Verification theorem related to a zero sum stochastic differential game via Fukushima-Dirichlet decomposition

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Abstract

We establish a verification theorem, inspired by those existing in stochastic control, to demonstrate how a pair of progressively measurable controls can form a Nash equilibrium in a stochastic zero-sum differential game. Specifically, we suppose that a pathwise-type Isaacs condition is satisfied together with the existence of what is termed a quasi-strong solution to the Bellman-Isaacs (BI) equations. In that case we are able to show that the value of the game is achieved and corresponds exactly to the unique solution of the BI equations. Those have also been applied for improving a well-known verification theorem in stochastic control theory. In so doing, we have implemented new techniques of stochastic calculus via regularizations, developing specific chain rules.

Key words and phrases: Stochastic differential games; verification theorem; stochastic control; weak Dirichlet processes.

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1 Introduction and problem formulation

The primary aim of this paper is to establish a verification theorem for a zero-sum stochastic differential game (SDG). Specifically, we demonstrate that, under certain conditions, a pair of controls forms a Nash equilibrium for the game, i.e. it constitutes a saddle point for payoff functional. Additionally we show that the game possesses a value as defined in Definition 1.1, which we provide. The game analyzed is a control versus control scenario, with the controls being progressively measurable processes. The game is defined as below.

We will deal with fixed horizon problem so that we fix $T \in [0, \infty[$, a finite dimensional Hilbert space, say $\mathbb{R}^d$ that will be the state space, a finite dimensional Hilbert space, say $\mathbb{R}^m$ (the noise space), two compact sets $U_1, U_2 \subseteq \mathbb{R}^k$ (the control spaces). We consider an initial time and state $(t, x) \in [0, T] \times \mathbb{R}^d$. Let us fix a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_s)_{s\in[t,T]}, \mathbb{P})$ satisfying the usual conditions. W is an $(\mathcal{F}_s)_{s\in[t,T]}$ d-dimensional Brownian motion.

The state process equation is

$$\begin{cases} dy(s) = f(s, y(s), z_1(s), z_2(s))ds + \sigma(s, y(s))dW_s \\ y(t) = x, \end{cases} \tag{1.1}$$

where f splits into

$$f(s, y(s), z_1(s), z_2(s)) = b(s, y(s)) + f_1(s, y(s), z_1(s), z_2(s)),$$

and the coefficients are defined as

$$\begin{aligned} &b : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d, \\ &f_1 : [0, T] \times \mathbb{R}^d \times U_1 \times U_2 \to \mathbb{R}^d, \\ &\sigma : [0, T] \times \mathbb{R}^d \to L(\mathbb{R}^m, \mathbb{R}^d). \end{aligned} \tag{1.2}$$

If E and F are finite dimensional spaces, $L(E, F)$ is the space of bounded linear operators from E to F. $\mathcal{Z}_i(t)$ is the set of *admissible control processes* $[t, T]\times\Omega \to U_i$, that is $(\mathcal{F}_s)_{s\in[t,T]}$-progressively measurable process taking values in U_i, $i \in \{1, 2\}$. The processes $\mathcal{Z}_i(t), i = \{1, 2\}$ are the controls employed by Player 1 and Player 2. The payoff function is defined by

$$J(t, x; z_1, z_2) = \mathbb{E}\left\{\int_t^T l(s, y(s; t, x, z_1, z_2); z_1(s), z_2(s))ds + g(y(T, t, x; z_1, z_2))\right\}. \tag{1.3}$$

More precise assumptions on f, σ, l, z_1, z_2 and g will be given in Section 3.1.

Hypothesis 3.4 provides sufficient conditions for the integral $\int_t^T l(s, y(s; t, x, z_1, z_2); z_1(s), z_2(s))ds$ to be always well-defined but, in general, its expectation may not exist. Therefore, we introduce two auxiliary payoff functions

$$J^\pm(t, x; z_1, z_2) = \begin{cases} J(t, x; z_1, z_2) & \text{if well-defined,} \\ \pm\infty & \text{otherwise.} \end{cases} \tag{1.4}$$

Player 1 has the objective to maximize the payoff (i.e. the amount of money she will receive from Player 2) and, on the opposite, Player 2 aims at minimizing it. There are two notions of value of the game, the upper value and the lower value, related to the Player 1 (resp. Player 2) who maximizes (resp. minimizes) on controls for which the integral and expectation in (1.3) exist.

Definition 1.1.

- The **upper value** V^+ and **lower value** V^- of the game (SDG) (1.1) and (1.3) with initial data (x,t) are given by

$$V^+(t,x) = \inf_{z_2 \in \mathcal{Z}_2(t)} \sup_{z_1 \in \mathcal{Z}_1(t)} J^+(t,x;z_1,z_2)$$
$$V^-(t,x) = \sup_{z_1 \in \mathcal{Z}_1(t)} \inf_{z_2 \in \mathcal{Z}_2(t)} J^-(t,x;z_1,z_2),$$

- If $V^+(t,x) = V^-(t,x)$, then we say that the SDG (1.1) and (1.3) has a *value* and we call $V(t,x) = V^+(t,x) = V^-(t,x)$ the *value of the game.*

Remark 1.2. Since $\sup\inf \leq \inf\sup$ and $J^-(t,x;z_1,z_2) \leq J^+(t,x;z_1,z_2)$, we observe that, in general $V^-(t,x) \leq V^+(t,x)$.

The motivation behind the upper and lower value functions stems from the inherent ambiguity in defining the value of the game. This ambiguity arises when the infimum (or supremum) is calculated prior to the supremum (or infimum) of the payoff function, denoted as (1.3). Such a process typically yields two distinct outcomes, designated as V^- and V^+, respectively. The concepts of upper and lower values were initially introduced in deterministic frameworks, as detailed in the works [18], [19], [14], and [38].

The PDE equations playing a similar role in literature to the Hamilton Jacobi Bellman equation (HJB) in case of control theory are the upper and lower Bellmann-Isaacs equation (BI) which are defined below in (1.6) and (1.7). These equations are associated with the stochastic differential game (1.1) and (1.3) and they were first formally derived in [26]. The current value Hamiltonian $H_{CV} : [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \times U_1 \times U_2 \to \mathbb{R}$ is defined as

$$H_{CV}(s,x,p,u_1,u_2) = \langle b(s,x), p\rangle + \langle f_1(s,x,u_1,u_2), p\rangle + l(s,x,u_1,u_2),$$

but, since the first term does not depend on the controls, we define

$$H^0_{CV}(s,x,p,u_1,u_2) = \langle f_1(s,x,u_1,u_2), p\rangle + l(s,x,u_1,u_2). \tag{1.5}$$

Defining formally the operator $\mathcal{L}_0$ as

$$\mathcal{L}_0 v(s,x) = \partial_s v(s,x) + \langle b(s,x), \partial_x v(s,x)\rangle + \frac{1}{2} Tr[\sigma^\top(s,x)\partial_{xx} v(s,x)\sigma(s,x)],$$

we can write the Bellmann-Isaacs equations associated with the problem (1.1) and (1.3) as

$$\begin{cases} \mathcal{L}_0 v(s,x) + H^{0,+}(s,x,\partial_x v(s,x)) = 0, \\ v(T,x) = g(x), \end{cases} \tag{1.6}$$

$$\begin{cases} \mathcal{L}_0 v(s,x) + H^{0,-}(s,x,\partial_x v(s,x)) = 0, \\ v(T,x) = g(x), \end{cases} \tag{1.7}$$

where $H^{0,-}, H^{0,+} : [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ are the so called *Hamiltonians* defined by

$$H^{0,-}(s,x,p) = \sup_{u_1 \in U_1} \inf_{u_2 \in U_2} H^0_{CV}(s,x,p,u_1,u_2), \tag{1.8}$$

$$H^{0,+}(s,x,p) = \inf_{u_2 \in U_2} \sup_{u_1 \in U_1} H^0_{CV}(s,x,p,u_1,u_2). \tag{1.9}$$

Equation (1.6) (resp. (1.7)) is usually called the *upper Bellmann-Isaacs (BI) equation* (resp. the *lower Bellmann-Isaacs (BI) equation*).

Definition 1.3. The SDG (1.1) and (1.3) is said to fulfill the Isaacs condition if $H^{0,+}(s,x,p) = H^{0,-}(s,x,p)$ for all $s \in [0,T]$, $x \in \mathbb{R}^d$, $p \in \mathbb{R}^d$.

To the best of our knowledge, all results proving that the game has a value (among which [20], [17], [6], [5], [7], [4], [12], [10], [24], [9], [25], [32]) assume, though formulated in different ways, the so called Isaacs condition. The usual formulation is the one of Definition 1.3.

If Isaacs condition holds, the couple of equations (1.6) and (1.7) collapse into only one equation. In fact our results require instead another type of assumption that we will call *Pathwise-Isaacs condition* stated in Theorem 3.13, see Remark 3.14. That Pathwise-Isaacs condition generalizes a (non-pathwise) analogous condition of [24], see Proposition 3.18. There are three main approaches to zero-sum stochastic differential games.

In the first one, a game is formulated in a control against strategy setting, in the sense that player acting first chooses a control and the other responds with a strategy. The player choosing a strategy has information also on the control used by his opponent. In that context, the definition of lower (resp. upper) value is similar to the one in Definition 1.1, where the supremum (resp. infimum) over controls is replaced by a supremum (resp. infimum) over strategies.

This formulation was introduced in a deterministic framework in [13]. The earliest work, which establishes the existence of a value in a stochastic differential game using this formulation appears to be [20]. Subsequently, the distinction between controls and strategies (appearing in [20]) is mitigated in [17]. In [20] and [17], it is proved that the upper (resp. lower) value function (see Definition 1.1) is a viscosity solutions to the upper (resp. lower) Bellman Isaacs equation, see (1.6) and (1.7). In particular, if the Isaacs condition hold, see Definition 1.3, then the game has a value, in the sense that lower and upper value coincide, whenever those equations are well-posed. In [6], the authors generalize the results of [20] expressing the cost functional through the solutions of a backward SDE (BSDE) in the sense of Pardoux and Peng [31] and thus simplifying the proof of [20]. They manage to prove a dynamic programming principle for the upper and lower value functions of the game in a straightforward way and without making use of approximations, thus simplifying the approach proposed in [20].

The second stream of literature also deals with stochastic differential games in the setting "strategy against control", but the formulation is weak, non-Markovian and the set of admissible controls and strategies are of feedback type. In [8], [12], [10] and in [11] necessary and sufficient conditions are provided for the existence of a value. They use a principle of optimality for non-Markovian controlled processes proved in [8], Theorem 4.1. In this context a verification theorem appears in [8], Theorem 7.2.

The third (more recent) approach makes use, as in the present paper, of a setting "control against control", is employed in [24], [9], [25], [32]. The authors formulate and prove a verification theorem, showing that a specific couple of feedback controls $(u^\star, v^\star)$, verifying some inequality of saddle point type, constitute a Nash equilibrium for SDG. Later we will denominate the aforementioned inequality, Hamadène-Lepeltier-Isaacs condition. For that task they use a comparison theorem for BSDEs.

We repeat that in the first and second approach, the games are asymmetrically formulated, namely, the player who plays last has information also on the control used by his opponent. In the third approach both players use feedback controls (control against control) and they play simultaneously: nevertheless, the first and third formulations are reconciled in [33].

In this paper, we prove that, if there exists a (what we call) quasi-strong solution (see Definition 3.7), of the upper/lower Bellmann-Isaacs equations, then this solution is exactly the upper/lower value of the SDG, and we also prove a verification theorem based on the Bellmann-

Isaacs PDEs (1.6) and (1.7). In our setting, controls are stochastic processes. We are not aware of any paper proving a verification theorem using a PDE argument with a control against control setting, where controls are not necessarily of feedback type and the players reveal their controls simultaneously. Nevertheless, verification theorems, in the different context of impulse games, are provided by [1] and references therein.

Concerning the hypotheses in our paper, we are not aware of any methodology dealing with strong formulation of the differential game problem which do not require time continuity of the coefficients. In our case, the Hamiltonian (1.5), all the coefficients of the state equation (1.1), the running cost and the BI equations (1.6) and (1.7) are only supposed to be continuous with respect to the space variable and neither with respect to the time nor with respect to the control variables. In particular we allow changes of regimes in the dynamics of the state equation.

Finally, we do not assume a priori the existence of the payoff functional J, defined in (1.3), for our class of admissible controls. We will prove that, for the optimal couple $(z_1^\star, z_2^\star)$ verifying Property $\pi(t, x; z_1^\star, z_2^\star)$ related to our Pathwise-Isaacs condition, J is automatically well-defined, i.e. the integrand is quasi-integrable.

This paper includes also a slight generalization of a verification theorem in a stochastic control problem making use of a quasi-strong solution of an Hamilton-Jacobi-Bellman (HJB) equation, as improvement of Theorem 4.9 of [22], which is realized in Section 4.

