REDUCED DISSIPATION EFFECT IN STOCHASTIC TRANSPORT BY GAUSSIAN NOISE WITH REGULARITY GREATER THAN $1/2$

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Diffusion with stochastic transport is investigated here when the random driving process is a very general Gaussian process including Fractional Brownian motion. The purpose is the comparison with a deterministic PDE which in certain cases represents the equation for the mean value. From this equation we observe a reduced dissipation property with respect to delta correlated noise when regularity is higher than the one of Brownian motion, a fact interpreted qualitatively here as a signature of the reduced dissipation observed for 2D turbulent fluids due to the inverse cascade. We give results also for the variance of the solution and for a scaling limit of a two-component noise input.

1. Introduction. This work investigates the dissipation properties of a stochastic transport term of Fractional Brownian Motion (FBM henceforth) type with Hurst parameter $H > \frac{1}{2}$, or more generally a Gaussian process with Hölder paths of exponents greater than $\gamma_0 > \frac{1}{2}$, compared to those of Brownian motion.

Starting from the paper [24], several works proved that a suitable scaling limit of a Brownian transport term lead to effects similar to those of an additional dissipation or viscosity, see for instance [1], [5], [11], [13], [14], [16], [15], [18], [17], [19], [20], [21], [25], [26], [28]. Interpreting stochatic transport as the effect of small-scale turbulence (the smallness of the scales being related to the scaling limit of the above quoted works) on a passive scalar or on the large scales of the fluid itself, these results support the intuition, expressed by Joseph Boussinesq [4], that small scale turbulence acts as a dissipation mechanism, similar to the molecular one (although being macroscopic).

However, the intuition of Boussinesq has limits of validity (even not so clear yet); see certain criticisms for instance in [31], [39], [40]. One mechanism which seems to work against it is the inverse cascade in 2D fluids. When turbulent small vortices, which by themselves would act as a dissipation, aggregate in larger vortex structures, their dissipation power decreases. In direct numerical simulations, where the inverse cascade happens, this depletion of dissipativity is observed [10]. In the simplified models mentioned above, where the fluid is replaced by a white noise, there is no inverse cascade by assumption (because the fluid model is not Navier-Stokes, producing the cascade but a simplified a priori given model with only small structures), hence the phenomenon is hidden.

Larger vortex structures not only have a greater spatial extension and smaller spatial variation, but also a longer time scale of relaxation, a longer memory, and positive time correlation, namely the velocity has a tendency to remain in the same direction for longer time, compared to the small vortex structures. A simplified phenomenological model of such larger vortex structures, therefore, and still in the philosophy of simple stochastic fluid models, may be the time derivative of FMB with $H > \frac{1}{2}$, with a large spatial scale, which we shall assume

MSC2020 subject classifications: 76M35, 60H07, 60H15, 60G22, 76F55.

Keywords and phrases: Stochastic turbulence modeling, Transport equation, Stochastic PDEs, Malliavin Calculus, Fractional Brownian motion, Stationary Gaussian noise.

constant for simplicity (like it may appear performing a local blow-up). Although FBM is the paradigmatic example, it may be useful to have more degrees of freedom in the statistical choice (see examples in [3], [2], [7]) and thus Gaussian process with Hölder paths of exponents greater than $\gamma_0 > \frac{1}{2}$ may be a convenient framework.

We therefore investigate the following model for the diffusion of a passive scalar $\theta_\epsilon(t,x)$ (e.g. the temperature of the fluid), with $t \geq 0$ and $x \in \mathbb{R}^2$:

$$\partial_t \theta_\epsilon(t,x) = \kappa \Delta \theta_\epsilon(t,x) + \sum_{k\in K} (\sigma_k(x) \cdot \nabla) \theta_\epsilon(t,x) \frac{d\mathcal{G}_t^{k,\epsilon}}{dt} \tag{1}$$

$$\theta_\epsilon|_{t=0} = \theta_0,$$

where $\kappa > 0$ is a (small) diffusion constant (in most part of this work also $\kappa = 0$ is admitted), K is a finite index set, $\sigma_k(x)$ are smooth divergence free vector fields and $\mathcal{G}_t^{k,\epsilon}$ are ϵ-regularization of stationary increment Gaussian processes G_t^k, $\epsilon > 0$:

$$\mathcal{G}_t^{k,\epsilon} = \int_0^t \frac{G_{s+\epsilon}^k - G_{(s-\epsilon)_+}^k}{2\epsilon} ds \tag{2}$$

where, here and below, we denote by $(s-\epsilon)_+$ the maximum between $s-\epsilon$ and zero. Hence the driving random field $\frac{d\mathcal{G}_t^{k,\epsilon}}{dt}$ is stationary. The choice to work in $\mathbb{R}^2$ is only due to the motivation of the inverse cascade, but the results proved here hold in any space dimension. The interpretation is that the velocity field $u_\epsilon(t,x)$ of the fluid is modeled by the stationary random field

$$u_\epsilon(t,x) = \sum_{k\in K} \sigma_k(x) \frac{d\mathcal{G}_t^{k,\epsilon}}{dt}.$$

Some of the works quoted above had in mind the case when the fluid structures $\sigma_k(x)$ where small, modeling small-space-scale turbulence; accordingly, it was natural to idealize the time-structure assuming very small time-correlation, hence $\frac{d\mathcal{G}_t^{k,\epsilon}}{dt}$ related to white noise. Here we have in mind larger space structures $\sigma_k(x)$ and longer time correlation of $\frac{d\mathcal{G}_t^{k,\epsilon}}{dt}$, as it is realized when $\mathcal{G}_t^{k,\epsilon}$ is related to FBM with $H > 1/2$.

We follow the philosophy that the physical model is the family of equations parametrized by $\epsilon > 0$. Of course it is mathematically interesting to investigate the limit as $\epsilon \to 0$ in itself but this is not the purpose of this work. On the contrary, we compute observations, especially mean values and take the limit as $\epsilon \to 0$ of their results, getting clean final expressions for the observed quantities.

Our aim is proving that the dissipation, for $H > 1/2$, is smaller than in the white noise case. Proving such a claim on the single realizations of $\theta_\epsilon(t,x)$ is very difficult. In the special regime investigated in [24], [16], [22] and other works this is possible, but in general the only information available could be on mean quantities. Assume that the Gaussian fields G_t^k are independent and equally distributed. We thus introduce the *mean field equation* associated to (1):

$$\partial_t \overline{\theta}(t,x) = \kappa \Delta \overline{\theta}(t,x) + \left(\mathcal{L}\overline{\theta}(t)\right)(x) \frac{d\gamma(t)}{dt} \tag{3}$$

$$\overline{\theta}|_{t=0} = \theta_0.$$

Here $\mathcal{L}$ is the elliptic operator (possibly non-uniformly elliptic)

$$(\mathcal{L}f)(x) = \operatorname{div}(Q(x,x)\nabla f(x)) \tag{4}$$

$$Q(x,y) = \sum_{k\in K} \sigma_k(x) \otimes \sigma_k(y)$$

and $\gamma(t)$ is the variance function of the G_t^k:

$$\gamma(t) = Var\left(G_t^k\right).$$

When we deduce equation (3) we restrict ourselves to $\frac{d\gamma(t)}{dt} \geq 0$ but this investigation opens the door to the possibility of negative viscosities, mentioned for instance by [40]. In relevant cases

$$\left(\mathcal{L}\overline{\theta}(t)\right)(x) = \kappa_T \Delta\overline{\theta}(t,x)$$

where $\kappa_T > 0$ may be called eddy dissipation constant, and

$$\frac{d\gamma(t)}{dt} \sim t^{2H-1}$$

for small t. The fact that $\frac{d\gamma(t)}{dt}$ is infinitesimal for small t when $H > 1/2$, compared to the Brownian case ($H = 1/2$) where it is constant, is the manifestation of depleted average dissipation identified by this model. For large t the asymptotics $\sim t^{2H-1}$ is not realistic having in mind fluids, see Remark 3 below.

The link between the stochastic equation (1) and what we have called its mean field equation (3) is however not always easy or completely clear. In the Brownian case, equation (3) is the equation for the limit as $\epsilon \to 0$ of the average of the solution of equation (1) and, as already mentioned, in several works it was shown, under suitable assumptions of small space-correlation, that the realizations of $\theta_\epsilon(t,x)$ are concentrated around their mean $\overline{\theta}(t,x)$. However, in the general Gaussian case treated here the link is less obvious.

We thus first analyze in detail, in Section 3, the simpler but very relevant[1] case of constant vector fields σ_k. We call it the commutative case since the differential operators $(\sigma_k(x)\cdot\nabla)$ commute. Thanks to Fourier analysis we show that equation (3) is the equation for the limit as $\epsilon \to 0$ of the average of the solution of equation (1). In addition, we get an equation for the limit as $\epsilon \to 0$ of the variance, which is

$$\partial_t V(t,x) = 2\kappa\Delta V(t,x) + \left((\mathcal{L}V(t))(x) + 2\sum_{k\in K}\left((\sigma_k\cdot\nabla)\overline{\theta}(t,x)\right)^2\right)\frac{d\gamma(t)}{dt}.$$

Let us indicate the relevance of this law. Heuristically speaking for simplicity, taking $\kappa = 0$ again to make a simple example, noticing that $V = 0$ at time $t = 0$, we get for small t

$$\overline{\theta}(t) \sim (\mathcal{L}\theta_0)\, t^{2H}$$

$$\sqrt{V(t)} \sim \sqrt{2\sum_{k\in K}\left((\sigma_k\cdot\nabla)\theta_0\right)^2} t^H.$$

Compare the Brownian case ($H = 1/2$) with the case $H > 1/2$. In the latter, not only the average dissipation is infinitesimal with respect to the Brownian case, but also a confidence interval around the average is small. Again, these results are meaningful for small time, see Remark 3 below.

Among the interests in the computation of the commutative case there is its generality in terms of noise, which in particular covers all $H \in (0,1)$ and even beyond. We refer in particular to a stochastic analysis related to a very singular covariance process, see e.g. [34].

In the last Section 4 we move a few steps in the direction of the non-commutative case. This case is very difficult and our understanding is only fragmentary. Subsection 4.1 describes

[1] Constant vector fields is the idealization - think also under a blow-up scaling - of large space structures.

an idealized model of turbulent 2D flow incorporating the idea of inverse cascade in the simplest possible way: the noise is divided into two components, one small-space-scale and white noise in time, the other larger-space-scale and correlated in time. In the larger one the space structures are constant in space. We prove that the smaller scales produce the effect predicted by the Boussinesq hypothesis, while the larger ones remain in their form. It is also an example of reduction to the commutative case. Subsection 4.1 is finally devoted to show the difficulty arising in the non-commutative case, where a commutator arises as a remainder in the link between the true expected value of the solution and the mean field equation.

