{1,2}(L_R(\mathbb{R}^d)) = \mathbb{D}^{1,2}(\mathbb{R}^d) \otimes L_R(\mathbb{R}^d)$. Therefore, for a given $-1 < r < 1$, we may write

$$\begin{aligned}\|Y_{\cdot+r} - Y_\cdot\|^2_{\mathbb{D}^{1,2}(L_R(\mathbb{R}^d))} &= \int_0^\infty \|Y_{t+r} - Y_t\|^2_{\mathbb{D}^{1,2}(\mathbb{R}^d)} \partial R(t, T) dt \\ &+ \frac{1}{2} \int_{\mathbb{R}^2_+ \setminus D} \|(Y_{t+r} - Y_t) - (Y_{s+r} - Y_s)\|^2_{\mathbb{D}^{1,2}(\mathbb{R}^d)} |\partial^2 R(s, t)| ds dt \\ &\lesssim r^{2\gamma} + \frac{1}{2} \int_{\mathbb{R}^2_+ \setminus D} \|(Y_{t+r} - Y_t) - (Y_{s+r} - Y_s)\|^2_{\mathbb{D}^{1,2}(\mathbb{R}^d)} |\partial^2 R(s, t)| ds dt,\end{aligned}$$

where $|\mu|(dv_1 dv_2) = |\partial^2 R(v_1, v_2)| dv_1 dv_2$ and $R(dt, T) = R(dt, \infty) = \partial R(t, T) dt$. At first, one can easily check Assumption SK yields

$$\|Y_{t+r} - Y_t - Y_{s+r} + Y_s\|^2_{\mathbb{D}^{1,2}(\mathbb{R}^d)} \lesssim \min\left\{|t - s|^{2\gamma}, |r|^{2\gamma}\right\}, \tag{41}$$

for $0 \le s < t \le T$ and $|r| \in (0, 1)$. Having said that, the idea is to split the region

$$\{(s, t) \in \mathbb{R}^2_+; 0 \le s < t < \infty\} = \{(s, t); 0 \le s < t < s + |r|\} \cup \{(s, t); 0 \le s < s + |r| \le t\}.$$

By symmetry and using Assumption SK and (41), we shall write

$$\begin{aligned}&\int_{\mathbb{R}^2_+ \setminus D} \|(Y_{t+r} - Y_t) - (Y_{s+r} - Y_s)\|^2_{\mathbb{D}^{1,2}(\mathbb{R}^d)} |\partial^2 R(s, t)| ds dt \\ &\lesssim \int_{0 \le s < t < s+|r|} |t - s|^{2\gamma} |\partial^2 R(s, t)| ds dt \\ &+ |r|^{2\gamma} \int_{0 \le s < s+|r| \le t} |\partial^2 R(s, t)| ds dt.\end{aligned} \tag{42}$$

By assumption C,

$$|\partial^2 R(s, t)| \lesssim |t - s|^\alpha + \phi(s, t); \ (s, t) \in [0, T]^2 \setminus D,$$

where ϕ is integrable over $[0, T]^2 \setminus D$. Fir this reason, without any loss of generality, we may assume $\phi = 0$. A direct computation yields

$$\int_{0 \le s < t < s+|r|} |t - s|^{2\gamma} |\partial^2 R(s, t)| ds dt \lesssim |r|^{2\gamma+\alpha+1}, \tag{43}$$

for every $|r| \in (0,1)$. We also have,

$$\int_{0\le s<s+|r|\le t} |\partial^2 R(s,t)| ds dt \lesssim \int_0^{T-|r|}\int_{s+|r|}^T (t-s)^\alpha dt ds \lesssim |r|^{\alpha+1}, \tag{44}$$

for every $|r|\in(0,1)$. Summing up, (42), (43) and (44), we have

$$\|Y_{\cdot+r}-Y\|^2_{\mathbb{D}^{1,2}(L_R(\mathbb{R}^d))} \lesssim |r|^{2\gamma+\alpha+1},$$

for every $|r|\in(0,1)$ and we conclude the proof. □

PROPOSITION 4.1. *Let X be a Gaussian process satisfying Assumption A, B, C and D with $-\frac{3}{2}<\alpha<-1$. Assume $Y\in\mathbb{D}^{1,2}\big(L_R(\mathbb{R}^d)\big)$ satisfies Assumptions SK with $2\gamma+\alpha+1>0$ and $\gamma\le\frac{\alpha}{2}+1$. Then,*

$$\mathbb{E}\bigg|\int_0^T \big(\bar{Y}^\epsilon_s - Y_s\big)\boldsymbol{\delta} X_u\bigg|^2 \lesssim \epsilon^{2\gamma+\alpha+1},$$

for every $\epsilon<\frac{T}{4}\wedge 1$.

PROOF. Assumption SK implies $t\mapsto Y_t\in\mathbb{D}^{1,2}(\mathbb{R}^d)$ is continuous and hence Lemma 4.2 allows us to state $\bar{Y}^\epsilon\in\mathbb{D}^{1,2}(L_R(\mathbb{R}^d))$. We may assume $\epsilon<\frac{T}{4}\wedge 1$, where $\epsilon\downarrow 0$. Let us denote

$$A_1(\epsilon):=[2\epsilon,T-2\epsilon],\ A_2(\epsilon):=[0,2\epsilon) \text{ and } A_3(\epsilon):=(T-2\epsilon,T].$$

By the continuity of $\boldsymbol{\delta}$ w.r.t $\mathbb{D}^{1,2}(L_R(\mathbb{R}^d))$-topology, we have

$$\begin{aligned}\mathbb{E}\bigg|\int_0^T \big(\bar{Y}^\epsilon_s - Y_s\big)\boldsymbol{\delta} X_u\bigg|^2 &\lesssim \|\bar{Y}^\epsilon - Y\|^2_{\mathbb{D}^{1,2}(L_R(\mathbb{R}^d))}\\ &= \mathbb{E}\|\bar{Y}^\epsilon - Y\|^2_{L_R(\mathbb{R}^d)} + \mathbb{E}\big\|\mathbf{D}\big(\bar{Y}^\epsilon - Y\big)\big\|^2_{L_{2,R}(\mathbb{R}^{d\times d})},\end{aligned}$$

for every $\epsilon<\frac{T}{4}\wedge 1$. In order to shorten notation, let us denote

$$f^\epsilon_t = \frac{1}{2\epsilon}\int_{-\epsilon}^{\epsilon}\big[Y_{t+r}-Y_t\big]dr; 0\le t\le T.$$

We observe $f^\epsilon_t = \bar{Y}^\epsilon_t - Y_t$ for $t\in A_1(\epsilon)$ and by applying Lemma 4.2, we have $f^\epsilon\in\mathbb{D}^{1,2}(L_R(\mathbb{R}^d))$ for every $\epsilon<\frac{T}{4}\wedge 1$. We observe

$$\bar{Y}^\epsilon_t = \frac{1}{2\epsilon}\int_\epsilon^{t+\epsilon} Y_r dr;\ t\in A_2(\epsilon) \quad \text{and} \quad \bar{Y}^\epsilon_t = \frac{1}{2\epsilon}\int_{t-\epsilon}^{T-\epsilon} Y_r dr;\ t\in A_3(\epsilon). \tag{45}$$

Clearly,

$$\begin{aligned}&\int_{A_1^2(\epsilon)\setminus D}\mathbb{E}|(\bar{Y}^\epsilon_t - Y_t)-(\bar{Y}^\epsilon_s - Y_s)|^2|\partial^2R(s,t)|dsdt\\ &\le \int_{[0,T]^2\setminus D}\mathbb{E}|f^\epsilon_t - f^\epsilon_s|^2|\partial^2R(s,t)|dsdt \lesssim \mathbb{E}\|f^\epsilon\|^2_{L_R(\mathbb{R}^d)}.\end{aligned}$$

By using Jensen's inequality on the Bochner integral (see e.g [27]) and Lemma 4.4, we get

$$\mathbb{E}\|f^\epsilon\|^2_{L_R(\mathbb{R}^d)} = \mathbb{E}\left\|\frac{1}{2\epsilon}\int_{-\epsilon}^{\epsilon}[Y_{\cdot+r} - Y_\cdot]dr\right\|^2_{L_R(\mathbb{R}^d)}$$
$$\leq \frac{1}{2\epsilon}\mathbb{E}\int_{-\epsilon}^{\epsilon}\|Y_{\cdot+r} - Y_\cdot\|^2_{L_R(\mathbb{R}^d)}dr \lesssim \epsilon^{2\gamma+\alpha+1}.$$

This shows

$$\int_{A_1^2(\epsilon)\setminus D}\mathbb{E}|(\bar{Y}^\epsilon_t - Y_t) - (\bar{Y}^\epsilon_s - Y_s)|^2|\partial^2 R(s,t)|dsdt \lesssim \epsilon^{2\gamma+\alpha+1}, \tag{46}$$

for every $\epsilon < \frac{T}{4}\wedge 1$. Next, we observe

$$\sup_{\epsilon<\frac{T}{4}\wedge 1}\sup_{(s,t)\in A_1^c(\epsilon)\times A_1(\epsilon)\setminus D}\mathbb{E}|(\bar{Y}^\epsilon_t - Y_t) - (\bar{Y}^\epsilon_s - Y_s)|^2 \lesssim \sup_{0\leq r\leq T}\mathbb{E}|Y_r|^2 < \infty,$$

where

$$\int_0^{2\epsilon}\int_{2\epsilon}^{T-2\epsilon}(t-s)^\alpha dtds + \int_{T-2\epsilon}^{T}\int_{2\epsilon}^{T-2\epsilon}(s-t)^\alpha dtds \lesssim \epsilon^{\alpha+2}.$$

Therefore,

$$\int_{A_1(\epsilon)\times A_1^c(\epsilon)\setminus D}\mathbb{E}|(\bar{Y}^\epsilon_t - Y_t) - (\bar{Y}^\epsilon_s - Y_s)|^2|\partial^2 R(s,t)|dsdt \lesssim \epsilon^{\alpha+2}, \tag{47}$$

for every $\epsilon < \frac{T}{4}\wedge 1$. By applying Jensen's inequality, using (45) and Assumption SK, we get

$$\mathbb{E}\big|(\bar{Y}^\epsilon_t - Y_t) - (\bar{Y}^\epsilon_s - Y_s)\big|^2 \leq (t-s)\mathbb{E}\int_{s+\epsilon}^{t+\epsilon}\left|Y_r\frac{1}{2\epsilon} - \frac{(Y_t - Y_s)}{t-s}\right|^2 dr$$
$$\lesssim \frac{(t-s)^2}{4\epsilon^2}\sup_{0\leq r\leq T}\mathbb{E}|Y_r|^2 + (t-s)^{2\gamma}; 0\leq s<t<2\epsilon.$$

Therefore,

$$\int_{A_2^2(\epsilon)\setminus D}\mathbb{E}|(\bar{Y}^\epsilon_t - Y_t) - (\bar{Y}^\epsilon_s - Y_s)|^2|\partial^2 R(s,t)|dsdt \lesssim \frac{1}{\epsilon^2}\int_0^{2\epsilon}\int_0^t (t-s)^{\alpha+2}dsdt$$
$$+ \int_0^{2\epsilon}\int_0^t (t-s)^{2\gamma+\alpha}dsdt$$
$$\lesssim \epsilon^{\alpha+2},$$

for every $\epsilon < \frac{T}{4}\wedge 1$. Similarly, by applying Jensen's inequality, using (45) and Assumption SK, we get

$$\mathbb{E}\big|(\bar{Y}^\epsilon_t - Y_t) - (\bar{Y}^\epsilon_s - Y_s)\big|^2 \leq (t-s)\mathbb{E}\int_{s-\epsilon}^{t-\epsilon}\left|Y_r\frac{1}{2\epsilon} + \frac{(Y_t - Y_s)}{t-s}\right|^2 dr$$
$$\lesssim \frac{(t-s)^2}{4\epsilon^2}\sup_{0\leq r\leq T}\mathbb{E}|Y_r|^2 + (t-s)^{2\gamma}; T-2\epsilon<s<t\leq T.$$

Therefore,

$$\int_{A_3^2(\epsilon)\setminus D}\mathbb{E}|(\bar{Y}_t^\epsilon - Y_t)-(\bar{Y}_s^\epsilon - Y_s)|^2|\partial^2 R(s,t)|dsdt \lesssim \frac{1}{\epsilon^2}\int_{T-2\epsilon}^T\int_{T-2\epsilon}^t (t-s)^{\alpha+2}dsdt$$
$$+\int_{T-2\epsilon}^T\int_{T-2\epsilon}^t (t-s)^{2\gamma+\alpha}dsdt$$
$$\lesssim \epsilon^{\alpha+2}, \tag{48}$$

for every $\epsilon < \frac{T}{4}\wedge 1$. By using (45), one can easily check

$$\sup_{t\in A_2(\epsilon)\cup A_3(\epsilon)}\mathbb{E}|\bar{Y}_t^\epsilon|^2 \le \sup_{0\le r\le T}\mathbb{E}|Y_r|^2<\infty.$$

Therefore, by using Assumption D and Jensen's inequality on the Bochner integral, we have

$$\mathbb{E}\int_0^T|\bar{Y}_t^\epsilon - Y_t|^2\partial R(t,T)dt = \sum_{i=1}^3\mathbb{E}\int_{A_i(\epsilon)}|\bar{Y}_t^\epsilon - Y_t|^2\partial R(t,T)dt$$
$$\lesssim \mathbb{E}\|f^\epsilon\|^2_{L_R(\mathbb{R}^d)}+\epsilon^{\alpha+2}$$
$$\lesssim \frac{1}{2\epsilon}\int_{-\epsilon}^{\epsilon}\big\|Y_{\cdot+r}-Y_r\big\|^2_{\mathbb{D}^{1,2}(L_R(\mathbb{R}^d))}dr+\epsilon^{\alpha+2}$$
$$\lesssim \epsilon^{2\gamma+\alpha+1}, \tag{49}$$

for every $\epsilon < \frac{T}{4}\wedge 1$. Summing up (46), (47), (48) and (49), we get

$$\mathbb{E}\|\bar{Y}^\epsilon - Y\|^2_{L_R(\mathbb{R}^d)}\lesssim \epsilon^{2\gamma+\alpha+2},$$

for every $\epsilon < \frac{T}{4}\wedge 1$. Next, we investigate

$$\mathbb{E}\big\|\mathbf{D}\big(\bar{Y}^\epsilon - Y\big)\big\|^2_{L_{2,R}(\mathbb{R}^{d\times d})} = \mathbb{E}\int_0^T\big\|\mathbf{D}\big(\bar{Y}_t^\epsilon - Y_t\big)\big\|^2_{L_R(\mathbb{R}^d)}\partial R(t,T)dt$$
$$+\frac{1}{2}\mathbb{E}\int_{[0,T]^2\setminus D}\big\|\mathbf{D}\big(\bar{Y}_t^\epsilon - Y_t\big)-\mathbf{D}\big(\bar{Y}_s^\epsilon - Y_s\big)\big\|^2_{L_R(\mathbb{R}^d)}|\partial^2R(s,t)|dsdt.$$

The analysis is similar to the first part so we omit some details. Indeed, by using (45) jointly with Jensen's inequality on the Bochner integral and Lemma 4.4, we get

$$\mathbb{E}\int_{A_1^2(\epsilon)\setminus D}\big\|\mathbf{D}\big(\bar{Y}_t^\epsilon - Y_t\big)-\mathbf{D}\big(\bar{Y}_s^\epsilon - Y_s\big)\big\|^2_{L_R(\mathbb{R}^d)}|\partial^2R(s,t)|dsdt$$
$$\le \|f^\epsilon\|^2_{\mathbb{D}^{1,2}(L_R(\mathbb{R}^d))} = \Big\|\frac{1}{2\epsilon}\int_{-\epsilon}^{\epsilon}[Y_{\cdot+r}-Y_\cdot]dr\Big\|^2_{\mathbb{D}^{1,2}(L_R(\mathbb{R}^d))}$$
$$\le \frac{1}{2\epsilon}\int_{-\epsilon}^{\epsilon}\|Y_{\cdot+r}-Y_\cdot\|^2_{\mathbb{D}^{1,2}(L_R(\mathbb{R}^d))}dr\lesssim \epsilon^{2\gamma+\alpha+1}, \tag{50}$$

for every $\epsilon < \frac{T}{4}\wedge 1$. Moreover,

(51) $$\sup_{\epsilon<\frac{T}{4}\wedge 1}\sup_{0\le t\le T}\mathbb{E}\|\mathbf{D}\bar{Y}^\epsilon_t\|^2_{L_R(\mathbb{R}^d)}+\sup_{0\le t\le T}\mathbb{E}\|\mathbf{D}Y_t\|^2_{L_R(\mathbb{R}^d)}\lesssim T^{2\gamma}+\|Y_0\|^2_{\mathbb{D}^{1,2}(\mathbb{R}^d)},$$

(52) $$\mathbb{E}\big\|\mathbf{D}\big(\bar{Y}^\epsilon_t-Y_t\big)-\mathbf{D}\big(\bar{Y}^\epsilon_s-Y_s\big)\big\|^2_{L_R(\mathbb{R}^d)}\lesssim\frac{(t-s)^2}{\epsilon^2}+(t-s)^{2\gamma},$$

for $0\le s<t<2\epsilon$ or $T-2\epsilon<s<t\le T$. The estimates (50), (51), (52) and Assumption SK yield

$$\mathbb{E}\big\|\mathbf{D}\big(\bar{Y}^\epsilon-Y\big)\big\|^2_{L_{2,R}(\mathbb{R}^{d\times d})}\lesssim\epsilon^{2\gamma+\alpha+1},$$

for every $\epsilon<\frac{T}{4}\wedge 1$. This concludes the proof.

□

In the sequel, we define

$$\text{Tr}\,(\mathbf{D}Y):=\lim_{\epsilon\downarrow 0}\frac{1}{2\epsilon}\int_\epsilon^{T-\epsilon}\sum_{i=1}^d\langle\mathbf{D}Y^i_s,e_i\mathbb{1}_{[s-\epsilon,s+\epsilon]}\rangle_{L_R(\mathbb{R}^d)}ds$$

when exists in $L^2(\mathbb{P})$. The following relation is a simple consequence of (39) and Proposition 4.1.