Our control problem inherits the assumptions of the game theory setting, and in particular the Hamiltonian (4.5), all the coefficients of the state equation (4.1), of the payoff functional (4.4), and of the HJB equation (4.6) are are again only supposed to be continuous with respect to the space variable and not with respect to the time and control variables, whereas in [22] they were required to be continuous with respect to all the entries. Also, in Theorem 4.9 and Lemma 4.10 of [22] one supposed the lower integrability condition in the running cost l, i.e.

$$\mathbb{E}\left(\int_t^T l^-(s, y(s, t, x, z); z(s)ds)\right) > -\infty,$$

where $l^- \doteqdot (l \wedge 0)$. This is not anymore required, see Hypothesis 3.4. Finally the terminal cost g is not required any more to be differentiable. We suppose the existence of quasi-strong solutions of the HJB equations (4.6). That notion is weaker than the corresponding concept of strong solution used in [22] and [23].

Our applications to game theory and stochastic control were made possible via some generalization of some stochastic calculus via regularization tools stated in [23] and they have also an independent interest. In particular this is motivated for applying a generalized Itô formula to non classical solutions, in our case quasi-strong solution of the Bellman Isaacs equations. respectively Hamilton-Jacobi-Bellman. For instance the Itô formula proved in Proposition 2.7 is an extension to $C^{0,2}_{ac}([t, T] \times \mathbb{R}^d)$ functions (see Definition 2.1), of the classical one based on the space $C^{1,2}$. With this generalization it is possible to get the Representation Theorem 2.24, which extends Theorem 4.5 in [23].

Our results are organized as follows.

In Section 2.1 we fix some notations and give some stochastic calculus preliminaries. Then in Section 2.2 we recall the definition of a weak Dirichlet process, prove the Itô formula and state the Fukushima-Dirichlet decomposition. We introduce in Section 2.3 the concept of a quasi-strong solution and prove that a mild solution to the BI equation is a quasi-strong solution. With these notions it is possible to state in Section 2.4 the improved (with respect to the one presented in [22]) version of the Representation Theorem 2.24 and its Corollary 2.26. In the whole Section 3 we switch to the zero-sum stochastic game theory setting. In Section 3.1 we state the main hypotheses for the state equation (1.1) and the integrand of the payoff (1.3) to be well-defined. Section 3.2 proves the so called Fundamental Lemma (see Lemma 3.10) that is a consequence of the Fukushima-Dirichlet decomposition. It is used in Section 3.3, which is the core of the paper and where it is proved that a Verification theorem for a zero-sum stochastic differential game has a value. The same theorem shows that the game has a value. In Section 4, from the game theory it is recovered the control theory scenario improving some of the results in [22].

2 Preliminaries and Stochastic calculus

First, we recall some basic definitions and fix some notations, then we recall the definition of the Fukushima-Dirichlet decomposition that plays a key role in the proof of Theorem 3.13.

2.1 Preliminaries

In this section, $0 \leq t < T < \infty$ will be fixed. The definition and conventions of this section will be in force for the whole paper.

By convention a continuous function $\phi(s)$, $s \in [t, T]$, is extended to the whole line setting
$$\phi(s) = \begin{cases} \phi(t) & s \leq t, \\ \phi(T) & s \geq T. \end{cases}$$

If E is a (in general finite-dimensional) Fréchet space, i.e. a complete metric space, $C^0(E)$ denotes the space of all continuous functions $f : E \to \mathbb{R}$. It is again a Fréchet space equipped with the topology of the uniform convergence of f on each compact. If (E, d) is a metric space and $f : E \to \mathbb{R}$ is a uniformly continuous function, we denote by $\gamma(f, \cdot)$ the modulus of continuity of f.

Le k be a non-negative integer. $C^k(\mathbb{R}^d)$ is the space of the functions such that all the derivatives up to order k exist and are continuous. It is again a Fréchet space equipped with the topology of the uniform convergence of f and all its derivatives on each compact. Let I be a compact real interval. $C^{1,2}(I \times \mathbb{R}^d)$ (respectively $C^{0,1}(I \times \mathbb{R}^d)$), is the space of continuous functions $f : I \times \mathbb{R}^d \to \mathbb{R}$, such that $\partial_s f, \partial_x f, \partial^2_{xx} f$ (respectively $\partial_x f$) are well-defined and continuous. This space is also a Fréchet space. In general $\mathbb{R}^d$-elements will be considered as column vector, with the exception of $\partial_x f$ which will by default a row vector.

Definition 2.1. $C^{0,2}_{ac}(I \times \mathbb{R}^d)$ will be the linear space of continuous functions $f : I \times \mathbb{R}^d \to \mathbb{R}$ such that the following holds.

1. $f(s, \cdot) \in C^2(\mathbb{R}^d)$ for all $s \in I$.

2. For any $x \in \mathbb{R}^d$ the function $s \mapsto f(s, x)$ is absolutely continuous and $\partial_s f(s, \cdot)$ is continuous for almost all $s \in I$.

3. For any compact $K \subset \mathbb{R}^d$, $\sup_{x \in K} |\partial_s f(\cdot, x)| \in L^1(I)$.

4. Let g be any second order space derivative of f.

   (a) g is continuous with respect to the space variable x varying on each compact K, uniformly with respect to $s \in I$;

   (b) *for every $x \in \mathbb{R}^d$, $s \mapsto g(s, x)$ is a.e. continuous.*

Suppose that f fulfills all previous items except 4.(b) which is replaced by

4.(b) Bis: *for every* $x \in \mathbb{R}^d$, $s \mapsto g(s,x)$ *has at most countable discontinuities.*

In this case f will be said to belong to $C^{0,2}_{ac,count}(I \times \mathbb{R}^d)$.

In the sequel $(\Omega, \mathcal{F}, (\mathcal{F}_s)_{s\geq t}, \mathbb{P})$ will be a given stochastic basis satisfying the usual conditions. $(W_s)_{s\geq t}$ will denote a classical $(\mathcal{F}_s)_{s\geq t}$-Brownian motion with values in $\mathbb{R}^d$. A sequence of processes (X^n_s) will be said to converge u.c.p. if the convergence holds in probability uniformly on compact intervals.

The space $C_{\mathcal{F}}(I \times \Omega; \mathbb{R})$ of all continuous $\mathbb{R}^d$-valued $(\mathcal{F}_s)_{s\in[t,T]}$-progressively measurable processes is a Fréchet space, if equipped with the metric of the u.c.p. convergence.

2.2 Weak Dirichlet processes and Fukushima Dirichlet decomposition

Definition 2.2. Let $(X_s)_{s\in[t,T]} = (X^1_s, \cdots, X^d_s)_{s\in[t,T]}$ be a vector of continuous processes and $(Y_s)_{s\in[t,T]} = (Y^1_s, \cdots, Y^d_s)_{s\in[t,T]}$ be vector process with integrable paths. We define

$$I^-(\varepsilon, Y, dX)(s) = \int_t^s Y_r \cdot \frac{X_{r+\varepsilon} - X_r}{\varepsilon} dr$$

where $\cdot$ denotes the inner product in $\mathbb{R}^d$. If $d = 1$ then, of course, we omit the $\cdot$. We also say that the **(vector) forward integral** of X with respect to Y exists if

1. $\lim_{\varepsilon\to 0} I^-(\varepsilon, Y, dX)(s)$ exists in probability for any $s \in [t,T]$,
2. the previous limit admits a continuous version.

That integral limit process, if exist, is denoted by $\int_t^s Y_r d^- X_r$, $s \in [t,T]$.

Definition 2.3. Let $(X_s)_{s\in[t,T]} = (X^1_s, \cdots, X^d_s)_{s\in[t,T]}$ and $(Y_s)_{s\in[t,T]} = (Y^1_s, \cdots, Y^d_s)_{s\in[t,T]}$ be two vector of continuous processes. We define

$$[X,Y]^\varepsilon(s) = \frac{1}{\varepsilon} \int_t^s (X_{r+\varepsilon} - X_r)(Y_{r+\varepsilon} - Y_r) dr, \tag{2.1}$$

and we say that the **(matrix) covariation or bracket** of X and Y exists if

1. $\lim_{\varepsilon\to 0} [X,Y]^\varepsilon(s)$ exists in probability for any $s \in [t,T]$,
2. the previous limit admits a continuous version.

That matrix-valued limit process, if it exists, is denoted by $[X,Y]_s$.

If X is a real process, $[X,X]$ will be also called **quadratic variation** of X.

The forward integral, the covariations and the finite quadratic variation processes were introduced in [34] and the first steps were performed in [35], see also the recent monograph [37].

Remark 2.4. . If X is a continuous $(\mathcal{F}_s)_{s\geq t}$-semimartingale and Y is an $(\mathcal{F}_s)_{s\geq t}$-progressively measurable cáglád process (resp an $(\mathcal{F}_s)_{s\geq t}$-semimartingale) then $\int_s^\cdot Y d^- S$ (resp. $[Y,S]$) coincides with $\int_s^\cdot Y dS$ (resp. the usual covariation). In particular if S is bounded variation process then $\int_s^\cdot Y d^- S$ is the usual Lebesgue integral $\int_s^\cdot Y dS$. See Proposition 2.1 of [35].

We define below a natural extension of the notion of Dirichlet processes. Dirichlet processes were defined by Föllmer, see also [15], [21], [23]. We remind that an $(\mathcal{F}_s)_{s\geq t}$-progressively measurable process X is called Dirichlet if it is the sum of $(\mathcal{F}_s)_{s\geq t}$-local martingale M and a zero quadratic variation process A, i.e. such that $[A, A] = 0$.

Definition 2.5. 1. An $(\mathcal{F}_s)_{s\in[t,T]}$-continuous progressively measurable process real-valued process X is called $(\mathcal{F}_s)_{s\in[t,T]}$-**weak Dirichlet process** if

$$X = M + A, \tag{2.2}$$

where

(a) $(M_s)_{s\in[t,T]}$ is a continuous $(\mathcal{F}_s)_{s\in[t,T]}$-local martingale,

(b) $A_0 = 0$ and A is an $(\mathcal{F}_s)_{s\in[t,T]}$-**martingale orthogonal process**, that is $[N, A] = 0$ for every $(\mathcal{F}_s)_{s\in[t,T]}$-continuous local martingale N.

2. A continuous $\mathbb{R}^d$ valued process $X = (X^1, \cdots, X^d)$ is said to be a $(\mathcal{F}_s)_{s\in[t,T]}$-weak Dirichlet (vector) (resp. $(\mathcal{F}_s)_{s\in[t,T]}$-**martingale orthogonal** process), if each component is $(\mathcal{F}_s)_{s\in[t,T]}$-weak Dirichlet (resp. martingale orthogonal process).

(2.2) was also called **Fukushima-Dirichlet** decomposition, see [23].

Remark 2.6. 1. The decomposition (2.2) is unique.

2. In the sequel, when self-explanatory, we will omit the filtration $(\mathcal{F}_s)_{s\in[t,T]}$.

3. A zero quadratic variation process is a martingale orthogonal process.

A proof can be found in [36] and usual properties of weak Dirichlet processes are given in [15], [16], [23], and in Chapter 15 of [37].

For the proof of the Propositions 2.10 and 2.21 we need a generalization of the Itô formula that we state here.

Proposition 2.7. Let $f \in C^{0,2}_{ac}([t, T] \times \mathbb{R}^d)$ and $(X_s)_{s\in[t,T]}$ be an $\mathbb{R}^d$-valued process that $[X, X]$ exists.

Suppose that one of the two conditions below is fulfilled.

1. $f \in C^{0,2}_{ac,count}([t, T] \times \mathbb{R}^d)$.

2. The mutual covariations $[X^i, X^j]$ are absolutely continuous, where $[X, X] = ([X^i, X^j])$.

Then the following Itô formula holds.

$$\begin{aligned} f(s, X_s) = f(t, X_t) &+ \int_t^s \partial_r f(r, X_r)dr + \int_t^s \partial_x f(r, X_r)d^- X_r \\ &+ \frac{1}{2}\sum_{i,j=1}^d \int_t^s \partial^2_{x_i x_j} f(r, X_r)d[X^i, X^j]_r. \end{aligned} \tag{2.3}$$

Remark 2.8. For shortness we denote the last sum term in (2.3) by $\int_t^s \partial^2_{xx} f(u, X_u)d[X, X]_u$.

Before the proof we want to say a few words about the preceding assumptions.

Remark 2.9. One consequence of $f \in C^{0,2}_{ac,count}$ (resp. $f \in C^{0,2}_{ac}$) is the following: for every second derivative (in space) g of f there is a countable (resp. Lebesgue null) subset D of $[t,T]$ such that the following holds.

1. $t \mapsto g(t,x)$ *is continuous for every* $t \in D^c$ *for every* $x \in \mathbb{R}^d$.
2. For every continuous function $k : [t,T] \to \mathbb{R}^d$ $t \mapsto g(t,k(t))$ *is continuous for every* $t \in D^c$.