Let us finally remark that the model presented here could be of interest also in connection with other research directions on stochastic transport, not necessarily related to Boussinesq assumption and inverse cascade. In particular, since we have modeled larger space structures, the connection with the general activity reported in [6], originated by the seminal work [29], see also [38], [8], [9], [12], [27], [30], [33].

2. Preparatory material. The material of this section is adapted from Chapter 1. of [35], see [32], in particular Section 10, for a more explicit formulation and summarized here for the reader's convenience. See also [36] for some more recent developments.

We consider a Gaussian process $G := \left(G^k; k \in K\right)$ in $\mathbb{R}^N$, $N = Card\,(K)$, whose components are independent and identically distributed. Denote by $\mathcal{H}$ the self-reproducing kernel space of G^1, with scalar product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$. Recall that $\left\langle 1_{[0,t]}, 1_{[0,s]} \right\rangle_{\mathcal{H}}$ gives us the covariance function of G^1.

EXAMPLE 1. *The FBM with $H > 1/2$ has covariance function given by*

$$R\,(t,s) = \frac{1}{2}\left(s^{2H} + t^{2H} - |t-s|^{2H}\right) = \alpha_H \int_0^t \int_0^s |r-u|^{2H-2}\, dudr$$

for a suitable constant $\alpha_H > 0$ and therefore

$$\langle f, g \rangle_{\mathcal{H}} = \alpha_H \int_0^T \int_0^T |r-u|^{2H-2} f\,(r)\, g\,(u)\, dudr.$$

EXAMPLE 2. *A model which seems to fit better our intuition of the intermediate vortex structures of a 2D turbulent fluid is*

$$R\,(t,s) = \alpha_{H,\lambda} \int_0^t \int_0^s |r-u|^{2H-2} e^{-\lambda|r-u|} dudr$$

$$\langle f, g \rangle_{\mathcal{H}} = \alpha_{H,\lambda} \int_0^T \int_0^T |r-u|^{2H-2} e^{-\lambda|r-u|} f\,(r)\, g\,(u)\, dudr$$

with $\lambda > 0$, for a suitable constant $\alpha_{H,\lambda} > 0$. Indeed,

$$\begin{aligned}
\mathbb{E}\left[\dot{\mathcal{G}}_t^{k,\epsilon}\, \dot{\mathcal{G}}_s^{k,\epsilon}\right] &= (2\epsilon)^{-2}\, \mathbb{E}\left[\left(G_{t+\epsilon} - G_{t-\epsilon}\right)\left(G_{s+\epsilon} - G_{s-\epsilon}\right)\right] \\
&= (2\epsilon)^{-2}\left(R\,(t+\epsilon, s+\epsilon) - R\,(t+\epsilon, s-\epsilon) - R\,(t-\epsilon, s+\epsilon) + R\,(t-\epsilon, s-\epsilon)\right) \\
&\to \partial_t \partial_s R\,(t,s) = |t-s|^{2H-2} e^{-\lambda|t-s|}.
\end{aligned}$$

This process develops, locally in time, the same correlation structure of FBM, but loses memory in the long time, closely to the fact that also large scale vortex structures are like a birth and death process, they do not persist to infinity.

REMARK 3. *In the case of the previous example, call $\tau > 0$ a time of decorrelation of the vortex structures we want to model and take $\lambda = 1/\tau$. We have*

$$\frac{d\gamma(t)}{dt} = 2\alpha_{H,\lambda} \int_0^t r^{2H-2} e^{-\lambda r} dr$$

which behaves like

$$\frac{d\gamma(t)}{dt} \sim t^{2H-1}$$

for small t but not for large ones. The limit as $t \to \infty$ of $\frac{d\gamma(t)}{dt}$ is given by ($\Gamma(r)$ denotes the Gamma function)

$$2\alpha_{H,\lambda} \int_0^\infty r^{2H-2} e^{-\lambda r} dr = 2\alpha_{H,\lambda} \Gamma(2H-1) \tau^{2H-1} \sim \tau^{2H-1}$$

so it remains small if τ is small.

Let $\Phi : \mathbb{R}^N \to \mathbb{R}$ be smooth an bounded and let $\varphi_k \in \mathcal{H}$, $k \in K$. Set

$$Y = \Phi\left(\int_0^T \varphi_k dG^k; k \in K\right)$$

where $\int_0^T \varphi_k dG^k$ are Wiener integrals. Then the Malliavin derivative

$$DY = \left(D^{(k)} Y; k \in K\right)$$

is a vector process given by

$$D_r^{(\ell)} Y = (\partial_\ell \Phi)\left(\int_0^T \varphi_k dG^k; k \in K\right) \varphi_\ell(r).$$

The integration by parts on Wiener space states that, if $Z = \left(Z^k; k \in K\right)$ is a Malliavin smooth vector of processes, then

$$\mathbb{E}\left[\langle DY, Z\rangle_{\mathcal{H}}\right] = \mathbb{E}\left[Y\delta(Z)\right],$$

where δ is the Skorohod integral (divergence operator), which has zero expectation, among other properties. For instance, if G is an N-dimensional Brownian motion, we get

$$\delta(Z) = \sum_{k \in K} \int_0^T Z^k dG^k.$$

If $\varphi \in \mathcal{H}$ then the Wiener integral $\int_0^T g dG$ coincides with the Skorohod integral $\delta(g)$. Key to our developments below is rewriting the terms

$$(\sigma_k(x) \cdot \nabla) \theta_\epsilon(t, x) \frac{G_{s+\epsilon}^k - G_{s-\epsilon}^k}{2\epsilon}$$

as a suitable mean zero term plus a term which remains when taking the mean and the limit as $\epsilon \to 0$ and possibly is "closed", namely expressed in terms of the mean of the limit of $\theta_\epsilon(t, x)$. The first step is identifying a suitable mean zero part and it will be the Skorohod integral. The second step is understanding the remaining term, which is the so-called trace.

If X is a Malliavin smooth stochastic process, we have

$$X_s \frac{G^k_{s+\epsilon} - G^k_{(s-\epsilon)_+}}{2\epsilon} = X_s \frac{1}{2\epsilon} \int_{(s-\epsilon)_+}^{s+\epsilon} \delta G^k_s$$
$$= \frac{1}{2\epsilon} \int_{(s-\epsilon)_+}^{s+\epsilon} X_s \delta G^k_s + \left\langle D^{(k)}_{\cdot} X_s, \frac{1}{2\epsilon} 1_{[(s-\epsilon)_+, s+\epsilon]} \right\rangle_{\mathcal{H}}$$

hence we get the formula

$$\int_0^t X_s \frac{G^k_{s+\epsilon} - G^k_{(s-\epsilon)_+}}{2\epsilon} ds = M_t + \int_0^t \left\langle D^{(k)}_{\cdot} X_s, \frac{1}{2\epsilon} 1_{[(s-\epsilon)_+, s+\epsilon]} \right\rangle_{\mathcal{H}} ds, \tag{5}$$

where M is the mean zero process

$$M_t = \int_0^t \left(\frac{1}{2\epsilon} \int_{(s-\epsilon)_+}^{s+\epsilon} X_s \delta G^k_s \right) ds.$$

Formula (5) is used several times below.

3. The commutative case. We consider the equation

$$\theta_\epsilon(t,x) = \theta_0(x) + \int_0^t \kappa \Delta \theta_\epsilon(s,x) ds + \sum_{k \in K} \int_0^t (\sigma_k \cdot \nabla) \theta_\epsilon(s,x) \frac{G^k_{s+\epsilon} - G^k_{(s-\epsilon)_+}}{2\epsilon} ds, \tag{6}$$

when the transport noise vector fields are constant, $\sigma_k \in \mathbb{R}^2$. We assume that G^k, $k \in K$, are independent mean zero Gaussian processes, starting at zero, equally distributed.

One can solve equation (6) by Fourier transform. We use the convention that

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i \xi \cdot x} f(x) dx$$
$$f(x) = \int_{\mathbb{R}^d} e^{2\pi i \xi \cdot x} \widehat{f}(\xi) d\xi,$$

when the notations are meaningful, in a classical or generalized sense. The equation in Fourier transform reads

$$\widehat{\theta_\epsilon}(t,\xi) = \widehat{\theta_0}(\xi) - \kappa |\xi|^2 \int_0^t \widehat{\theta_\epsilon}(s,\xi) ds + i \sum_{k \in K} (\sigma_k \cdot \xi) \int_0^t \widehat{\theta_\epsilon}(s,\xi) \frac{G^k_{s+\epsilon} - G^k_{(s-\epsilon)_+}}{2\epsilon} ds \tag{7}$$

and, being decoupled with respect to the frequency variable ξ, it is pointwise meaningful as a complex-valued ordinary differential equation parametrized by ξ. In fact it can be explicitly solved:

$$\widehat{\theta_\epsilon}(t,\xi) = \widehat{\theta_0}(\xi) \exp\left(-\kappa |\xi|^2 t + i \sum_{k \in K} (\sigma_k \cdot \xi) \mathcal{G}^{k,\epsilon}_t \right), \tag{8}$$

where $\mathcal{G}^{k,\epsilon}_t$ is defined by (2). We introduce the following assumption.

Assumption A

i) G is a Gaussian continuous process with stationary increments, vanishing at zero.
ii) $\gamma(t) = Var\left(G^1_t\right)$ is a bounded variation function.

REMARK 4. *The property of stationary increments can be relaxed, but we keep it as it is to avoid complications.*

REMARK 5. *Of course this includes the case of G being a Fractional Brownian motion with any Hurst index H.*

LEMMA 6. *Suppose Assumption A. We denote*

$$\dot{\mathcal{V}}_\epsilon(t) = \frac{1}{(2\epsilon)^2}\int_0^t \left\langle 1_{[(t-\epsilon)_+,t+\epsilon]}, 1_{[(s-\epsilon)_+,s+\epsilon]}\right\rangle_{\mathcal{H}} ds$$

and

$$\mathcal{V}_\epsilon(\tau) = \int_0^\tau \dot{\mathcal{V}}_\epsilon(t)\,dt. \tag{9}$$

Then the measure $d\mathcal{V}_\epsilon(t)$ converges weak star to $d\gamma(t)$, namely

$$\int_0^T \varphi(t)\,d\mathcal{V}_\epsilon(t) \to \int_0^T \varphi(t)\,d\gamma(t),$$

for every $\varphi \in C([0,T])$.