LEMMA 4.5. *Let Y be an $\mathbb{R}^d$-valued process which satisfies: There exists $q>2$ such that $\sup_{0\le t\le T}\mathbb{E}|Y_t|^q<\infty$, Assumption SK and Tr $(\mathbf{D}Y)$ exists. Then, the symmetric integral $\int_0^t Y\,d^0X$ exists and it is equal to*

(53) $$\int_0^T Y_s d^0 X_s=\int_0^T Y_s\boldsymbol{\delta}X_s+(Tr\mathbf{D}Y).$$

PROPOSITION 4.2. *Assume X is a d-dimensional Gaussian process (with iid components), where Assumptions A, B, C and D are fulfilled. Then, the $\mathbb{R}^{d\times d}$-valued two-parameter process*

$$\mathbb{X}^{ij}_{s,t}=\begin{cases}\int_s^t(X^i_r-X^i_s)d^0X^j_r;\ \text{if } i\neq j\\ \frac{1}{2}(X^i_t-X^j_s)^2;\ \text{if } i=j.\end{cases}$$

is geometric, it satisfies the Chen's relation and we have the representation

(54) $$\mathbb{X}^{ij}_{s,t}=\begin{cases}\boldsymbol{\delta}\big(X^i_{s,\cdot}\mathbb{1}_{[s,t]}e_j\big);\ \text{if } i\neq j\\ \frac{1}{2}(X^i_t-X^j_s)^2;\ \text{if } i=j.\end{cases}$$

PROOF. Let us fix $s<t$ and let $X^1,\ldots,X^d$ independent copies of a Gaussian process satisfying Assumptions A, B, C and D. We may assume that $X^i_r=X^i_t$ if $r>t$ for $1\le i\le d$. In order to define $\mathbb{X}$ outside the diagonal, one has to be careful. Let us take $g_{st}(s_i,s_j)=\mathbb{1}_{[s\wedge s_i,s_i\wedge t]}(s_j)=\mathbb{1}_{[0,s_i\wedge t]}(s_j)-\mathbb{1}_{[0,s_i\wedge s)}(s_j)$. By (31), we have

$$\int_0^\infty\mathbb{1}_{[a,b]}dX^j=-\int_0^\infty X^j d\mathbb{1}_{[a,b]}=X^j_b-X^j_a,$$

for every $a < b$. By Assumption C, we may apply Lemma 3.3 to get

$$I_2^{ij}(g) = \int_0^t \Big(\int_0^t g_{st}(s_i, s_j) dX^j_{s_j} \Big) dX^i_{s_i} = \int_0^t \big(X^j_{s_i \wedge t} - X^j_{s_i \wedge s} \big) dX^i_{s_i}$$

$$= \int_s^t \big(X^j_{s_i} - X^j_s \big) dX^i_{s_i}$$

for every $i \neq j$. Since X^j is independent of X^i for $i \neq j$ and the sample paths of X satisfies (34) and (35), we can apply Lemma 3.2 to state that

$$\int_s^t \big(X^j_{s_i} - X^j_s \big) dX^i_{s_i} = \int_s^t \big(X^j_{s_i} - X^j_s \big) d^0 X^i_{s_i} \ a.s.$$

We then set

$$\mathbb{X}^{ij}_{s,t} = \int_s^t \big(X^j_r - X^j_s \big) d^0 X^i_r \ a.s$$

for $i \neq j$. The Chen's relation is obvious because Stratonovich integrals (in the sense of [29]) are constructed by regularization via limits of Riemann's integrals. A simple integration by parts arguments yields $\mathbb{X}$ is geometric. Finally, for $i \neq j$, we set

$$f^{ij,\epsilon}_{\cdot} = \frac{1}{2\epsilon} \int_{(\cdot - \epsilon)\vee 0}^{(\cdot + \epsilon)\wedge T} (X^i_r - X^i_s) \mathbb{1}_{[s,t]}(r) dr e_j.$$

By Lemmas 4.1 and 4.2, $f^{ij,\epsilon} \in \mathbb{D}^{1,2}(L_R(\mathbb{R}^d))$. Independence of (X^i, X^j) yields

$$\Big\langle \mathbf{D}_{\cdot}(X^i_r - X^i_s)\mathbb{1}_{[s,t]}(r), \mathbb{1}_{[(r-\epsilon)\vee 0, (r+\epsilon)\wedge T]}(\cdot) e_j \Big\rangle_{L_R(\mathbb{R}^d)} = 0 \ a.s,$$

for $r \in [0, T]$. Therefore, multiplication rule of Skorohod integral and Fubini's theorem yield

$$\frac{1}{2\epsilon} \int_0^{+\infty} (X^i_r - X^i_s) \mathbb{1}_{[s,t]}(r) \Big(X^j_{(r+\epsilon)\wedge T} - X^j_{(r-\epsilon)\vee 0} \Big) dr = \boldsymbol{\delta}(f^{ij,\epsilon}).$$

Assumption A, B, C, D and and Lemma 4.1 yield $Y = (X^i_{\cdot} - X^i_s)\mathbb{1}_{[s,t]}(\cdot) \in \mathbb{D}^{1,2}(L_R(\mathbb{R}^d))$ and $\|Y_{\cdot+r} - Y\|^2_{\mathbb{D}^{1,2}(L_R(\mathbb{R}^d))} \to 0$ as $r \to 0$. Therefore, by applying Th 13.5 in [21], we get (54) for $i \neq j$.

□

In order to integrate a controlled rough path in the sense of [19], one has to check $\mathbb{X} \in \mathcal{C}_2^{2\gamma}$ a.s. Next, we give two examples in this direction. In the sequel, if g is an E-valued two-parameter continuous function, $\alpha > 0$ and $p \geq 1$, we write

$$U_{\alpha,p}(g) := \left[\int_0^T \int_0^T \frac{|g_{s,t}|^p}{|t-s|^{\alpha p + 2}} ds dt \right]^{\frac{1}{p}}.$$

In order to estimate $\mathbb{E}[U^p_{2\gamma,p}(\mathbb{X})] < \infty$ for $p > 1$, one may use the fundamental inequality of the Skorohod operator (for $i \neq j$)

$$\|\mathbb{X}^{ij}_{s,t}\|_{L^p(\Omega)} = \big\|\boldsymbol{\delta}\big(X^i_{s,\cdot}\mathbb{1}_{[s,t]}e_j\big)\big\|_{L^p(\Omega)}$$
$$\lesssim \|\mathbb{E}\big(X^i_{s,\cdot}\mathbb{1}_{[s,t]}e_j\big)\|_{L_R(\mathbb{R}^d)} + \big\|\mathbf{D}\big(X^i_{s,\cdot}\mathbb{1}_{[s,t]}e_j\big)\big\|_{L^p(\Omega;L_{2,R}(\mathbb{R}^{d\times d}))}$$
$$= \big\|\mathbf{D}\big(X^i_{s,\cdot}\mathbb{1}_{[s,t]}e_j\big)\big\|_{L^p(\Omega;L_{2,R}(\mathbb{R}^{d\times d}))}. \tag{55}$$

See e.g Prop 1.5.8 in [24] and (115). Based on (55), one can easily get the following example.

EXAMPLE 5. If X is a d-dimensional fractional Brownian motion with exponent $\frac{1}{4} < H < \frac{1}{2}$, then $\mathbb{X}$ given by (54) satisfies

$$\mathbb{E}[U^p_{2\gamma,p}(\mathbb{X})] \lesssim \int_{[0,T]^2} \frac{|t-s|^{2pH}}{|t-s|^{2\gamma p+2}} dsdt < \infty,$$

whenever $0 < \gamma < H$ and $p > \frac{1}{2H-2\gamma}$. By [[19]; Th 3.1], this implies $\mathbb{X} \in \mathcal{C}^{2\gamma}_2$ a.s for every $\gamma < H$. If X is a bifractional Brownian motion with parameter $\frac{1}{4} < HK < \frac{1}{2}$, then $\mathbb{X}$ given by (54) satisfies

$$\mathbb{E}[U^p_{2\gamma,p}(\mathbb{X})] \lesssim \int_{[0,T]^2} \frac{|t-s|^{2pKH}}{|t-s|^{2\gamma p+2}} dsdt < \infty,$$

whenever $0 < \gamma < HK$ and $p > \frac{1}{2HK-2\gamma}$. This implies $\mathbb{X} \in \mathcal{C}^{2\gamma}_2$ a.s for every $\gamma < HK$.

5. Equivalence of stochastic rough path and symmetric-Stratonovich integral. We are now in position to state the following result. The proof of Theorem 5.1 is presented in Appendix A.

THEOREM 5.1. *Let X be a Gaussian process satisfying assumptions A, B, C and D with $-\frac{4}{3} < \alpha < -1$. Let $\mathbf{X} = (X, \mathbb{X})$ be the geometric process given by (54). Assume that $(Y, Y') \in \mathcal{D}_X(\mathbb{R}^d)$, where Y' satisfies the properties below.*

1. *$s \mapsto \mathbf{D}_v Y'_s$ is continuous a.s on $(0,T) \setminus \{v\}$ for Lebesgue a.a v.*
2. *There exist $p, q > 2$ such that $t \mapsto Y'_t$ is a $\mathbb{D}^{1,p}$-valued continuous function and*

$$\int_0^T \int_{v_2}^T \sup_{s \ge v_1 \text{ or } s < v_2} \|\mathbf{D}_{v_1} Y'_s - \mathbf{D}_{v_2} Y'_s\|^q_{L^q(\mathbb{P})} |\partial^2 R(v_1, v_2)|^{\frac{q}{2}} dv_1 dv_2 < \infty. \tag{56}$$

3. *There exists $p > 2$ such that*

$$\sup_{0 \le t \le T} \mathbb{E}|Y'_t|^p + \sup_{0 \le t, r \le T} \mathbb{E}|\mathbf{D}_t Y'_r|^p < \infty. \tag{57}$$

Then, $(Y, Y') \in \mathcal{D}_X(\mathbb{R}^d)$ is rough (stochastically) integrable if, and only if, Y is symmetric-Stratonovich integrable and, in this case, both integrals coincide

$$\int_0^t Y_s d\mathbf{X}_s = \int_0^t Y_s d^0 X_s; 0 \le t \le T. \tag{58}$$

REMARK 5.1. *Under assumptions (1), (2) and (3) for a pair $(Y, Y') \in \mathcal{D}_X(\mathbb{R}^d)$ in Theorem 5.1, the symmetric-Stratonovich integral behaves like a stochastic rough path integral driven by a reduced geometric process $\mathbb{X} = (X, Sym(\mathbb{X}))$. See (105) for details.*

EXAMPLE 6. If $f:\mathbb{R}^d\rightarrow\mathbb{R}^d\in C^2_b$ and X is a Gaussian process satisfying assumptions A, B, C and D with $-\frac{4}{3}<\alpha<-1$. Then, $(f(X),\nabla f(X))\in\mathcal{D}_X(\mathbb{R}^d)$ satisfies the assumptions in Theorem 5.1.

EXAMPLE 7. Assume that $V\in C^3_b(\mathbb{R}^d,\mathcal{L}(\mathbb{R}^d,\mathbb{R}^d))$, $\xi\in\mathbb{R}^d$ and let X be a Gaussian process satisfying Assumptions A, B, C and D with $-\frac{4}{3}<\alpha<-1$. In addition, we assume the second order process (54) satisfies $\mathbb{X}\in\mathcal{C}^{2\gamma}_2$ a.s ($\frac{1}{3}<\gamma<\frac{\alpha}{2}+1$) and R has finite two-dimensional ρ-variation for $1\le\rho<\frac{3}{2}$ (see e.g Def. 5.50 in [14]). Let Y be the solution of the rough differential equation

$$Y_t=\xi+\int_0^t V(Y_s)d\mathbf{X}_s; 0\le t\le T. \tag{59}$$

Then, Y satisfies the assumptions in Theorem 5.1.

PROOF. Let $V=(V^1,\ldots,V^d)$ where $V^i:\mathbb{R}^d\rightarrow\mathbb{R}^d$ are $C^3_b(\mathbb{R}^d;\mathbb{R}^d)$ vector fields. It is known that $Y'_t=V(Y_t)$ (see e.g Prop 8.3 in [13]) and hence chain rule yields $\mathbf{D}V(Y_t)=\big(\mathbf{D}V^1(Y_t),\ldots,\mathbf{D}V^d(Y_t)\big)=\big(\nabla V^1(Y_t)\circ\mathbf{D}Y_t,\ldots,\nabla V^d(Y_t)\circ\mathbf{D}Y_t\big)$. It is well-known (see e.g [3]) that $\mathbf{D}_sY_t=J_t\circ J_s^{-1}\circ V(Y_s)1\!\!1_{[0,t]}(s)$, where J_t denotes the Jacobian of the solution Y_t where $Y_0=\xi$. Here, J^{-1}_s is the inverse of the matrix-valued Jacobian J_s. We fix $\frac{1}{3}<\gamma<\frac{\alpha}{2}+1$. Then,

$$\begin{aligned}\sup_{s\ge v_1\vee v_2 \text{ or } s<v_1\wedge v_2}|\mathbf{D}_{v_1}Y^{'}_s-\mathbf{D}_{v_2}Y^{'}_s|&\le\max_{1\le i\le d}\|\nabla V^i(Y)\|_\infty\|J_\cdot\|_\infty\|J^{-1}\|_\gamma\|V(Y)\|_\infty|v_1-v_2|^\gamma\\&+\max_{1\le i\le d}\|\nabla V^i(Y)\|_\infty\|J_\cdot\|_\infty\|J_\cdot^{-1}\|_\infty\|V(Y)\|_\gamma|v_1-v_2|^\gamma\end{aligned}$$

for $\frac{1}{3}<\gamma<\frac{1}{2}$. By using [6], we know that

$$\big(\|J\|_{p-var},\|J^{-1}\|_{p-var}\big)\in\bigcap_{q\ge1}L^q(\mathbb{P})$$

for $2<p<3$. Here, $\|\cdot\|_{p-var}$ denotes the p-variation norm. If R has finite two-dimensional ρ-variation, it is actually possible to prove (see e.g Remark 7.3 in [3])

$$\Big\{\|J\|_\infty,\|J^{-1}\|_\infty,\|J\|_{\frac{1}{p}},\|J^{-1}\|_{\frac{1}{p}}\Big\}\subset\bigcap_{q\ge1}L^q(\mathbb{P}). \tag{60}$$

Under Assumption C, assumptions (56) and (57) hold true. Indeed, since $-\frac{4}{3}<\alpha<-1$ and $\frac{1}{3}<\gamma<\frac{1}{2}$, then $\frac{-1}{\gamma+\frac{\alpha}{2}}>3$ so that $p\gamma+\alpha\frac{p}{2}+1>0$ as long as $2<p<\frac{-1}{\gamma+\frac{\alpha}{2}}$.

□

Theorem 5.1 and Example 6 implies the following existence result for stochastic differential equations driven by Gaussian processes satisfying Assumptions A, B, C and D with $-\frac{4}{3}<\alpha<-1$. In short, solutions Y of rough differential equations driven by $\mathbf{X}=(X,\mathbb{X})$ with $\mathbb{X}\in\mathcal{C}^{2\gamma}_2$ a.s for $\frac{1}{3}<\gamma<\frac{\alpha}{2}+1$ are actually solutions of Stratonovich differential equations. The proof of Corollary 5.1 follows by routine arguments based on chain rule and application of Theorem 5.1 to $V(Y)$, so we omit details.

COROLLARY 5.1. *Assume the same hypotheses of Example 7. Let Y be the solution of the rough differential equation in the sense of [19]*

$$Y_t = \xi + \int_0^t V(Y_s)d\mathbf{X}_s; 0 \le t \le T.$$

Then, Y is a solution to the Stratonovich differential equation interpreted in the sense of [29]

$$Y_t = \xi + \int_0^t V(Y_s)d^0 X_s; 0 \le t \le T. \tag{61}$$

Next, it is instructive to compare our analysis with the recent work [22].

REMARK 5.2. *In [22], if one restricts the analysis to a γ-Hölder geometric rough path $\mathbf{X} = (X, \mathbb{X})$ for $\frac{1}{3} < \gamma < \frac{1}{2}$ and a first-order controlled process (Y, Y') with remainder $R^Y \in \mathcal{C}_2^{2\gamma}$ in the sense of [19], then the authors decompose*

$$\frac{1}{2}\langle Y_{t_k} + Y_{t_{k+1}}, X_{t_k,t_{k+1}}\rangle = \langle Y_{t_k}, X_{t_k,t_{k+1}}\rangle + Y'_{t_k}\mathbb{X}_{t_k,t_{k+1}} \\ - \langle Y'_{t_k}, \textit{Anti}\ (\mathbb{X}_{t_k,t_{k+1}})\rangle + \frac{1}{2}\langle R^Y_{t_k,t_{k+1}}, X_{t_k,t_{k+1}}\rangle.$$

The key point in [22] is the convergence

$$\sum_{t_k \in \Pi} \langle Y'_{t_k}, \textit{Anti}\ (\mathbb{X}_{t_k,t_{k+1}})\rangle \to 0, \tag{62}$$

in probability as the mesh of partitions $|\Pi| \to 0$. In [22], in order to show (62), the authors make a fundamental use of Lemma 3.9 in [22] which, in the context of the case $\frac{1}{3} < \gamma < \frac{1}{2}$, requires a higher-order expansion

$$Y'_{s,t} = Y''_s X_{s,t} + r_{s,t}, \quad Y_{s,t} = Y'_s X_{s,t} + R^Y_{s,t}, \tag{63}$$

where $Y, Y', Y'' \in \mathcal{C}^\gamma$ and $R^Y, r \in \mathcal{C}_2^{2\gamma}$.

EXAMPLE 8. We give an example of a pair $(Y, Y') \in \mathcal{D}_X(\mathbb{R}^d)$, where Y' is not a controlled rough path in the sense of [19]. Let X be a d-dimensional fractional Brownian motion with parameter $\frac{1}{3} < H < \frac{1}{2}$. Let $f : [0,T] \to \mathbb{R}^{d\times d}$ be an arbitrary deterministic β-Hölder continuous function such that $\beta + 2H - 1 > 0$. By Lemma 3.2, $Y_t = \int_0^t f_s d^0 X_s$ is a well-defined Gaussian process. One can check

$$R_{s,t} = Y_t - Y_s - f_s(X_t - X_s)$$

satisfies

$$\mathbb{E}\left|\frac{1}{\epsilon}\int_0^t \langle R_{s,s+\epsilon}, X_{s+\epsilon} - X_s\rangle ds\right| \lesssim \epsilon^{\beta+2H-1} \to 0,$$

as $\epsilon \to 0^+$ and $(Y, f) \in \mathcal{D}_X(\mathbb{R}^d)$ fulfills the assumptions of Theorem 5.1. In particular, if $1 - 2H < \beta \le \frac{1}{3} < \gamma < H$ and f is at best β-Hölder continuous, then it is not possible the existence of a path $g \in \mathcal{C}^\theta$ such that $R_{s,t} = O(|t-s|^{\theta+\gamma})$ and

$$f_t - f_s = g_s(X_t - X_s) + R_{s,t}; s < t \le T.$$

6. The trace component. In this section, although it is possible to present a complete analysis on $\mathrm{Tr}(\mathbf{D}Y)$ under the general assumption C, in order to keep presentation simple, we restrict the analysis to the concrete case of the fractional Brownian motion $\frac{1}{4} < H < \frac{1}{2}$.

In this section, we study

$$\mathrm{Tr}\big(\mathbf{D}Y\big)_\epsilon := \frac{1}{2\epsilon}\int_\epsilon^{T-\epsilon}\sum_{i=1}^d \langle \mathbf{D}Y^i_s, e_i \mathbb{1}_{[s-\epsilon,s+\epsilon]}\rangle_{L_R(\mathbb{R}^d)} ds. \tag{64}$$

For simplicity, we assume that Y is adapted. Therefore, we can decompose

$$\begin{aligned}\mathrm{Tr}\big(\mathbf{D}Y\big)_\epsilon &= \frac{1}{2\epsilon}\int_\epsilon^{T-\epsilon} \mathrm{tr}[\mathbf{D}_{s-}Y_s]\big\langle \mathbb{1}_{[0,s]}, \mathbb{1}_{[s-\epsilon,s+\epsilon]}\big\rangle_{L_R} ds \\ &+ \frac{1}{2\epsilon}\int_\epsilon^{T-\epsilon} \Big\langle \mathrm{tr}[(\mathbf{D}Y_s - \mathbf{D}_{s-}Y_s)]\mathbb{1}_{[0,s]}, \mathbb{1}_{[s-\epsilon,s+\epsilon]}\Big\rangle_{L_R} ds \\ &=: \mathrm{Tr}_1\big(\mathbf{D}Y\big)_\epsilon + \mathrm{Tr}_2\big(\mathbf{D}Y\big)_\epsilon. \end{aligned}\tag{65}$$

LEMMA 6.1. *Assume that X be a d-dimensional fractional Brownian motion $\frac{1}{4} < H < \frac{1}{2}$. Assume that*

$$\sup_{0\le s\le T} \mathbb{E}|tr[\mathbf{D}_{s-}Y_s]|^2 < \infty. \tag{66}$$

Then, there exists a constant C which depends on (66) and T such that

$$\mathbb{E}\Big|Tr_1(\mathbf{D}Y)_\epsilon - \int_0^T tr[\mathbf{D}_{s-}Y_s]d\boldsymbol{v}_s\Big|^2 \le C\epsilon^{\frac{6H-1}{2}} \tag{67}$$

for every $\epsilon > 0$ such that $\epsilon^{0.75} + 2\epsilon < T$, where $\boldsymbol{v}(s) = s^{2H}; s \ge 0$.