We explain below the reasons. Of course we can substitute $\mathbb{R}^d$ in previous sentence with a fixed compact K of $\mathbb{R}^d$.

Concerning item 1., let now $\varepsilon > 0$. Since K is compact, it is included in a finite union of balls $B(x_i,\varepsilon)$, where $\{x_i : i = 1,\dots,N\}$ are elements of K. By item 4(b) Bis (resp. item 4(b)) there is a countable (resp. zero Lebesgue measure) set $D \subset [t,T]$ such that $s \mapsto g(s,x_i)$, $i = \{1,\dots,N\}$ is continuous outside D. Let $s_0 \notin D$ and $x \in K$. We show that $g(\cdot,x)$ is continuous in s_0.

There is $i \in \{1,\dots,N\}$ such that $|x - x_i| < \varepsilon$ so that, for $s \in [t,T]$, we have

$$\begin{aligned}|g(s,x) - g(s_0,x)| &\le |g(s,x) - g(s,x_i)| + |g(s,x_i) - g(s_0,x_i)| + |g(s_0,x_i) - g(s_0,x)| \\ &\le \gamma(g(s,\cdot)_{|K};\varepsilon) + |g(s,x_i) - g(s_0,x_i)| + \gamma(g_{|K}(s_0,\cdot);\varepsilon),\end{aligned}$$

where we recall that γ is the modulus of continuity. Taking on both sides the $\limsup_{s\to s_0}$, we get

$$\limsup_{s\to s_0} |g(s,x) - g(s_0,x)| \le 2 \sup_{s\in[t,T]} \gamma(g(s,\cdot);\varepsilon).$$

We now take the limit when $\varepsilon \to 0$ getting

$$\limsup_{s\to s_0} |g(s,x) - g(s_0,x)| = 0.$$

Concerning item 2., let $s_0 \in D^c$ and (s_n) be a sequence converging to s_0. We have

$$\begin{aligned}|g(s_n,k(s_n)) - g(s_0,k(s_0))| &\le |g(s_n,k(s_n)) - g(s_n,k(s_0))| + |g(s_n,k(s_0)) - g(s_0,k(s_0))| \\ &\le \sup_{s\in[t,T]} \gamma(g(s,\cdot)_{|K}, |k(s_n) - k(s_0)|) + |g(s_n,k(s_0)) - g(s_0,k(s_0))|.\end{aligned}$$

The first term goes to zero because of item 4. (a) of Definition 2.1 and the second term converges to zero by item 4 (b) Bis (resp. 4. (b)) of Definition 2.1.

Proof of Proposition 2.7. For simplicity of notations we write the proof in the case $d = 1$.

For $r \in [t,T]$ we have

$$\begin{aligned}f(r+\varepsilon, X_{r+\varepsilon}) =& f(r,X_r) + f(r+\varepsilon, X_{r+\varepsilon}) - f(r, X_{r+\varepsilon}) \\ &+ \partial_x f(r,X_r)(X_{r+\varepsilon} - X_r) + \partial^2_{xx} f(r,X_r)\frac{(X_{r+\varepsilon} - X_r)^2}{2} \\ &+ \frac{1}{2}\int_0^1 da\, [\partial^2_{xx} f(X_r + a(X_{r+\varepsilon} - X_r)) - \partial^2_{xx} f(X_r)](X_{r+\varepsilon} - X_r)^2.\end{aligned}$$

Integrating from t to s and dividing by ε, we get

$$I_0(s,\varepsilon) := \frac{1}{\varepsilon}\int_t^s f(r+\varepsilon, X_{r+\varepsilon}) - f(r,X_r)dr = I_1(s,\varepsilon) + I_2(s,\varepsilon) + I_3(s,\varepsilon) + I_4(s,\varepsilon),$$

where

$$I_1(s,\varepsilon) = \frac{1}{\varepsilon}\int_t^s f(r+\varepsilon, X_{r+\varepsilon}) - f(r, X_{r+\varepsilon})dr,$$
$$I_2(s,\varepsilon) = \frac{1}{\varepsilon}\int_t^s \partial_x f(r, X_r)(X_{r+\varepsilon} - X_r)dr,$$
$$I_3(s,\varepsilon) = \frac{1}{\varepsilon}\int_t^s \partial_{xx}^2 f(r, X_r)\frac{(X_{r+\varepsilon} - X_r)^2}{2}dr,$$
$$I_4(s,\varepsilon) = \frac{1}{2\varepsilon}\int_t^s \int_0^1 da\ [\partial_{xx}^2 f(r, X_r + a(X_{r+\varepsilon} - X_r)) - \partial_{xx}^2 f(r, X_r)]\ (X_{r+\varepsilon} - X_r)^2 dr.$$

Of course, as $\varepsilon \to 0$,

$$I_0(s,\varepsilon) = \frac{1}{\varepsilon}\int_s^{s+\varepsilon} f(r, X_r)dr - \frac{1}{\varepsilon}\int_t^{t+\varepsilon} f(r, X_r)dr \to f(s, X_s) - f(t, X_t), \quad s \in [t,T], \quad \text{u.c.p.}$$

Let $s \in [t,T]$. We have

$$I_1(s,\varepsilon) = \frac{1}{\varepsilon}\int_t^s dr \ \int_r^{r+\varepsilon} da\ \partial_a f(a, X_{r+\varepsilon}) = I_{1,1}(s,\varepsilon) + I_{1,2}(s,\varepsilon),$$

where

$$I_{1,1}(s,\varepsilon) = \frac{1}{\varepsilon}\int_t^{s+\varepsilon} da \ \int_{(a-\varepsilon)\vee t}^{a\wedge s} dr\ \partial_a f(a, X_a);$$
$$I_{1,2}(s,\varepsilon) = \frac{1}{\varepsilon}\int_t^{s+\varepsilon} da \ \int_{(a-\varepsilon)\vee t}^{a\wedge s} dr\ [\ \partial_a f(a, X_{s+\varepsilon}) - \partial_a f(a, X_a)\ ].$$

It is straightforward that

$$I_{1,1}(s,\varepsilon) = \int_t^{s+\varepsilon} da\ \partial_a f(a, X_a)\frac{(a\wedge s) - [(a-\varepsilon)\vee t]}{\varepsilon},$$

hence as $\varepsilon \to 0$,

$$I_{1,1}(s,\varepsilon) \to \int_t^s da\ \partial_a f(a, X_a)\ a.s.,$$

(even uniformly) because of the Lebesgue dominated convergence theorem. Now we investigate the convergence of the other term

$$I_{1,2}(s,\varepsilon) = \int_t^{s+\varepsilon} J_{1,2}(a,\varepsilon)\ da,$$

with

$$J_{1,2}(a,\varepsilon) = \frac{1}{\varepsilon}\int_{(a-\varepsilon)\vee t}^{a\wedge s} dr\ [\partial_a f(a, X_{r+\varepsilon})) - \partial_a f(a, X_a)].$$

We fix $\omega \in \Omega$. We prove the convergence for fixed $a \in [t,s]$ We have

$$J_{1,2}(a,\varepsilon) = \frac{1}{\varepsilon}\int_{(a-\varepsilon)\vee t}^{a\wedge s} dr\ [\partial_a f(a, \xi(a,\varepsilon)) - \partial_a f(a, X_a)],$$

where $\xi(a,\varepsilon) \in [\min_{r\in[a,a+\varepsilon]} X_r, \max_{r\in[a,a+\varepsilon]} X_r]$ and

$$Leb\ ([\min_{r\in[a,a+\varepsilon]} X_r, \max_{r\in[a,a+\varepsilon]} X_r]) \le \gamma(X;\varepsilon).$$

In particular $|X_a - X_r| < \gamma(X;\varepsilon)$ for all $r \in [a, a+\varepsilon]$. Consequently, for almost all $a \in [t,s]$, it holds $|J_{1,2}(a,\varepsilon)| \le \gamma(\partial_a f(a,\cdot);\gamma(X;\varepsilon))) \to 0$ as $\varepsilon \to 0$ since $\partial_a f(a,\cdot)$ is uniformly continuous on each compact.

Moreover

$$|J_{1,2}(a,\varepsilon)| \le 2 \sup_{x\in\mathbb{K}_t} |\partial_a f(a,x)|, \tag{2.4}$$

where $\mathbb{K}_t := [\min_{r\in[t,T]} X_r, \max_{r\in t,T]} X_r]$. Since the right-hand side of (2.4) is integrable by item 3. of Definition 2.1 on $[t,T]$, then

$$\lim_{\varepsilon\to 0} I_{1,2}(s,\varepsilon) = 0, \quad \text{a.s.},$$

by Lebesgue dominated convergence theorem. This implies that

$$\lim_{\varepsilon\to 0} I_1(s,\varepsilon) = \int_t^s da\ \partial_a f(a, X_a),$$

in probability for any $s \in [t,T]$.

We want the third integral

$$I_3(s,\varepsilon) = \frac{1}{\varepsilon}\int_t^s \partial^2_{xx} f(r,X_r)\frac{(X_{r+\varepsilon}-X_r)^2}{2}dr$$

to converge in probability to $\int_t^s \partial^2_{xx} f(r,X_r)d[X,X]_r$, $\forall s \in [t,T]$. We will actually prove that the convergence holds u.c.p.

By Lemma 3.1 in [35]

$$\frac{1}{\varepsilon}\int_t^s (X_{r+\varepsilon}-X_r)^2 dr \to [X,X]_s, \quad s \in [t,T], \quad \text{u.c.p.},$$

therefore for any sequence ε_n there exists a subsequence still denoted by ε_n such that, setting

$$\mu_{\varepsilon_n}(s) = [X,X]^{\varepsilon_n}(s),$$

where the right-hand side was defined in (2.1), then

$$\lim_{\varepsilon\to 0}[\sup_{s\in[t,T]} |\mu_{\varepsilon_n}(s) - [X,X]_s|] = 0, \quad \text{a.s.}$$

Let M be a null set such that for $\omega \notin M$ the sequence of real functions $\mu_{\varepsilon_n}(\omega,\cdot) \to [X,X].(\omega)$ uniformly on $[t,T]$, which implies that $d\mu_{\varepsilon_n}(\omega,\cdot) \Rightarrow d[X,X].(\omega)$. Suppose the validity of item 1. (resp. item 2.) Given a continuous function $k : [t,T] \mapsto \mathbb{R}^d$, using Remark 2.9, the set of discontinuities of $r \mapsto \partial_{xx} f(r,k(r))$ is countable (resp. has zero Lebesgue measure). This implies that (for the fixed ω), for every $s \in [t,T]$,

$$\int_t^s \partial^2_{xx} f(r,X_r(\omega))d\mu_{\varepsilon_n}(\omega,r) \to \int_t^s \partial^2_{xx} f(r,X_r(\omega))d[X,X](\omega,r),$$

making use of Portemanteau theorem. The convergence can be shown to be uniform in s by decomposing the second derivative(s) in difference of positive and negative part.

We have proved that for every sequence (ε_n) there is a subsequence (ε_{n_k}) that converges uniformly out of a null set, that is as $\varepsilon \to 0$,

$$I_3(\cdot,\varepsilon) = \frac{1}{\varepsilon}\int_t^{\cdot} \partial^2_{xx} f(r,X_r)\frac{(X_{r+\varepsilon}-X_r)^2}{2}dr \to \int_t^{\cdot} \partial^2_{xx} f(r,X_r)d[X,X](r) \text{ u.c.p.}$$

Now, let us investigate the convergence of the fourth integral I_4. We have

$$I_4(s,\varepsilon) \le \frac{1}{2}\int_t^T \eta(\varepsilon,r)d[X,X]^{\varepsilon}(r),$$

where $[X,X]^{\varepsilon}$ was introduced in (2.1), and

$$\begin{aligned}\eta(\varepsilon,r) &= \int_0^1 \frac{1}{2}\int_t^s |\partial^2_{xx} f(r,X_r + a(X_{r+\varepsilon}-X_r)) - \partial^2_{xx} f(r,X_r)|da \\ &\le T \sup_{r\in[t,T]} \gamma(\partial^2_{xx} f_K(r,\gamma(X,\varepsilon)).\end{aligned}$$

Since $[X,X]^{\varepsilon}$ converges u.c.p. to $[X,X]$ we obtain that $I_4(\cdot,\varepsilon) \to 0$ u.c.p.

At this point the limit in probability of $I_2(s,\varepsilon)$ is forced to exist and to be continuous because the other terms are. This will be of course $\int_t^s f(r,X_r)d^-X_r$ and the proof is complete. □

Proposition 2.10. Suppose $(X_s)_{s\in[t,T]}$ is an $(\mathcal{F}_s)_{s\in[t,T]}$-weak Dirichlet continuous process such that $[X,X]$ exists (resp. exists and it is absolutely continuous). Let $X = M + A$ be its decomposition.

There is a continuous linear map $\mathcal{B}^X : C^{0,1}([t,T]\times\mathbb{R}^d) \to C_{\mathcal{F}}([t,T]\times\Omega;\mathbb{R}^d)$ such that the following holds.