REMARK 7. *The integral in (9) is a well defined Bochner integral in $\mathcal{H}$.*

REMARK 8. *One has $D_r^{(k)}\mathcal{G}_t^{k',\epsilon} = 0$ for $k' \neq k$ and*

$$D_r^{(k)}\mathcal{G}_t^{k,\epsilon} = \frac{1}{2\epsilon}\int_0^t 1_{[(s-\epsilon)_+,s+\epsilon]}(r)\,ds. \tag{10}$$

Indeed (the case $k' \neq k$ is similar),

$$D_r^{(k)}\int_{(s-\epsilon)_+}^{s+\epsilon}\delta G_u^k = 1_{[(s-\epsilon)_+,s+\epsilon]}(r).$$

PROOF. We have, for $\tau \geq \epsilon$,

$$\begin{aligned}\mathcal{V}_\epsilon(\tau) &= \int_0^\tau \frac{1}{(2\epsilon)^2}\int_0^t \left\langle 1_{[(t-\epsilon)_+,t+\epsilon]}, 1_{[(s-\epsilon)_+,s+\epsilon]}\right\rangle_{\mathcal{H}} dsdt\\ &=: \mathcal{V}_\epsilon(\epsilon) + \widetilde{\mathcal{V}}_\epsilon(\tau),\end{aligned}$$

where $\widetilde{\mathcal{V}}_\epsilon(\tau) = 0$ for $\tau \leq \epsilon$,

$$\widetilde{\mathcal{V}}_\epsilon(\tau) = \int_\epsilon^\tau \frac{1}{(2\epsilon)^2}\int_\epsilon^t \left\langle 1_{[(t-\epsilon)_+,t+\epsilon]}, 1_{[(s-\epsilon)_+,s+\epsilon]}\right\rangle_{\mathcal{H}} dsdt,$$

for $\tau \geq \epsilon$. It is not difficult to show that $\mathcal{V}_\epsilon(\epsilon) \to 0$ as $\epsilon \to 0$. It remains to show that $d\widetilde{\mathcal{V}}_\epsilon(t)$ converges weak star to $d\gamma(t)$. For this it is enough to show that

$$\widetilde{\mathcal{V}}_\epsilon(\tau) \to \frac{1}{2}\gamma(\tau) \text{ for every } \tau \in [0,T] \tag{11}$$

and that $\sup_{\epsilon\in(0,1)}\int_0^T \left|\dot{\widetilde{\mathcal{V}}}_\epsilon(t)\right| dt < \infty$, hence that

$$\sup_{\epsilon\in(0,1)}\int_\epsilon^T \frac{1}{(2\epsilon)^2}\left|\int_\epsilon^t \left\langle 1_{[(t-\epsilon)_+,t+\epsilon]}, 1_{[(s-\epsilon)_+,s+\epsilon]}\right\rangle_{\mathcal{H}} ds\right| dt < \infty. \tag{12}$$

At this point, for $t \geq s \geq \epsilon$ (denote any one of the G^k by G), using stationarity of the increments,

$$\begin{aligned}
&\left\langle 1_{[(t-\epsilon)_+,t+\epsilon]}, 1_{[(s-\epsilon)_+,s+\epsilon]} \right\rangle_{\mathcal{H}} \\
&= Cov\left(G_{t+\epsilon} - G_{(t-\epsilon)_+}, G_{s+\epsilon} - G_{(s-\epsilon)_+}\right) \\
&= Cov\left(G_{t-s+2\epsilon} - G_{t-s}, G_{2\epsilon}\right) \\
&= Cov\left(G_{t-s+2\epsilon}, G_{2\epsilon}\right) - Cov\left(G_{t-s}, G_{2\epsilon}\right) \\
&= \frac{1}{2}\left(\gamma\left(t-s+2\epsilon\right) + \gamma\left(2\epsilon\right) - \gamma\left(t-s\right)\right) \\
&= \frac{1}{2}\left(\gamma\left(2\epsilon\right) + \gamma\left(t-s\right) - \gamma\left(t-s-2\epsilon\right)\right),
\end{aligned}$$

with the convention that γ is extended by parity for negative arguments. So

$$\begin{aligned}
&\left\langle 1_{[(t-\epsilon)_+,t+\epsilon]}, 1_{[(s-\epsilon)_+,s+\epsilon]} \right\rangle_{\mathcal{H}} \\
&= \frac{1}{2}\left(\gamma\left(t-s+2\epsilon\right) - \gamma\left(t-s\right)\right) \\
&- \frac{1}{2}\left(\gamma\left(t-s\right) - \gamma\left(t-s-2\epsilon\right)\right).
\end{aligned}$$

Now, for $\tau \geq \epsilon$, by telescopy,

$$\begin{aligned}
\widetilde{\mathcal{V}}_\epsilon\left(\tau\right) &= \int_\epsilon^\tau \frac{1}{8\epsilon^2}\int_0^t \left(\gamma\left(t-s+2\epsilon\right) - \gamma\left(t-s\right)\right) dsdt \\
&- \int_\epsilon^\tau \frac{1}{8\epsilon^2}\int_0^t \left(\gamma\left(t-s\right) - \gamma\left(t-s-2\epsilon\right)\right) dsdt \\
&= \int_\epsilon^\tau \frac{1}{8\epsilon^2}\int_0^{t-\epsilon} \left(\gamma\left(s+2\epsilon\right) - \gamma\left(s\right)\right) dsdt \\
&- \int_\epsilon^\tau \frac{1}{8\epsilon^2}\int_0^{t-\epsilon} \left(\gamma\left(s\right) - \gamma\left(s-2\epsilon\right)\right) dsdt \\
&= \int_\epsilon^\tau \frac{1}{8\epsilon^2}\int_\epsilon^t \left(\gamma\left(s+\epsilon\right) - \gamma\left(s-\epsilon\right)\right) dsdt \\
&- \int_\epsilon^\tau \frac{1}{8\epsilon^2}\int_{-\epsilon}^{t-\epsilon} \left(\gamma\left(s+\epsilon\right) - \gamma\left(s-\epsilon\right)\right) dsdt.
\end{aligned}$$

So, by telescopy, for $\tau \geq \epsilon$,

$$\widetilde{\mathcal{V}}_\epsilon\left(\tau\right) = \frac{1}{2}\left(I_1\left(\tau,\epsilon\right) - I_2\left(\tau,\epsilon\right)\right),$$

where

$$\begin{aligned}
I_1\left(\tau,\epsilon\right) &= \int_\epsilon^\tau \frac{1}{2\epsilon}\int_{t-\epsilon}^{t+\epsilon} \frac{\gamma\left(s+\epsilon\right) - \gamma\left(s-\epsilon\right)}{2\epsilon} dsdt \\
I_2\left(\tau,\epsilon\right) &= \int_\epsilon^\tau \frac{1}{2\epsilon}\int_{-\epsilon}^{0} \frac{\gamma\left(s+\epsilon\right) - \gamma\left(s-\epsilon\right)}{2\epsilon} dsdt.
\end{aligned}$$

By Fubini theorem, using that

$$\gamma(s+\epsilon)-\gamma(s-\epsilon)=\int_{s-\epsilon}^{s+\epsilon}d\gamma(r),$$

we can easily show that

$$I_1(\tau,\epsilon)\to\gamma(\tau)$$
$$I_2(\tau,\epsilon)\to 0.$$

This shows (11). Concerning (12), we proceed similarly. By the same arguments as before we show that

$$\begin{aligned}&\int_\epsilon^T\frac{1}{(2\epsilon)^2}\left|\int_\epsilon^t\left\langle 1_{[(t-\epsilon)_+,t+\epsilon]},1_{[(s-\epsilon)_+,s+\epsilon]}\right\rangle_{\mathcal{H}}ds\right|dt\\ &\le\int_\epsilon^\tau\frac{1}{2\epsilon}\left|\int_{t-\epsilon}^{t+\epsilon}\frac{\gamma(s+\epsilon)-\gamma(s-\epsilon)}{2\epsilon}ds\right|dt\\ &+\int_\epsilon^\tau\frac{1}{2\epsilon}\left|\int_{-\epsilon}^{0}\frac{\gamma(s+\epsilon)-\gamma(s-\epsilon)}{2\epsilon}ds\right|dt.\end{aligned}$$

We proceed as above, using that

$$|\gamma(s+\epsilon)-\gamma(s-\epsilon)|\le\int_{s-\epsilon}^{s+\epsilon}d\|\gamma\|(r),$$

where $\|\gamma\|$ is the total variation function. □

LEMMA 9. *$D_r^{(k)}\widehat{\theta}_\epsilon(t,\xi)$ exists and it is given by*

$$D_r^{(k)}\widehat{\theta}_\epsilon(t,\xi)=\widehat{\theta}_\epsilon(t,\xi)\,i(\sigma_k\cdot\xi)\frac{1}{2\epsilon}\int_0^t 1_{[(s-\epsilon)_+,s+\epsilon]}(r)\,ds. \tag{13}$$

PROOF. We could deduce this formula from equation (7) and its uniqueness property. For shortness, let us use the explicit formula (8). It gives us

$$\begin{aligned}D_r^{(k)}\widehat{\theta}_\epsilon(t,\xi)&=\widehat{\theta}_\epsilon(t,\xi)\,i\sum_{k'\in K}(\sigma_{k'}\cdot\xi)\,D_r^{(k)}\mathcal{G}_t^{k',\epsilon}\\ &=\widehat{\theta}_\epsilon(t,\xi)\,i(\sigma_k\cdot\xi)\frac{1}{2\epsilon}\int_0^t 1_{[(s-\epsilon)_+,s+\epsilon]}(r)\,ds,\end{aligned}$$

where we have used (10). □

We come back to equation (7). Using (5) we have

$$\begin{aligned}\widehat{\theta}_\epsilon(t,\xi)&=\widehat{\theta}_0(\xi)-\kappa|\xi|^2\int_0^t\widehat{\theta}_\epsilon(s,\xi)\,ds+M_t\\ &+i\sum_{k\in K}(\sigma_k\cdot\xi)\int_0^t\left\langle D_\cdot^{(k)}\widehat{\theta}_\epsilon(s,\xi),\frac{1}{2\epsilon}1_{[(s-\epsilon)_+,s+\epsilon]}\right\rangle_{\mathcal{H}}ds,\end{aligned}$$

where M is a mean zero process, and using (13)

$$\begin{aligned}&=\widehat{\theta}_0(\xi)-\kappa|\xi|^2\int_0^t\widehat{\theta}_\epsilon(s,\xi)\,ds+M_t\\ &-\sum_{k\in K}(\sigma_k\cdot\xi)^2\int_0^t\widehat{\theta}_\epsilon(s,\xi)\frac{1}{(2\epsilon)^2}\int_0^s\left\langle 1_{[(r-\epsilon)_+,r+\epsilon]},1_{[(s-\epsilon)_+,s+\epsilon]}\right\rangle_{\mathcal{H}}drds.\end{aligned}$$