PROOF. Let us denote $\boldsymbol{v}(s) = s^{2H}; s \ge 0$. We observe it satisfies the following properties: $s \mapsto \boldsymbol{v}(s)$ is a $C^3(0,T)$ non-decreasing map and $s \mapsto |\boldsymbol{v}^{(3)}(s)|$ is non-increasing. In addition, there exists $\beta \in (0,1)$ such that $|\boldsymbol{v}^{(3)}(\epsilon^\beta)|\epsilon^2 \to 0$ and $\epsilon^{\beta(2H+1)-1} \to 0$ as $\epsilon \to 0^+$. Indeed, $\boldsymbol{v}^{(3)}(s) = c_H s^{2H-3}$ for a positive constant c_H and notice

$$\frac{1}{1+2HK} < \frac{2}{3-2HK}$$

for $H > \frac{1}{6}$. Therefore, we can take any β realizing

$$0 < \frac{1}{1+2H} < \beta < \frac{2}{3-2H} < 1, \tag{68}$$

and for any such choice, we have $\epsilon^{\beta(2H-3)+2} \to 0, \quad \epsilon^{\beta(2H+1)-1} \to 0$, as $\epsilon \to 0^+$. Having said that, we can write

$$\frac{1}{2\epsilon}\int_\epsilon^{T-\epsilon} \Big\langle \mathrm{tr}[\mathbf{D}_{s-}Y_s]\mathbb{1}_{[0,s]}, \mathbb{1}_{[s-\epsilon,s+\epsilon]}\Big\rangle_{L_R} ds = \int_\epsilon^{T-\epsilon} \mathrm{tr}\big[\mathbf{D}_{s-}Y_s\big]dF_\epsilon(s),$$

where

$$F_\epsilon(x) = \frac{1}{2\epsilon}\int_0^x \langle \mathbb{1}_{[0,r]}, \mathbb{1}_{[r-\epsilon,r+\epsilon]}\rangle_{L_R} dr$$
$$= \frac{1}{2}\int_0^x \Big[\frac{\boldsymbol{v}(r+\epsilon)-\boldsymbol{v}(r-\epsilon)}{2\epsilon}\Big]dr; x\ge 0.$$

We denote

$$V_\epsilon(s) = \frac{\boldsymbol{v}(s+\epsilon)-\boldsymbol{v}(s-\epsilon)}{2\epsilon} - \boldsymbol{v}^{(1)}(s); \epsilon < s < T-\epsilon.$$

Taylor formula and mean value theorem yield

$$V_\epsilon(s) = \frac{\epsilon^2}{6}\boldsymbol{v}^{(3)}(a(s,\epsilon)), \tag{69}$$

where $a(s,\epsilon)\in(s-\epsilon,s+\epsilon)$ and $\epsilon < s < T-\epsilon$. Fix $0<\beta<1$ according to (68). We split

$$\int_\epsilon^{T-\epsilon} \text{tr}\big[\mathbf{D}_{s-}Y_s\big]\Big[F_\epsilon^{(1)}(s) - \frac{1}{2}\boldsymbol{v}^{(1)}(s)\Big]ds = \frac{1}{2}\int_{\epsilon^\beta+\epsilon}^{T-\epsilon}\text{tr}\big[\mathbf{D}_{s-}Y_s\big]V_\epsilon(s)ds$$
$$+\frac{1}{2}\int_\epsilon^{\epsilon^\beta+\epsilon}\text{tr}\big[\mathbf{D}_{s-}Y_s\big]V_\epsilon(s)ds,$$

where we may assume $\epsilon^\beta + 2\epsilon < T$. By (69), we observe

$$\big|V_\epsilon(s)\big| = \frac{\epsilon^2}{6}|\boldsymbol{v}^{(3)}(a(s,\epsilon))| \lesssim \epsilon^2|\boldsymbol{v}^{(3)}(\epsilon^\beta)|, \tag{70}$$

for every $s\in(\epsilon^\beta+\epsilon, T-\epsilon)$. By applying Jensen's inequality, (66) and (70), there exists a constant C which depends on T such that

$$\mathbb{E}\Big|\frac{1}{2}\int_{\epsilon^\beta+\epsilon}^{T-\epsilon}\text{tr}[\mathbf{D}_{s-}Y_s]V_\epsilon(s)ds\Big|^2 \le C\big(\epsilon^2|\boldsymbol{v}^{(3)}(\epsilon^\beta)|\big)^2 \to 0, \tag{71}$$

as $\epsilon\to 0^+$. Fubini's theorem and (66) yield

$$\mathbb{E}\Big|\int_\epsilon^{\epsilon^\beta+\epsilon}\text{tr}[\mathbf{D}_{s-}Y_s]V_\epsilon(s)ds\Big|^2$$
$$= \mathbb{E}\int_\epsilon^{\epsilon^\beta+\epsilon}\int_\epsilon^{\epsilon^\beta+\epsilon}\text{tr}[\mathbf{D}_{s-}Y_s]\text{tr}[\mathbf{D}_{t-}Y_t]V_\epsilon(s)V_\epsilon(t)dsdt$$
$$\le C\int_\epsilon^{\epsilon^\beta+\epsilon}\int_\epsilon^{\epsilon^\beta+\epsilon}|V_\epsilon(s)||V_\epsilon(t)|dsdt$$
$$\le C\epsilon^{2[\beta(2H+1)-1]}, \tag{72}$$

for every $\epsilon > 0$ sufficiently small. By (66), we have

$$\mathbb{E}\Big|\int_{T-\epsilon}^{T} \text{tr}\big[\mathbf{D}_{s-}Y_s\big]d\boldsymbol{v}_s\Big|^2 + \mathbb{E}\Big|\int_0^{\epsilon} \text{tr}\big[\mathbf{D}_{s-}Y_s\big]d\boldsymbol{v}_s\Big|^2 \lesssim \epsilon^{4H}, \tag{73}$$

for every ϵ sufficiently small. Summing up the estimates (71), (72) and (73), we obtain

$$\mathbb{E}\Big|\text{Tr}_1(\mathbf{D}Y)_\epsilon - \int_0^T \text{tr}[\mathbf{D}_{s-}Y_s]d\boldsymbol{v}_s\Big|^2 \le C\Big\{(\epsilon^2\boldsymbol{v}^{(3)}(\epsilon^\beta))^2 + \epsilon^{2[\beta(2H+1)-1]}\Big\}, \tag{74}$$

for every $\epsilon > 0$ such that $\epsilon^\beta + 2\epsilon < T$. Now, we will optimize the righ-hand side of (74). Let us consider the following bound for the right-hand side of (74):

$$\epsilon^{2\big(2\beta H+2-3\beta\big)} + \epsilon^{2\big(2\beta H+\beta-1\big)} \le 2\max\Big\{\epsilon^{2\big(2\beta H+2-3\beta\big)}, \epsilon^{2\big(2\beta H+\beta-1\big)}\Big\},$$

where $\beta \in (\frac{1}{1+2H}, \frac{2}{3-2H})$. Next, we aim to compute

$$\arg\min_{\beta\in\big(\frac{1}{1+2H},\frac{2}{3-2H}\big)} \max\Big\{\epsilon^{\big(2\beta H+2-3\beta\big)}, \epsilon^{\big(2\beta H+\beta-1\big)}\Big\}. \tag{75}$$

We observe

$$\frac{1}{2} < \frac{1}{1+2H} < \frac{2}{3} < 0.80 < \frac{2}{3-2H} < 1,$$

and

$$2\beta H + 2 - 3\beta \ge 2\beta H + \beta - 1,$$

whenever $\frac{1}{1+2H} < \beta \le 0.75 < \frac{2}{3-2H}$ and

$$2\beta H + 2 - 3\beta < 2\beta H + \beta - 1,$$

whenever $0.75 < \beta < \frac{2}{3-2H}$. Moreover,

$$2\beta H - 3\beta + 2 = 2\beta H + \beta - 1 \Longleftrightarrow \beta = 0.75.$$

The fact that $\beta \mapsto 2\beta H - 3\beta + 2$ is strictly decreasing and the constant C which appears in (67) does not depend on β allow us to choose $\beta^* = 0.75$ and this is the optimal choice realizing (75). Therefore,

$$\epsilon^{2\big(2\beta H+2-3\beta\big)} + \epsilon^{2\big(2\beta H+\beta-1\big)} \le 2\epsilon^{2\big(2\times 0.75H+0.75-1\big)} = 2\epsilon^{\frac{6H-1}{2}}.$$

This concludes the proof. □

Next, we devote our attention to the component $\text{Tr}_2\big(\mathbf{D}Y\big)_\epsilon$.

PROPOSITION 6.1. *Let $\alpha = 2H - 2$ with $\frac{1}{4} < H < \frac{1}{2}$. Assume there exists $\eta > 0$ such that $\eta + \alpha + 1 > 0$ and*

$$\mathbb{E}\big|tr[\mathbf{D}_{r_1}Y_s - \mathbf{D}_{r_2}Y_s]\big|^2 \lesssim |r_2 - r_1|^{2\eta}, \tag{76}$$

for every $0 < r_1 < r_2 \leq s \leq T$. Assume that tr$[\mathbf{D}.Y_s]$ has continuous paths on $[0,s]$ for every $s \leq T$. Then,

$$\mathbb{E}\Big|Tr_2(\mathbf{D}Y)_\epsilon - \int_{0\leq r_1<r_2\leq T} tr[\mathbf{D}_{r_1}Y_{r_2} - \mathbf{D}_{r_2-}Y_{r_2}]\partial^2 R(r_1,r_2)dr_1dr_2\Big|^2 \lesssim \epsilon^{2(\eta+\alpha+1)} \to 0,$$

as $\epsilon \to 0^+$.

PROOF. Since both $\partial^2 R$ and h_ϵ are symmetric functions and $\partial^2 R \leq 0$, then we shall write

$$\mathrm{Tr}_2(\mathbf{D}Y)_\epsilon = \frac{1}{2\epsilon}\int_\epsilon^{T-\epsilon}\int_0^\infty \mathrm{tr}[\mathbf{D}_rY_s - \mathbf{D}_{s-}Y_s]\mathbb{1}_{[0,s]}(r)\mathbb{1}_{[s-\epsilon,s+\epsilon]}(r)\partial_r R(r,T)drds$$

$$-\frac{1}{2\epsilon}\int_\epsilon^{T-\epsilon}\int_{\Delta_1} h_\epsilon(r_1,r_2;s)\partial^2 R(r_1,r_2)dr_1dr_2ds =: I_{1,\epsilon} + I_{2,\epsilon}, \tag{77}$$

where $\Delta_1 = \{(a,b) \in \mathbb{R}^2_+ \setminus D; a < b\}$ and

$$h_\epsilon(r_1,r_2;s) := \Big\{\mathrm{tr}[\mathbf{D}_{r_1}Y_s - \mathbf{D}_{s-}Y_s]\mathbb{1}_{[0,s]}(r_1) - \mathrm{tr}[\mathbf{D}_{r_2}Y_s - \mathbf{D}_{s-}Y_s]\mathbb{1}_{[0,s]}(r_2)\Big\}$$

$$\times\Big\{\mathbb{1}_{[s-\epsilon,s+\epsilon]}(r_1) - \mathbb{1}_{[s-\epsilon,s+\epsilon]}(r_2)\Big\}.$$

We can write

$$I_{1,\epsilon} = \frac{1}{2\epsilon}\int_\epsilon^{T-\epsilon}\int_{s-\epsilon}^s \mathrm{tr}[\mathbf{D}_rY_s - \mathbf{D}_{s-}Y_s]\partial_r R(r,T)drds.$$

Jensen's inequality and (76) yield

$$\mathbb{E}\Big|\frac{1}{2\epsilon}\int_\epsilon^{T-\epsilon}\int_{s-\epsilon}^s \mathrm{tr}[\mathbf{D}_rY_s - \mathbf{D}_{s-}Y_s]\partial_r R(r,T)drds\Big|^2$$

$$\lesssim \int_\epsilon^{T-\epsilon}\frac{1}{\epsilon}\int_{s-\epsilon}^s (s-r)^{2\eta}|\partial R(r,T)|^2drds$$

$$\lesssim \epsilon^{2(\eta+\alpha+1)} \to 0, \tag{78}$$

as $\epsilon \to 0^+$.

Now, we deal with the second term in (77). In case $\epsilon \leq s$, we observe

$$h_\epsilon(r_1,r_2;s) = \begin{cases} -\mathrm{tr}\Big[\mathbf{D}_{r_1}Y_s - \mathbf{D}_{r_2}Y_s\Big]; \text{if } 0 < r_1 < s-\epsilon, s-\epsilon \leq r_2 \leq s \\ -\mathrm{tr}\Big[\mathbf{D}_{r_1}Y_s - \mathbf{D}_{s-}Y_s\Big]; \text{if } 0 < r_1 < s-\epsilon, s < r_2 \leq s+\epsilon \\ \mathrm{tr}\Big[\mathbf{D}_{r_1}Y_s - \mathbf{D}_{s-}Y_s\Big]; \text{if } 0 < s+\epsilon < r_2, s-\epsilon \leq r_1 \leq s. \end{cases} \tag{79}$$

As a result, we can write $I_{2,\epsilon}$ as

$$\frac{-1}{2\epsilon}\int_\epsilon^{T-\epsilon}\int_{\Delta_1} h_\epsilon(r_1,r_2;s)\partial^2 R(r_1,r_2)dr_1dr_2ds$$

$$= \frac{1}{2\epsilon}\int_{\epsilon}^{T-\epsilon}\int_{0}^{s-\epsilon}\int_{s}^{s+\epsilon} \mathrm{tr}[\mathbf{D}_{r_1}Y_s - \mathbf{D}_{s-}Y_s]\partial^2 R(r_1,r_2)dr_2dr_1ds$$

$$+\frac{1}{2\epsilon}\int_{\epsilon}^{T-\epsilon}\int_{0}^{s-\epsilon}\int_{s-\epsilon}^{s} \mathrm{tr}[\mathbf{D}_{r_1}Y_s - \mathbf{D}_{r_2}Y_s]\partial^2 R(r_1,r_2)dr_2dr_1ds$$

$$-\frac{1}{2\epsilon}\int_{\epsilon}^{T-\epsilon}\int_{s+\epsilon}^{\infty}\int_{s-\epsilon}^{s} \mathrm{tr}[\mathbf{D}_{r_1}Y_s - \mathbf{D}_{s-}Y_s]\partial^2 R(r_1,r_2)dr_1dr_2ds$$

$$=: I_{2,\epsilon,1} + I_{2,\epsilon,2} + I_{2,\epsilon,3}.$$

At first, we estimate $I_{2,\epsilon,3}$. By using (76), Fubini's theorem and Cauchy-Schwarz's inequality, we have

$$\mathbb{E}|I_{2,\epsilon,3}|^2 \lesssim \left(\frac{1}{\epsilon}\int_{\epsilon}^{T}\int_{s+\epsilon}^{T}\int_{s-\epsilon}^{s}(s-r_1)^{\eta}(r_2-r_1)^{\alpha}dr_1dr_2ds\right)^2.$$

A direct computation shows that

$$\frac{1}{\epsilon}\int_{\epsilon}^{T}\int_{s+\epsilon}^{T}\int_{s-\epsilon}^{s}(s-r_1)^{\eta}(r_2-r_1)^{\alpha}dr_1dr_2ds \lesssim \epsilon^{\eta+\alpha+1}.$$

Therefore,

$$\mathbb{E}|I_{2,\epsilon,3}|^2 \lesssim \epsilon^{2(\eta+\alpha+1)}. \tag{80}$$

We now investigate

$$I_{2,\epsilon,1} - \frac{1}{2}\int_{0}^{T}\int_{0}^{s} \mathrm{tr}[\mathbf{D}_{r_1}Y_s - \mathbf{D}_{s-}Y_s]\partial^2 R(r_1,s)dr_1ds$$

$$+I_{2,\epsilon,2} - \frac{1}{2}\int_{0}^{T}\int_{0}^{s} \mathrm{tr}[\mathbf{D}_{r_1}Y_s - \mathbf{D}_{s-}Y_s]\partial^2 R(r_1,s)dr_1ds.$$

It is convenient to split it as

$$\frac{1}{2}\int_{0}^{T}\int_{0}^{s} \mathrm{tr}[\mathbf{D}_{r_1}Y_s - \mathbf{D}_{s-}Y_s]\partial^2 R(r_1,s)dr_1ds =$$

$$= \frac{1}{2}\int_{0}^{\epsilon}\int_{0}^{s} \mathrm{tr}[\mathbf{D}_{r_1}Y_s - \mathbf{D}_{s-}Y_s]\partial^2 R(r_1,s)dr_1ds$$

$$+\frac{1}{2}\int_{\epsilon}^{T-\epsilon}\int_{0}^{s-\epsilon} \mathrm{tr}[\mathbf{D}_{r_1}Y_s - \mathbf{D}_{s-}Y_s]\partial^2 R(r_1,s)dr_1ds$$

$$+\frac{1}{2}\int_{\epsilon}^{T-\epsilon}\int_{s-\epsilon}^{s} \mathrm{tr}[\mathbf{D}_{r_1}Y_s - \mathbf{D}_{s-}Y_s]\partial^2 R(r_1,s)dr_1ds$$

$$+\frac{1}{2}\int_{T-\epsilon}^{T}\int_0^s \text{tr}[\mathbf{D}_{r_1}Y_s - \mathbf{D}_{s-}Y_s]\partial^2 R(r_1,s)dr_1 ds$$

$$=: J_{1,\epsilon} + J_{2,\epsilon} + J_{3,\epsilon} + J_{4,\epsilon}.$$

At first, we observe Fubini's theorem, assumption (76) and Cauchy-Schwartz's inequality yield

$$\mathbb{E}|J_{3,\epsilon}|^2 \lesssim \left(\int_{\epsilon}^{T-\epsilon}\int_{s-\epsilon}^{s}(s-r_1)^{\eta}|\partial^2 R(r_1,s)|dr_1 ds\right)^2$$

$$\lesssim \Big(\int_{\epsilon}^{T}\int_{s-\epsilon}^{s}(s-r_1)^{\eta+\alpha}dr_1 ds\Big)^2 \lesssim \epsilon^{2(\eta+\alpha+1)}, \tag{81}$$

as $\epsilon \to 0^+$. Similarly,

$$\mathbb{E}|J_{1,\epsilon}|^2 \le \Big(\int_0^{\epsilon}\int_0^s (s-r_1)^{\eta}|\partial^2 R(r_1,s)|dr_1 ds\Big)^2$$

$$\lesssim \epsilon^{2(\alpha+\eta+2)} \tag{82}$$

and

$$\mathbb{E}|J_{4,\epsilon}|^2 \le \Big(\int_{T-\epsilon}^{T}\int_0^s (s-r_1)^{\eta}|\partial^2 R(r_1,s)|dr_1 ds\Big)^2$$

$$\lesssim \epsilon^{2(\alpha+\eta+2)}. \tag{83}$$

Now, we observe we can write

$$I_{2,\epsilon,1} - J_{2,\epsilon} = \frac{1}{2}\int_{\epsilon}^{T-\epsilon}\int_0^{s-\epsilon}\text{tr}[\mathbf{D}_{r_1}Y_s - \mathbf{D}_{s-}Y_s]g_\epsilon(r_1,s)dr_1 ds$$

where we denote $g_\epsilon(r_1,s) := \frac{1}{\epsilon}\int_s^{s+\epsilon}\partial^2 R(r_1,r_2)dr_2 - \partial^2 R(r_1,s)$ for $0 \le r_1 < s-\epsilon$. By Fubini's theorem and using assumption (76) jointly with Cauchy-Schwartz's inequality, we have

$$\mathbb{E}|I_{2,\epsilon,1} - J_{2,\epsilon}|^2 = \mathbb{E}\int_{Q_\epsilon \times Q_\epsilon}\text{tr}[\mathbf{D}_{r_1}Y_s - \mathbf{D}_{s-}Y_s]\text{tr}[\mathbf{D}_{v_1}Y_z - \mathbf{D}_{z-}Y_z]g_\epsilon(r_1,s)$$
$$\times g_\epsilon(v_1,z)dr_1 ds dv_1 dz$$
$$\lesssim \left(\int_{Q_\epsilon}(s-r_1)^{\eta}|g_\epsilon(r_1,s)|dr_1 ds\right)^2,$$

where $Q_\epsilon = \{(x,y); 0 \le x < y-\epsilon, \epsilon < y < T-\epsilon\}$. By using the fact that $s \mapsto \frac{\partial^3 R}{\partial s^2 \partial r}(r_1,s)$ is continuous on (r_1,T), we can make use of Taylor expansion to estimate g_ϵ. We observe for each $r_1 < s < s+\epsilon$, there exists $\bar{s}_\epsilon$ with $r_1 < s < \bar{s}_\epsilon < s+\epsilon$ realizing

$$g_\epsilon(r_1,s) = \frac{1}{2}\frac{\partial^3 R}{\partial s^2 \partial r_1}(r_1,\bar{s}_\epsilon)\epsilon; \quad r_1 < s < \bar{s}_\epsilon < s+\epsilon < T.$$