1. For every $u \in C^{0,1}([t,T]\times\mathbb{R})$ we have

$$u(s,X_s) = u(t,X_t) + \int_t^s \partial_x u(r,X_r)dM_r + \mathcal{B}^X(u)_s,\ s \ge t,$$

   and $\mathcal{B}^X(u)_s$ is a martingale orthogonal process.

2. If $u \in C^{0,2}_{ac,count}$ (resp. $u \in C^{0,2}_{ac}$) we have

$$\mathcal{B}^X(u)_s = \int_t^s \partial_r u(r,X_r)dr + \int_t^s \partial_{xx} u(r,X_r)d[M,M]_r + \int_t^s \partial_x u(r,X_r)d^-A_r.$$

3. If $u \in C^{0,2}_{ac,count}$ (resp.$u \in C^{0,2}_{ac}$), then $\int_0^s \partial_x u(r,X_r)d^-A_r$ is a martingale orthogonal process.

Proof. Item 1. was the object of one part of Proposition 3.10 in [23]. Items 2. and 3. are proved is the same way as Proposition 3.10 in [23], whereas, instead of using the Itô formula for finite quadratic variation process, see e.g. Proposition 2.4 in [23], we make use of the Itô formula (2.3) in Proposition 2.7. □

Corollary 2.11. Suppose $(D_s)_{s\in[t,T]}$ is an $(\mathcal{F}_s)_{s\in[t,T]}$-weak Dirichlet (vector) process such that $[D,D]$ exists. For every $u \in C^{0,1}(\mathbb{R}_+\times\mathbb{R}^d)$, $(u(s,D_s), s\in[t,T])$ is a $(\mathcal{F}_s)_{s\in[t,T]}$-weak Dirichlet process with martingale part $M_s = \int_t^s \nabla_x u(r,D_r)dM_r$, $s \in [t,T]$.

2.3 Concept of solution

Proposition 2.10 has an interesting development that will concern us closely. First of all, we introduce a parabolic partial differential equation that will be used in all the work.

In the sequel $L(\mathbb{R}^m, \mathbb{R}^d)$ will stand for the linear space of $d \times m$ real matrices. $b : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$, $\sigma : [0,T] \times \mathbb{R}^m \to L(\mathbb{R}^m, \mathbb{R}^d)$ will be locally bounded Borel functions, i.e. for all K compact, $\sup_{r\in[0,T],x\in K}(|b(r,x)| + |\sigma(r,x)|) < \infty$. and the linear parabolic operator given formally by

$$\mathcal{L}_0 u(s,x) = \partial_s u(s,x) + \langle b(s,x), \partial_x u(s,x)\rangle + \frac{1}{2} Tr[\sigma^\top(s,x)\partial_{xx}u(s,x)\sigma(s,x)].$$

In this section we will fix Borel functions $g : \mathbb{R}^d \to \mathbb{R}$ and $h : [t,T] \times \mathbb{R}^d \to \mathbb{R}$ such that

$$\int_t^T \sup_{x\in K} |h(s,x)| ds < \infty, \tag{2.5}$$

for every compact $K \subset \mathbb{R}^d$.

We will consider the inhomogeneous backward parabolic problem

$$\begin{cases} \mathcal{L}_0 u(s,x) = h(s,x), & s \in [t,T],\ x \in \mathbb{R}^d, \\ u(T,x) = g(x), & x \in \mathbb{R}^d. \end{cases} \tag{2.6}$$

The definition below generalizes the notion of strict solution, that appears in [22], Definition 4.1. For $s \in [t,T]$, we set

$$\mathcal{A}_s f(x) \doteqdot \langle b(s,x), \partial_x f(x)\rangle + \frac{1}{2} Tr[\sigma^\top(s,x)\partial_{xx}u(s,x)\sigma(s,x)]. \tag{2.7}$$

Definition 2.12. We say that $u : [t,T] \times \mathbb{R}^d \to \mathbb{R}$, $u \in C^{0,2}_{ac}([t,T] \times \mathbb{R}^d)$ is a *quasi-strict solution* to the backward Cauchy problem (2.6) if

$$u(s,x) = g(x) - \int_s^T h(r,x)dr + \int_s^T (\mathcal{A}_r u)(r,x)dr. \tag{2.8}$$

Remark 2.13. In this case, for every $x \in \mathbb{R}^d$,

$$\partial_s u(s,x) = h(s,x) - (\mathcal{A}_s u)(s,x) \qquad a.e. \tag{2.9}$$

where $\partial_s u$ stands for the distributional derivative of u.

The notion of quasi-strict solution allows to consider (somehow classical) solutions of (2.6) even though h, σ, b are not continuous in time.

The definition below is a relaxation of the notion of strong solution defined for instance in [23], Definition 4.2., which is based on approximation of strict (classical) solutions.

Definition 2.14. $u \in C^0([t,T] \times \mathbb{R}^d)$ is a *quasi-strong solution* (with *approximating sequence* (u_n)) to the backward Cauchy problem (2.6) if there exists sequence $u_n \in C^{0,2}_{ac}([t,T] \times \mathbb{R}^d)$ and two sequences $(g_n) : \mathbb{R}^d \to \mathbb{R}$, $(h_n) : [t,T] \times \mathbb{R}^d \to \mathbb{R}$ such that $\int_t^T \sup_{x\in K} |h_n(s,x)|ds < \infty$, for every compact $K \subset \mathbb{R}^d$ realizing the following.

1. $\forall n \in \mathbb{N}$, u_n is a quasi-strict solution of the problem

$$\begin{cases} \mathcal{L}_0 u_n(s,x) = h_n(s,x) & s \in [t,T],\ x \in \mathbb{R}^d, \\ u_n(T,x) = g_n(x) & x \in \mathbb{R}^d. \end{cases}$$

2. For each compact $K \subset \mathbb{R}^d$

$$\begin{cases} \sup_{(s,x)\in[t,T]\times K} |u_n - u|(s,x) \to 0, \\ \sup_{x\in K} |h_n - h|(\cdot,x) \to 0 \qquad \text{in } L^1([t,T]), \end{cases}$$

We define now the notion of a mild solution. We suppose here for simplicity that σ and b are of polynomial growth.

Suppose moreover that the SDE

$$\begin{cases} dY_s = b(s,Y_s) + \sigma(s,Y_s)\, dW_s, \\ Y_t = x, \end{cases} \tag{2.10}$$

admits existence and uniqueness in law for every initial condition for any $0 \le t \le s \le T$ and any $x \in \mathbb{R}^d$. Let $\check{\Omega} := C([t,T])$ be the canonical path space equipped with its Borel σ-algebra and let $X = (X_s)_{s\in[t,T]}$ be the canonical process.

For the notion of mild solution we introduce the associated inhomogeneous semigroup $(\mathcal{P}_{t,s})_{0\le t\le s\le T}$. For this we denote $(\mathbb{P}^{t,x}), 0 \le t \le T, x \in \mathbb{R}^d$ the associated Markov canonical class, see e.g. [3, 2], with associated expectation $(\mathbb{E}^{t,x}_s)_{0\le t\le s\le T}$. $\mathbb{P}^{t,x}$ will be the solution of the (unique) solution in law of (2.10).

For $f : \mathbb{R}^d \to \mathbb{R}^d$ continuous with polynomial growth we set

$$\mathcal{P}_{t,s} f(x) = \mathbb{E}^{t,x} f(X_s). \tag{2.11}$$

$(\mathcal{P}_{t,s})_{0\le t\le s\le T}$ is then the time-inhomogeneous semigroup associated with the generator (2.7).

Definition 2.15. Suppose g and h with polynomial growth. A function $u : [t,T] \times \mathbb{R}^d$ is said to be a *mild* solution to (2.6) if

$$u(s,x) = \mathcal{P}_{s,T} g(x) + \int_s^T \mathcal{P}_{s,r} h(r,x) dr, \ (s,x) \in [t,T] \times \mathbb{R}^d.$$

Remark 2.16.

1. By the usual techniques of stochastic calculus, see e.g. Burkholder-David-Gundy and Jensen's inequalities we can get

$$\sup_{t\in[0,T], x\in\mathbb{R}^d} \mathbb{E}^{t,x}\left(\sup_{t\le s\le T} |X_s|^p \right) \le c|x|^p.$$

2. Taking into account item 1. and (2.11), since h and g have polynomial growth a mild solution has necessarily polynomial growth.

Definition 2.17. We say that σ to be **non-degenerate** we mean that there is a a constant $c > 0$ such that for all $(s,x) \in [0,T] \times \mathbb{R}^d$ and $\xi \in \mathbb{R}^d$ we have

$$\xi^\top \sigma(s,x)^\top \sigma(s,x) \xi \ge c|\xi|^2. \tag{2.12}$$

Lemma 2.18. Suppose the following.

1. σ, b are Hölder continuous in space (uniformly in time) and σ is non-degenerate. Moreover $\sigma, b : [t,T] \times \mathbb{R}^d \to \mathbb{R}$ a.e. continuous in time for all $x \in \mathbb{R}^d$.

2. h has polynomial growth in space (uniformly in time). Moreover $h : [t, T] \times \mathbb{R}^d \to \mathbb{R}$ a.e. continuous in time for all $x \in \mathbb{R}^d$.

3. g is continuous with polynomial growth.

4. For every compact K of $\mathbb{R}^d$, h is Hölder continuous in space (uniformly in time).

Then there is a quasi-strict solution of (2.6).

Remark 2.19. 1. The proof of Lemma 2.18 makes essentially use of Proposition 4.2 in [29].

2. Indeed, the context of [29] applies to our case, $B = 0$. Therein, one shows existence of a so called strong Lie solution u, which fulfills (2.8). In particular one gets

$$\partial_s u(s,x) = h(s,x) + (\mathcal{A}_s u)(s,x), \forall x \in \mathbb{R}^d, \quad ds \text{ a.e.} \tag{2.13}$$

3. However, the notion of quasi-strict solution requires also u to belong to $C^{0,2}_{ac}(t,T] \times \mathbb{R}^d)$. The spatial first and second order spatial derivatives of u in [29] are (even Hölder) continuous in time, for fixed x. Therefore u fulfills items 1., 3. and 4. of Definition 2.1.

   Moreover (2.8) and (2.13) imply that u also fulfills item 2. of Definition 2.1.

Remark 2.20. If $h \in C^{0,\gamma}([t,T] \times \mathbb{R}^d)$ and g is Hölder continuous, the result appears in Theorem 5.1.9 of [30] and in this case we even have a strict solution.

Below, for simplicity of the formulation we set $t = 0$ as initial time of the interval $[t, T]$ in the PDEs. We will need a supplementary time t in the proof.

Proposition 2.21. Let us suppose σ, b, h with polynomial growth. A quasi-strict solution u of (2.6) with polynomial growth is also a mild solution.

Proof. Let u be a quasi-strict solution of (2.6) with polynomial growth. In particular, for every $t \in [0, T]$, $u \in C^{0,2}_{ac}([t,T] \times \mathbb{R}^d)$. We fix $t \in [0, T]$ and we can apply the Itô type formula (2.3) in Proposition 2.7. Clearly the canonical process X is a solution of (2.10) under $\mathbb{P}^{t,x}$. Explicitly, for $s \in [t, T]$, we get

$$\begin{aligned}
u(s, X_s) &= u(t,x) + \int_t^s (\partial_r u(r, X_r)dr + \partial_x u(r, X_r)[b(r, X_r)dr + \sigma(r, X_r)dW_r]) \\
&\quad + \frac{1}{2}\int_t^s Tr[\sigma^\top(r, X_r)\partial^2_{xx} u(r, X_r)\sigma(r, X_r)]dr \\
&= u(t,x) + \int_t^s \partial_x u(r, X_r)\sigma(r, X_r)dW_r + \int_t^s \partial_r u(r, X_r)dr + \int_t^s (\mathcal{A}_r u)(r, X_r)dr \\
&= u(t,x) + M_s + \int_t^s h(r, X_r)dr,
\end{aligned} \tag{2.14}$$

where

$$M_s = \int_t^s \partial_x u(r, X_r)\sigma(r, X_r)dW_r, \quad s \in [t, T].$$

The last equality in (2.14) holds because u is a quasi-strict solution of (2.6) so by (2.8), for all $x \in \mathbb{R}^d$, $u(\cdot, x)$, is absolutely continuous and (2.9) holds. Since u and h have polynomial growth, Remark 2.16 implies that the local martingale vanishing at zero M is such that $\sup_{s\in[t,T]} |M_s|$ is an integrable r.v., which implies that it is a martingale. Being u a solution of (2.6) $u(T,\cdot) = g$ and so (2.14) implies

$$g(X_T) = u(t,x) + M_T + \int_t^T h(r, X_r)dr.$$

Taking the expectation under $\mathbb{P}^{t,x}$, being M is a martingale, therefore it has a null expectation, gives

$$\mathcal{P}_{t,T}g(x) = u(t,x) + \int_t^T \mathcal{P}_{t,r}h(r,x)dr,$$

taking into account (2.11) and Fubini's theorem. Finally u is a mild solution of (2.6).