Recalling the definition of $\mathcal{V}_\epsilon$ and taking expectation we get for $e_\epsilon(t,\xi) = \mathbb{E}\left[\widehat{\theta}_\epsilon(t,\xi)\right]$

$$e_\epsilon(t,\xi) = \widehat{\theta}_0(\xi) - \kappa|\xi|^2 \int_0^t e_\epsilon(s,\xi)\,ds - \sigma^2(\xi)\int_0^t e_\epsilon(s,\xi)\,d\mathcal{V}_\epsilon(s), \tag{14}$$

where

$$\sigma^2(\xi) := \sum_{k\in K} (\sigma_k \cdot \xi)^2 .$$

The limit

$$e(t,\xi) = \lim_{\epsilon\to 0} e_\epsilon(t,\xi)$$

exists (it can be deduced by a stability argument on the differential equation and Lemma 6, but for shortness let us invoke here the explicit formula (8)). Taking the limit as $\epsilon \to 0$ into the previous equation, by Lemma 6 we get the result of the following corollary.

COROLLARY 10. *Suppose Assumption A. Then the function $e(t,\xi)$ satisfies the closed form equation*

$$e(t,\xi) = \widehat{\theta}_0(\xi) - \kappa|\xi|^2 \int_0^t e(s,\xi)\,ds - \sigma^2(\xi)\int_0^t e(s,\xi)\,d\gamma(s).$$

Now we go back in physical space by inverse Fourier transform. Until now we have assumed only Assumption A.

REMARK 11. *Without additional assumptions the Fourier coefficients $e(t,\xi)$ could not have easy decay properties for large ξ, in the case when the measure $d\gamma(s)$ has a negative component, and as a consequence the inverse Fourier transform could give us a true distribution, which solves in the distributional sense the equation written below but should require a closer investigation, due to its singularity. Similarly, if $d\gamma(s)/ds$ is well defined, non negative, but it is not bounded above, like in the case of FBM with $H < 1/2$ where it is diverges at $s = 0$, we are faced - in the inverse Fourier transform - with a parabolic equation equation with singular second order coefficients, which is uncommon and also requires special theory.*

REMARK 12. *In addition, the present work has a precise motivation from 2D inverse cascade turbulence and, in that framework, we expect the large vortex structures being positively correlated in time, as it is for $H > 1/2$, not negatively as it is for $H < 1/2$. Therefore we prefer to assume $d\gamma(s)/ds$ bounded from above for reasons of coherence with the purposes of this work. On the contrary, the case when $d\gamma(s)$ has a negative component could correspond to negative viscosity, a debated phenomenon for turbulent fluids [40], perhaps also associated with the 2D inverse cascade. However, it must be better understood and thus we postpone to future works.*

For the reasons highlighted in the previous two remarks, we introduce the following additional assumption:

Assumption B

i) the measure $d\gamma(s)$ has a non-negative density $d\gamma(s)/ds$ with respect to Lebesgue measure

ii) and there exists $C > 0$ such that $d\gamma(s)/ds \leq C$ for a.e. $s \geq 0$.

We may call "regular" the case when Assumption B is satisfied and "singular" the other case, which covers measures with negative components of viscosity and unbounded positive viscosities.

Under Assumption B, $|e(t,\xi)| \leq \left|\widehat{\theta}_0(\xi)\right|$, hence the following.

COROLLARY 13. *Suppose Assumptions A and B and $\theta_0 \in L^2\left(\mathbb{R}^2\right)$. Then*

$$\overline{\theta}(t,x) := \int_{\mathbb{R}^d} e^{2\pi i\xi\cdot x} e(t,\xi)\, d\xi$$

has the property $\overline{\theta}(t) \in L^2\left(\mathbb{R}^2\right)$, it is weakly continuous in time in $L^2\left(\mathbb{R}^2\right)$ and it satisfies, in the sense of distribution,

$$\overline{\theta}(t,x) = \theta_0(x) + \int_0^t \kappa\Delta\overline{\theta}(s,x)\, ds + \int_0^t \left(\mathcal{L}\overline{\theta}(s)\right)(x)\, d\gamma(s),$$

where $\mathcal{L}$ is the differential operator defined by (4).

The only part of the statement we have to clarify is the form of $\mathcal{L}$. Until now it is

$$\widehat{(\mathcal{L}f)}(\xi) = -\sigma^2(\xi)\,\widehat{f}(\xi).$$

Then

$$\begin{aligned}(\mathcal{L}f)(x) &= \sum_{k\in K}\int_{\mathbb{R}^d} e^{2\pi i\xi\cdot x}(i\sigma_k\cdot\xi)(i\sigma_k\cdot\xi)\,\widehat{f}(\xi)\, d\xi\\ &= \sum_{k\in K}(\sigma_k\cdot\nabla)(\sigma_k\cdot\nabla f(x))\\ &= \sum_{k\in K}\operatorname{div}\left((\sigma_k\otimes\sigma_k)\nabla f(x)\right)\\ &= \operatorname{div}\left(Q\nabla f(x)\right),\end{aligned}$$

where $Q = \sum_{k\in K}(\sigma_k\otimes\sigma_k)$.

REMARK 14. *Assume, for instance, that $N = 2$, $K = \{1,2\}$, e_1, e_1 canonical basis of $\mathbb{R}^2$,*

$$\sigma_k = \sqrt{\kappa_T}e_1,$$

where $\kappa_T > 0$ is a constant (with the physical meaning of turbulent kinetic energy). Then

$$\sigma^2(\xi) = \kappa_T|\xi|^2$$
$$(\mathcal{L}f)(x) = \kappa_T\Delta f(x)$$

and the equations take the form

$$e(t,\xi) = \widehat{\theta}_0(\xi) - \kappa|\xi|^2\int_0^t e(s,\xi)\, ds - \kappa_T|\xi|^2\int_0^t e(s,\xi)\, d\gamma(s)$$

$$\overline{\theta}(t,x) = \theta_0(x) + \int_0^t \kappa\Delta\overline{\theta}(s,x)\, ds + \int_0^t \kappa_T\Delta\overline{\theta}(s,x)\, d\gamma(s).$$

The dissipation $\kappa\Delta\overline{\theta}(s,x)$ is enhanced by the term $\kappa_T\Delta\overline{\theta}(s,x)\,d\gamma(s)$, on average. However, compared to the Brownian case $H = 1/2$, where

$$\overline{\theta}(t,x) = \theta_0(x) + \int_0^t (\kappa+\kappa_T)\Delta\overline{\theta}(s,x)\, ds,$$

the case when $H > 1/2$ is slower for short times, because $d\gamma(s) \sim s^{H-\frac{1}{2}}ds$ which is infinitesimal for small s. Positively correlated noise decreases the dissipation power with respect to the incorrelated case, which is constituted by Gaussian white noise.

We go on investigating the variance in closed form.

3.1. *Variance-covariance of the solution.* In order to evaluate the variance of the solution one needs to understand the covariance structure of $\left(\widehat{\theta}_\epsilon(t,\xi)\right)_{\xi\in\mathbb{R}^2}$. Indeed, if $\widehat{\theta}_\epsilon$ is the solution of equation (6) then

$$\theta_\epsilon(t,x) - \mathbb{E}\left[\theta_\epsilon(t,x)\right] = \int e^{2\pi i\xi\cdot x}\left(\widehat{\theta}_\epsilon(t,\xi) - e_\epsilon(t,\xi)\right)d\xi$$

$$\begin{aligned} Var\left(\theta_\epsilon(t,x)\right) &= \mathbb{E}\left[\left(\theta_\epsilon(t,x) - \mathbb{E}\left[\theta_\epsilon(t,x)\right]\right)\overline{\left(\theta_\epsilon(t,x) - \mathbb{E}\left[\theta_\epsilon(t,x)\right]\right)}\right] \\ &= \int\int e^{2\pi i(\xi-\eta)\cdot x} C_\epsilon(t,\xi,\eta)\,d\xi d\eta, \end{aligned}$$

where $C_\epsilon(t,\xi,\eta)$ is the covariance function

$$C_\epsilon(t,\xi,\eta) = \mathbb{E}\left[\left(\left(\widehat{\theta}_\epsilon(t,\xi) - e_\epsilon(t,\xi)\right)\right)\overline{\left(\widehat{\theta}_\epsilon(t,\eta) - e_\epsilon(t,\eta)\right)}\right].$$

We come back to equations (7) and (14). We set

$$\widetilde{\theta}_\epsilon(t,\xi) = \widehat{\theta}_\epsilon(t,\xi) - e_\epsilon(t,\xi)$$

and have

$$\begin{aligned} \widetilde{\theta}_\epsilon(t,\xi) = &-\kappa\left|\xi\right|^2\int_0^t \widetilde{\theta}_\epsilon(s,\xi)\,ds \\ &+ i\sum_{k\in K}(\sigma_k\cdot\xi)\int_0^t \widetilde{\theta}_\epsilon(s,\xi)\,\frac{G^k_{s+\epsilon} - G^k_{(s-\epsilon)_+}}{2\epsilon}ds \\ &+ i\sum_{k\in K}(\sigma_k\cdot\xi)\int_0^t e_\epsilon(s,\xi)\,\frac{G^k_{s+\epsilon} - G^k_{(s-\epsilon)_+}}{2\epsilon}ds \\ &- \sigma^2(\xi)\int_0^t e_\epsilon(s,\xi)\,d\mathcal{V}_\epsilon(s). \end{aligned}$$