The function $(\cdot - r_1)^{\alpha-1}$ is decreasing and hence

$$|g_\epsilon(r_1,s)| \le \frac{1}{2}\Big|\frac{\partial^3 R}{\partial s^2 \partial r_1}(r_1,\bar{s}_\epsilon)\Big|\epsilon \lesssim (s-r_1)^{\alpha-1}\epsilon,$$

for every $(r_1,s)\in Q_\epsilon$. Therefore,

$$\mathbb{E}|I_{2,\epsilon,1} - J_{2,\epsilon}|^2 \lesssim \left(\epsilon\int_{Q_\epsilon}(s-r_1)^{\eta+\alpha-1}dr_1 ds\right)^2 \lesssim \epsilon^{2(\eta+\alpha+1)}, \tag{84}$$

for every $\epsilon > 0$ sufficiently small. In view of (78), (80), (81), (82), (83) and (84), it remains to estimate $I_{2,\epsilon,2} - J_{2,\epsilon}$. For this purpose, we write

$$I_{2,\epsilon,2} = \frac{1}{2\epsilon}\int_\epsilon^{T-\epsilon}\int_0^{s-\epsilon}\int_{s-\epsilon}^s \mathrm{tr}[\mathbf{D}_{r_1}Y_s - \mathbf{D}_{r_2}Y_s]\big\{\partial^2 R(r_1,r_2) - \partial^2 R(r_1,s)\big\}dr_2 dr_1 ds$$

$$+\frac{1}{2\epsilon}\int_\epsilon^{T-\epsilon}\int_0^{s-\epsilon}\int_{s-\epsilon}^s \mathrm{tr}[\mathbf{D}_{r_1}Y_s - \mathbf{D}_{r_2}Y_s]\partial^2 R(r_1,s)dr_2 dr_1 ds.$$

Mean value theorem yields

$$\partial^2 R(r_1,s) - \partial^2 R(r_1,r_2) = \frac{\partial^3 R}{\partial s^2\partial r_1}(r_1,\bar{r})(s-r_2),$$

on $r_1 < s-\epsilon < r_2 < \bar{r} < s$. Therefore,

$$|\partial^2 R(r_1,s) - \partial^2 R(r_1,r_2)| \le \Big|\frac{\partial^3 R}{\partial s^2\partial r_1}(r_1,\bar{r})\Big|\epsilon \lesssim \epsilon(r_2-r_1)^{\alpha-1}, \tag{85}$$

on $r_1 < s-\epsilon < r_2 < \bar{r} < s$. Denote $\Delta(r_1,r_2,s) = \partial^2 R(r_1,s) - \partial^2 R(r_1,r_2)$. Therefore, assumption (76) and (85) allow us to apply Fubini's theorem and we get

$$\mathbb{E}\Big|\frac{1}{2\epsilon}\int_\epsilon^{T-\epsilon}\int_0^{s-\epsilon}\int_{s-\epsilon}^s \mathrm{tr}[\mathbf{D}_{r_1}Y_s - \mathbf{D}_{r_2}Y_s]\Delta(r_1,r_2,s)dr_2dr_1ds\Big|^2$$

$$\lesssim \left(\int_\epsilon^{T-\epsilon}\int_0^{s-\epsilon}\int_{s-\epsilon}^s (r_2-r_1)^{\eta+\alpha-1}dr_2dr_1ds\right)^2$$

$$\lesssim \epsilon^{2(\eta+\alpha+1)}, \tag{86}$$

as $\epsilon\to 0^+$. Next, we observe

$$\frac{1}{2\epsilon}\int_\epsilon^{T-\epsilon}\int_0^{s-\epsilon}\int_{s-\epsilon}^s \mathrm{tr}[\mathbf{D}_{r_1}Y_s - \mathbf{D}_{r_2}Y_s]\partial^2 R(r_1,s)dr_2dr_1ds - J_{2,\epsilon}$$

$$= \frac{1}{2}\int_\epsilon^{T-\epsilon}\int_0^{s-\epsilon}\Big\{\mathrm{tr}[\mathbf{D}_{s-}Y_s] - \frac{1}{\epsilon}\int_{s-\epsilon}^s \mathrm{tr}[\mathbf{D}_{r_2}Y_s]dr_2\Big\}\partial^2 R(r_1,s)dr_1ds. \tag{87}$$

By mean value theorem, we can write

$$\text{tr}[\mathbf{D}_{s-}Y_s] - \frac{1}{\epsilon}\int_{s-\epsilon}^{s} \text{tr}[\mathbf{D}_{r_2}Y_s]dr_2 = \text{tr}[\mathbf{D}_{s-}Y_s] - \text{tr}[\mathbf{D}_{s'_\epsilon}Y_s], \tag{88}$$

on $r_1 < s-\epsilon < s'_\epsilon < s$. By (87), (88) and again by using Fubini's theorem and assumption (76), we arrive at

$$\mathbb{E}\left|\frac{1}{2\epsilon}\int_{\epsilon}^{T-\epsilon}\int_0^{s-\epsilon}\int_{s-\epsilon}^{s} \text{tr}[\mathbf{D}_{r_1}Y_s - \mathbf{D}_{r_2}Y_s]\partial^2 R(r_1,s)dr_2dr_1ds - J_{2,\epsilon}\right|^2$$

$$\lesssim \left(\epsilon^\eta \int_\epsilon^{T-\epsilon}\int_0^{s-\epsilon}(s-r_1)^\alpha dr_1 ds\right)^2 \lesssim \epsilon^{2(\eta+\alpha+1)}, \tag{89}$$

as $\epsilon \to 0^+$. The estimates (86) and (89) show

$$\mathbb{E}|I_{2,\epsilon,2} - J_{2,\epsilon}|^2 \lesssim \epsilon^{2(\alpha+\eta+1)} \to 0, \tag{90}$$

as $\epsilon \to 0^+$. The estimates (78), (80), (81), (82), (83), (84), (90) allow us to conclude the proof. □

We are now able to present the main abstract result concerning the symmetric-Stratonovich integral. It is a consequence of Propositions 4.1 and 6.1, Lemma 6.1 and decomposition (39). The final connection with the rough path integral is established by Theorem 5.1.

THEOREM 6.2. *Let X be a d-dimensional fractional Brownian motion with $\frac{1}{4} < H < \frac{1}{2}$. Assume $Y \in \mathbb{D}^{1,2}(L_R(\mathbb{R}^d))$ is adapted w.r.t X and it satisfies the following regularity conditions.*

- *There exists $q > 2$ such that $\sup_{0\le t\le T}\mathbb{E}|Y_t|^q < \infty$.*
- *tr$[\mathbf{D}_\cdot Y_s]$ has continuous paths on $[0,s]$ for every $s \le T$.*
- *There exists $\eta > 0$ such that $\eta + 2H - 1 > 0$ and (76) is fulfilled.*
- *Assumption SK is fulfilled for $0 < \gamma \le H$ such that $2\gamma + 2H - 1 > 0$.*

Then, Y is symmetric-Stratonovich integrable w.r.t X and we have the representation

$$\int_0^T Y_s d^0 X_s = \int_0^T Y_s \boldsymbol{\delta} X_s + H\int_0^T tr[\mathbf{D}_{s-}Y_s]s^{2H-1}ds$$

$$+ \int_{0\le r_1<r_2\le T} tr[\mathbf{D}_{r_1}Y_{r_2} - \mathbf{D}_{r_2-}Y_{r_2}]\partial^2 R(r_1,r_2)dr_1dr_2. \tag{91}$$

In addition, there exists a constant C which depends on (40) and (76) such that

$$\mathbb{E}\left|\int_0^T Y_s d^0X_s - I^0(\epsilon, Y, dX)(T)\right|^2 \le C\{\epsilon^{2\gamma+2H-1} + \epsilon^{2(\eta+2H-1)}\}, \tag{92}$$

for every $\epsilon > 0$ sufficiently small. Moreover, in case $\frac{1}{3} < H < \frac{1}{2}$ and $(Y, Y') \in \mathcal{D}_X(\mathbb{R}^d)$ satisfies (1), (2) and (3) in Theorem 5.1, then $(Y,Y') \in \mathcal{D}_X(\mathbb{R}^d)$ it is rough stochastically integrable and the estimate (92) holds for the stochastic rough integral as well.

REMARK 6.1. *(Let γ and η be the exponents in Assumption SK and (76), respectively. As far as the exponent γ in Assumption SK is concerned, typically, we expect $\gamma \leq H$. In case $\gamma = H$ and $\eta \geq \frac{1}{2}$, the rate becomes ϵ^{4H-1}. In case $\eta \uparrow \frac{1}{2}$ and either $\gamma \uparrow H$ or $\gamma = H$, then the rate becomes $\epsilon^{(4H-1)-}$. In case $\eta \uparrow H$ and either $\gamma \uparrow H$ or $\gamma = H$, then the rate becomes $\epsilon^{(6H-2)-}$.*

REMARK 6.2. *The fact that the Gaussian process in Proposition 6.1 and Theorem 6.2 is the fractional Brownian motion with $\frac{1}{4} < H < \frac{1}{2}$ can be extended to a more general class of Gaussian process (such as bifractional Brownian motion) satisfying*

$$|\partial^2 R(s,t)| \lesssim |t-s|^{\alpha} + \phi(s,t), \tag{93}$$

where $-\frac{3}{2} < \alpha < -1$ and $s \mapsto \frac{\partial^3 R}{\partial s^2 \partial r}(r,s)$ is continuous on (r,T) such that

$$\Big|\frac{\partial^3 R}{\partial s^2 \partial r}(r,s)\Big| \lesssim |s-r|^{\alpha-1} + z(r,s),$$

for $0 < r < s \leq T$, where $\phi, z : [0,T]^2 \setminus D \to \mathbb{R}_+$ satisfy some regularity conditions compatible with α and η given in (93) and (76), respectively.

We now present two classes of significant examples which illustrate Theorem 6.2 and its relation with Theorem 5.1.

6.1. *The case $Y = f(X)$.*

LEMMA 6.3. *Fix $\frac{1}{4} < H < \frac{1}{2}$ and let $f : \mathbb{R}^d \to \mathbb{R}^d$ be a continuously differentiable function such that f and ∇f are θ-Hölder continuous functions with $\frac{1}{2H} - 1 < \theta \leq 1$. Then, $f(X) \in \mathbb{D}^{1,2}(L_R(\mathbb{R}^d))$ and Assumption SK is fulfilled as*

$$\|f(X_t) - f(X_s)\|^2_{\mathbb{D}^{1,2}(\mathbb{R}^d)} \lesssim |t-s|^{2H\theta}, \tag{94}$$

for $s,t \geq 0$. Therefore, $f(X)$ it is symmetric-Stratonovich integrable. Moreover, if $\frac{1}{3} < H < \frac{1}{2}$ and $\frac{1}{2H} - 1 < \theta \leq \frac{1}{H} - 2$, then $\nabla f(X)$ is $\theta\gamma$-Hölder continuous for every $\gamma < H$ and

$$f(X_t) - f(X_s) - \nabla f(X_s)(X_t - X_s) = O(|t-s|^{(\theta+1)\gamma}), \tag{95}$$

where $(\theta+1)\gamma + \gamma < 1$ for every $\gamma < H$. In particular, the classical Sewing lemma fails.

PROOF. The proof follows from routine arguments as summarized here. Choose an orthonormal basis $\{e_n; n \geq 1\}$ of $L_R(\mathbb{R}^d)$ of continuous functions (see Prop 6.2 in [21]). The conditions imposed on $(f, \nabla f)$ yields $f(X) \in L_R(\mathbb{R}^d)$ $a.s$ and we can define

$$F_n := \sum_{\ell=1}^{n} \langle f(X), e_\ell \rangle_{L_R(\mathbb{R}^d)} e_\ell; n \geq 1,$$

in such way that $F_n \to f(X)$ in $L^2(\Omega, L_R(\mathbb{R}^d))$ as $n \to +\infty$. By Prop 8.12-8.14 in [21], the subexponential behavior of ∇f and the assumption $f \in C^1$ imply $f(X_s) \in \mathbb{D}^{1,2}$ and $\mathbf{D} f(X_s) = \nabla f(X_s) \mathbb{1}_{[0,s]}$ for every $s \in [0,T]$. Moreover, by using Lemma 9.13 in [21], one can easily check $\langle f(X), e_n \rangle_{L_R(\mathbb{R}^d)} \in \mathbb{D}^{1,2}$ and hence $F_n \in \mathbb{D}^{1,2}\big(L_R(\mathbb{R}^d)\big)$ for every $n \geq 1$. By using the θ-Hölder regularity of ∇f, we can check

$$\sup_{n \geq 1} \mathbb{E}\|\mathbf{D} F_n\|^2_{L_{2,R}(\mathbb{R}^{d\times d})} < \infty.$$

This shows that $f(X)\in \mathbb{D}^{1,2}(L_R(\mathbb{R}^d))$. Clearly,

$$\mathbb{E}|f(X_t)-f(X_s)|^2\lesssim \|f\|^2_\theta|t-s|^{2\theta H},$$

for every $0\le s,t<\infty$, and

$$|\nabla f(X_a)|\le \|\nabla f\|_\theta |X_a|^\theta + |\nabla f(0)|;\ a\ge 0.$$

Then, there exists a constant C which depends on T and H such that

$$\sup_{s\ge 0}\mathbb{E}\big|\nabla f(X_s)\big|^2\le C\max\{\|\nabla f\|^2_\theta, |\nabla f(0)|^2\}<\infty.$$

By definition,

$$\begin{aligned}\|f(X_t)-f(X_s)\|^2_{\mathbb{D}^{1,2}(\mathbb{R}^d)} &= \mathbb{E}|f(X_t)-f(X_s)|^2\\ &\quad + \mathbb{E}\|\nabla f(X_t)\mathbb{1}_{[0,t]}-\nabla f(X_s)\mathbb{1}_{[0,s]}\|^2_{L_R(\mathbb{R}^{d\times d})},\end{aligned}$$

and triangle inequality yields

$$\begin{aligned}\mathbb{E}\|\nabla f(X_t)\mathbb{1}_{[0,t]}-\nabla f(X_s)\mathbb{1}_{[0,s]}\|^2_{L_R(\mathbb{R}^{d\times d})} &\lesssim \mathbb{E}|\nabla f(X_t)-\nabla f(X_s)|^2\|\mathbb{1}_{[0,t]}\|^2_{L_R}\\ &\quad + \mathbb{E}|\nabla f(X_s)|^2\|\mathbb{1}_{[0,t]}-\mathbb{1}_{[0,s]}\|^2_{L_R}\\ &\le T^{2H}\|\nabla f\|^2_\theta|t-s|^{2\theta H}+C|t-s|^{2H}.\end{aligned}$$

Therefore,

$$\|f(X_t)-f(X_s)\|^2_{\mathbb{D}^{1,2}(\mathbb{R}^d)}\le C\big\{|t-s|^{2\theta H}+|t-s|^{2H}\big\},$$

for every $0\le s,t<\infty$. Then, $f(X)$ satisfies the assumptions of Theorem 6.2 with $\eta=1$. Now, it is known that (see e.g Exercise 13.2 in [13])

$$f(y)=f(x)+\nabla f(x)(y-x)+O(|y-x|^{\theta+1});\ y,x\in\mathbb{R}^d. \tag{96}$$

Fix an arbitrary $\gamma<H$. Expansion (96) immediately implies that $\nabla f(X)$ is $\theta\gamma$-Hölder continuous and (95) holds. This concludes the proof.

□

We now present the proof of Corollary 1.1.

Proof of Corollary 1.1. Since

$$\mathbf{D}_{r_1}f(X_s)-\mathbf{D}_{r_2}f(X_s)=0,$$

for every $0\le r_1<r_2\le s\le T$, we can take any $\eta=1$ in (76). A direct application of Lemma 6.3, Theorems 5.1 and 6.2 and Example 6 yields

$$\mathbb{E}\bigg|\int_0^T f(X_s)d\mathbf{X}_s - I^0(2^{-n},f(X),dX)(T)\bigg|^2\lesssim \max\{\|\nabla f\|^2_\theta,\|f\|^2_\theta,|\nabla f(0)|^2\}2^{-n(4H-1)} \tag{97}$$

for every $n\ge 1$ sufficiently large. By applying a routine Borel-Cantelli argument and Chebyshev's inequality, (14) is a consequence of (97).

6.2. *The case of rough differential equations.* In this section, we investigate the following class of rough differential equations:

$$dU^{\mathbf{X}}_{t\leftarrow 0}(y_0) = dV(U^{\mathbf{X}}_{t\leftarrow 0}(y_0))d\mathbf{X}_t \tag{98}$$

where $U^{\mathbf{X}}_{0\leftarrow 0}(y_0) = y_0$, $V \in C^3_b\big(\mathbb{R}^d, \mathcal{L}(\mathbb{R}^d, \mathbb{R}^d)\big)$, $y_0 \in \mathbb{R}^d$ and $\mathbf{X}$ is a γ-Hölder geometric rough path lift for the fractional Brownian motion with parameter $\frac{1}{3} < H < \frac{1}{2}$ and $\gamma < H$. It is well-known that $y_0 \mapsto U^{\mathbf{X}}_{t\leftarrow 0}(y_0)$ is differentiable with derivative $J^{\mathbf{X}}_{t\leftarrow 0}(y_0) \in \mathbb{R}^{d\times d}$, $J^{\mathbf{X}}_{t\leftarrow 0}(y_0)$ is invertible a.s as a linear map with inverse denoted by $J^{-1,\mathbf{X}}_{0\leftarrow t}(y_0)$.

Throughout this section, we fix once and for all a deterministic initial condition $y_0 \in \mathbb{R}^d$ and a random γ-Hölder geometric rough path lift for X. We will denote

$$Y_t = U^{\mathbf{X}}_{t\leftarrow 0}(y_0),\ J_t = J^{\mathbf{X}}_{t\leftarrow 0}(y_0),\ J^{-1}_t = J^{-1,\mathbf{X}}_{t\leftarrow 0}(y_0).$$

We recall the following fundamental result due to [6] and [3]:

$$\|J\|_{\frac{1}{\gamma}-var}, \|J^{-1}\|_{\frac{1}{\gamma}-var} \in \cap_{q\ge 1} L^q(\mathbb{P}) \tag{99}$$

and

$$\|Y\|_\gamma, \|J\|_\gamma, \|J^{-1}\|_\gamma \in \cap_{q\ge 1} L^q(\mathbb{P}), \tag{100}$$

whenever $2 < \frac{1}{H} < \frac{1}{\gamma} < 4$, where $\|\cdot\|_{\frac{1}{\gamma}-var}$ denotes the $\frac{1}{\gamma}$-variation semi-norm. See also Remark 2.7 in [3]. Of course, (100) implies

$$\|Y\|_\infty, \|J\|_\infty, \|J^{-1}\|_\infty \in \cap_{q\ge 1} L^q(\mathbb{P}). \tag{101}$$

In this section, the Hölder-type estimates (100) and (101) play a key role in our analysis.

At first, it is convenient to work with the norms

$$\|f\|_{\infty,\kappa} := \|f\|_\infty + \|f\|_\kappa,$$

for a function $f : [0, T] \to E$ taking values on a finite-dimensional space E and $0 < \kappa \le 1$.

We need some technical lemmas which we describe as follows. They are straightforward consequences of the regularity of the vector field V, (100) and (101), so we omit the details.

LEMMA 6.4. *For a given $\frac{1}{3} < \gamma < H < \frac{1}{2}$, there exists a constant C which depends on T, H and γ such that*

$$\Big\|J^{-1}_\cdot \circ V(Y)1\!\!1_{[0,M]}\Big\|^2_{L_R(\mathbb{R}^d)} \le C \max\Big\{\|J^{-1}\|_{\infty,\gamma}\|\nabla V\|^2_\infty \|Y\|^2_\gamma; \|V(Y)\|_\infty \|J^{-1}\|^2_{\infty,\gamma}\Big\}\ a.s,$$

for every $M > 0$.

LEMMA 6.5. *For a given $\frac{1}{3} < \gamma < H < \frac{1}{2}$, there exists a constant C which depends on T, H and γ such that*

$$\Big\|J^{-1}_\cdot \circ V(Y)1\!\!1_{(N,M]}\Big\|^2_{L_R(\mathbb{R}^d)} \le C \max\Big\{\|J^{-1}\|^2_{\infty,\gamma}\|V(Y)\|^2_\infty; \|\nabla V\|^2_\infty\|Y\|^2_\gamma; \|V(Y)\|^2_\infty\Big\}$$
$$\times \Big\{|T\wedge M - T\wedge N|^{2H} + |T\wedge M - T\wedge N|^{2\gamma+2H}\Big\}\ a.s,$$

for every $N < M < \infty$.

Lemmas 6.4 and 6.5 yield the following result.

LEMMA 6.6. *For a given $\frac{1}{3} < \gamma < H < \frac{1}{2}$, there exists a constant C which depends on the moments of $\|J\|_{\infty,\gamma}, \|J^{-1}\|_{\infty,\gamma}, \|Y\|_{\infty,\gamma}, \|\nabla V\|_\infty, H$ and T such that*

$$\|Y_t - Y_s\|^2_{\mathbb{D}^{1,2}(\mathbb{R}^d)} \le C|t-s|^{2\gamma}, \tag{102}$$

for every $s,t \ge 0$.

We are now in position to present the proof of Corollary 1.2.