□

Now we prove that under certain assumptions a mild solution is a quasi-strong solution. Again we will take $t = 0$ as initial time of the PDE, for simplicity.

Proposition 2.22. Assume item 1., 2. and 3. of Lemma 2.18 and that the restriction of h to each compact is continuous in space, uniformly in $s \in [0,T]$.

Then a mild solution is a quasi-strong solution.

Remark 2.23.

- The assumptions of Proposition 2.22 is therefore slightly weaker than those of Lemma 2.18.
- We insist on the fact that h does not need to by bicontinuous in previous statement.

Proof of Proposition 2.22. Let $(\mathbb{P}^{t,x})$ be the family of probability measures introduced after (2.10).

We write

$$u := v^0 + v,$$

where

$$v^0(t,x) = \mathbb{E}^{t,x}(g(X_T)),$$
$$v(t,x) = \mathbb{E}^{t,x}\left(\int_t^T h(r, X_r)dr\right).$$

Without restriction of generality we can suppose $g = 0$, since the general case can be treated similarly. We prove that there exists a sequence of quasi-strict solutions v_n of (2.6) with $g = 0$, according to Definition 2.14. We proceed in two steps:

1. truncation,
2. regularization.

1. <u>Truncation</u>

This step consists in reducing the problem to the case when h and g are bounded with compact support. Let h as in the assumption and $h_n(t,x) = h(t,x)\chi(x)_{[-n,n]^d}$, where $\chi_{[-n,n]^d}$ is a smooth function bounded by 1 which support is $[-(n+1), n+1]^d$ and equal to 1 on $[-n,n]^d$. We define

$$v_n(t,x) = \mathbb{E}^{t,x}\left(\int_t^T h_n(r, X_r)dr\right). \tag{2.15}$$

Obviously v_n is a mild solution of (2.6) with h replaced with h_n.

By hypothesis, there exists $p \geq 1$ and a positive constant C such that $|h(r,x)| \leq C(1+|x|^p)$, for all $(r,x) \in [0,T] \times \mathbb{R}^d$. For every $(t,x) \in [0,T] \times \mathbb{R}^d$, taking $X = (X_{s \geq t})$ has the canonical process solution of (2.10), with $X_t = x$. Its law is of course $P^{t,x}$ and we have

$$\begin{aligned}|v_n(t,x) - v(t,x)| &\leq \int_t^T \mathbb{E}^{t,x} |h(r,X_r)| 1_{\{X_r \notin [-n,n]^d\}} dr \\ &\leq C \int_t^T \mathbb{E}^{t,x}((1+|X_r|^p) 1_{\{X_r \notin [-n,n]^d\}}) dr \\ &\leq 2C \int_t^T (\mathbb{E}^{t,x}((1+|X_r|)^{2p}))^{\frac{1}{2}} \mathbb{P}^{t,x}\{|X_r| \geq n\}^{\frac{1}{2}} dr.\end{aligned}$$

By Chebyshev this is smaller than

$$\int_t^T 2C(\mathbb{E}^{t,x}((1+|X_r|)^{2p}))^{\frac{1}{2}} \frac{\mathbb{E}^{t,x}(|X_r|)^{\frac{1}{2}}}{n^{\frac{1}{2}}} dr \leq \int_0^T 2C(\mathbb{E}^{t,x}((1+|X_r|)^{2p}))^{\frac{1}{2}} \frac{\mathbb{E}^{t,x}(|X_r|)^{\frac{1}{2}}}{n^{\frac{1}{2}}} dr.$$

By Remark 2.16, for all $p \geq 1$

$$\sup_{0 \leq t \leq s \leq T} \mathbb{E}^{t,x}(|X_s|^p) \leq C(1+|x|^p).$$

At this point the partial result $\sup_{(t,x)\in[0,T]\times K} |v_n(t,x) - v(t,x)| \to 0$ is established.

The sequence (h_n) and (v_n) are compatible with 2. of Definition 2.14, which concludes the step 1.

2)Regularization

We have now reduced the problem to the case when h has compact support. We define h_n as

$$h_n(r,y) = \int_{\mathbb{R}^d} h(r,\xi) n\phi(n(y-\xi)) d\xi = \int_{\mathbb{R}^d} \phi(\xi) h(r, y - \frac{\xi}{n}) d\xi, \tag{2.16}$$

where ϕ is a non-negative mollifier on $\mathbb{R}^d$ with compact support. h_n is smooth in space with all the space derivatives being bounded. In particular, each h_n is Hölder continuous in space uniformly time. By Lemma 2.18 there exists a quasi-strict solution $u = v_n$ to the problem (2.6) with h replaced by h_n and again $g = 0$. Since a quasi-strict solution is a mild solution by Proposition 2.21, we can represent it as in (2.15), with h_n as in (2.16). We denote by $P(t,s;x,dz)$ the (marginal) law of X_s under $\mathbb{P}^{t,x}$.

It is now possible to write

$$\begin{aligned}|v_n(t,x) - v(t,x)| &\leq \mathbb{E}^{t,x} \left| \int_{\mathbb{R}^d} d\xi\ \phi(\xi) \int_t^T \left(h(r,X_r) - h(r, X_r - \frac{\xi}{n}) \right) dr \right| \\ &\leq \int_{\mathbb{R}^d} d\xi\ \phi(\xi) \int_t^T dr \int_{\mathbb{R}^d} \left| h(r,a) - h(r, a - \frac{\xi}{n}) \right| P(t,r;x,da) \\ &\leq \int_{\mathbb{R}^d} d\xi\ \phi(\xi) \int_t^T dr\ \gamma\left(h(r,\cdot); \frac{diam\ (supp\ \phi)}{n} \right) \int_{\mathbb{R}^d} P(t,r;x,da) \\ &\leq \sup_{r\in[t,T]} \gamma\left(h(r,\cdot); \frac{diam\ (supp\ \phi)}{n} \right) \cdot \int_{\mathbb{R}^d} d\xi\ \phi(\xi) \int_t^T dr \int_{\mathbb{R}^d} P(t,r;x,da) \\ &\leq T \sup_{r\in[t,T]} \gamma\left(h(r); \frac{diam\ (supp\ \phi)}{n} \right)\end{aligned}$$

and this converges to zero by assumption, uniformly in $t \in [0,T]$. We have now proved that the sequence v_n converges to v uniformly on each compact. The convergence of (h_n) is even stronger than required in item 2. of Definition 2.14. Finally v is a quasi-strong solution to (2.6). This completes the proof.

□

2.4 The representation result

The following result is well-known when $u \in C^{1,2}$. When $u \in C^{0,1}$ is a strong solution, it was the object of Theorem 4.5 of [23]. We drive the attention that in the present case the coefficients σ and b are not continuous, in particular with respect to time.

Theorem 2.24. Let $b : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : [0,T] \times \mathbb{R}^d \to L(\mathbb{R}^m, \mathbb{R}^d)$ be Borel functions, continuous with respect to the space variable, and locally bounded.

Let $t \in [0,T]$. Let $h : [t,T] \times \mathbb{R}^d \to \mathbb{R}$ and $g : \mathbb{R}^d \to \mathbb{R}$ as in the lines before (2.6). Consider $u : [t,T] \times \mathbb{R}^d \to \mathbb{R}$ supposed to be of class $C^{0,1}([t,T] \times \mathbb{R}^d)$ be a quasi-strong solution of the Cauchy problem (2.6).

Fix $x \in \mathbb{R}^d$ and let $(S_s)_{s\in[t,T]}$ be a process of the form

$$S_s = x + \int_t^s \sigma(r, S_r) dW_r + A_s,$$

where $(A_s)_{s\in[t,T]}$ is an $(\mathcal{F}_s)_{s\in[t,T]}$-martingale orthogonal process such $[A,A]$ exists.

Let us suppose that the assumptions below are verified.

1. $\int_t^s \partial_x u(r, S_r) d^- A_r,\ s \in [t,T]$ exists and it is a martingale orthogonal process.
2. $$\lim_{n\to\infty} \int_t^\cdot (\partial_x u_n(r, S_r) - \partial_x u(r, S_r)) d^- A_r - \lim_{n\to\infty} \int_t^\cdot (\partial_x u_n(r, S_r) - \partial_x u(r, S_r)) b(r, S_r) dr = 0 \qquad s \in [t,T] \text{ u.c.p.}, \tag{2.17}$$

   where the sequence $\{u_n\}$ is the *approximating sequence* of the quasi strong solution u.

Then

$$u(s, S_s) = u(t, S_t) + \int_t^s \partial_x u(r, S_r) \sigma(r, S_r) dW_r + \mathcal{B}^S(u)_s,$$

where, for $s \in [t,T]$,

$$\mathcal{B}^S(u)_s = \int_t^s h(r, S_r) dr + \int_t^s \partial_x u(r, S_r) d^- A_r - \int_t^s \partial_x u(r, S_r), b(r, S_r) dr, \tag{2.18}$$

and $\mathcal{B}^S(u)$ is a martingale orthogonal process.

Remark 2.25. $\int_t^\cdot \partial_x u(r, S_r) d^- A_r$ exists and it is a martingale orthogonal process in the following two cases.

1. A is a bounded variation process so that S is a semimartingale. In this case $\int_t^\cdot \partial_x u(r, S_r) d^- A_r$ is the Lebesgue-Stieltjes integral $\int_t^\cdot \partial_x u(r, S_r) dA_r$, which is particular a bounded variation process and therefore a martingale orthogonal process.

2. $u \in C^{0,2}_{ac}([t,T], \mathbb{R}^d)$. By Proposition 2.7 and Remark 2.4

$$\begin{aligned}\int_t^s \partial_x u(r, S_r) d^- A_r &= u(s, S_s) - u(t, x) - \int_t^s \partial_x u(r, S_r)\sigma(r, S_r) dW_r - \int_t^s \partial_r u(r, S_r) dr \\ &- \frac{1}{2}\int_t^s \partial^2_{xx} u(r, S_r) d[S, S]_r, \ s \in [t, T].\end{aligned}$$

In particular the aforementioned forward integral exists. It remains to show that it is a martingale orthogonal process. For notational simplicity, we set $d = 1$. By the usual stability properties for the covariation, see e.g. Proposition 6.1 of [37], the covariation of the right-hand side of (2.19) with a continuous local martingale N gives

$$\int_t^s \partial_x u(r, S_r) d[S, N]_r - \int_t^s \partial_x u(r, S_r)\sigma(r, S_r) d[W, N]_r = 0.$$

Proof of Theorem 2.24. For simplicity of the formulation we set $d = 1$ and without restriction of generality we set $t = 0$. Let $u_n \to u$ be a sequence as in Definition 2.14. By Proposition 2.7 and Remark 2.4, we get

$$\begin{aligned}u_n(s, S_s) = u_n(0, S_0) &+ \int_0^s \mathcal{L}_0 u_n(r, S_r) dr - \int_0^s \partial_x u_n(r, S_r) b(r, S_r) dr \\ &+ \int_0^s \partial_x u_n(r, S_r) d^- A_r + N^n_s,\end{aligned}$$

where

$$N^n_s \doteqdot \int_0^s \partial_x u_n(r, S_r)\sigma(r, S_r) dW_r,$$

which is in particular a local martingale vanishing at zero. By assumption (2.17), the process N^n converges u.c.p. to

$$N_s \doteqdot u(s, S_s) - u(0, S_0) - \int_0^s h(r, S_r) dr + \int_0^s \partial_x u(r, S_r) b(r, S_r) dr - \int_0^s \partial_x u(r, S_r) d^- A_r.$$

At this point we have the decomposition

$$u(s, S_s) = u(0, S_0) + N_s + \mathcal{B}^S(u)_s, \qquad s \in [0, T],$$

where $\mathcal{B}^S(u)$ is defined by the right-hand side of (2.18). Now, the space of the space of continuous local martingales vanishing at zero, as a linear subspace of $C_{\mathcal{F}}([0, T] \times \Omega; \mathbb{R})$, is closed (under the u.c.p. convergence topology), see e.g. Proposition 4.4 in [23]. Consequently N is a continuous local martingale vanishing at zero. It remains to prove the following.

1. $\mathcal{B}^S(u)$ is a martingale orthogonal process.

2. $$N_s = \int_0^s \partial_x u(r, S_r)\sigma(r, S_r) dW_r. \tag{2.20}$$

1. follows because by additivity since $\int_0^{\cdot} \partial_x u(r, S_r) d^- A_r$ and all bounded variation processes are martingale orthogonal processes.