Therefore

$$\begin{aligned} &\widetilde{\theta}_\epsilon(t,\xi)\,\overline{\widetilde{\theta}_\epsilon(t,\eta)} \\ &\quad = \int_0^t \widetilde{\theta}_\epsilon(ds,\xi)\,\overline{\widetilde{\theta}_\epsilon(s,\eta)} + \int_0^t \widetilde{\theta}_\epsilon(s,\xi)\,\overline{\widetilde{\theta}_\epsilon(ds,\eta)} \end{aligned}$$

$$\begin{aligned} = &-\kappa\left(|\xi|^2 + |\eta|^2\right)\int_0^t \widetilde{\theta}_\epsilon(s,\xi)\,\overline{\widetilde{\theta}_\epsilon(s,\eta)}ds \\ &+ i\sum_{k\in K}(\sigma_k\cdot(\xi-\eta))\int_0^t \widetilde{\theta}_\epsilon(s,\xi)\,\overline{\widetilde{\theta}_\epsilon(s,\eta)}\frac{G^k_{s+\epsilon} - G^k_{(s-\epsilon)_+}}{2\epsilon}ds \\ &+ i\sum_{k\in K}(\sigma_k\cdot\xi)\int_0^t e_\epsilon(s,\xi)\,\overline{\widetilde{\theta}_\epsilon(s,\eta)}\frac{G^k_{s+\epsilon} - G^k_{(s-\epsilon)_+}}{2\epsilon}ds \\ &- i\sum_{k\in K}(\sigma_k\cdot\eta)\int_0^t \widetilde{\theta}_\epsilon(s,\xi)\,\overline{e_\epsilon(s,\eta)}\frac{G^k_{s+\epsilon} - G^k_{(s-\epsilon)_+}}{2\epsilon}ds \end{aligned}$$

$$
\begin{aligned}
&-\sigma^2(\xi)\int_0^t e_\epsilon(s,\xi)\overline{\widetilde{\theta}_\epsilon(s,\eta)}d\mathcal{V}_\epsilon(s)\\
&-\sigma^2(\eta)\int_0^t \widetilde{\theta}_\epsilon(s,\xi)\overline{e_\epsilon(s,\eta)}d\mathcal{V}_\epsilon(s).
\end{aligned}
$$

So setting

$$R_\epsilon(t,\xi,\eta)=\widetilde{\theta}_\epsilon(t,\xi)\overline{\widetilde{\theta}_\epsilon(t,\eta)},$$

we get

$$
\begin{aligned}
R_\epsilon(t,\xi,\eta)=&-\kappa\left(|\xi|^2+|\eta|^2\right)\int_0^t R_\epsilon(s,\xi,\eta)\,ds\\
&+i\sum_{k\in K}(\sigma_k\cdot(\xi-\eta))\int_0^t R_\epsilon(s,\xi,\eta)\frac{G^k_{s+\epsilon}-G^k_{(s-\epsilon)_+}}{2\epsilon}ds\\
&+i\sum_{k\in K}(\sigma_k\cdot\xi)\int_0^t e_\epsilon(s,\xi)\overline{\widetilde{\theta}_\epsilon(s,\eta)}\frac{G^k_{s+\epsilon}-G^k_{(s-\epsilon)_+}}{2\epsilon}ds\\
&-i\sum_{k\in K}(\sigma_k\cdot\eta)\int_0^t \widetilde{\theta}_\epsilon(s,\xi)\overline{e_\epsilon(s,\eta)}\frac{G^k_{s+\epsilon}-G^k_{(s-\epsilon)_+}}{2\epsilon}ds\\
&+M_t,
\end{aligned}
$$

where M is a mean zero process.

Now we need to express the three terms on the right-hand-side of this identity which involve the noise by means of mean zero processes plus a trace. Let us treat each one of them. Denoting again by M a generic mean zero process, we re-express the first one of the previous terms, using in particular (5), (13) and (9):

$$\int_0^t R_\epsilon(s,\xi,\eta)\frac{G^k_{s+\epsilon}-G^k_{(s-\epsilon)_+}}{2\epsilon}ds=M_t+\int_0^t\left\langle D^{(k)}_{\cdot}R_\epsilon(s,\xi,\eta),\frac{1}{2\epsilon}1_{[(s-\epsilon)_+,s+\epsilon]}\right\rangle_{\mathcal{H}}ds.$$

From (13),

$$
\begin{aligned}
D^{(k)}_r R_\epsilon(t,\xi,\eta)&=D^{(k)}_r\widehat{\theta}_\epsilon(t,\xi)\cdot\overline{\widetilde{\theta}_\epsilon(t,\eta)}+\widetilde{\theta}_\epsilon(t,\xi)\cdot\overline{D^{(k)}_r\widehat{\theta}_\epsilon(t,\eta)}\\
&=\widehat{\theta}_\epsilon(t,\xi)\overline{\widetilde{\theta}_\epsilon(t,\eta)}i(\sigma_k\cdot\xi)\frac{1}{2\epsilon}\int_0^t 1_{[(s-\epsilon)_+,s+\epsilon]}(r)\,ds\\
&\quad-\widetilde{\theta}_\epsilon(t,\xi)\overline{\widehat{\theta}_\epsilon(t,\eta)}i(\sigma_k\cdot\eta)\frac{1}{2\epsilon}\int_0^t 1_{[(s-\epsilon)_+,s+\epsilon]}(r)\,ds\\
&=M_t+R_\epsilon(t,\xi,\eta)\,i(\sigma_k\cdot(\xi-\eta))\frac{1}{2\epsilon}\int_0^t 1_{[(s-\epsilon)_+,s+\epsilon]}(r)\,ds
\end{aligned}
$$

and thus

$$\int_0^t\left\langle D^{(k)}_{\cdot}R_\epsilon(s,\xi,\eta),1_{[(s-\epsilon)_+,s+\epsilon]}\right\rangle_{\mathcal{H}}ds=M_t+i(\sigma_k\cdot(\xi-\eta))\int_0^t R_\epsilon(s,\xi,\eta)\,d\mathcal{V}_\epsilon(s).$$

Concerning the third one of the previous terms we have

$$\begin{aligned}
&\int_0^t \widetilde{\theta}_\epsilon(s,\xi)\overline{e_\epsilon(s,\eta)}\frac{G^k_{s+\epsilon}-G^k_{(s-\epsilon)_+}}{2\epsilon}ds \\
&= M_t + \int_0^t \overline{e_\epsilon(s,\eta)}\left\langle D^{(k)}_{\cdot}\widehat{\theta}_\epsilon(s,\xi), \frac{1}{2\epsilon}1_{[(s-\epsilon)_+,s+\epsilon]}\right\rangle_{\mathcal{H}} ds \\
&= M_t + i(\sigma_k\cdot\xi)\int_0^t \widehat{\theta}_\epsilon(s,\xi)\overline{e_\epsilon(s,\eta)}\frac{1}{(2\epsilon)^2}\int_0^s \left\langle 1_{[(r-\epsilon)_+,r+\epsilon]}, 1_{[(s-\epsilon)_+,s+\epsilon]}\right\rangle_{\mathcal{H}} dr ds \\
&= M_t + i(\sigma_k\cdot\xi)\int_0^t \widehat{\theta}_\epsilon(s,\xi)\overline{e_\epsilon(s,\eta)}d\mathcal{V}_\epsilon(s)
\end{aligned}$$

hence

$$\begin{aligned}
&-i\sum_{k\in K}(\sigma_k\cdot\eta)\int_0^t \widetilde{\theta}_\epsilon(s,\xi)\overline{e_\epsilon(s,\eta)}\frac{G^k_{s+\epsilon}-G^k_{(s-\epsilon)_+}}{2\epsilon}ds \\
&= M_t + \sum_{k\in K}(\sigma_k\cdot\xi)(\sigma_k\cdot\eta)\int_0^t \widehat{\theta}_\epsilon(s,\xi)\overline{e_\epsilon(s,\eta)}d\mathcal{V}_\epsilon(s).
\end{aligned}$$

Similarly, concerning the second one of the terms,

$$\int_0^t e_\epsilon(s,\xi)\overline{\widetilde{\theta}_\epsilon(s,\eta)}\frac{G^k_{s+\epsilon}-G^k_{(s-\epsilon)_+}}{2\epsilon}ds = M_t - i(\sigma_k\cdot\eta)\int_0^t e_\epsilon(s,\xi)\overline{\widehat{\theta}_\epsilon(s,\eta)}d\mathcal{V}_\epsilon(s)$$

and therefore

$$\begin{aligned}
&i\sum_{k\in K}(\sigma_k\cdot\xi)\int_0^t e_\epsilon(s,\xi)\overline{\widetilde{\theta}_\epsilon(s,\eta)}\frac{G^k_{s+\epsilon}-G^k_{(s-\epsilon)_+}}{2\epsilon}ds \\
&= M_t + \sum_{k\in K}(\sigma_k\cdot\xi)(\sigma_k\cdot\eta)\int_0^t e_\epsilon(s,\xi)\overline{\widehat{\theta}_\epsilon(s,\eta)}d\mathcal{V}_\epsilon(s).
\end{aligned}$$

Summarizing the previous identities, we get

$$\begin{aligned}
R_\epsilon(t,\xi,\eta) = &-\kappa\left(|\xi|^2+|\eta|^2\right)\int_0^t R_\epsilon(s,\xi,\eta)\,ds \\
&-\sum_{k\in K}(\sigma_k\cdot(\xi-\eta))^2\int_0^t R_\epsilon(s,\xi,\eta)\,d\mathcal{V}_\epsilon(s) \\
&+\sum_{k\in K}(\sigma_k\cdot\xi)(\sigma_k\cdot\eta)\int_0^t e_\epsilon(s,\xi)\overline{\widehat{\theta}_\epsilon(s,\eta)}d\mathcal{V}_\epsilon(s) \\
&+\sum_{k\in K}(\sigma_k\cdot\xi)(\sigma_k\cdot\eta)\int_0^t \widehat{\theta}_\epsilon(s,\xi)\overline{e_\epsilon(s,\eta)}d\mathcal{V}_\epsilon(s) \\
&+M_t.
\end{aligned}$$

Taking expectation (recall $C_\epsilon(t,\xi,\eta) = \mathbb{E}[R_\epsilon(t,\xi,\eta)]$), we get

$$C_\epsilon(t,\xi,\eta) = -\kappa\left(|\xi|^2+|\eta|^2\right)\int_0^t C_\epsilon(s,\xi,\eta)\,ds$$

$$
\begin{aligned}
&- \sum_{k\in K} (\sigma_k \cdot (\xi - \eta))^2 \int_0^t C_\epsilon (s, \xi, \eta)\, d\mathcal{V}_\epsilon (s) \\
&+ 2 \sum_{k\in K} (\sigma_k \cdot \xi) (\sigma_k \cdot \eta) \int_0^t e_\epsilon (s, \xi) \overline{e_\epsilon (s, \eta)} d\mathcal{V}_\epsilon (s) .
\end{aligned}
$$

Let us introduce the notation

$$
\rho (\xi, \eta) := \sum_{k\in K} (\sigma_k \cdot \xi) (\sigma_k \cdot \eta) .
$$

Taking the limit (it existence is like above for $e(t, \xi)$)

$$
C (t, \xi, \eta) := \lim_{\epsilon\to 0} C_\epsilon (t, \xi, \eta) ,
$$

from Lemma 6, we establish the following.