Proof of Corollary 1.2. At first, we observe the solution of the rough differential equation (98) belongs to $\mathbb{D}^{1,2}(L_R(\mathbb{R}^d))$. Indeed, the proof follows the same lines of Lemma 6.3 and the well-known facts $Y_t \in \mathbb{D}^{1,2}(\mathbb{R}^d)$ for every $t \ge 0$, $\mathbf{D}_s Y_t = J_t \circ J_s^{-1} \circ V(Y_s) \mathbb{1}_{[0,t]}(s)$, (100) and (101). Therefore, we omit the details. Moreover, (100) and (101) imply

$$\mathbb{E}|\mathbf{D}_{r_1} Y_s - \mathbf{D}_{r_2} Y_s|^2 \lesssim |r_1 - r_2|^{2\eta},$$

on $0 \le r_1 < r_2 \le s \le T$, for any η such that $\frac{1}{3} < \eta < H < \frac{1}{2}$. By applying Example 7, Lemma 6.6, Theorems 5.1 and 6.2, we get

$$\mathbb{E}\left| \int_0^T Y_s d\mathbf{X}_s - I^0(2^{-n}, Y, dX)(T) \right|^2 \lesssim 2^{-n(2(\eta+2H-1))} \tag{103}$$

for every $n \ge 1$ sufficiently large. By applying a routine Borel-Cantelli argument and Chebyshev's inequality, (15) is a consequence of (103).

The remainder of this paper is devoted to the proof of Theorem 5.1.

APPENDIX A: PROOF OF THEOREM 5.1

In this section, we present the proof of Theorem 5.1. Before we present it, it is convenient to summarize the main idea. Under the assumptions of Theorem 5.1, it is enough to prove that

$$\lim_{\epsilon \to 0^+} \frac{1}{\epsilon} \int_0^t \Big\langle Y'_s, \mathrm{Anti}(\mathbb{X}_{s,s+\epsilon}) \Big\rangle_{\mathbf{F}} ds = 0 \tag{104}$$

in probability, where $\langle \cdot, \cdot \rangle_{\mathbf{F}}$ denotes the Frobenius inner product on the space of $d \times d$-matrices. Indeed, if $(Y, Y') \in \mathcal{D}_X(\mathbb{R}^d)$, then we can take advantage of decomposition (21) and the geometric property of $\mathbb{X}$ to write

$$\begin{aligned}
\frac{1}{\epsilon}\Big\langle \frac{Y_s + Y_{s+\epsilon}}{2}, X_{s,s+\epsilon} \Big\rangle &= \frac{1}{\epsilon}\Big\langle Y_s, X_{s,s+\epsilon} \Big\rangle \\
&\quad + \frac{1}{2\epsilon}\big\langle Y'_s X_{s,s+\epsilon}, X_{s,s+\epsilon} \big\rangle + \frac{1}{2\epsilon}\big\langle R^Y_{s,s+\epsilon}, X_{s,s+\epsilon} \big\rangle \\
&= \frac{1}{\epsilon}\Big\langle Y_s, X_{s,s+\epsilon} \Big\rangle + \frac{1}{2\epsilon}\big\langle Y'_s, X_{s,s+\epsilon} \otimes X_{s,s+\epsilon} \big\rangle_{\mathbf{F}} + o_{\mathbb{P}}(1) \\
&= \frac{1}{\epsilon}\Big\langle Y_s, X_{s,s+\epsilon} \Big\rangle + \frac{1}{\epsilon}\big\langle Y'_s, \mathrm{Sym}(\mathbb{X}_{s,s+\epsilon}) \big\rangle_{\mathbf{F}} + o_{\mathbb{P}}(1) \\
&= \frac{1}{\epsilon}\Big\langle Y_s, X_{s,s+\epsilon} \Big\rangle + \frac{1}{\epsilon}\big\langle Y'_s, \mathbb{X}_{s,s+\epsilon} \big\rangle_{\mathbf{F}} - \frac{1}{\epsilon}\big\langle Y'_s, \mathrm{Anti}(\mathbb{X}_{s,s+\epsilon}) \big\rangle_{\mathbf{F}} + o_{\mathbb{P}}(1)
\end{aligned} \tag{105}$$

where $\langle \cdot, \cdot \rangle$ above denotes the standard inner product on $\mathbb{R}^d$.

A.1. Preliminary estimates. In this section, we provide some preliminary estimates towards the obtention of (104). In the sequel, we fix $(Y,Y')\in\mathcal{D}_X(\mathbb{R}^d)$ where

$$Y'_t=\big(Y_t^{',ji}\big);\ 1\le i,j\le d,$$

is a $d\times d$-matrix. For a given $\epsilon>0$, the (i,j)-th element of the $d\times d$-matrix $(Y'_s)^\top\text{Anti}(\mathbb{X}_{s,s+\epsilon})$ is given by

$$\frac{1}{2}\sum_{\ell=1}^d Y_s^{',i\ell}\bigg\{\int_s^{s+\epsilon}(X_r^j-X_s^j)d^0X_r^\ell-\int_s^{s+\epsilon}(X_r^\ell-X_s^\ell)d^0X_r^j\bigg\}.$$

Proposition 4.2 yields

$$\int_s^{s+\epsilon}(X_r^j-X_s^j)d^0X_r^\ell-\int_s^{s+\epsilon}(X_r^\ell-X_s^\ell)d^0X_r^j=$$

$$\boldsymbol{\delta}\Big((X^j-X_s^j)\mathbb{1}_{[s,s+\epsilon]}e_\ell-(X^\ell-X_s^\ell)\mathbb{1}_{[s,s+\epsilon]}e_j\Big)$$

and by Proposition 10.2 in [21], we can write

$$Y_s^{',i\ell}\boldsymbol{\delta}\Big((X^j-X_s^j)\mathbb{1}_{[s,s+\epsilon]}e_\ell-(X^\ell-X_s^\ell)\mathbb{1}_{[s,s+\epsilon]}e_j\Big)=$$

$$\boldsymbol{\delta}\Big(Y_s^{',i\ell}(X^j-X_s^j)\mathbb{1}_{[s,s+\epsilon]}e_\ell-Y_s^{',i\ell}(X^\ell-X_s^\ell)\mathbb{1}_{[s,s+\epsilon]}e_j\Big)$$

$$+\Big\langle\mathbf{D}_\cdot Y_s^{',i\ell},\big[(X_\cdot^j-X_s^j)e_\ell-(X_\cdot^\ell-X_s^\ell)e_j\big]\mathbb{1}_{[s,s+\epsilon]}(\cdot)\Big\rangle_{L_R(\mathbb{R}^d)}\ a.s.$$

We will analyze

$$\epsilon^{-1}\int_0^t\langle Y'_s,\text{Anti}(\mathbb{X}_{s,s+\epsilon})\rangle_{\mathbf{F}}ds=\epsilon^{-1}\sum_{i=1}^d\int_0^t\Big((Y'_s)^\top\text{Anti}(\mathbb{X}_{s,s+\epsilon})\Big)_{ii}ds.$$

By applying the Fubini's Theorem 10.3 in [21], we arrive at the following representation for the (i,j)-th element of the matrix $\epsilon^{-1}\int_0^t(Y'_s)^\top\text{Anti}(\mathbb{X}_{s,s+\epsilon})ds$:

$$\sum_{\ell=1}^d\frac{1}{\epsilon}\int_0^t Y_s^{',i\ell}\big(\text{Anti}(\mathbb{X}_{s,s+\epsilon})\big)_{\ell j}ds=\frac{1}{2\epsilon}\sum_{\ell=1}^d\int_0^t\bigg(\int_{r-\epsilon}^r Y_s^{',i\ell}\big\{X_{s,r}^je_\ell-X_{s,r}^\ell e_j\big\}ds\bigg)\boldsymbol{\delta}X_r$$

(106)

$$+\frac{1}{2\epsilon}\sum_{\ell=1}^d\int_0^t\Big\langle\mathbf{D}_\cdot Y_s^{',i\ell},[X_{s,\cdot}^je_\ell-X_{s,\cdot}^\ell e_j]\mathbb{1}_{[s,s+\epsilon]}(\cdot)\Big\rangle_{L_R(\mathbb{R}^d)}ds,$$

for every $t\in[0,T]$, $\epsilon>0$ and $i,j\in\{1,\ldots,d\}$.

In the sequel, we are going to fix $i,\ell,j\in\{1,\ldots,d\}$ and $t\in[0,T]$ and prove that the second part in the right-hand side of $\epsilon^{-1}\int_0^t Y_s^{',i\ell}(\text{Anti}(\mathbb{X}_{s,s+\epsilon}))^{\ell j}ds$ in (106) vanishes in $L^1(\mathbb{P})$ as $\epsilon\downarrow0$. In the sequel, we recall

$$|\mu|(dv_1dv_2)=\big|\partial^2R(v_1,v_2)\big|dv_1dv_2.$$

To keep notation, variables in $\mathbb{R}^2_+ \setminus D$ are denoted by $\mathbf{r} = (r_1, r_2)$.

Let us write $\mathbf{D}_r Y_s^{',i\ell} = (\mathbf{D}^1_r Y_s^{',i\ell}, \ldots, \mathbf{D}^d_r Y_s^{',i\ell})$ in $L_R(\mathbb{R}^d)$. Then, we have

$$\Big\langle \mathbf{D}_\cdot Y_s^{',i\ell}, \big[(X^j_\cdot - X^j_s)e_\ell - (X^\ell_\cdot - X^\ell_s)e_j\big]\mathbb{1}_{[s,s+\epsilon]}(\cdot)\Big\rangle_{L_R(\mathbb{R}^d)}$$

$$= \int_s^{s+\epsilon} \mathbf{D}^\ell_r Y_s^{',i\ell} X^j_{s,r}\partial_r R(r,T)dr$$

$$- \int_s^{s+\epsilon} \mathbf{D}^j_r Y_s^{',i\ell} X^\ell_{s,r}\partial_r R(r,T)dr$$

$$+\frac{1}{2}\int_{[0,T]^2\setminus D}\Big(\mathbf{D}^\ell_{r_1} Y_s^{',i\ell} - \mathbf{D}^\ell_{r_2} Y_s^{',i\ell}\Big)\Big(X^j_{s,r_1}\mathbb{1}_{[s,s+\epsilon]}(r_1) - X^j_{s,r_2}\mathbb{1}_{[s,s+\epsilon]}(r_2)\Big)|\mu|(dr_1 dr_2)$$

$$-\frac{1}{2}\int_{[0,T]^2\setminus D}\Big(\mathbf{D}^j_{r_1} Y_s^{',i\ell} - \mathbf{D}^j_{r_2} Y_s^{',i\ell}\Big)\Big(X^\ell_{s,r_1}\mathbb{1}_{[s,s+\epsilon]}(r_1) - X^\ell_{s,r_2}\mathbb{1}_{[s,s+\epsilon]}(r_2)\Big)|\mu|(dr_1 dr_2)$$

$$=: I^1_s + I^2_s + I^3_s + I^4_s \quad a.s.$$

The components I^1 and I^2 can be estimated as follows. In order to keep notation simple, we set $\beta = \frac{\alpha}{2} + 1 \in \big(\frac{1}{3}, \frac{1}{2}\big)$. By using (57), Assumption D(4) and Hölder's inequality, we get

$$\mathbb{E}\Big|\frac{1}{\epsilon}\int_0^t I^1_s ds\Big| \lesssim \epsilon^\beta \int_0^t \left(\frac{1}{\epsilon}\int_s^{s+\epsilon}\partial R(r,T)dr\right)ds \to 0$$

as $\epsilon \to 0^+$. The term I^2 is similar. By symmetry, the analysis of the term I^3 is similar to I^4. Again, by using (57), Assumptions D, C and Hölder's inequality, we get

$$\mathbb{E}\Big|\frac{1}{\epsilon}\int_0^t I^3_s ds\Big| \lesssim \frac{1}{\epsilon}\int_0^t\int_{\{s<r_2<r_1<s+\epsilon\}}\Big\{(r_1-r_2)^{\beta+\alpha} + (r_1-r_2)^\beta\phi(r_1,r_2)\Big\}dr_1dr_2ds$$

$$+\frac{1}{\epsilon}\int_0^t\int_{\{r_2\le s<r_1<s+\epsilon\}}\Big\{(r_1-s)^\beta(r_1-r_2)^\alpha + (r_1-s)^\beta\phi(r_1,r_2)\Big\}dr_1dr_2ds$$

$$+\frac{1}{\epsilon}\int_0^t\int_{\{s<r_2<s+\epsilon\le r_1\}}\Big\{(r_2-s)^\beta(r_1-r_2)^\alpha + (r_2-s)^\beta\phi(r_1,r_2)\Big\}dr_1dr_2ds,$$

for $\epsilon > 0$. By invoking Assumption D, we observe

$$\epsilon^{-1}\int_{\{s<r_2<r_1<s+\epsilon\}}(r_1-r_2)^\beta\phi(r_1,r_2)dr_1dr_2 \lesssim \epsilon^{\alpha+1+\beta}, \tag{107}$$

for every $s\in[0,t]$. Moreover,

$$\begin{aligned}\epsilon^{-1}\int_{\{s<r_2<s+\epsilon\le r_1\}}(r_2-s)^\beta\phi(r_1,r_2)dr_1dr_2 &= \epsilon^{-1}\int_s^{s+\epsilon}\int_{s+\epsilon}^T(r_2-s)^\beta\phi(r_1,r_2)dr_1dr_2\\ &\lesssim \epsilon^{-1}[T^{\frac{\alpha+2}{2}} - (s+\epsilon)^{\frac{\alpha+2}{2}}]\varphi(s)\int_s^{s+\epsilon}(r_2-s)^\beta dr_2\\ &= \epsilon^\beta[T^{\frac{\alpha+2}{2}} - (s+\epsilon)^{\frac{\alpha+2}{2}}]\varphi(s)\\ &\lesssim \epsilon^\beta T^{\frac{\alpha+2}{2}}\varphi(s), \end{aligned}\tag{108}$$

and

$$\begin{aligned}\epsilon^{-1}\int_{\{r_2\le s<r_1<s+\epsilon\}}(r_1-s)^\beta\phi(r_1,r_2)dr_1dr_2 &= \epsilon^{-1}\int_s^{s+\epsilon}\int_0^s(r_1-s)^\beta\phi(r_1,r_2)dr_2dr_1\\ &\lesssim \epsilon^{-1}\int_s^{s+\epsilon}(r_1-s)^\beta s^{\frac{\alpha+2}{2}}\varphi(r_1)dr_1\\ &\lesssim s^{\frac{\alpha+2}{2}}\varphi(s)\epsilon^{-1}\int_s^{s+\epsilon}(r_1-s)^\beta dr_1\\ &= s^{\frac{\alpha+2}{2}}\varphi(s)\epsilon^\beta, \qquad (109)\end{aligned}$$

for each $s\in[0,t]$. Moreover,

$$\int_{\{s<r_2<r_1<s+\epsilon\}}(r_1-r_2)^{\alpha+\beta}dr_1dr_2\lesssim\epsilon^{\alpha+2+\beta},$$

$$\int_{\{s<r_2<s+\epsilon\le r_1\}}(r_1-r_2)^{\alpha}dr_1dr_2\lesssim\epsilon^{\alpha+2}, \qquad (110)$$

and

$$\begin{aligned}\int_{\{r_2\le s<r_1<s+\epsilon\}}(r_1-r_2)^\alpha d\mathbf{r} &= \frac{1}{(1-(\alpha+2))(\alpha+2)}\left\{s^{\alpha+2}+\epsilon^{\alpha+2}-(s+\epsilon)^{\alpha+2}\right\}\\ &\lesssim \epsilon^{\alpha+2}, \qquad (111)\end{aligned}$$

for every $s\in[0,t]$. Then, (110), (111), (107), (108) and (109) allow us to conclude there exists a constant C such that

$$\mathbb{E}\frac{1}{\epsilon}\left|\int_0^t I_s^3ds\right|\le C\epsilon^{\beta+\alpha+1}\to0$$

as $\epsilon\downarrow0$, because $\alpha+1+\beta>0$. This shows that the second part of (106) vanishes.

A.2. Estimating the Skorohod integral in (106). Let us now devote our attention to the first component in the right-hand side of (106), namely the Skorohod integral. In the sequel, we are going to fix $i,\ell,j\in\{1,\dots,d\}$ and $t\in[0,T]$ and prove that the first part in the right-hand side of (106) vanishes in $L^1(\mathbb{P})$ as $\epsilon\downarrow0$.

In the sequel, to keep notation simple, we set

$$u^{i\ell,j}_{r-\epsilon,r}:=\int_{r-\epsilon}^r Y_s^{',i\ell}(X_r^j-X_s^j)ds=\int_0^\infty(X_r^j-X_s^j)\mathbb{1}_{(r-\epsilon,r)}(s)Y_s^{',i\ell}ds.$$

The following technical result is an almost immediate consequence of Assumptions (1) and (2) in Theorem 5.1. Indeed, it is an application of Lemma 4.2. We left the details of the proof to the reader.

LEMMA A.1. *Assume that assumptions (2) and (3) in Theorem 5.1 hold. Then, for every $i,\ell,j\in\{1,\dots,d\}$ and $\epsilon>0$,*

$$\big(u^{i\ell,j}_{\cdot-\epsilon,\cdot}e_\ell - u^{i\ell,\ell}_{\cdot-\epsilon,\cdot}e_j\big) \in \mathbb{D}^{1,2}(L_R(\mathbb{R}^d)). \tag{112}$$

In particular, the (only) non-null ℓ-th column of $\mathbf{D}_v u^{i\ell,j}_{r-\epsilon,r}e_\ell$ equals to

$$\int_{r-\epsilon}^{r}\Big\{(X^j_r - X^j_s)\mathbf{D}_v Y^{',i\ell}_s + Y^{',i\ell}_s \mathbb{1}_{[s,r]}(v)e_j\Big\}ds \tag{113}$$

and the (only) non-null j-th column of $\mathbf{D}_v u^{i\ell,\ell}_{r-\epsilon,r}e_j$ equals to

$$\int_{r-\epsilon}^{r}\Big\{(X^\ell_r - X^\ell_s)\mathbf{D}_v Y^{',i\ell}_s + Y^{',i\ell}_s \mathbb{1}_{[s,r]}(v)e_\ell\Big\}ds \tag{114}$$

a.s for every $v, r \in [0,T]$ and $\epsilon > 0$.

The Skorohod operator $(\text{dom}\,\boldsymbol{\delta}, \boldsymbol{\delta})$ is defined on $\text{dom}\,\boldsymbol{\delta} \subset L^2(\Omega; L_R(\mathbb{R}^d))$, where

$$\frac{1}{\epsilon}\int_0^t \left(\int_{r-\epsilon}^r Y^{',i\ell}_s\{X^j_{s,r}e_\ell - X^\ell_{s,r}e_j\}ds\right)\boldsymbol{\delta} X_r = \boldsymbol{\delta}\Big(\frac{1}{\epsilon}\big(u^{i\ell,j}_{\cdot-\epsilon,\cdot}e_\ell - u^{i\ell,\ell}_{\cdot-\epsilon,\cdot}e_j\big)\mathbb{1}_{[0,t]}\Big).$$

By recalling that

$$\|F - \mathbb{E}[F]\|_{L^2(\Omega, L_R(\mathbb{R}^d))} \lesssim \|\mathbf{D}F\|_{L^2(\Omega; L_{2,R}(\mathbb{R}^{d\times d}))} \tag{115}$$

for every $F \in \mathbb{D}^{1,2}\big(L_R(\mathbb{R}^d)\big)$, we may apply Proposition 1.5.8 in [24] (see also Prop 12.5 in [21]) and Lemma A.1 to infer the existence of an universal constant c_2 such that

$$\Big\|\boldsymbol{\delta}\Big(\frac{1}{\epsilon}\big(u^{i\ell,j}_{\cdot-\epsilon,\cdot}e_\ell - u^{i\ell,\ell}_{\cdot-\epsilon,\cdot}e_j\big)\mathbb{1}_{[0,t]}\Big)\Big\|_{L^2(\mathbb{P})} \le c_2\Bigg(\Big\|\mathbb{E}\Big[\frac{1}{\epsilon}\big(u^{i\ell,j}_{\cdot-\epsilon,\cdot}e_\ell - u^{i\ell,\ell}_{\cdot-\epsilon,\cdot}e_j\big)\mathbb{1}_{[0,t]}\Big]\Big\|_{L_R(\mathbb{R}^d)}$$

$$+\Big\|\mathbf{D}_\cdot\Big[\frac{1}{\epsilon}\big(u^{i\ell,j}_{\cdot-\epsilon,\cdot}\mathbb{1}_{[0,t]}e_\ell - u^{i\ell,\ell}_{\cdot-\epsilon,\cdot}\mathbb{1}_{[0,t]}e_j\big)\Big]\Big\|_{L^2(\Omega;L_{2,R}(\mathbb{R}^{d\times d}))}\Bigg) =: J_1(\epsilon,t) + J_2(\epsilon,t), \tag{116}$$

for every $t \in [0,T]$ and $\epsilon > 0$.