Concerning 2., setting $M_s = \int_0^s \sigma(r,S_r)dW_r$, Proposition 2.10 1. provides another weak Dirichlet decomposition

$$u(s,S_s) = u(0,S_0) + \int_0^s \partial_x u(r,S_r)\sigma(r,S_r)dW_r + \bar{\mathcal{B}}^S(u)_s,$$

where $\bar{\mathcal{B}}^S(u)_s$ is a martingale orthogonal process. By the uniqueness of the weak Dirichlet decomposition we get finally (2.20). The proof is now complete.

□

In the next corollary we will apply Theorem 2.24 to a solution S of a non-Markovian SDE. That result extends Corollary 4.6 of [23] where the coefficients σ, b were continuous. Here, we also assume $u \in C^{0,1}([t,T[\times\mathbb{R}^d) \cap C^0([t,T]\times\mathbb{R}^d)$, that is, $u(T,\cdot)$ may not be differentiable in the space variable. This allows lower regularity in the terminal condition of the PDE (2.6).

Corollary 2.26. Let $f : \Omega \times [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ be a progressively measurable field, continuous in x and a.s. locally bounded. Let $b : [0,T]\times\mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : [0,T]\times\mathbb{R}^d \to L(\mathbb{R}^m,\mathbb{R}^d)$ be Borel functions, continuous with respect to x, and locally bounded.

Let $t \in [0,T]$ and $u \in C^{0,1}([t,T[\times\mathbb{R}^d) \cap C^0([t,T]\times\mathbb{R}^d)$ be a quasi-strong solution of the Cauchy problem (2.6) fulfilling

$$\int_t^T \sup_{x\in K} |\partial_x u(s,x)|^2 ds < \infty. \tag{2.21}$$

Fix $x \in \mathbb{R}^d$ and let $(S_s)_{s\in[t,T]}$ be a solution to the SDE

$$dS_s = f(s,S_s)ds + \sigma(s,S_s)dW_s,$$

with initial condition $S_t = x \in \mathbb{R}^d$. Moreover we suppose the validity of one of the following items.

1. The approximating sequence $\{u_n\}$ of Definition 2.14 fulfills

$$\lim_{n\to\infty} \int_t^{\cdot} (\partial_x u_n(r,S_r) - \partial_x u(r,S_r))(f(r,S_r) - b(r,S_r))dr = 0 \quad \text{u.c.p.} \tag{2.22}$$

2. $\sigma\sigma^\top(r,S_r)$ is invertible for every $r \in [t,T]$ and denote the right pseudo-inverse $\sigma^{-1}(r,S_r) := \sigma^\top(\sigma\sigma^{-1})(r,S_r)$. Moreover the Novikov condition

$$\mathbb{E}\left(\exp\left(\frac{1}{2}\int_t^T |\sigma^{-1}(f-b)|^2(r,S_r)dr\right)\right) < \infty, \tag{2.23}$$

holds.

Then

$$u(s,S_s) = u(t,S_t) + \int_t^s \partial_x u(r,S_r)\sigma(r,S_r)dW_r + \mathcal{B}^S(u)_s, \tag{2.24}$$

where, for $s \in [t,T]$,

$$\mathcal{B}^S(u)_s = \int_t^s h(r,S_r)dr + \int_t^s \partial_x u(r,S_r)f(r,S_r)dr - \int_t^s \partial_x u(r,S_r)b(r,S_r)dr. \tag{2.25}$$

Remark 2.27. Clearly, whenever $(\omega, s, x) \mapsto \sigma^{-1}(s,x)(f(\omega, s, x) - b(s,x))$ is bounded, then (2.23) is fulfilled.

Proof of Corollary 2.26. In the proof, we set $t = 0$ without loss of generality. We first prove the result assuming that item 1. holds. We set

$$A = \int_0^{\cdot} f(r, S_r)dr.$$

Let $\varepsilon > 0$. The idea is to apply Theorem 2.24 with T replaced by $T - \varepsilon$. The function u restricted to $[0, T-\varepsilon] \times \mathbb{R}^d$ is trivially a quasi-strong solution to

$$\mathcal{L}_0 u(s,x) = h(s,x), \ u(T-\varepsilon, \cdot) = g_\varepsilon,$$

where $g_\varepsilon = u(T-\varepsilon, \cdot)$. By Theorem 2.24, taking into account Remark 2.4, we get the decomposition (2.24) and (2.25) for $s \in [0, T-\varepsilon]$. The result follows letting $\varepsilon \to 0$ and using in particular condition (2.21).

Now let us assume that item 2. is verified. Let $\mathbb{Q}$ be the probability measure on $(\Omega, \mathcal{F}_T)$ defined

$$d\mathbb{Q} = Z_T(\eta)d\mathbb{P},$$

where

$$Z_s(\eta) = e^{\left(\int_0^s \eta_r dW_r - \frac{1}{2}\int_0^s |\eta_r|^2 dr\right)}, \ s \in [0,T]$$

and

$$\eta_s = \sigma^{-1}(s, S_s)(b(s, S_s) - f(s, S_s)), \ s \in [0,T].$$

Then, the Novikov condition implies that $Z(\eta)$ is a martingale. Hence by Girsanov's Theorem, the process

$$\tilde{W}_s = W_s - \int_0^s \sigma^{-1}(r, S_r)(b(r, S_r) - f(\omega, r, S_r))dr, \ s \in [0,T],$$

is an $(\mathcal{F}_s, \mathbb{Q})$-Brownian motion and $(S_s)_{s\in[0,T]}$ satisfies the stochastic differential equation

$$dS_s = b(s, S_s)ds + \sigma(s, S_s)d\tilde{W}_s.$$

Then, by item 1. in the statement (here $\mathbb{Q}$ replaces $\mathbb{P}$, $f = b$ and $\tilde{W}$ replaces W), we obtain

$$u(s, S_s) = u(0, S_0) + \int_0^s \partial_x u(r, S_r)\sigma(r, S_r)d\tilde{W}_r + \int_0^s h(r, S_r)dr, \ s \in [t,T],$$

and the conclusion follows. □

Remark 2.28. If $\lim_{n\to\infty} \partial_x u_n = \partial_x u$ in $C^0([t,T] \times \mathbb{R}^d)$, then assumption (2.22) is trivially verified.

3 Application to game theory

In this section we prove the verification theorem. It states a sufficient condition for a pair of stochastic controls $(z_1, z_2) \in \mathcal{Z}_1(t) \times \mathcal{Z}_2(t)$, for a given $t \in [0, T[$, to constitute a *Nash equilibrium* in a zero-sum stochastic differential game, as formulated as Theorem 3.13, item 3.

3.1 Basic setting

In this section we give precise assumptions on the coefficients of the state equation and the payoff function.

Throughout the paper, the following assumptions hold.

Hypothesis 3.1. The functions $b(t,x), f_1(t,x,u_1,u_2), \sigma(t,x)$ are Borel and continuous in x for any $(t,u_1,u_2) \in [0,T] \times U_1 \times U_2$.
Moreover, we assume that for every $t \in [0,T]$, there exists $K > 0$ such that $\forall x, x_1, x_2 \in \mathbb{R}^d$, $\forall (u_1,u_2) \in U_1 \times U_2$, the properties below hold.

1. $\langle b(t,x_1) - b(t,x_2), x_1 - x_2 \rangle + ||\sigma(t,x_1) - \sigma(t,x_2)||^2 \leq K|x_1 - x_2|^2$,
2. $\langle f_1(t,x_1,u_1,u_2) - f_1(t,x_2,u_1,u_2), x_1 - x_2 \rangle + ||\sigma(t,x_1) - \sigma(t,x_2)||^2 \leq K|x_1 - x_2|^2$,
3. $|b(t,x)| + |f_1(t,x,u_1,u_2)| + \|\sigma(t,x)\| \leq K(1 + |x|)$.

Proposition 3.2. Let $t \in [0,T[$. Under Hypothesis 3.1, for every $z_1 \in \mathcal{Z}_1(t)$ and $z_2 \in \mathcal{Z}_2(t)$ there is a unique strong solution (up to indistinguishability) to the equation (1.1).

Proof. The result follows taking into account the fact that the coefficients of the SDE (1.1) verify the assumptions of Theorem 1.2 in [27]. □

We denote the solution of (1.1) with

$$y(s;t,x,z_1,z_2) \text{ or } y(s), \ \ s \in [t,T], x \in \mathbb{R}^d, z_1 \in \mathcal{Z}_1(t), z_2 \in \mathcal{Z}_2(t). \tag{3.1}$$

It is worth to remark that Hypothesis 3.1 is stronger than the corresponding monotonicity condition (1.3) in Theorem 1.2 of [27], since item 3. is a linear growth condition. This is because, on the top of having an existence and uniqueness result, we also need bounds on the moments of the solution. Related to this, we now state a direct consequence of Theorem 4.6 in [27].

Theorem 3.3. Let $t \in [0,T[$. Assume Hypothesis 3.1 and let $y(s;t,x,z_1,z_2)$ be defined in (3.1). Then $\forall p > 0$, $s \in [t,T]$ and $\forall z_1 \in \mathcal{Z}_1(t)$ and $\forall z_2 \in \mathcal{Z}_2(t)$, there is a constant $N = N(p,K) > 0$ such that

$$\mathbb{E} \sup_{t \leq r \leq T} |y(s;t,x,z_1(r),z_2(r))|^p \leq Ne^{NT}(1 + |x|^p). \tag{3.2}$$

For the expected payoff (1.3) to be well-defined, we need some supplementary hypotheses. That payoff is the quantity that Player 1 wishes to maximize (it is the amount she will earn from Player 2) and Player 2 wishes to minimize (it is the amount he will pay to Player 1).

Later, we will also suppose the hypothesis below.

Hypothesis 3.4. Let $l : [0,T] \times \mathbb{R}^d \times U_1 \times U_2 \to \mathbb{R}$ be a Borel function, $g : \mathbb{R}^d \to \mathbb{R}$ continuous. Moreover, for any $t \in [0,T]$ we assume that for all $z_1 \in \mathcal{Z}_1(t)$, $z_2 \in \mathcal{Z}_2(t)$ the function $s \mapsto l(s, y(s;t,x,z_1,z_2); z_1(s), z_2(s))$ is integrable in $[t,T]$, ω a.s. In particular $-\infty < \int_t^T l(s, y(s;t,x,z_1,z_2), z_1(s), z_2(s))ds < \infty$ a.s.

Definition 3.5. Let $t \in [0,T[$. Under Hypothesis 3.4, fixing a $z_1 \in \mathcal{Z}_1(t)$ and a $z_2 \in \mathcal{Z}_2(t)$, we define $\tilde{J} : [t,T] \times \mathbb{R}^d \times \mathcal{Z}_1(t) \times \mathcal{Z}_2(t) \times \Omega \to \mathbb{R}$ by

$$\tilde{J}(t,x;z_1,z_2) \doteqdot \int_t^T l(s, y(s;t,x,z_1,z_2); z_1(s), z_2(s))ds + g(y(T;t,x,z_1,z_2)). \tag{3.3}$$

At this point, the functional $\tilde{J}$ is well-defined for any $(t,x) \in [0,T] \times \mathbb{R}^d$ and $(z_1, z_2) \in \mathcal{Z}_1(t)\times\mathcal{Z}_2(t)$. We will prove in Theorem 3.13 that, if the aforementioned Pathwise-Isaacs condition is verified for a couple of controls $(z_1^\star, z_2^\star) \in \mathcal{Z}_1(t) \times \mathcal{Z}_2(t)$, then $J(t,x;z_1^\star,z_2^\star)$ is well-defined, that is $\tilde{J}$ is quasi-integrable. In particular

$$J(t,x;z_1^\star,z_2^\star) = J^+(t,x;z_1^\star,z_2^\star) = J^-(t,x;z_1^\star,z_2^\star).$$

Moreover, we will prove that, assuming the same Pathwise-Isaacs condition,

$$V^-(t,x) = V^+(t,x) = v^+(t,x) = v^-(t,x).$$

We also need some regularity hypotheses on the upper and lower Hamiltonians $H^{0,+}, H^{0,-}$. They will be more technical and will rely on the notion of *quasi-strong solution* and they will be stated in Definition 3.7.

3.2 The fundamental lemma

Similarly to the quasi-strong and quasi-strict solutions defined in Definition 2.12, we now define the notion of quasi-strong and quasi-strict solution of equations (1.6) and (1.7). This concept of solution is necessary to prove the so called Fundamental Lemma 3.10 below.

Let $H = H^{0,-}$ (resp. $H^{0,+}$) be the function defined in (1.8) (resp. (1.9)) supposed to Borel.

Definition 3.6. Let $v \in C^{0,2}_{ac}([0,T],\mathbb{R}^d)$. We set $h(r,x) = -H(r,x,\partial_x v(r,x))$, $r \in [0,T]$, $x \in \mathbb{R}^d$. We say that v is a *quasi-strict solution* of (1.6) (resp. (1.7)) if the following holds.

1. $\int_0^T \sup_{x\in K} |h(r,x)| dr < \infty$ for every compact $K \in \mathbb{R}^d$,
2. $u = v$ is a quasi-strict solution of (2.6) with $t = 0$.