PROPOSITION 15. *Suppose Assumption A. Then the function $C(t, \xi, \eta)$ satisfies, together with e, the identity*

$$
\begin{aligned}
C (t, \xi, \eta) &= -\kappa \left(|\xi|^2 + |\eta|^2 \right) \int_0^t C (s, \xi, \eta)\, ds \\
&- \sigma^2 (\xi - \eta) \int_0^t C (s, \xi, \eta)\, d\gamma (s) \\
&+ 2\rho (\xi, \eta) \int_0^t e (s, \xi) \overline{e (s, \eta)} d\gamma (s) .
\end{aligned}
$$

Let us go back to the computation of the function

$$
V (t, x) = \int \int e^{2\pi i (\xi - \eta)\cdot x} C (t, \xi, \eta)\, d\xi d\eta,
$$

which is limit of $Var (\theta_\epsilon (t, x))$.

PROPOSITION 16. *Suppose Assumptions A and B. Then, in the sense of distributions,*

$$
V (t, x) = \int_0^t 2\kappa \Delta V (s, x)\, ds + \int_0^t (\mathcal{L} V (s)) (x)\, d\gamma (s) + 2 \sum_{k\in K} \int_0^t \left((\sigma_k \cdot \nabla) \overline{\theta} (s, x) \right)^2 d\gamma (s) .
$$

PROOF. The proof proceeds by applying $\int \int e^{2\pi i (\xi - \eta)\cdot x} \cdots d\xi d\eta$ to each term of the identity of Proposition 15. Its application to the first term on the right-hand-side gives us $\int_0^t 2\Delta V (s, x)\, ds$ because

$$
\begin{aligned}
&\int \int e^{2\pi i (\xi - \eta)\cdot x} |\xi|^2 C (s, \xi, \eta)\, d\xi d\eta \\
&= \int e^{2\pi i \xi\cdot x} |\xi|^2 \left(\int e^{-2\pi i \eta\cdot x} C (s, \xi, \eta)\, d\eta \right) d\xi \\
&= -\Delta \int e^{2\pi i \xi\cdot x} \left(\int e^{-2\pi i \eta\cdot x} C (s, \xi, \eta)\, d\eta \right) d\xi \\
&= -\Delta V (t, x)
\end{aligned}
$$

and a similar computation on the conjugate holds for the other term.

Let us come to the second term on the right-hand-side of the identity of Proposition 15. To make the computation more transparent, let us write $V(t,x)$ in Fourier form as

$$\begin{aligned} V(t,x) &= \int\int e^{2\pi i(\xi-\eta)\cdot x} C(t,\xi,\eta)\, d\xi d\eta \\ &\overset{\xi'=\xi-\eta}{=} \int\left(\int e^{2\pi i\xi'\cdot x} C\left(s,\xi'+\eta,\eta\right) d\xi'\right) d\eta \\ &= \int e^{2\pi i\xi'\cdot x}\left(\int C\left(s,\xi'+\eta,\eta\right) d\eta\right) d\xi' = \int e^{2\pi i\xi\cdot x}\widehat{V}(t,\xi)\, d\xi, \end{aligned}$$

where

$$\widehat{V}(t,\xi) = \int C(s,\xi+\eta,\eta)\, d\eta.$$

Then

$$\begin{aligned} (\widehat{\mathcal{L}V(t)})(\xi) &= \sigma^2(\xi)\,\widehat{V}(t,\xi) \\ &= \sigma^2(\xi)\int C(s,\xi+\eta,\eta)\, d\eta \end{aligned}$$

$$\begin{aligned} &(\mathcal{L}V(s))(x) \\ &= \int e^{2\pi i\xi\cdot x}(\widehat{\mathcal{L}V(s)})(\xi)\, d\xi \\ &= \int e^{2\pi i\xi\cdot x}\sigma^2(\xi)\int C(s,\xi+\eta,\eta)\, d\eta d\xi \\ &\overset{\xi'=\xi+\eta}{=} \int\int e^{2\pi i(\xi'-\eta)\cdot x}\sigma^2\left(\xi'-\eta\right) C\left(s,\xi',\eta\right) d\xi' d\eta. \end{aligned}$$

Thus also the second term is checked.

Finally, let us treat the third term on the right-hand-side of the identity of Proposition 15. Here we simply have

$$\begin{aligned} &\int\int e^{2\pi i(\xi-\eta)\cdot x}(\sigma_k\cdot\xi)(\sigma_k\cdot\eta)\, e(s,\xi)\,\overline{e(s,\eta)}\, d\xi d\eta \\ &= \left(\int e^{2\pi i\xi\cdot x} i(\sigma_k\cdot\xi)\, e(s,\xi)\, d\xi\right)\left(\overline{\int e^{2\pi i\eta\cdot x} i(\sigma_k\cdot\eta)\, e(s,\eta)\, d\eta}\right) \\ &= \left((\sigma_k\cdot\nabla)\,\overline{\theta}(s,x)\right)^2 \end{aligned}$$

which leads to the claimed identity. □

4. The non commutative case. The case when the stochastic transport terms do not have constant-in-space coefficients and do not commute between themselves and with the Laplacian, is admittedly very difficult and still obscure, from the viewpoint of theoretical quantitative results. We may only present two subsections with side remarks on this topic.

The first subsection idealizes a turbulent 2D fluid undergoing inverse cascade by prescribing two families of stochastic transport terms: a small-space-scale component modeling the smallest turbulent scales, maintained, white noise (namely uncorrelated) in time, and a larger-space-scale component modeling the larger structures which appear and disappear by inverse cascade. The latter are constant in space, idealization of their relative size with respect to the smallest ones, and correlated in time. The result we prove is that the smaller scales produce the effect predicted by the Boussinesq hypothesis, while the larger ones are maintained in their form. The system then reduces to the commutative case.

The second subsection is only aimed to explain as clearly as possible the technical difficulty arising in a truly non-commutative case. A commutator appears which spoils the simple link with the mean field equation found in Section 3. Nevertheless, a link up to a remainder exists and could be important in future investigations.

4.1. *Two-scale system and reduction to the commutative case.* Consider a more complete fluid dynamic model than the one introduced in Section 1, further parametrized by a parameter $N \in \mathbb{N}$:

$$\begin{aligned}\partial_t \theta_{\epsilon,N}(t,x) &= \kappa \Delta \theta_{\epsilon,N}(t,x) \\ &\quad + \mathcal{L}_N^0 \theta_{\epsilon,N}(t,x) + \sum_{j \in J_N} \left(v_{j,N}(x) \cdot \nabla\right) \theta_{\epsilon,N}(t,x) \frac{dW_t^j}{dt} \\ &\quad + \sum_{k \in K} \left(\sigma_k(x) \cdot \nabla\right) \theta_{\epsilon,N}(t,x) \frac{d\mathcal{G}_t^{k,\epsilon}}{dt} \\ \theta_{\epsilon,N}|_{t=0} &= \theta_0,\end{aligned}$$

where now $\sigma_k(x)$ are smooth divergence free fields, the finite index set J_N may vary with N, the vector fields $v_{j,N}(x)$ too, as well as the associated covariance function $Q_N^0(x,y)$ defined as

$$Q_N^0(x,y) = \sum_{j \in J_N} v_{j,N}(x) \otimes v_{j,N}(y)$$

covariance operator $\mathbb{Q}_N^0$ on vector fields $v, w \in L^2\left(\mathbb{R}^2\right)$ defined as

$$\left\langle \mathbb{Q}_N^0 v, w \right\rangle_{L^2} = \int\int w(x)^T \cdot Q_N^0(x,y) \cdot v(y)\, dx dy$$

and differential operator $\mathcal{L}_N^0$ defined as

$$\left(\mathcal{L}_N^0 f\right)(x) = \operatorname{div}\left(Q_N^0(x,x) \nabla f(x)\right).$$

The sum

$$\mathcal{L}_N^0 \theta_{\epsilon,N}(t,x) + \sum_{j \in J_N} \left(v_{j,N}(x) \cdot \nabla\right) \theta_{\epsilon,N}(t,x) \frac{dW_t^j}{dt}$$

stands for the Itô formulation (easier to define) of the Stratonovich integral

$$\sum_{j \in J_N} \left(v_{j,N}(x) \cdot \nabla\right) \theta_{\epsilon,N}(t,x) \circ \frac{dW_t^j}{dt}.$$

The main aim of this section is proving that, under suitable assumptions, we may reduce the model to the commutative case. This requires that the Itô integrals go to zero and that the corrector goes to $\kappa_T \Delta \theta_{\epsilon,N}(t,x)$. Since we want to interpret rigorously the equation in mild form, in order to reduce details we assume that the diagonal $Q_N^0(x,x)$ is independent of N and already equal to $\kappa_T Id$:

$$Q_N^0(x,x) = \kappa_T Id$$

for every $N \in \mathbb{N}$ and $x \in \mathbb{R}^2$. Moreover, we assume that $\kappa + \kappa_T > 0$. Let A be the infinitesimal generator of analytic semigroup in $L^2(\mathbb{R}^2)$ (see [37], Chapter 7), defined on $W^{2,2}(\mathbb{R}^2)$ as

$$(Af)(x) = (\kappa + \kappa_T)\Delta f(x).$$

Since we assume independence of $(W^j; j \in J_N)$ from $\left(\mathcal{G}_t^{k,\epsilon}; k \in K\right)$ and the analysis in this section is pathwise with respect to $\left(\mathcal{G}_t^{k,\epsilon}; k \in K\right)$, we replace the above equation by

$$d\theta_{\epsilon,N}(t) = (A\theta_{\epsilon,N}(t) + (v(t)\cdot\nabla)\theta_{\epsilon,N}(t))\,dt + \sum_{j\in J_N}(v_{j,N}\cdot\nabla)\theta_{\epsilon,N}(t)\,dW_t^j$$

$$\theta_{\epsilon,N}|_{t=0} = \theta_0,$$

where $v(t,x)$ is a single path of $\sum_{k\in K}\sigma_k(x)\frac{d\mathcal{G}_t^{k,\epsilon}}{dt}$. This equation, when $\theta_0 \in L^2(\mathbb{R}^2)$, can be solved, in mild form

$$\theta_{\epsilon,N}(t) = e^{tA}\theta_0 + \sum_{j\in J_N}\int_0^t e^{(t-s)A}(v_{j,N}\cdot\nabla)\theta_{\epsilon,N}(s)\,dW_s^j + \int_0^t e^{(t-s)A}(v(s)\cdot\nabla)\theta_{\epsilon,N}(s)\,ds$$

as in the case $v(t) = 0$ as in [23], Chapter 3. Here e^{tA}, $t \geq 0$, denotes the analytic semigroup generated by A on $L^2(\mathbb{R}^2)$. The solution is an adapted process with paths of class

$$\theta_{\epsilon,N} \in C\left([0,T]; L^2(\mathbb{R}^2)\right) \cap L^2\left(0,T; W^{1,2}(\mathbb{R}^2)\right),$$

it satisfies a.s.