A.3. Analysis of $J_1(\epsilon, t)$. In the sequel, we set $\beta = \frac{\alpha}{2} + 1$, where $-\frac{4}{3} < \alpha < -1$. To shorten notation, we set

$$U^{i\ell,j,\epsilon}_{s_1} := \mathbb{E}[u^{i\ell,j}_{s_1-\epsilon,s_1}] = \int_{s_1-\epsilon}^{s_1}\mathbb{E}[Y^{',i\ell}_s(X^j_{s_1} - X^j_s)]ds$$

and

$$\Delta_{(\mathbf{s};t)}U^{i\ell,j,\epsilon} := U^{i\ell,j,\epsilon}_{s_1}\mathbb{1}_{[0,t]}(s_1) - U^{i\ell,j,\epsilon}_{s_2}\mathbb{1}_{[0,t]}(s_2)$$

for $\mathbf{s} = (s_1, s_2) \in [0,T]^2 \setminus D$. Then, for $\ell \neq j$, we have

$$\mathbb{E}\Big[\frac{1}{\epsilon}\big(u^{i\ell,j}_{s_1-\epsilon,s_1}e_\ell - u^{i\ell,\ell}_{s_1-\epsilon,s_1}e_j\big)\Big]\mathbb{1}_{[0,t]}(s_1) = \frac{1}{\epsilon}\Big(U^{i\ell,j,\epsilon}_{s_1}e_\ell - U^{i\ell,\ell,\epsilon}_{s_1}e_j\Big)\mathbb{1}_{[0,t]}(s_1)$$

and

$$\Big\|\mathbb{E}\Big[\frac{1}{\epsilon}\big(u^{i\ell,j}e_\ell-u^{i\ell,\ell}e_j\big)\mathbb{1}_{[0,t]}\Big]\Big\|^2_{L_R(\mathbb{R}^d)}\lesssim\int_0^t\Big|\frac{1}{\epsilon}\int_{r-\epsilon}^r\mathbb{E}[Y_s^{',i\ell}(X_r^j-X_s^j)]ds\Big|^2\big|\partial_rR(r,T)\big|dr$$

(117)

$$+\int_0^t\Big|\frac{1}{\epsilon}\int_{r-\epsilon}^r\mathbb{E}[Y_s^{',i\ell}(X_r^\ell-X_s^\ell)]ds\Big|^2\big|\partial_rR(r,T)\big|dr$$

$$+\int_{[0,T]^2\setminus D}\Big|\frac{1}{\epsilon}\Delta_{(\mathbf{s};t)}U^{i\ell,j,\epsilon}\Big|^2|\mu|(ds_1ds_2)$$

$$+\int_{[0,T]^2\setminus D}\Big|\frac{1}{\epsilon}\Delta_{(\mathbf{s};t)}U^{i\ell,\ell,\epsilon}\Big|^2|\mu|(ds_1ds_2).$$

By Hölder's inequality, assumption (57) and Assumption C (ii), we have

$$\int_0^t\Big|\frac{1}{\epsilon}\int_{r-\epsilon}^r\mathbb{E}[Y_s^{',i\ell}(X_r^j-X_s^j)]ds\Big|^2\big|\partial R_r(r,T)\big|dr\le\epsilon^{-2}\int_0^t\Big(\int_{r-\epsilon}^r(r-s)^\beta ds\Big)^2\big|\partial R_r(r,T)\big|dr$$

(118)

$$\lesssim\epsilon^{-2}\epsilon^{2(\beta+1)}\int_0^T\big|\partial R_r(r,T)\big|dr.$$

By symmetry, the estimate (118) also holds for the second term in the right-hand side of (117). Now, we split

$$\int_{[0,T]^2\setminus D}\Big|\frac{1}{\epsilon}\Delta_{(\mathbf{s};t)}U^{i\ell,j,\epsilon}\Big|^2|\mu|(ds_1ds_2)=2\int_{0<s_1<t<s_2\le T}\Big|\frac{1}{\epsilon}\Delta_{(\mathbf{s};t)}U^{i\ell,j,\epsilon}\Big|^2|\mu|(ds_1ds_2)$$

$$+\int_{[0,t]^2\setminus D}\Big|\frac{1}{\epsilon}\Delta_{(\mathbf{s};t)}U^{i\ell,j,\epsilon}\Big|^2|\mu|(ds_1ds_2). \tag{119}$$

In the sequel, we will take advantage of assumption (25). In case, $s_1<t<s_2$, mean value theorem, assumption (57), Hölder's inequality and Assumption C (ii) yield

$$\Big|\frac{1}{\epsilon}\Delta_{(\mathbf{s};t)}U^{i\ell,j,\epsilon}\Big|^2=\Big|\frac{1}{\epsilon}\int_{s_1-\epsilon}^{s_1}\mathbb{E}[Y_s^{',i\ell}(X_{s_1}^j-X_s^j)]ds\Big|^2=\Big|\mathbb{E}[Y_{r_1}^{',i\ell}(X_{s_1}^j-X_{r_1}^j)]\Big|^2$$

$$\lesssim(s_1-r_1)^{2\beta}\le\epsilon^{2\beta},$$

for some r_1 satisfying $s_1-\epsilon<r_1<s_1<t<s_2$. Then,

$$2\int_{0<s_1<t<s_2\le T}\Big|\frac{1}{\epsilon}\Delta_{(\mathbf{s};t)}U^{i\ell,j,\epsilon}\Big|^2|s_1-s_2|^\alpha ds_1ds_2$$

$$\lesssim\epsilon^{2\beta}\int_t^T\int_0^t(s_2-s_1)^\alpha ds_1ds_2$$

$$\lesssim \epsilon^{2\beta}\int_t^T \{(s_2-t)^{\alpha+1} - s_2^{\alpha+1}\}ds_2 \to 0, \tag{120}$$

as $\epsilon \downarrow 0$. In addition,

$$\int_{0<s_1<t<s_2\le T}\Big|\frac{1}{\epsilon}\Delta_{(\mathbf{s};t)}U^{i\ell,j,\epsilon}\Big|^2\phi(s_1,s_2)ds_1ds_2 \lesssim \epsilon^{2\beta}\int_{0<s_1<t<s_2\le T}\phi(s_1,s_2)ds_1ds_2 \to 0,$$

as $\epsilon \downarrow 0$.

The case $s_1 < t$ and $s_2 < t$ is trickier. At first, we observe $a \mapsto \mathbb{E}[Y_a^{',i\ell}(X_b^j - X_a^j)]$ is continuous for every b. Hence,

$$\lim_{\epsilon\downarrow 0}\Big|\frac{1}{\epsilon}\Delta_{(\mathbf{s};t)}U^{i\ell,j,\epsilon}\Big|^2 = 0, \tag{121}$$

for each $\mathbf{s} = (s_1,s_2) \in [0,t]^2 \setminus D$. If $s_2 < s_1 < t$, then we shall write

$$\frac{1}{\epsilon}\int_{s_1-\epsilon}^{s_1}\mathbb{E}[Y_s^{',i\ell}(X_{s_1}^j - X_s^j)]ds = \frac{1}{\epsilon}\int_{s_2-\epsilon}^{s_2}\mathbb{E}[Y_s^{',i\ell}(X_{s_1}^j - X_s^j)]ds$$

$$+\frac{1}{\epsilon}\int_{s_2}^{s_1}\mathbb{E}[Y_s^{',i\ell}(X_{s_1}^j - X_s^j)]ds - \frac{1}{\epsilon}\int_{s_2-\epsilon}^{s_1-\epsilon}\mathbb{E}[Y_s^{',i\ell}(X_{s_1}^j - X_s^j)]ds,$$

and we arrive at

$$\Big|\frac{1}{\epsilon}\Delta_{(\mathbf{s};t)}U^{i\ell,j,\epsilon}\Big|^2 = \Big|\frac{1}{\epsilon}\int_{s_1-\epsilon}^{s_1}\mathbb{E}[Y_s^{',i\ell}(X_{s_1}^j - X_s^j)]ds - \frac{1}{\epsilon}\int_{s_2-\epsilon}^{s_2}\mathbb{E}[Y_s^{',i\ell}(X_{s_2}^j - X_s^j)]ds\Big|^2$$

$$= \Big|\frac{1}{\epsilon}\int_{s_2-\epsilon}^{s_2}\mathbb{E}[Y_s^{',i\ell}(X_{s_1}^j - X_{s_2}^j)]ds + \frac{1}{\epsilon}\int_{s_2}^{s_1}\mathbb{E}[Y_s^{',i\ell}(X_{s_1}^j - X_s^j)]ds \tag{122}$$

$$-\frac{1}{\epsilon}\int_{s_2-\epsilon}^{s_1-\epsilon}\mathbb{E}[Y_s^{',i\ell}(X_{s_1}^j - X_s^j)]ds\Big|^2.$$

Mean value theorem, assumption (57), Hölder's inequality and Assumption C (ii) yield

$$\Big|\frac{1}{\epsilon}\int_{s_2-\epsilon}^{s_2}\mathbb{E}[Y_s^{',i\ell}(X_{s_1}^j - X_{s_2}^j)]ds\Big|^2 \lesssim (s_1-s_2)^{2\beta}, \tag{123}$$

for every $s_2 < s_1 < t$. In addition, the same argument yields

$$\Big|\frac{1}{\epsilon}\int_{s_2}^{s_1}\mathbb{E}[Y_s^{',i\ell}(X_{s_1}^j - X_s^j)]ds\Big|^2 \lesssim \left(\int_{s_2}^{s_1}(s_1-s)^{\beta}ds\epsilon^{-1}\right)^2$$

$$\lesssim (s_1-s_2)^{2\beta}, \tag{124}$$

whenever $(s_1-s_2) < \epsilon$ and $s_2 < s_1 < t$. Similarly,

$$\Big|\frac{1}{\epsilon}\int_{s_2-\epsilon}^{s_1-\epsilon}\mathbb{E}[Y^{',i\ell}_s(X^j_{s_1}-X^j_s)]ds\Big|^2 \lesssim \epsilon^{-2}\Bigg(\int_{s_2-\epsilon}^{s_1-\epsilon}(s_1-s)^\beta ds\Bigg)^2$$

$$\lesssim \epsilon^{-2}(s_1-s_2+\epsilon)^{2(\beta+1)}$$

$$\lesssim (s_1-s_2+\epsilon)^{2\beta}, \tag{125}$$

whenever $(s_1-s_2)<\epsilon$ and $s_2<s_1<t$.

We observe $|s_1-s_2|^{2\beta}$ is integrable w.r.t the positive measures $|s_1-s_2|^\alpha ds_1ds_2 + \phi(s_1,s_2)ds_1ds_2$ (recall $2\beta+\alpha+1>0$). Then, (121), the estimates (122), (123), (124), (125) and assumption (25) allow us to apply bounded convergence theorem to get

$$\int_{\{\mathbf{s};s_2<s_1<t,(s_1-s_2)<\epsilon\}}\Big|\frac{1}{\epsilon}\Delta_{(\mathbf{s};t)}U^{i\ell,j,\epsilon}\Big|^2|\mu|(ds_1ds_2)\rightarrow 0, \tag{126}$$

as $\epsilon\downarrow 0$. Now, Mean Value theorem yields

$$\Big|\frac{1}{\epsilon}\Delta_{(\mathbf{s};t)}U^{i\ell,j,\epsilon}\Big|^2\mathbb{1}_{\{\mathbf{s};s_2<s_1<t,(s_1-s_2)\geq\epsilon\}}$$

$$=\Big|\mathbb{E}[Y^{',i\ell}_{\bar{s}_1(\epsilon)}(X^j_{s_1}-X^j_{\bar{s}_1(\epsilon)})]-\mathbb{E}[Y^{',i\ell}_{\bar{s}_2(\epsilon)}(X^j_{s_2}-X^j_{\bar{s}_2(\epsilon)})]\Big|^2\mathbb{1}_{\{\mathbf{s};s_2<s_1<t,(s_1-s_2)\geq\epsilon\}} \tag{127}$$

for some $(\bar{s}_1(\epsilon),\bar{s}_2(\epsilon))$ satisfying $s_1-\epsilon<\bar{s}_1(\epsilon)<s_1$ and $s_2-\epsilon<\bar{s}_2(\epsilon)<s_2$. Jensen's inequality, (57) and Assumption C (ii) yield

$$|\mathbb{E}[Y^{',i\ell}_{\bar{s}_1(\epsilon)}(X^j_{s_1}-X^j_{\bar{s}_1(\epsilon)})]|^2\mathbb{1}_{\{\mathbf{s};s_2<s_1<t,(s_1-s_2)\geq\epsilon\}}$$

$$\lesssim (s_1-\bar{s}_1(\epsilon))^{2\beta}\mathbb{1}_{\{\mathbf{s};s_2<s_1<t,(s_1-s_2)\geq\epsilon\}}$$

$$\lesssim (s_1-s_2)^{2\beta} \tag{128}$$

and

$$|\mathbb{E}[Y^{',i\ell}_{\bar{s}_2(\epsilon)}(X^j_{s_2}-X^j_{\bar{s}_2(\epsilon)})]|^2\mathbb{1}_{\{\mathbf{s};s_2<s_1<t,(s_1-s_2)\geq\epsilon\}}$$

$$\lesssim (s_1-\bar{s}_2(\epsilon))^{2\beta}\mathbb{1}_{\{\mathbf{s};s_2<s_1<t,(s_1-s_2)\geq\epsilon\}}$$

$$\lesssim (s_1-s_2+\epsilon)^{2\beta}\mathbb{1}_{\{\mathbf{s};(s_1-s_2)\geq\epsilon\}} \tag{129}$$

$$\lesssim (s_1-s_2)^{2\beta}.$$

Summing up (121), (127), (128), (129) and invoking bounded convergence theorem and (25), we conclude

$$\int_{\{\mathbf{s};s_2<s_1<t,(s_1-s_2)\geq\epsilon\}}\Big|\frac{1}{\epsilon}\Delta_{(\mathbf{s};t)}U^{i\ell,j,\epsilon}\Big|^2|\mu|(ds_1ds_2)\rightarrow 0, \tag{130}$$

as $\epsilon\downarrow 0$.

Summing up (117), (118), (119), (120), (126 and (130) and using symmetry of the terms in (117), we conclude $\lim_{\epsilon\downarrow 0}J_1(\epsilon,t)=0$ in (116) for each $t\in[0,T]$.

A.4. Analysis of $J_2(\epsilon,t)$. In order to finish the proof of the theorem, we now need to estimate $J_2(\epsilon,t)$. With a slight abuse of notation, when no confusion is possible, we write $|\cdot| = \|\cdot\|_{\mathbb{R}^{d\times d}}$. Let us fix $r\neq v$, $i,\ell,j\in\{1,\dots,d\}$ with $\ell\neq j$.

LEMMA A.2. *If $Y^{'}$ satisfies the assumptions of Theorem 5.1, then*

$$\lim_{\epsilon\to 0^+}\frac{1}{\epsilon}\int_{r-\epsilon}^{r}\Big\{X^j_{s,r}\mathbf{D}_vY^{',i\ell}_s + Y^{',i\ell}_s\mathbb{1}_{[s,r]}(v)e_j\Big\}ds = 0 \tag{131}$$

almost surely, for Lebesgue almost all $(r,v)\in[0,T]^2\setminus D$.

PROOF. If $r<v$, then $\mathbb{1}_{[s,r]}(v)=0$ whenever $r-\epsilon<s<r$. Then, for Lebesgue almost all (r,v) with $r<v$, we have

$$\Big|\frac{1}{\epsilon}\int_{r-\epsilon}^{r}\Big\{X^j_{s,r}\mathbf{D}_vY^{',i\ell}_s + Y^{',i\ell}_s\mathbb{1}_{[s,r]}(v)e_j\Big\}ds\Big| = \Big|\frac{1}{\epsilon}\int_{r-\epsilon}^{r}X^j_{s,r}\mathbf{D}_vY^{',i\ell}_s ds\Big|\to 0 \tag{132}$$

almost surely as $\epsilon\downarrow 0$. In case $v<r$, we observe $v<r-\epsilon<r$ for every ϵ sufficiently small and $\mathbb{1}_{[s,r]}(v)=0$ whenever $v<r-\epsilon<s<r$. Then, for each (r,v) with $v<r$, one can take $\epsilon=\epsilon(r,v)$ sufficiently small such that the estimate (132) holds true as well. Then, we do have the almost sure convergence (131) pointwise in $[0,T]^2\setminus D$. □

Next, we provide the analysis of $J_2(\epsilon,t)$. In this section, we will analyze

$$J_2(\epsilon,t) = \mathbb{E}\|h_\epsilon\|^2_{L_{2,R}(\mathbb{R}^{d\times d}))},$$

where h_ϵ is given by

$$h_\epsilon(v,r) = \mathbf{D}_v\Big[\frac{1}{\epsilon}\big(u^{i\ell,j}_{r-\epsilon,r}\mathbb{1}_{[0,t]}(r)e_\ell - u^{i\ell,\ell}_{r-\epsilon,r}\mathbb{1}_{[0,t]}(r)e_j\big)\Big],$$

where

$$\begin{aligned}
\|h_\epsilon\|^2_{L_{2,R}(\mathbb{R}^{d\times d})} &= \int_0^\infty\int_0^\infty |h_\epsilon(v,r)|^2|\partial_vR(v,T)|ds|\partial_rR(r,T)|dr\\
&+\frac{1}{2}\int_0^\infty\int_{\mathbb{R}^2_+\setminus D}|h_\epsilon(v_1,r)-h_\epsilon(v_2,r)|^2|\mu|(dv_1dv_2)|\partial_rR(r,T)|dr\\
&+\frac{1}{2}\int_{\mathbb{R}^2_+\setminus D}\int_0^\infty|h_\epsilon(v,r_1)-h_\epsilon(v,r_2)|^2|\partial_vR(v,T)|dv|\mu|(dr_1dr_2)\\
&+\frac{1}{4}\int_{\mathbb{R}^2_+\setminus D}\int_{\mathbb{R}^2_+\setminus D}|\Delta\Delta h_\epsilon(\mathbf{v},\mathbf{r})|^2|\mu|(dv_1dv_2)|\mu|(dr_1dr_2)\\
&=: L_1(\epsilon)+L_2(\epsilon)+L_3(\epsilon)+L_4(\epsilon),
\end{aligned}$$

and

$$\Delta\Delta h_\epsilon(\mathbf{v},\mathbf{r}) = h_\epsilon(v_1,r_1)-h_\epsilon(v_1,r_2)-h_\epsilon(v_2,r_1)+h_\epsilon(v_2,r_2)$$

for $\mathbf{v}=(v_1,v_2),\mathbf{r}=(r_1,r_2)\in\mathbb{R}^2_+\setminus D$.

Analysis of $L_1(\epsilon)$. By using Jensen's inequality, Lemma A.1, Gaussian moments and Assumptions A and (57), one can easily check there exists $p>1$ such that

$$\sup_{0<\epsilon<1}\mathbb{E}\int_0^T\int_0^T|h_\epsilon(r,v)|^{2p}|\partial R(r,T)\partial R(v,T)|dv<\infty.$$

Lemma A.2 and Vitali's theorem allow us to conclude $\mathbb{E}[L_1(\epsilon)]\to 0$ as $\epsilon\downarrow 0$.