Definition 3.7. Let $v \in C^{0,1}([0,T[\times\mathbb{R}^d)\cap C^0([0,T]\times\mathbb{R}^d)$. We set again $h(r,x) = -H(r,x,\partial_x v(r,x))$, $r \in [0,T]$, $x \in \mathbb{R}^d$. We say that v is a *quasi-strong solution* of (1.6) (resp. (1.7)) if $u := v$ is a quasi-strong solution of (2.6) with $t = 0$.

We introduce now an assumption related to two functions $v^+, v^- : [0,T]\times\mathbb{R}^d \to \mathbb{R}^d$, which will be used in the sequel. We recall that $y(s) = y(s;t,x,z_1,z_2)$, $s \in [t,T]$, is the process introduced in (3.1), being the solution (1.1), for every admissible control $z_1 \in \mathcal{Z}_1(t)$, $z_2 \in \mathcal{Z}_2(t)$. It exists thanks to Hypothesis 3.1 and Proposition 3.2.

Given a couple of functions $v^-, v^+ : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$, we will consider the following assumption.

Hypothesis 3.8. $v = v^-$ (resp. v^+) belongs to $C^{0,1}([0,T[\times\mathbb{R}^d)\cap C^0([0,T]\times\mathbb{R}^d)$ which is a quasi-strong solution of the Bellmann-Isaacs equation (1.7) (resp. (1.6)) with approximating sequences (v_n).

Moreover, $\partial_x v$ has polynomial growth and one of the two below conditions holds true.

1. For every $t \in [0,T]$ we have

$$\lim_{n\to\infty} \int_t^\cdot (\partial_x v_n(r,y(r)) - \partial_x v(r,y(r))) f_1(r,y(r),z_1(r),z_2(r)) dr = 0, u.c.p.,$$

   where $y(r) = y(r;t,x,z_1,z_2), r \in [t,T]$.

2. The function

$$(r,x,u_1,u_2) \mapsto \sigma^{-1}(r,x) f_1(r,x,u_1,u_2)$$

is well-defined and bounded on $[0,T]\times\mathbb{R}^d\times U_1\times U_2$ (σ^{-1} being the right pseudo-inverse $\sigma^\top(\sigma\sigma^\top)^{-1}$ provided $\sigma\sigma^\top$ is invertible).

Remark 3.9. 1. By definition of quasi-strong solution of (1.7) $h(r,y) := -H^{0,-}(r,y,\partial_x v(r,y))$, fulfills (2.5).

2. Since $\partial_x v$ with polynomial growth, the condition (2.21) is verified.

3. $\partial_x v$ with polynomial growth implies that v has polynomial growth. This in turn implies that g has polynomial growth.

Similarly to Lemma 4.10 of [23] we state a Fundamental Lemma which is based on a Fukushima-Dirichlet decomposition. The Hamiltonian H^0_{CV} below was defined in (1.5).

Lemma 3.10. We assume Hypotheses 3.1, 3.4. We also suppose the existence of functions v^-, v^+ satisfying Hypothesis 3.8.

Then $\forall (t,x)\in[0,T]\times\mathbb{R}^d$ and $\forall z_1\in\mathcal{Z}_1(t)$, $\forall z_2\in\mathcal{Z}_2(t)$, we have

$$\tilde{J}(t,x;z_1,z_2) = v^{-(+)}(t,x) + \int_t^T \Big(H^0_{CV}(r,y(r),\partial_x v^{-(+)}(r,y(r)),z_1(r),z_2(r)) \\ - H^{0,-(+)}(r,y(r),\partial_x v^{-(+)}(r,y(r)))\Big)\,dr + M_T,$$

where M is a (square integrable) martingale on $[t,T]$, vanishing at t, and $\tilde{J}$ is defined in (3.3),

Proof. We formulate the proof in the case of lower BI equation. We set $v := v^-$. It is not possible to use Itô's formula because $\partial_s v, \partial_{xx} v$ do not necessarily exist. To overcome this difficulty we use the representation Corollary 2.26 for $u = v$. By Hypothesis 3.8 we know that v is a quasi-strong solution of

$$\mathcal{L}_0 v(t,x) = -H^{0,-}(t,x,\partial_x v(t,x)), \\ v(T,x) = g(x),$$

with polynomial growth, by Remark 3.9 2. By the same Remark 3.9 2., (2.21) is verified for $u = v$. The process y in the statement fulfills the moments inequality (3.2) in Theorem 3.3.

Taking also into account the second part of Hypothesis 3.8, we can apply Corollary 2.26, again with $u = v$, which implies

$$g(y(T)) = v(t,x) + \int_t^T \partial_x v(r,y(r))\sigma(r,y(r))dW_r + \mathcal{B}^S(v)_T, \tag{3.4}$$

where

$$\mathcal{B}^S(v)_s = \int_t^s -H^{0,-}(r,y(r),\partial_x v(r,y(r)))dr + \int_t^s \partial_x v(r,y(r)) f_1(r,y(r),z_1(r),z_2(r))dr, \;\; s\in[t,T].$$

Now, $\int_t^T l(r, y(r; t, x, z_1, z_2), z_1(r), z_2(r))dr$ is a.s. finite by Hypothesis 3.4 so we can add it to both sides of the equality (3.4)

$$\int_t^T l(r, y(r), z_1(r), z_2(r))dr + g(y(T)) = v(t, x) + \int_t^T \Big(-H^{0,-}(r, y(r), \partial_x v(r, y(r)))+ \\ +H^0_{CV}(r, y(r), \partial_x v(r, y(r)), z_1(r), z_2(r))\Big)\ dr + \int_t^T \partial_x v(r, y(r))\sigma(r, y(r))dW_r.$$

Cauchy-Schwarz inequality together with the moments inequality (3.2) in Theorem 3.3 imply that

$$\mathbb{E}\int_t^T |(\partial_x v(r, y(r))\sigma(r, y(r))|^2 dr < \infty,$$

so the stochastic integral is a square integrable martingale. Defining

$$M_s := \int_t^s \partial_x v(r, y(r))\sigma(r, y(r))dW_r,\ \ s \geq t,$$

by the definition of $\tilde{J}$ in (3.3), the conclusion follows. □

At this point, we remark that $\tilde{J}$ is a.s. finite though could not be integrable.

3.3 Verification theorem and value of the game

In this section, we formulate our Pathwise-Isaacs condition, which is inspired from the Isaacs type condition formulated in [24] and [25]. In Proposition 3.18, we will show that our Pathiwse-Isaacs condition extends the one given in [24] and [25], in the zero-sum games.

We state first an equivalence statement between the saddle point and the min-max properties, which will be used in the proof of the verification Theorem 3.13, see Theorem 4.1.1 in [28].

Lemma 3.11. Let $G : U_1 \times U_2 \to \mathbb{R}$ be Borel and $(u_1^\star, u_2^\star) \in U_1 \times U_2$. Then the following properties 1. and 2. are equivalent.

1. $$G(u_1, u_2^\star) \leq G(u_1^\star, u_2^\star) \leq G(u_1^\star, u_2), \qquad \forall u_1 \in U_1, \forall u_2 \in U_2.$$

2. $$\sup_{u_1 \in U_1} \inf_{u_2 \in U_2} G(u_1, u_2) = \inf_{u_2 \in U_2} \sup_{u_1 \in U_1} G(u_1, u_2). \tag{3.5}$$

   and

   $$\begin{cases} \sup_{u_1 \in U_1} G(u_1, u_2^\star) = \inf_{u_2 \in U_2} \sup_{u_1 \in U_1} G(u_1, u_2), \\ \inf_{u_2 \in U_2} G(u_1^\star, u_2) = \sup_{u_1 \in U_1} \inf_{u_2 \in U_2} G(u_1, u_2). \end{cases} \tag{3.6}$$

In particular, if 1. or 2. are verified then $G(u_1^\star, u_2^\star)$ is equal to both (3.5) and (3.6).

Theorem 3.13 and Corollary 3.15, under our Pathwise-Isaacs condition, link quasi-strong solutions of the BI equations with the values of the game. Before stating the aforementioned theorem we formulate a remark.

Remark 3.12. 1. The payoff function J, defined in (1.3), is connected with $\tilde{J}$, defined in (3.3), by

$$J(t, x; z_1, z_2) = \mathbb{E}(\tilde{J}(t, x; z_1, z_2)), \qquad t \in [0, T], x \in \mathbb{R}^d, z_1 \in \mathcal{Z}_1(t), z_2 \in \mathcal{Z}_2(t),$$

provided previous expectation makes sense. We highlight the fact that we do not need any integrability assumption for every $z_1 \in \mathcal{Z}_1(t)$ and $z_2 \in \mathcal{Z}_2(t)$.

2. The upper and lower value V^-, V^+ for the stochastic differential game with payoff J and state equation (1.1) are defined in Definition 1.1 of Section 1.

3. For a fixed $t \in [0, T]$, we recall that $y(s) = y(s; t, x, z_1, z_2)$, for $(s, x, z_1, z_2) \in [t, T] \times \mathbb{R}^d \times \mathcal{Z}_1(t) \times \mathcal{Z}_2(t)$, is a solution to the SDE (1.1), see also (3.1).

Theorem 3.13. We assume that Hypotheses 3.1, 3.4 are satisfied. We suppose moreover the existence of functions $v^{-/+}$ fulfilling Hypothesis 3.8. Let $t \in [0, T], x \in \mathbb{R}^d$. We suppose the existence of a couple $(z_1^\star, z_2^\star) \in \mathcal{Z}_1(t) \times \mathcal{Z}_2(t)$ for which the following is verified.

Property $\pi(t, x; z_1^\star, z_2^\star)$**.**
For both $v = v^+$ and $v = v^-$, for a.e. $r \in [t, T]$, $\mathbb{P}$-a.s. we have the following.

$$\begin{aligned} H^0_{CV}(r, y(r), \partial_x v(r, y(r)), z_1^\star(r), z_2^\star(r)) &\geq H^0_{CV}(r, y(r), \partial_x v(r, y(r)), u_1, z_2^\star(r)), \qquad \forall u_1 \in U_1, \\ H^0_{CV}(r, y(r), \partial_x v(r, y(r)), z_1^\star(r), z_2^\star(r)) &\leq H^0_{CV}(r, y(r), \partial_x v(r, y(r)), z_1^\star(r), u_2), \qquad \forall u_2 \in U_2. \end{aligned} \tag{3.7}$$

Then the following holds true.

1. (a) $J(t, x; z_1^\star, z_2^\star)$ is well-defined and finite.
   (b) For any other $(z_1, z_2) \in \mathcal{Z}_1(t) \times \mathcal{Z}_2(t)$, $J(t, x; z_1^\star, z_2)$ (resp. $J(t, x; z_1, z_2^\star)$) is well-defined and greater than $-\infty$ (resp. smaller than $+\infty$).

2. $J(t, x; z_1^\star, z_2^\star) = v^-(t, x) = v^+(t, x)$.

3. The couple $(z_1^\star, z_2^\star)$ is a saddle point for the game: this means that, for any $(z_1, z_2) \in \mathcal{Z}_1(t) \times \mathcal{Z}_2(t)$ we have $J(t, x; z_1, z_2^\star) \leq J(t, x; z_1^\star, z_2^\star) \leq J(t, x; z_1^\star, z_2)$.

4. The payoff functional evaluated at $(z_1^\star, z_2^\star)$ is equal to both the upper and lower value of the game, i.e. $J(t, x; z_1^\star, z_2^\star) = V^+(t, x) = V^-(t, x)$. In particular the game admits a value.

Remark 3.14. As already mentioned in the Introduction, the existence of an optimal couple $(z_1^\star, z_2^\star)$ for which **Property** $\pi(t, x; z_1^\star, z_2^\star)$, for every (t, x), constitutes for us what we call *Pathwise-Isaacs condition*.

Corollary 3.15. Under the assumptions of Theorem 3.13, for every (t, x), we have

$$v^-(t, x) = V^-(t, x) = V^+(t, x) = v^+(t, x),$$

for every (t, x).
In particular $V := V^- = V^+$ is the unique quasi-strong solution of both (1.6) and (1.7).

Before the proof of the theorem, we mention the following technical point.

Proposition 3.16. We assume that Hypotheses 3.1, 3.4, hold. Let $v^{-/+}$ be a quasi-strong solution of the lower/upper Bellmann-Isaacs equation, whenever they exist. Then, for every $(t,x) \in [0,T] \times \mathbb{R}^d$ and for a couple of controls $(z_1^\star, z_2^\star) \in \mathcal{Z}_1(t) \times \mathcal{Z}_2(t)$, the following are equivalent.

1. Property $\pi(t,x; z_1^\star, z_2^\star)$ is verified.
2. Define $y(r) := y(r; t, x, z_1^\star(r), z_2^\star(r))$ as in (3.1). For both $v = v^+$ and $v = v^-$, the following statements hold for a.e. $(r,x) \in [t,T] \times \mathbb{R}^d$, $\mathbb{P}$-a.s.