$$\sup_{t\in[0,T]}\|\theta_{\epsilon,N}(t)\|^2_{L^2(\mathbb{R}^2)} + \kappa\int_0^T\|\nabla\theta_{\epsilon,N}(s)\|^2_{L^2(\mathbb{R}^2)}\,ds \leq \|\theta_0\|^2_{L^2(\mathbb{R}^2)}.$$

Moreover, it satisfies the maximum principle ([23], Chapter 3; see also [22])

$$\sup_{t\in[0,T]}\|\theta_{\epsilon,N}(t)\|_{L^\infty(\mathbb{R}^2)} \leq \|\theta_0\|_{L^\infty(\mathbb{R}^2)} \tag{15}$$

However, with some easy technical work, we may also interpret the equation in the alternative mild form

$$\theta_{\epsilon,N}(t) = U(t,0)\theta_0 + \sum_{j\in J_N}\int_0^t U(t,s)(v_{j,N}\cdot\nabla)\theta_{\epsilon,N}(s)\,dW_s^j \tag{16}$$

where $U(t,s)$ is the evolution operator defined as follows. For every $s \geq 0$, consider the deterministic equation

$$u(t) = e^{(t-s)A}u_0 + \int_s^t e^{(t-r)A}(v(r)\cdot\nabla)u(r)\,dr$$

for $t \in [s,\infty)$.

For every $u_0 \in L^2(\mathbb{R}^2)$ and $T > s$, let

$$u \in C\left([s,T]; L^2(\mathbb{R}^2)\right) \cap L^2\left(s,T; W^{1,2}(\mathbb{R}^2)\right)$$

be its unique solution. Then we set

$$U(t,s)u_0 = u(t),$$

for $t \in [s,T]$ and extend to all t in an obvious way using the uniqueness. We construct a family of bounded linear operators $\{U(t,s); 0 \le s \le t\}$ on $L^2(\mathbb{R}^2)$. With a little work one can show that $(t,s) \longmapsto U(t,s) u_0$ is continuous, for every $u_0 \in L^2(\mathbb{R}^2)$ and that the mild formulation (16) based on $U(t,s)$ holds true. Finally, it is easy to see that $U^*(t,s)$ is the analogous evolution operator associated to the equation

$$z(t) = e^{(t-s)A} z_0 - \int_s^t e^{(t-r)A} (v(r) \cdot \nabla) z(r) \, dr$$

for $t \in [s, \infty)$.

In particular it satisfies the inequality

$$\sup_{t \in [s,T]} \|U^*(t,s) \phi\|_{L^2(\mathbb{R}^2)}^2 + \kappa \int_s^T \|\nabla U^*(t,s) \phi\|_{L^2(\mathbb{R}^2)}^2 \, ds \le \|\phi\|_{L^2(\mathbb{R}^2)}^2 . \tag{17}$$

Consider also the reduced problem

$$\partial_t \theta_\epsilon(t,x) = (\kappa + \kappa_T) \Delta \theta_\epsilon(t,x) + \sum_{k \in K} (\sigma_k \cdot \nabla) \theta_\epsilon(t,x) \frac{d\mathcal{G}_t^{k,\epsilon}}{dt}$$

$$\theta_\epsilon|_{t=0} = \theta_0.$$

We simply have

$$\theta_\epsilon(t) = U(t,0) \theta_0.$$

THEOREM 17. *We have*

$$\mathbb{E}\left[\langle \theta_{\epsilon,N}(t) - \theta_\epsilon(t), \phi \rangle^2\right] \le T \left\|\mathbb{Q}_N^0\right\|_{L^2 \to L^2} \|\theta_0\|_{L^\infty}^2 \|\phi\|_{L^2}^2$$

for every $\phi \in L^2(\mathbb{R}^2)$ *and* $t \in [0,T]$. *Therefore*

$$\lim_{N \to \infty} \sup_{t \in [0,T]} \mathbb{E}\left[\langle \theta_{\epsilon,N}(t) - \theta_\epsilon(t), \phi \rangle^2\right] = 0$$

if $\lim_{N \to \infty} \left\|\mathbb{Q}_N^0\right\|_{L^2 \to L^2} = 0$.

PROOF. One has

$$\langle \theta_{\epsilon,N}(t) - \theta_\epsilon(t), \phi \rangle = \sum_{j \in J_N} \int_0^t \langle U(t,s) (v_{j,N} \cdot \nabla) \theta_{\epsilon,N}(s), \phi \rangle \, dW_s^j$$

$$\begin{aligned}\mathbb{E}\left[\langle \theta_{\epsilon,N}(t) - \theta_\epsilon(t), \phi \rangle^2\right] &= \sum_{j \in J_N} \mathbb{E} \int_0^t \langle U(t,s) (v_{j,N} \cdot \nabla) \theta_{\epsilon,N}(s), \phi \rangle^2 \, ds \\ &= \sum_{j \in J_N} \mathbb{E} \int_0^t \langle \theta_{\epsilon,N}(s), (v_{j,N} \cdot \nabla) U^*(t,s) \phi \rangle^2 \, ds.\end{aligned}$$

Now, called $g_t(s) := U^*(t,s) \phi$ for shortness,

$$\begin{aligned}&\sum_{j \in J_N} \langle \theta_{\epsilon,N}(s), (v_{j,N} \cdot \nabla) g_t(s) \rangle^2 \\ &= \sum_{j \in J_N} \int \int \theta_{\epsilon,N}(s,x) (v_{j,N}(x) \cdot \nabla) g_t(s,x) \theta_{\epsilon,N}(s,y) (v_{j,N}(y) \cdot \nabla) g_t(s,y) \, dx dy\end{aligned}$$

$$= \sum_{\alpha,\beta=1}^{2} \sum_{j \in J_N} \int\int \theta_{\epsilon,N}(s,x) v_{j,N}^{\alpha}(x) \partial_\alpha g_t(s,x) \theta_{\epsilon,N}(s,y) v_{j,N}^{\beta}(y) \partial_\beta g_t(s,y) \, dxdy$$

$$= \sum_{\alpha,\beta=1}^{2} \int\int \theta_{\epsilon,N}(s,x) \left(\sum_{j \in J_N} v_{j,N}^{\alpha}(x) v_{j,N}^{\beta}(y) \right) \partial_\alpha g_t(s,x) \theta_{\epsilon,N}(s,y) \partial_\beta g_t(s,y) \, dxdy$$

$$= \sum_{i,j=1}^{2} \int\int \theta_{\epsilon,N}(s,x) Q_N^{0,\alpha,\beta}(x,y) \partial_\alpha g_t(s,x) \theta_{\epsilon,N}(s,y) \partial_\beta g_t(s,y) \, dxdy$$

$$= \int\int \theta_{\epsilon,N}(s,x) \nabla g_t(s,x)^T \cdot Q_N^0(x,y) \cdot \nabla g_t(s,y) \theta_{\epsilon,N}(s,y) \, dxdy$$

$$= \left\langle \mathbb{Q}_N^0 \nabla g_t(s) \theta_{\epsilon,N}(s), \nabla g_t(s) \theta_{\epsilon,N}(s) \right\rangle$$
$$\leq \left\| \mathbb{Q}_N^0 \right\|_{L^2 \to L^2} \|\theta_{\epsilon,N}(s)\|_{L^\infty}^2 \|\nabla U^*(t,s) \phi\|_{L^2}^2$$
$$\leq \left\| \mathbb{Q}_N^0 \right\|_{L^2 \to L^2} \|\theta_0\|_{L^\infty}^2 \|\phi\|_{L^2}^2$$

by (15) and (17). We conclude that

$$\mathbb{E}\left[\langle \theta_{\epsilon,N}(t) - \theta_\epsilon(t), \phi \rangle^2 \right] \leq \mathbb{E} \int_0^t \left\| \mathbb{Q}_N^0 \right\|_{L^2 \to L^2} \|\theta_0\|_{L^\infty}^2 \|\phi\|_{L^2}^2 \, ds$$
$$= T \left\| \mathbb{Q}_N^0 \right\|_{L^2 \to L^2} \|\theta_0\|_{L^\infty}^2 \|\phi\|_{L^2}^2$$

for $t \in [0,T]$. □

4.2. *Link with the mean field equation, up to a commutator.* Consider now equation (1) without the assumption that the vector fields σ_k are constant; assume them smooth, bounded and divergence free. Assume $\theta_0 \in L^2(\mathbb{R}^2)$. As outlined in the previous subsection, introducing the operator A as above but with $\kappa_T = 0$ (assuming therefore $\kappa > 0$) and the associated analytic semigroup e^{tA}, $t \geq 0$, one can study pathwise the equation in mild form

$$\theta_\epsilon(t) = e^{tA}\theta_0 + \sum_{k \in K} \int_0^t e^{(t-s)A} (\sigma_k \cdot \nabla) \theta_\epsilon(s) \frac{d\mathcal{G}_s^{k,\epsilon}}{ds} ds \tag{18}$$

and prove that there exists a unique solution of class

$$\theta_\epsilon \in C\left([0,T]; L^2(\mathbb{R}^2)\right) \cap L^2\left(0,T; W^{1,2}(\mathbb{R}^2)\right).$$

Moreover, it is measurable in the random parameter. Moreover, it holds

$$\sup_{t \in [0,T]} \|\theta_\epsilon(t)\|_H^2 + \kappa \int_0^T \|\theta_\epsilon(s)\|_V^2 \, ds \leq \|\theta_0\|_H^2.$$

Solving the equation from a generic initial time $s \geq 0$ as indicated in the previous subsection, the solution defines a family $U_\epsilon(t,s,\omega)$ of bounded linear operators on $L^2(\mathbb{R}^2)$, for $t \geq s \geq 0$, satisfying

$$U_\epsilon(t,s,\omega) U_\epsilon(s,0,\omega) = U_\epsilon(t,0,\omega)$$

$$U_\epsilon(s,s,\omega) = Id$$

$$\theta_\epsilon(t) = U_\epsilon(t,0)\theta_0.$$

Precisely, $U_\epsilon(t,s)\psi$ satisfies

$$U_\epsilon(t,s)\psi = e^{tA}\psi + \sum_{k\in K}\int_s^t e^{(t-r)A}(\sigma_k\cdot\nabla)U_\epsilon(r,s)\psi\frac{d\mathcal{G}_r^{k,\epsilon}}{dr}dr.$$

In this case we are not able to close the equation for the expected value $\mathbb{E}[\theta_\epsilon(t)]$ and our aim therefore is only to estimate its distance from the solution of the mean field equation (3).