Analysis of $L_2(\epsilon)$. Next, we analyze

(133)
$$\mathbb{E}\int_0^t\int_{[0,T]^2\setminus D}\Big|\mathbf{D}_{v_1}\big[\frac{1}{\epsilon}\big(u^{i\ell,j}_{r-\epsilon,r}e_\ell-u^{i\ell,\ell}_{r-\epsilon,r}e_j\big)\big]-\mathbf{D}_{v_2}\big[\frac{1}{\epsilon}\big(u^{i\ell,j}_{r-\epsilon,r}e_\ell-u^{i\ell,\ell}_{r-\epsilon,r}e_j\big)\big]\Big|^2|\mu|(dv_1dv_2)|R(dr,T)|.$$

For this purpose, by symmetry and Lemma A.1, it is sufficient to bound

$$\Big|\frac{1}{\epsilon}\int_{r-\epsilon}^r X^j_{s,r}\Big(\mathbf{D}^m_{v_1}Y^{',i\ell}_s-\mathbf{D}^m_{v_2}Y^{',i\ell}_s\Big)ds\Big|^2 \tag{134}$$

for $m\neq j$ and

$$\Big|\frac{1}{\epsilon}\int_{r-\epsilon}^r\Big\{X^j_{s,r}\Big(\mathbf{D}^j_{v_1}Y^{',i\ell}_s-\mathbf{D}^j_{v_2}Y^{',i\ell}_s\Big)+Y^{',i\ell}_s\big[\mathbb{1}_{[s,r]}(v_1)-\mathbb{1}_{[s,r]}(v_2)\big]\Big\}ds\Big|^2. \tag{135}$$

Clearly, we only need to check (135) because the term (134) is totally analogous. In the sequel, to shorten notation, we denote $A_\epsilon(r,v_1,v_2)$ as the square root of (135). By using the same argument given in the proof of Lemma A.2, we can safely state that

$$\lim_{\epsilon\to 0^+}A_\epsilon(r,v_1,v_2)=0\ a.s, \tag{136}$$

for each $v_1\neq v_2$ and $r\in[0,T]$. In the sequel, let us write

$$A_\epsilon(r,v_1,v_2)=\sum_{i=1}^6 A_\epsilon(r,v_1,v_2)\mathbb{1}_{E_i(\epsilon)}$$

for $v_1<v_2$ (without any loss of generality), where

- $E_1(\epsilon)=\{(r,v_1,v_2);v_1<v_2<r-\epsilon\}$
- $E_2(\epsilon)=\{(r,v_1,v_2);r<v_1<v_2\}$
- $E_3(\epsilon)=\{(r,v_1,v_2);v_1<r-\epsilon<v_2<r\}$
- $E_4(\epsilon)=\{(r,v_1,v_2);r-\epsilon<v_1<v_2<r\}$
- $E_5(\epsilon)=\{(r,v_1,v_2);r-\epsilon<v_1<r<v_2\}$
- $E_6(\epsilon)=\{(r,v_1,v_2);v_1<r-\epsilon<r<v_2\}$.

Here, for each positive small ϵ, $\{E_i(\epsilon);1\le i\le 6\}$ constitutes a partition of $[0,T]\times\{(v_1,v_2)\in[0,T]^2\setminus D;v_1<v_2\}$. By using Jensen, Hölder's inequalities and Assumption A, C(ii) and (56), there exists $q>1$ such that

$$\mathbb{E}\int_{E_1(\epsilon)}|A_\epsilon(r,v_1,v_2)|^2|\mu|(dv_1dv_2)|R(dr,T)|$$

$$\lesssim \epsilon^{\alpha+2}\int_0^T\int_{v_1}\sup_{s\geq v_2}\|\mathbf{D}_{v_1}Y_s^{'}-\mathbf{D}_{v_2}Y_s^{'}\|^2_{L^{2q}(\mathbb{P})}|\mu|(dv_1dv_2)\rightarrow 0,$$

as $\epsilon\downarrow 0$. Similarly, there exists $q>1$ such that

$$\mathbb{E}\int_{E_2(\epsilon)}|A_\epsilon(r,v_1,v_2)|^2|\mu|(dv_1dv_2)|R(dr,T)|$$

$$\lesssim \epsilon^{\alpha+2}\int_0^T\int_{v_1}\sup_{s< v_1}\|\mathbf{D}_{v_1}Y_s^{'}-\mathbf{D}_{v_2}Y_s^{'}\|^2_{L^{2q}(\mathbb{P})}|\mu|(dv_1dv_2)\rightarrow 0,$$

as $\epsilon\downarrow 0$. Similar analysis can be made for $E_i(\epsilon)$ for $3\le i\le 6$. Indeed, one can show that for each $i=3,4,5,6$,

$$\{|A_\epsilon|^2\mathbb{1}_{E_i(\epsilon)}|\partial^2 R|;0<\epsilon<1\}$$

is uniformly integrable w.r.t $\mathbb{P}\times|R(\cdot,T)|\times Leb$. Vitali's theorem combined with (136) yield $\mathbb{E}[L_2(\epsilon)]\rightarrow 0$ as $\epsilon\downarrow 0$.

Analysis of $L_3(\epsilon)$ and $L_4(\epsilon)$. In order to shorten notation, we now set

$$\Xi^{i\ell,j,\epsilon}_{r,v,t}:=\mathbf{D}_v\Big[\frac{1}{\epsilon}\big(u^{i\ell,j}_{r-\epsilon,r}\mathbb{1}_{[0,t]}(r)e_\ell-u^{i\ell,\ell}_{r-\epsilon,r}\mathbb{1}_{[0,t]}(r)e_j\big)\Big],$$

$$\Delta_{\mathbf{r}}\Xi^{i\ell,j,\epsilon}(\mathbf{r},v,t):=\Xi^{i\ell,j,\epsilon}_{r_1,v,t}-\Xi^{i\ell,j,\epsilon}_{r_2,v,t}=\frac{1}{\epsilon}\mathbf{D}_v\Big[\big(u^{i\ell,j}_{r_1-\epsilon,r_1}\mathbb{1}_{[0,t]}(r_1)-u^{i\ell,j}_{r_2-\epsilon,r_2}\mathbb{1}_{[0,t]}(r_2)\big)e_\ell$$

(137)

$$-\ \big(u^{i\ell,\ell}_{r_1-\epsilon,r_1}\mathbb{1}_{[0,t]}(r_1)-u^{i\ell,\ell}_{r_2-\epsilon,r_2}\mathbb{1}_{[0,t]}(r_2)\big)e_j\big)\Big],$$

(138) $$\Delta_{\mathbf{v}}\Delta_{\mathbf{r}}\Xi^{i\ell,j,\epsilon}(\mathbf{r},\mathbf{v},t):=\Delta_{\mathbf{r}}\Xi^{i\ell,j,\epsilon}(\mathbf{r},v_1,t)-\Delta_{\mathbf{r}}\Xi^{i\ell,j,\epsilon}(\mathbf{r},v_2,t).$$

Of course, we recall that the above multi-parameter processes take values on the space of $d\times d$-matrices. It remains to estimate

$$\mathbb{E}\int_{[0,T]^2\setminus D}\Big\|\mathbf{D}_\cdot\Big[\frac{1}{\epsilon}\big(u^{i\ell,j}_{r_1-\epsilon,r_1}\mathbb{1}_{[0,t]}(r_1)e_\ell-u^{i\ell,\ell}_{r_1-\epsilon,r_1}\mathbb{1}_{[0,t]}(r_1)e_j\big)\Big]$$

$$-\mathbf{D}_\cdot\Big[\frac{1}{\epsilon}\big(u^{i\ell,j}_{r_2-\epsilon,r_2}\mathbb{1}_{[0,t]}(r_2)e_\ell-u^{i\ell,\ell}_{r_2-\epsilon,r_2}\mathbb{1}_{[0,t]}(r_2)e_j\big)\Big]\Big\|^2_{L_R(\mathbb{R}^{d\times d})}|\mu|(dr_1dr_2)$$

$$=\mathbb{E}\int_{[0,T]^2\setminus D}\int_0^T|\Delta_{\mathbf{r}}\Xi^{i\ell,j,\epsilon}(\mathbf{r},v,t)|^2|R(dv,T)||\mu|(dr_1dr_2)$$

$$+\frac{1}{2}\mathbb{E}\int_{[0,T]^2\setminus D}\int_{[0,T]^2\setminus D}|\Delta_{\mathbf{v}}\Delta_{\mathbf{r}}\Xi^{i\ell,j,\epsilon}(\mathbf{r},\mathbf{v},t)|^2|\mu|(dv_1dv_2)|\mu|(dr_1dr_2)=L_3(\epsilon)+L_4(\epsilon).$$

Analysis of $L_3(\epsilon)$. Since $\ell\neq j$, by symmetry, Lemma A.1 and the definition of (137), we only need to check convergence to zero in $L^2(\mathbb{P}\times|R(\cdot,T)|\times d|\mu|)$ of the ℓ-th column (the only non-null column) of $\frac{1}{\epsilon}\mathbf{D}_v\Big[\big(u^{i\ell,j}_{r_1-\epsilon,r_1}-u^{i\ell,j}_{r_2-\epsilon,r_2}\big)e_\ell\Big]$.

LEMMA A.3. *Assume that $Y^{'}$ satisfies the assumptions in Theorem 5.1. Then, for each $\ell \neq j$ and $t \in (0,T]$,*

$$\lim_{\epsilon \to 0^+} \frac{1}{\epsilon}\mathbf{D}_v\Big[\big(u^{i\ell,j}_{r_1-\epsilon,r_1}\mathbb{1}_{[0,t]}(r_1) - u^{i\ell,j}_{r_2-\epsilon,r_2}\mathbb{1}_{[0,t]}(r_2)\big)e_\ell\Big] = 0 \; a.s$$

for almost all $(v,r_1,r_2) \in [0,T]\times[0,T]^2 \setminus D$ w.r.t the product measure $|R(\cdot,T)| \times d|\mu|$.

PROOF. The (only) non-null ℓ-th column of

$$\epsilon^{-1}\mathbf{D}_v u^{i\ell,j}_{r_1-\epsilon,r_1}e_\ell \mathbb{1}_{[0,t]}(r_1) - \epsilon^{-1}\mathbf{D}_v u^{i\ell,j}_{r_2-\epsilon,r_2}e_\ell \mathbb{1}_{[0,t]}(r_2)$$

equals to

$$\frac{1}{\epsilon}\int_{r_1-\epsilon}^{r_1}\Big\{X^j_{s,r_1}\mathbf{D}_v Y^{',i\ell}_s + Y^{',i\ell}_s \mathbb{1}_{[s,r_1]}(v)e_j\Big\}ds \mathbb{1}_{[0,t]}(r_1)$$
$$-\frac{1}{\epsilon}\int_{r_2-\epsilon}^{r_2}\Big\{X^j_{s,r_2}\mathbf{D}_v Y^{',i\ell}_s + Y^{',i\ell}_s \mathbb{1}_{[s,r_2]}(v)e_j\Big\}ds \mathbb{1}_{[0,t]}(r_2) \tag{139}$$

a.s for Lebesgue almost all $v, r_1, r_2 \in [0,T]$ and $\epsilon > 0$. Then, the argument is the same as the one applied in the proof of Lemma A.2. □

We need to investigate convergence to zero of (139) in $L^2(\mathbb{P}\times|R(\cdot,T)|\times d|\mu|)$. Again, the idea is to explore almost sure convergence stated in Lemma A.3 and uniform integrability. By symmetry, we may restrict $r_2 < r_1 \le t$. The case $r_2 \le t < r_1 \le T$ is trivial because no singularity appears in $\partial^2 R(r_1,r_2)$. We split $[0,T]\times\{(r_1,r_2); r_2 < r_1 \le t\}$ into three cases

$$F_1 = \{(v,r_1,r_2); 0 \le v < r_2 < r_1 \le t\}, \quad F_2 = \{(v,r_1,r_2); 0 \le r_2 < v < r_1 \le t\}$$
$$F_3 = \{(v,r_1,r_2); 0 \le r_2 < r_1 \le v \le T\}.$$

We will check that

$$\left|\frac{1}{\epsilon}\mathbf{D}_v\Big[\big(u^{i\ell,j}_{r_1-\epsilon,r_1}\mathbb{1}_{[0,t]}(r_1) - u^{i\ell,j}_{r_2-\epsilon,r_2}\mathbb{1}_{[0,t]}(r_2)\big)e_\ell\Big]\right|^2 |\partial^2 R(r_1,r_2)|\mathbb{1}_{F_z}$$

is uniformly integrable (along the parameter $\epsilon \in (0,1)$) over the measure space $\mathbb{P}\times|R(\cdot,T)|\times$ Leb, for each $z = 1,2,3$.

The process (139) at the region F_2 can be easily estimated by using (25), (57), assumption A and the fact that no singularity appears in $\partial^2 R$. Indeed, there exists $p > 1$ such that

$$\mathbb{E}\int_{F_2}\left|\frac{1}{\epsilon}\int_{r_m-\epsilon}^{r_m}\Big\{X^j_{s,r_m}\mathbf{D}_v Y^{',i\ell}_s + Y^{',i\ell}_s\mathbb{1}_{[s,r_m]}(v)e_j\Big\}ds\right|^{2p}|\partial^2 R(r_1,r_2)|^p|\partial R(v,T)|d\mathbf{r}dv$$
$$\lesssim \int_{r_2<v<r_1\le t}\{(r_1-r_2)^{\alpha p} + \phi(r_1,r_2)^p\}|\partial R(v,T)|d\mathbf{r}dv < \infty,$$

for every $\epsilon \in (0,1)$ and $m = 1,2$. At the region F_3 (we may suppose $r_1 < v$), (139) reduces to

$$\frac{1}{\epsilon}\int_{r_1-\epsilon}^{r_1} X^j_{s,r_1}\mathbf{D}_v Y^{',i\ell}_s ds - \frac{1}{\epsilon}\int_{r_2-\epsilon}^{r_2} X^j_{s,r_2}\mathbf{D}_v Y^{',i\ell}_s ds. \tag{140}$$

We split $\{(v,r_1,r_2); 0\le r_2<r_1<v\} = \{(v,r_1,r_2); 0\le r_2<r_1<v, r_1-r_2<\epsilon\}\cup\{(v,r_1,r_2); 0\le r_2<r_1<v, r_1-r_2\ge\epsilon\} =: K_1\cup K_2$. On K_1, we can write (140) as

$$\frac{1}{\epsilon}\int_{r_1-\epsilon}^{r_2} X^j_{r_2,r_1}\mathbf{D}_v Y^{',i\ell}_s ds + \frac{1}{\epsilon}\int_{r_2}^{r_1} X^j_{s,r_1}\mathbf{D}_v Y^{',i\ell}_s ds - \frac{1}{\epsilon}\int_{r_2-\epsilon}^{r_1-\epsilon} X^j_{s,r_2}\mathbf{D}_v Y^{',i\ell}_s ds$$

and hence Assumption C yield

$$\mathbb{E}\int_{K_1}\left|\frac{1}{\epsilon}\int_{r_2-\epsilon}^{r_1-\epsilon} X^j_{s,r_2}\mathbf{D}_v Y^{',i\ell}_s ds\right|^2|\mu|(dr_1dr_2)|\partial R(v,T)|dv$$

$$+\mathbb{E}\int_{K_1}\left|\frac{1}{\epsilon}\int_{r_2}^{r_1} X^j_{s,r_1}\mathbf{D}_v Y^{',i\ell}_s ds\right|^2|\mu|(dr_1dr_2)|\partial R(v,T)|dv$$

$$+\mathbb{E}\int_{K_1}\left|\frac{1}{\epsilon}\int_{r_1-\epsilon}^{r_2} X^j_{r_2,r_1}\mathbf{D}_v Y^{',i\ell}_s ds\right|^2|\mu|(dr_1dr_2)|\partial R(v,T)|dv$$

$$\lesssim \int_0^T\int_{r_1-\epsilon}^{r_1}(r_1-r_2)^{2\alpha+2}dr_2dr_1 \lesssim \epsilon^{2\alpha+3}\to 0,$$

as $\epsilon\downarrow 0$, because $2\alpha+3>0$. On K_2, we estimate (140) as follows: We take $1<p<\frac{1}{-2\alpha-2}$ and again by Assumption C, we have

$$\mathbb{E}\int_{K_2}\left|\frac{1}{\epsilon}\int_{r_1-\epsilon}^{r_1} X^j_{s,r_1}\mathbf{D}_v Y^{',i\ell}_s ds\right|^{2p}|\partial^2 R(r_1,r_2)|^p dr_1dr_2|\partial R(v,T)|dv$$

$$\lesssim \int_0^T\int_{r_2}^T (r_1-r_2)^{p(2\alpha+2)}dr_1dr_2<\infty,$$

for every $\epsilon\in(0,1)$.

For the analysis on F_1, we write $F_1=\cup_{i=1}^7 F_{1,i}$, where

$$F_{1,1}=\{v<r_2-\epsilon<r_1-\epsilon<r_2<r_1\},\ F_{1,2}=\{v<r_2-\epsilon<r_2<r_1-\epsilon<r_1\},$$

$$F_{1,3}=\{r_2-\epsilon<v<r_1-\epsilon<r_2<r_1\},\ F_{1,4}=\{r_2-\epsilon<v<r_2<r_1-\epsilon<r_1\},$$

$$F_{1,5}=\{r_2-\epsilon<r_1-\epsilon<v<r_2<r_1\},\ F_{1,6}=\{r_2-\epsilon<v<r_2<r_1-\epsilon<r_1\},$$

$$F_{1,7}=\{r_2-\epsilon<v<r_1-\epsilon<r_2<r_1\}.$$

We observe (57), Assumption C, Jensen and Hölder'inequality allow us to choose $1<q<\frac{\alpha+3}{-\alpha}$ such that

$$\mathbb{E}\int_{F_{1,z}}\left|\frac{1}{\epsilon}\int_{r_m-\epsilon}^{r_m}\left\{X^j_{s,r_m}\mathbf{D}_vY^{',i\ell}_s+Y^{',i\ell}_s\mathbb{1}_{[s,r_m]}(v)e_j\right\}ds\right|^{2q}|\partial^2R(r_1,r_2)|^q|\partial R(v,T)|d\mathbf{r}dv$$

$$\lesssim\int_{r_2<r_1}(r_1-r_2)^{\alpha+2+q\alpha}d\mathbf{r}<\infty$$

for every $\epsilon\in(0,1)$, $m=1,2$ and $z=3,4,6,7$. Next, we analyze the set $F_{1,5}$. In this case, we may write (139) equals to

$$\frac{1}{\epsilon}\int_{r_2-\epsilon}^{r_1-\epsilon}Y^{',i\ell}_sdse_j+\frac{1}{\epsilon}\int_v^{r_1}X^j_{s,r_1}\mathbf{D}_vY^{',i\ell}_sds-\frac{1}{\epsilon}\int_v^{r_2}X^j_{s,r_2}\mathbf{D}_vY^{',i\ell}_sds$$

$$+\frac{1}{\epsilon}\int_{r_1-\epsilon}^{v}X^j_{r_2,r_1}\mathbf{D}_vY^{',i\ell}_sds-\frac{1}{\epsilon}\int_{r_2-\epsilon}^{r_1-\epsilon}X^j_{s,r_2}\mathbf{D}_vY^{',i\ell}_sds$$

on $F_{1,5}$. At this point, we use Assumptions C, E, (57) and Fubini's theorem to get

$$\mathbb{E}\int_{F_{1,5}}\left|\frac{1}{\epsilon}\int_{r_2-\epsilon}^{r_1-\epsilon}Y^{',i\ell}_sdse_j\right|^2|\partial R(v,T)|dv|\partial R(r_1,r_2)|dr_1dr_2$$

$$\lesssim\epsilon^{-2}\int_0^T\int_{r_1-\epsilon}^{r_1}\big(r_2-(r_1-\epsilon)\big)^{\alpha+2}(r_1-r_2)^{\alpha+2}dr_2dr_1\lesssim\epsilon^{2\alpha+3}\to 0$$

as $\epsilon\downarrow 0$. We can write

$$\frac{1}{\epsilon}\int_v^{r_1}X^j_{s,r_1}\mathbf{D}_vY^{',i\ell}_sds-\frac{1}{\epsilon}\int_v^{r_2}X^j_{s,r_2}\mathbf{D}_vY^{',i\ell}_sds=\frac{1}{\epsilon}\int_v^{r_2}X^j_{r_2,r_1}\mathbf{D}_vY^{',i\ell}_sds$$
$$+\frac{1}{\epsilon}\int_{r_2}^{r_1}X^j_{s,r_1}\mathbf{D}_vY^{',i\ell}_sds$$

on $F_{1,5}$. Repeat the same argument used above to conclude

$$\lim_{\epsilon\to0^+}\mathbb{E}\int_{F_{1,5}}\left|\frac{1}{\epsilon}\int_{r_2}^{r_1}X^j_{s,r_1}\mathbf{D}_vY^{',i\ell}_sds\right|^2|\partial R(v,T)|dv|\mu|(dr_1dr_2)=0,$$

$$\lim_{\epsilon\to0^+}\mathbb{E}\int_{F_{1,5}}\left|\frac{1}{\epsilon}\int_v^{r_2}X^j_{r_2,r_1}\mathbf{D}_vY^{',i\ell}_sds\right|^2|\partial R(v,T)|dv|\mu|(dr_1dr_2)=0,$$

$$\lim_{\epsilon\to0^+}\mathbb{E}\int_{F_{1,5}}\left|\frac{1}{\epsilon}\int_{r_1-\epsilon}^{v}X^j_{r_2,r_1}\mathbf{D}_vY^{',i\ell}_sds\right|^2|\partial R(v,T)|dv|\mu|(dr_1dr_2)=0,$$

$$\lim_{\epsilon\to0^+}\mathbb{E}\int_{F_{1,5}}\left|\frac{1}{\epsilon}\int_{r_2-\epsilon}^{r_1-\epsilon}X^j_{s,r_2}\mathbf{D}_vY^{',i\ell}_sds\right|^2|\partial R(v,T)|dv|\mu|(dr_1dr_2)=0.$$

By using Jensen's inequality, Assumptions A, C and (57), we can repeat the same argument given in the analysis of (140) to conclude

$$\lim_{\epsilon\to 0^+}\mathbb{E}\int_{F_{1,1}}\Big|\Delta_{\mathbf{r}}\Xi^{i\ell,j,\epsilon}(\mathbf{r},v,t)\Big|^2|\partial R(v,T)|dv|\mu|(dr_1dr_2)=0$$

and there exists $p>2$ such that

$$\sup_{0<\epsilon<1}\mathbb{E}\int_{F_{1,2}}\Big|\Delta_{\mathbf{r}}\Xi^{i\ell,j,\epsilon}(\mathbf{r},v,t)\Big|^p|\partial R(v,T)|dv|\partial^2R(r_1,r_2)|^{\frac{p}{2}}d\mathbf{r}<\infty.$$

Vitali's theorem combined with Lemma A.3 allow us to conclude $\mathbb{E}[L_3(t)]\to 0$ as $\epsilon\to 0^+$.

Analysis of $L_4(\epsilon)$. In the sequel, in view of assumption (25), we may suppose that $\phi=0$, i.e.,

$$\big|\partial^2R(r_1,r_2)\big|\lesssim|r_1-r_2|^\alpha;(r_1,r_2)\in[0,T]^2\setminus D.$$

The main difficulty lies on the singularity of the kernel $|r_1-r_2|^\alpha$ on $[0,T]^2\setminus D$. Indeed, by Assumption C, we recall there exists $L>1$ such that ϕ is p-integrable on $[0,T]^2\setminus D$ for every $p\in(1,L)$. Then, we may restrict the analysis to the case $\phi=0$.

Since $\ell\neq j$, by symmetry, Lemma A.1 and the definition of (138), we only need to check convergence to zero in $L^2(\mathbb{P}\times|\mu|\times|\mu|)$ of the ℓ-th column (the only non-null column) of

(141)
$$\frac{1}{\epsilon}\Bigg\{\mathbf{D}_{v_1}\Big[\big(u^{i\ell,j}_{r_1-\epsilon,r_1}\mathbb{1}_{[0,t]}(r_1)-u^{i\ell,j}_{r_2-\epsilon,r_2}\mathbb{1}_{[0,t]}(r_2)\big)e_\ell\Big]-\mathbf{D}_{v_2}\Big[\big(u^{i\ell,j}_{r_1-\epsilon,r_1}\mathbb{1}_{[0,t]}(r_1)-u^{i\ell,j}_{r_2-\epsilon,r_2}\mathbb{1}_{[0,t]}(r_2)\big)e_\ell\Big]\Bigg\}.$$

Without any loss of generality, we may assume $0\le r_2<r_1\le t$, $v_2<v_1\le T$. We also observe the case $r_2<t<r_1$ can be easily treated because, in this case, no singularity appears in $|r_1-r_2|^\alpha$. We can write (141) as

$$\frac{1}{\epsilon}\int_{r_1-\epsilon}^{r_1}X^j_{s,r_1}\Big(\mathbf{D}_{v_1}Y^{',i\ell}_s-\mathbf{D}_{v_2}Y^{',i\ell}_s\Big)ds$$

$$-\frac{1}{\epsilon}\int_{r_2-\epsilon}^{r_2}X^j_{s,r_2}\Big(\mathbf{D}_{v_1}Y^{',i\ell}_s-\mathbf{D}_{v_2}Y^{',i\ell}_s\Big)ds$$

$$+\frac{1}{\epsilon}\int_{r_1-\epsilon}^{r_1}Y^{',i\ell}_s\big(\mathbb{1}_{[s,r_1]}(v_1)-\mathbb{1}_{[s,r_1]}(v_2)\big)e_jds$$

$$-\frac{1}{\epsilon}\int_{r_2-\epsilon}^{r_2}Y^{',i\ell}_s\big(\mathbb{1}_{[s,r_2]}(v_1)-\mathbb{1}_{[s,r_2]}(v_2)\big)e_jds$$

$$=:a_1(\mathbf{r},\mathbf{v},\epsilon)-a_2(\mathbf{r},\mathbf{v},\epsilon)+b_1(\mathbf{r},\mathbf{v},\epsilon)-b_2(\mathbf{r},\mathbf{v},\epsilon).$$

To shorten notation, we denote $a(\mathbf{r},\mathbf{v},\epsilon)=a_1(\mathbf{r},\mathbf{v},\epsilon)-a_2(\mathbf{r},\mathbf{v},\epsilon)$, $b(\mathbf{r},\mathbf{v},\epsilon)=b_1(\mathbf{r},\mathbf{v},\epsilon)-b_2(\mathbf{r},\mathbf{v},\epsilon)$.

LEMMA A.4. *We have* $\lim_{\epsilon\downarrow 0}a_i(\mathbf{r},\mathbf{v},\epsilon)=\lim_{\epsilon\downarrow 0}b_i(\mathbf{r},\mathbf{v},\epsilon)=\mathbf{0}$ *a.s for Lebesgue almost all* $(\mathbf{r},\mathbf{v})\in[0,T]^2\setminus D\times[0,T]^2\setminus D$*, for each* $i=1,2$.

PROOF. The same argument given in Lemmas A.2 and A.3 applies here. □

In the sequel, we will check that

$$|b(\mathbf{r},\mathbf{v},\epsilon)|^2|\partial^2 R(\mathbf{r})|\partial^2 R(\mathbf{v})|$$

is an uniformly integrable family (in $0<\epsilon<1$) w.r.t the measure $\mathbb{P}\times$ Leb and hence Vitali's theorem combined with Lemma A.4 will imply

$$\lim_{\epsilon\to 0^+}\mathbb{E}\int_{v_1>v_2,t\geq r_1>r_2}|b(\mathbf{r},\mathbf{v},\epsilon)|^2|\partial^2 R(r_1,r_2)||\partial^2 R(v_1,v_2)|d\mathbf{r}d\mathbf{v}=0.$$

We observe $b=\mathbf{0}$ on $\{r_2<r_1<v_2<v_1\}$ so that we only need to analyze b on $r_1>v_2$. We split $\{(\mathbf{r},\mathbf{v});0\leq r_2<r_1\leq t,0\leq v_2<v_1\leq T,r_1>v_2\}$ in terms of the partition

$$G_1=\{v_2<v_1<r_2<r_1\},\ G_2=\{r_2<v_2<v_1<r_1\}$$

$$G_3=\{v_2<r_2<v_1<r_1\},\ G_4=\{v_2<r_2<r_1<v_1\},\ G_5=\{r_2<v_2<r_1<v_1\}$$

The most delicate cases are G_1 and G_2. We split G_1 in terms of the partition

$$G_{11}=\{r_2-\epsilon<r_1-\epsilon<v_2<v_1<r_2<r_1\},G_{12}=\{v_2<v_1<r_2-\epsilon<r_1-\epsilon<r_2<r_1\}$$

$$G_{13}=\{v_2<v_1<r_2-\epsilon<r_2<r_1-\epsilon<r_1\},G_{14}=\{r_2-\epsilon<v_2<r_1-\epsilon<v_1<r_2<r_1\}$$

$$G_{15}=\{v_2<r_2-\epsilon<v_1<r_1-\epsilon<r_2<r_1\},G_{16}=\{v_2<r_2-\epsilon<v_1<r_2<r_1-\epsilon<r_1\}$$

$$G_{17}=\{r_2-\epsilon<v_2<v_1<r_1-\epsilon<r_2<r_1\},\ G_{18}=\{r_2-\epsilon<v_2<v_1<r_2<r_1-\epsilon<r_1\}$$

$$G_{19}=\{v_2<r_2-\epsilon<r-1-\epsilon<v_1<r_2<r_1\}.$$

We observe $b=\mathbf{0}$ on $\cup_{\ell=1}^3 G_{1\ell}$. Jensen's inequality and assumption (57) yield

$$\mathbb{E}\int_{\cup_{\ell=4}^6 G_{1\ell}}|b(\mathbf{r},\mathbf{v},\epsilon)|^p|\partial^2 R(\mathbf{r})|^{\frac{p}{2}}|\partial^2 R(\mathbf{v})|^{\frac{p}{2}}d\mathbf{r}d\mathbf{v}\lesssim\int_{\cup_{\ell=4}^6 G_{1\ell}}|\partial^2 R(\mathbf{r})|^{\frac{p}{2}}|\partial^2 R(\mathbf{v})|^{\frac{p}{2}}d\mathbf{r}d\mathbf{v}.$$

Next, by using Assumption C and choosing $2<p<\frac{18}{4}$, we have

$$\int_{G_{14}}|\partial^2 R(\mathbf{r})|^{\frac{p}{2}}|\partial^2 R(\mathbf{v})|^{\frac{p}{2}}d\mathbf{r}d\mathbf{v}\lesssim\int_{r_2<r_1}\int_{r_1-\epsilon}^{r_2}\int_{r_2-\epsilon}^{r_1-\epsilon}(r_1-r_2)^{\frac{\alpha p}{2}}(v_1-v_2)^{\frac{\alpha p}{2}}dv_2dv_1d\mathbf{r}$$

$$\lesssim\int_{r_2<r_1}(r_1-r_2)^{\frac{\alpha p}{2}+2}d\mathbf{r}<\infty$$

for every $\epsilon\in(0,1)$. Similar analysis can be made on G_{15} and G_{16}.

We can choose $0<\beta<1$ such that $0<-(\alpha+1)<\frac{1}{3}<\beta<\frac{2}{3}<\alpha+2<1$. Then, $\epsilon^{-2}(r_1-r_2)^2\leq\epsilon^{-\beta}(r_1-r_2)^{\beta}$ on G_{19}. Then,

$$\mathbb{E}\int_{G_{19}}|b(\mathbf{r},\mathbf{v},\epsilon)|^2|\partial^2 R(\mathbf{r})||\partial^2 R(\mathbf{v})|d\mathbf{r}d\mathbf{v}\lesssim\frac{1}{\epsilon^2}\int_{G_{19}}(r_1-r_2)^2(r_1-r_2)^{\alpha}(v_1-v_2)^{\alpha}d\mathbf{r}d\mathbf{v}.$$

$$\lesssim \epsilon^{-\beta} \int_{r_2<r_1} \int_0^{r_2-\epsilon} \int_{r_1-\epsilon}^{r_2} (v_1 - v_2)^{\alpha} (r_1 - r_2)^{\alpha+\beta} d\mathbf{r}$$

$$\lesssim \epsilon^{\alpha+\beta-2} \int_{r_2<r_1} (r_1 - r_2)^{\alpha+\beta} d\mathbf{r} \to 0$$

as $\epsilon \downarrow 0$. The analysis of the sets G_{17} and G_{18} is easy, so we omit the details. Next, we split the set G_2 into

$$G_{21} = \{r_2 - \epsilon < r_2 < v_2 < v_1 < r_1 - \epsilon < r_1\},\ G_{22} = \{r_2 - \epsilon < r_2 < v_2 < r_1 - \epsilon < v_1 < r_1\}$$

$$G_{23} = \{r_2 - \epsilon < r_2 < r_1 - \epsilon < v_2 < v_1 < r_1\},\ G_{24} = \{r_2 - \epsilon < r_1 - \epsilon < r_2 < v_2 < v_1 < r_1\}$$

We observe $b = \mathbf{0}$ on G_{21} and, for each $i = 2, 3, 4$, one can easily check we can take $2 < p < \frac{-3}{\alpha}$ such that

$$\mathbb{E} \int_{G_{2i}} |b(\mathbf{r}, \mathbf{v}, \epsilon)|^p |\partial^2 R(\mathbf{r})|^{\frac{p}{2}} |\partial^2 R(\mathbf{v})|^{\frac{p}{2}} d\mathbf{r} d\mathbf{v} \lesssim \int_{r_2<r_1} (r_1 - r_2)^{\alpha p+2} d\mathbf{r}$$

for every $\epsilon \in (0, 1)$. The analysis over G_4 is similar to G_2. The analysis of G_3 and G_5 is straightforward. By symmetry, we conclude

$$\lim_{\epsilon \to 0^+} \mathbb{E} \int_{[0,T]^2 \setminus D} |b(\mathbf{r}, \mathbf{v}, \epsilon)|^2 |\partial^2 R(\mathbf{r})| \partial^2 R(\mathbf{v})| d\mathbf{r} d\mathbf{v} = 0.$$

The analysis of the term $a(\mathbf{r}, \mathbf{v}, \epsilon)$ is similar to b, so we may omit the details. Indeed, we need to combine assumptions **C** and (56) to check uniform integrability of

$$|a(\mathbf{r}, \mathbf{v}, \epsilon)|^2 |\partial^2 R(\mathbf{r})| |\partial^2 R(\mathbf{v})|$$

just like we did for the term b. For the subset $\{(\mathbf{r}, \mathbf{v}); 0 \le r_2 < r_1 \le t, 0 \le v_2 < v_2 \le T, r_1 > v_2\}$, we make the analysis over the same partition $\cup_{z=1}^5 G_z$. For the subset $\{(\mathbf{r}, \mathbf{v}); 0 \le r_2 < r_1 \le t, 0 \le r_2 < r_1 < v_2 < v_1 \le T\}$, we decompose just like G_1 and use assumptions **C** and (56). By using symmetry and Vitali's theorem, we conclude

$$\lim_{\epsilon \to 0^+} \mathbb{E} \int_{[0,T]^2 \setminus D} |a(\mathbf{r}, \mathbf{v}, \epsilon)|^2 |\partial^2 R(\mathbf{r})| \partial^2 R(\mathbf{v})| d\mathbf{r} d\mathbf{v} = 0.$$

This concludes the proof of Theorem 5.1.

Funding. This research was supported by the Regional Program MATH-AmSud 2018 (grant 88887.197425/2018-00) and Fundação de Apoio a Pesquisa do Destrito Federal (FAPDF grant 00193-00000229/2021-21). A.O.acknowledges the support of CNPq Bolsa de Produtividade Pesquisa grant 303443/2018-9.

REFERENCES

[1] ALOS, E., LEÓN, J. A. and NUALART, D. (2001). Stochastic Stratonovich calculus fBm for fractional Brownian motion with Hurst parameter less than 1/2. *Taiwanese Journal of Mathematics* **5** 609–632.

[2] ALÒS, E. and NUALART, D. (2003). Stochastic integration with respect to the fractional Brownian motion. *Stochastics and Stochastic Reports* **75** 129–152.

[3] CASS, T., HAIRER, M., LITTERER, C. and TINDEL, S. (2015). Smoothness of the density for solutions to Gaussian rough differential equations. *Annals of Probability* **43** 188–239.

[4] CASS, T. and LIM, N. (2019). A Stratonovich-Skorohod integral formula for Gaussian rough paths. *Annals of Probability* **47** 1–60.
[5] CASS, T. and LIM, N. (2021). Skorohod and rough integration for stochastic differential equations driven by Volterra processes. In *Annales de l'Institut Henri Poincaré, Probabilités et Statistiques* **57** 132–168. Institut Henri Poincaré.
[6] CASS, T., LITTERER, C. and LYONS, T. (2013). Integrability and tail estimates for Gaussian rough differential equations. *Annals of Probability* **41** 3026–3050.
[7] CHERIDITO, P. and NUALART, D. (2005). Stochastic integral of divergence type with respect to fractional Brownian motion with Hurst parameter $H \in (0, \frac{1}{2})$. *Ann. Inst. Henri Poincaré, Probab. Stat.* **41** 1049–1081.
[8] COUTIN, L. and QIAN, Z. (2002). Stochastic analysis, rough path analysis and fractional Brownian motions. *Probability theory and related fields* **122** 108–140.
[9] COVIELLO, R., DI GIROLAMI, C. and RUSSO, F. (2011). On stochastic calculus related to financial assets without semimartingales. *Bull. Sci. Math.* **135** 733–774.
[10] COVIELLO, R. and RUSSO, F. (2007). Nonsemimartingales: stochastic differential equations and weak Dirichlet processes. *Ann. Probab.* **35** 255–308.
[11] ERRAMI, M. and RUSSO, F. (2003). n-covariation, generalized Dirichlet processes and calculus with respect to finite cubic variation processes. *Stochastic Process. Appl.* **104** 259–299.
[12] FEYEL, D. and DE LA PRADELLE, A. (2006). Curvilinear integrals along enriched paths. *Electronic Journal of Probability* **11** 860–892.
[13] FRIZ, P. K. and HAIRER, M. (2014). *A course on rough paths*. *Universitext*. Springer, Cham With an introduction to regularity structures. https://doi.org/10.1007/978-3-319-08332-2 MR3289027
[14] FRIZ, P. K. and VICTOIR, N. B. (2010). *Multidimensional stochastic processes as rough paths*. *Cambridge Studies in Advanced Mathematics* **120**. Cambridge University Press, Cambridge. Theory and applications.
[15] GOMES, A. O., OHASHI, A., RUSSO, F. and TEIXEIRA, A. (2021). Rough paths and regularization. *Journal of Stochastic Analysis (JOSA).* **2 (4)** 1–21. https://doi.org/10.31390/josa.2.4.01
[16] GRADINARU, M. and NOURDIN, I. (2003). Approximation at first and second order of m-order integrals of the fractional Brownian motion and of certain semimartingales. *Electron. J. Probab.* **8** no. 18, 26 pp. (electronic).
[17] GRADINARU, M., NOURDIN, I., RUSSO, F. and VALLOIS, P. (2005). m-order integrals and generalized Itô's formula: the case of a fractional Brownian motion with any Hurst index. *Ann. Inst. H. Poincaré Probab. Statist.* **41** 781–806.
[18] GRADINARU, M., RUSSO, F. and VALLOIS, P. (2003). Generalized covariations, local time and Stratonovich Itô's formula for fractional Brownian motion with Hurst index $H \geq \frac{1}{4}$. *Ann. Probab.* **31** 1772–1820.
[19] GUBINELLI, M. (2004). Controlling rough paths. *Journal of Functional Analysis* **216** 86–140.
[20] HU, Y., JOLIS, M. and TINDEL, S. (2013). On Stratonovich and Skorohod stochastic calculus for Gaussian processes. *The Annals of Probability* **41** 1656–1693.
[21] KRUK, I. and RUSSO, F. (2010). Malliavin-Skorohod calculus and Paley-Wiener integral for covariance singular processes. *ArXiv preprint arxiv:1011.6478v1*.
[22] LIU, Y., SELK, Z. and TINDEL, S. (2020). Convergence of trapezoid rule to rough integrals. *ArXiv preprint arXiv:2005.06500*.
[23] LYONS, T. J. (1998). Differential equations driven by rough signals. *Rev. Mat. Iberoamericana* **14** 215–310.
[24] NUALART, D. (2006). *The Malliavin calculus and related topics* **1995**. Springer.
[25] NUALART, D. and PARDOUX, É. (1988). Stochastic calculus with anticipating integrands. *Probability Theory and Related Fields* **78** 535–581.
[26] PERKOWSKI, N. and PRÖMEL, D. J. (2016). Pathwise stochastic integrals for model free finance. *Bernoulli* **22** 2486–2520.
[27] PERLMAN, M. D. (1974). Jensen's inequality for a convex vector-valued function on an infinite-dimensional space. *Journal of Multivariate Analysis* **4** 52–65.
[28] RUSSO, F. and TUDOR, C. (2006). Bifractional Brownian motion. *Stochastic Processes and their Applications* **116**.
[29] RUSSO, F. and VALLOIS, P. (1993). Forward, backward and symmetric stochastic integration. *Probability theory and related fields* **97** 403–421.

[30] RUSSO, F. and VALLOIS, P. (2007). Elements of stochastic calculus via regularization. In *Séminaire de Probabilités XL* 147–185. Springer.

[31] SONG, J. and TINDEL, S. Skorohod and Stratonovich integrals for controlled processes. *ArXiv preprint arXiv:2102.02693*.

[32] WONG, E. and ZAKAI, M. (1965). On the relation between ordinary and stochastic differential equations. *International Journal of Engineering Science* **3** 213–229.