In order to see the difficulty, let us consider equation (18) in weak form on a test function $\phi\in C_c^\infty(\mathbb{R}^2)$

$$\langle\theta_\epsilon(t),\phi\rangle = \langle e^{tA}\theta_0,\phi\rangle - \sum_{k\in K}\int_0^t\left\langle\theta_\epsilon(s),(\sigma_k\cdot\nabla)e^{(t-s)A}\phi\right\rangle\frac{d\mathcal{G}_s^{k,\epsilon}}{ds}ds.$$

Then, similarly to the strategy described in Section 3, by means of formula (5) we rewrite the stochastic integral a a Skorohod integral (mean zero) plus a trace

$$\begin{aligned}\langle\theta_\epsilon(t),\phi\rangle &= \langle e^{tA}\theta_0,\phi\rangle + M_t\\ &-\sum_{k\in K}\int_0^t\left\langle\left\langle D_\cdot^{(k)}\theta_\epsilon(s),\frac{1}{2\epsilon}1_{[(s-\epsilon)_+,s+\epsilon]}\right\rangle_{\mathcal{H}},(\sigma_k\cdot\nabla)e^{(t-s)A}\phi\right\rangle ds,\end{aligned}$$

where M_t has zero mean. Therefore

$$\begin{aligned}\langle\mathbb{E}[\theta_\epsilon(t)],\phi\rangle &= \langle e^{tA}\theta_0,\phi\rangle\\ &-\sum_{k\in K}\int_0^t\left\langle\left\langle\mathbb{E}\left[D_\cdot^{(k)}\theta_\epsilon(s)\right],\frac{1}{2\epsilon}1_{[(s-\epsilon)_+,s+\epsilon]}\right\rangle_{\mathcal{H}},(\sigma_k\cdot\nabla)e^{(t-s)A}\phi\right\rangle ds.\end{aligned}$$

In the commutative case $D_\cdot^{(k)}\theta_\epsilon(s)$ can be expressed by means of $\theta_\epsilon(s)$ and we find a closed equation for $\mathbb{E}[\theta_\epsilon(t)]$. Indeed, by Lemma 9, we have

$$\begin{aligned}D_r^{(k)}\theta_\epsilon(t) &= \chi_\epsilon(t,r)(\sigma_k\cdot\nabla)\theta_\epsilon(t)\\ \chi_\epsilon(t,r) &= \frac{1}{2\epsilon}\int_0^t 1_{[(s-\epsilon)_+,s+\epsilon]}(r)\,ds.\end{aligned}$$

Now, without commutation, we have only the following result.

LEMMA 18.

$$D_r^{(k)}\theta_\epsilon(t) = \chi_\epsilon(t,r)U_\epsilon(t,r)(\sigma_k\cdot\nabla)\theta_\epsilon(r).$$

PROOF. Indeed, from (18) we get

$$\begin{aligned}D_r^{(k)}\theta_\epsilon(t) &= \sum_{k'\in K}\int_r^t e^{(t-s)A}(\sigma_{k'}\cdot\nabla)D_r^{(k)}\theta_\epsilon(s)\frac{d\mathcal{G}_s^{k',\epsilon}}{ds}ds\\ &+\int_0^t e^{(t-s)A}(\sigma_k\cdot\nabla)\theta_\epsilon(s)\frac{dD_r^{(k)}\mathcal{G}_s^{k,\epsilon}}{ds}ds.\end{aligned}$$

Then we use (10) to express $D_r^{(k)}\mathcal{G}_s^{k,\epsilon}$ and get

$$\begin{aligned}D_r^{(k)}\theta_\epsilon(t) &= \sum_{k'\in K}\int_r^t e^{(t-s)A}(\sigma_{k'}\cdot\nabla)D_r^{(k)}\theta_\epsilon(s)\frac{d\mathcal{G}_s^{k',\epsilon}}{ds}ds\\ &+e^{(t-r)A}(\sigma_k\cdot\nabla)\theta_\epsilon(r)\chi_\epsilon(t,r)\end{aligned}$$

which leads to the result by uniqueness for the equation defining $U_\epsilon(t,r)$. □

The problem is that we cannot commute $U_\epsilon(t,r)(\sigma_k\cdot\nabla)$ with $(\sigma_k\cdot\nabla)U_\epsilon(t,r)$, otherwise we would have

$$\begin{aligned} D_r^{(k)}\theta_\epsilon(t) &= \chi_\epsilon(t,r)(\sigma_k\cdot\nabla)U_\epsilon(t,r)\theta_\epsilon(r) \\ &= \chi_\epsilon(t,r)(\sigma_k\cdot\nabla)\theta_\epsilon(t) \end{aligned}$$

(due to $\theta_\epsilon(r)=U_\epsilon(r,0)\theta_0$ and $U_\epsilon(t,r)U_\epsilon(r,0)=U_\epsilon(t,0)$) like in the commuting case. Summarizing, until now we have found:

LEMMA 19.

$$\langle\mathbb{E}[\theta_\epsilon(t)],\phi\rangle=\left\langle e^{tA}\theta_0,\phi\right\rangle$$

$$-\sum_{k\in K}\int_0^t\left\langle\left\langle\chi_\epsilon(s,\cdot)\,\mathbb{E}[U_\epsilon(s,\cdot)(\sigma_k\cdot\nabla)\theta_\epsilon(\cdot)],\frac{1}{2\epsilon}1_{[(s-\epsilon)_+,s+\epsilon]}\right\rangle_{\mathcal{H}},(\sigma_k\cdot\nabla)e^{(t-s)A}\phi\right\rangle ds.$$

Adding and subtracting the term with $(\sigma_k\cdot\nabla)U_\epsilon(t,r)$ in place of $U_\epsilon(t,r)(\sigma_k\cdot\nabla)$ we have

$$\langle\mathbb{E}[\theta_\epsilon(t)],\phi\rangle=\left\langle e^{tA}\theta_0,\phi\right\rangle$$

$$\begin{aligned} &-\sum_{k\in K}\int_0^t\left\langle\left\langle\chi_\epsilon(s,\cdot),\frac{1}{2\epsilon}1_{[(s-\epsilon)_+,s+\epsilon]}\right\rangle_{\mathcal{H}}(\sigma_k\cdot\nabla)\mathbb{E}[\theta_\epsilon(s)],(\sigma_k\cdot\nabla)e^{(t-s)A}\phi\right\rangle ds \\ &+\sum_{k\in K}\int_0^t\left\langle\left\langle\chi_\epsilon(s,\cdot)R_{\epsilon,k}(s,\cdot),\frac{1}{2\epsilon}1_{[(s-\epsilon)_+,s+\epsilon]}\right\rangle_{\mathcal{H}},(\sigma_k\cdot\nabla)e^{(t-s)A}\phi\right\rangle ds \end{aligned}$$

where for shortness of notations we have set

$$R_{\epsilon,k}(s,r)=\mathbb{E}[(U_\epsilon(s,r)(\sigma_k\cdot\nabla)-(\sigma_k\cdot\nabla)U_\epsilon(s,r))\theta_\epsilon(r)]. \tag{19}$$

With the notations of Section 3, we have

$$\langle\mathbb{E}[\theta_\epsilon(t)],\phi\rangle=\left\langle e^{tA}\theta_0+\int_0^t e^{(t-s)A}\mathcal{L}\mathbb{E}[\theta_\epsilon(s)]\,d\mathcal{V}_\epsilon(s),\phi\right\rangle$$

$$+\sum_{k\in K}\int_0^t\left\langle\left\langle\chi_\epsilon(s,\cdot)R_{\epsilon,k}(s,\cdot),\frac{1}{2\epsilon}1_{[(s-\epsilon)_+,s+\epsilon]}\right\rangle_{\mathcal{H}},(\sigma_k\cdot\nabla)e^{(t-s)A}\phi\right\rangle ds$$

or

$$\begin{aligned} \mathbb{E}[\theta_\epsilon(t)] &= e^{tA}\theta_0+\int_0^t e^{(t-s)A}\mathcal{L}\mathbb{E}[\theta_\epsilon(s)]\,d\mathcal{V}_\epsilon(s) \\ &\quad -\sum_{k\in K}\int_0^t e^{(t-s)A}(\sigma_k\cdot\nabla)\left\langle\chi_\epsilon(s,\cdot)R_\epsilon(s,\cdot),\frac{1}{2\epsilon}1_{[(s-\epsilon)_+,s+\epsilon]}\right\rangle_{\mathcal{H}}ds. \end{aligned}$$

Notice that the mean field equation is

$$\overline{\theta}(t)=e^{tA}\theta_0+\int_0^t e^{(t-s)A}\mathcal{L}\overline{\theta}(s)\,d\gamma(s).$$

The closedness of $\mathbb{E}[\theta_\epsilon(t)]$ to $\overline{\theta}(t)$ depends on the smallness of the average commutator $R_{\epsilon,k}(s,r)$. Estimates on $R_{\epsilon,k}(s,r)$ seem possible but those we have found until now do not deserve to be reported, so we postpone this subject to future research.

Acknowledgments. This research was influnced by discussions with several experts and in particular, for the first author, by James Michael Leahy and Umberto Pappalettera.

Funding. The research of the first author is funded by the European Union (ERC, NoisyFluid, No. 101053472). Views and opinions expressed are however those of the authors only and do not necessarily reflect those of the European Union or the European Research Council. Neither the European Union nor the granting authority can be held responsible for them. The research of the second named author was partially supported by the ANR-22-CE40-0015-01 (SDAIM).

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