   (a)
$$\begin{aligned} &\sup\nolimits_{u_1 \in U_1} \inf\nolimits_{u_2 \in U_2} H^0_{CV}(r, y(r), \partial_x v(r, y(r)), u_1, u_2) \\ &\quad = \inf\nolimits_{u_2 \in U_2} \sup\nolimits_{u_1 \in U_1} H^0_{CV}(r, y(r), \partial_x v(r, y(r)), u_1, u_2). \end{aligned} \tag{3.8}$$

   (b)
$$\begin{cases} \sup_{u_1 \in U_1} H^0_{CV}(r, y(r), \partial_x v(r, y(r)), u_1, z_2^\star(r)), \\ \qquad = \inf_{u_2 \in U_2} \sup_{u_1 \in U_1} H^0_{CV}(r, y(r), \partial_x v(r, y(r)), u_1, u_2), \\ \inf_{u_2 \in U_2} H^0_{CV}(r, y(r), \partial_x v(r, y(r)), z_1^\star(r), u_2) \\ \qquad = \sup_{u_1 \in U_1} \inf_{u_2 \in U_2} H^0_{CV}(r, y(r), \partial_x v(r, y(r)), u_1, u_2). \end{cases} \tag{3.9}$$

In particular
$$H^0_{CV}(r, y(r), \partial_x v(r, y(r)), z_1^\star(r), z_2^\star(r)) \tag{3.10}$$
is equal to both (3.8) and (3.9).

Proof. Without losing the generality, for a fixed $\omega \in \Omega$, and $r \in [t,T]$, let $y(r) := y(r; t, x, z_1^\star(r), z_2^\star(r)) \in \mathbb{R}^d$ and $\partial_x v(r, y(r)) \in \mathbb{R}^d$. We set $G(u_1, u_2) := H^0_{CV}(r, y(r), \partial_x v(r, y(r)), u_1, u_2)$, $u_1^\star := z_1^\star(r, \omega) \in U_1$, and $u_2^\star := z_2^\star(r, \omega) \in U_2$. Hence Property $\pi(t, x, z_1^\star(r,\omega), z_2^\star(r,\omega))$ reads as
$$\begin{aligned} &G(u_1^\star, u_2^\star) \geq G(u_1, u_2^\star), \qquad \forall u_1 \in U_1, \\ &G(u_1^\star, u_2^\star) \leq G(u_1^\star, u_2), \qquad \forall u_2 \in U_2. \end{aligned}$$
Similarly, (3.8) and (3.9) read as
$$\sup_{u_1 \in U_1} \inf_{u_2 \in U_2} G(u_1, u_2) = \inf_{u_2 \in U_2} \sup_{u_1 \in U_1} G(u_1, u_2),$$
$$\begin{cases} \sup_{u_1 \in U_1} G(u_1, u_2^\star) = \inf_{u_2 \in U_2} \sup_{u_1 \in U_1} G(u_1, u_2), \\ \inf_{u_2 \in U_2} G(u_1^\star, u_2) = \sup_{u_1 \in U_1} \inf_{u_2 \in U_2} G(u_1, u_2). \end{cases}$$
The conclusion follows after applying Lemma 3.11. □

Proof of Theorem 3.13. Let v^- be a solution of the lower Bellmann-Isaacs equation (1.7). Applying Lemma 3.10 to a generic couple (z_1, z_2) we obtain
$$\begin{aligned} &\tilde{J}(t, x, z_1, z_2) = \\ &\quad v^-(t,x) + \int_t^T \Big(H^0_{CV}(r, y(r; t, x, z_1(r), z_2(r)), \partial_x v^-(r, y(r; t, x, z_1(r), z_2(r))), z_1, z_2) \\ &\quad - \sup_{u_1 \in U_1} \inf_{u_2 \in U_2} H^0_{CV}(r, y(r; t, x, z_1(r), z_2(r)), \partial_x v^-(r, y(r; t, x, z_1(r), z_2(r))), u_1, u_2) \Big) \, dr \\ &\quad + M_T(z_1, z_2), \end{aligned} \tag{3.11}$$

where $M(z_1, z_2)$ is a square integrable martingale indexed by $[t, T]$, vanishing at t. We apply (3.11) for $(z_1, z_2) = (z_1^\star, z_2^\star)$. By making use of (3.10), the expression inside the integral vanishes, so we get

$$\tilde{J}(t, x, z_1^\star, z_2^\star) = v^-(t, x) + M_T(z_1^\star, z_2^\star), \tag{3.12}$$

and taking the expectation we get

$$v^-(t, x) = J(t, x; z_1^\star, z_2^\star).$$

With a similar argument, we can show that $v^+(t, x) = J(t, x; z_1^\star, z_2^\star)$, hence item 1. a) and item 2. are established.

In the sequel, we will drop the dependency on r of a generic $(z_1, z_2) \in \mathcal{Z}_1 \times \mathcal{Z}_2$. Furthermore, we prove that the couple $(z_1^\star, z_2^\star)$ is a saddle point for the game, i.e. item 3. For a control $z_2 \in \mathcal{Z}_2(t)$, subtracting (3.11) from (3.12) evaluated at $z_1 = z_1^\star$ and z_2, we have

$$\begin{aligned} &\tilde{J}(t, x, z_1^\star, z_2^\star) - \tilde{J}(t, x, z_1^\star, z_2) \\ &= -\int_t^T \Big(H^0_{CV}(r, y(r; t, x, z_1^\star, z_2), \partial_x v^-(r, y(r; t, x, z_1^\star, z_2)), z_1^\star, z_2) \\ &\quad - \sup_{u_1 \in U_1} \inf_{u_2 \in U_2} H^0_{CV}(r, y(r; t, x, z_1^\star, z_2), \partial_x v^-(r, y(r; t, x, z_1^\star, z_2)), u_1, u_2) \Big) dr \\ &\quad + M_T(z_1^\star, z_2^\star) - M_T(z_1^\star, z_2) \\ &\leq -\int_t^T \inf_{u_2 \in U_2} \Big(H^0_{CV}(r, y(r; t, x, z_1^\star, z_2), \partial_x v^-(r, y(r; t, x, z_1^\star, z_2)), z_1^\star, u_2) \\ &\quad - \sup_{u_1 \in U_1} \inf_{u_2 \in U_2} H^0_{CV}(r, y(r; t, x, z_1^\star, z_2), \partial_x v^-(r, y(r; t, x, z_1^\star, z_2)), u_1, u_2) \Big) dr \\ &\quad + M_T(z_1^\star, z_2^\star) - M_T(z_1^\star, z_2). \end{aligned} \tag{3.13}$$

Proposition 3.16 tells us that Property $\pi(t, x; z_1^\star, z_2^\star)$ is equivalent to item 2. of the aforementioned proposition. Consequently, by (b) of the same item 2., the integral in the right-hand side of the inequality of (3.13), vanishes. Therefore

$$\tilde{J}(t, x; z_1^\star, z_2^\star) - \tilde{J}(t, x; z_1^\star, z_2) \leq M_T(z_1^\star, z_2^\star)) - M_T(z_1^\star, z_2). \tag{3.14}$$

It follows that $\tilde{J}(t, x, z_1^\star, z_2)$ is integrable since all the other terms are; in particular $\tilde{J}(t, x; z_1^\star, z_2^\star)$ is integrable by item 2. Taking expectation of (3.14)

$$J(t, x; z_1^\star, z_2^\star) - J(t, x; z_1^\star, z_2) \leq 0. \tag{3.15}$$

Similarly, to (3.13), by taking a control $z_1 \in \mathcal{Z}_1(t)$,

$$\begin{aligned}
&\tilde{J}(t,x;z_1^\star,z_2^\star) - \tilde{J}(t,x;z_1,z_2^\star) \\
&\quad = -\int_t^T \Big(H^0_{CV}(r, y(r;t,x,z_1,z_2^\star), \partial_x v^+(r, y(r;t,x,z_1,z_2^\star)), z_1, z_2^\star) \\
&\qquad - \inf_{u_2\in U_2}\sup_{u_1\in U_1} H^0_{CV}(r, y(r;t,x,z_1,z_2^\star), \partial_x v^+(r, y(r;t,x,z_1,z_2^\star)), u_1, u_2) \Big)\, dr \\
&\qquad + M_T(z_1^\star,z_2^\star) - M_T(z_1,z_2^\star) \\
&\quad \geq -\int_t^T \sup_{u_1\in U_1} \Big(H^0_{CV}(r, y(r;t,x,z_1,z_2^\star), \partial_x v^+(r, y(r;t,x,z_1,z_2^\star)), u_1, z_2^\star) \\
&\qquad - \inf_{u_2\in U_2}\sup_{u_1\in U_1} H^0_{CV}(r, y(r;t,x,z_1,z_2^\star), \partial_x v^+(r, y(r;t,x,z_1,z_2^\star)), u_1, u_2) \Big)\, dr \\
&\qquad + M_T(z_1^\star,z_2^\star) - M_T(z_1,z_2^\star).
\end{aligned}$$

We recall that Property $\pi(t,x;z_1^\star,z_2^\star)$ is equivalent to item 2. of the Proposition 3.16. Consequently by item 2.(b) also the integral in the right-hand side of the inequality of (3.13) vanishes. Therefore

$$\tilde{J}(t,x;z_1^\star,z_2^\star) - \tilde{J}(t,x;z_1,z_2^\star) \geq M_T(z_1^\star,z_2^\star) - M_T(z_1,z_2^\star). \tag{3.16}$$

We recall that, by item 2. $J(t,x;z_1^\star,z_2^\star)$ is integrable, so $\tilde{J}(t,x;z_1,z_2^\star)$ is integrable since all the other terms are. Taking expectation of (3.16)

$$J(t,x;z_1^\star,z_2^\star) - J(t,x;z_1,z_2^\star) \geq 0. \tag{3.17}$$

Inequalities (3.15) and (3.17) prove item 3. of the statement.

We continue with the proof of item 4. We recall that, by Definition 1.1,

$$V^-(t,x) = \sup_{z_1\in\mathcal{Z}_1(t)} \inf_{z_2\in\mathcal{Z}_2(t)} J^-(t,x;z_1,z_2)$$

and observe that, trivially we have

$$\inf_{z_2\in\mathcal{Z}_2(t)} \sup_{z_1\in\mathcal{Z}_1(t)} J^-(t,x;z_1,z_2) \leq \sup_{z_1\in\mathcal{Z}_1(t)} J^-(t,x;z_1,z_2^\star). \tag{3.18}$$

Since, by item 1.(b), $J(t,x;z_1,z_2^\star)$ is well-defined for every $z_1 \in \mathcal{Z}_1(t)$, (1.4) yields

$$\sup_{z_1\in\mathcal{Z}_1(t)} J^-(t,x;z_1,z_2^\star) = \sup_{z_1\in\mathcal{Z}_1(t)} J(t,x;z_1,z_2^\star). \tag{3.19}$$

$$\sup_{z_1\in\mathcal{Z}_1(t)} J(t,x;z_1,z_2^\star) \leq J(t,x;z_1^\star,z_2^\star) \leq \inf_{z_2\in\mathcal{Z}_2(t)} J(t,x;z_1^\star,z_2). \tag{3.20}$$

Therefore, using (3.18), (3.19) and (3.20), we obtain

$$\inf_{z_2\in\mathcal{Z}_2(t)} \sup_{z_1\in\mathcal{Z}_1} J^-(t,x;z_1,z_2) \leq \inf_{z_2\in\mathcal{Z}_2(t)} J(t,x;z_1^\star,z_2). \tag{3.21}$$

By item 1.(b) it also follows that $J(t,x;z_1^\star,z_2)$ is well-defined for every $z_2 \in \mathcal{Z}_2(t)$, hence

$$\inf_{z_2\in\mathcal{Z}_2(t)} J(t,x;z_1^\star,z_2) = \inf_{z_2\in\mathcal{Z}_2(t)} J^-(t,x;z_1^\star,z_2) \tag{3.22}$$

and trivially we have

$$\inf_{z_2\in\mathcal{Z}_2(t)} J^-(t,x;z_1^\star,z_2) \le \sup_{z_1\in\mathcal{Z}_1(t)} \inf_{z_2\in\mathcal{Z}_2(t)} J^-(t,x;z_1,z_2). \tag{3.23}$$

By (3.21), (3.22) and (3.23) we obtain

$$\inf_{z_2\in\mathcal{Z}_2(t)} \sup_{z_1\in\mathcal{Z}_1(t)} J^-(t,x;z_1,z_2) \le \sup_{z_1\in\mathcal{Z}_1(t)} \inf_{z_2\in\mathcal{Z}_2(t)} J^-(t,x;z_1,z_2).$$

Since

$$\sup_{z_1\in\mathcal{Z}_1(t)} \inf_{z_2\in\mathcal{Z}_2(t)} J^-(t,x;z_1,z_2) \le \inf_{z_2\in\mathcal{Z}_2(t)} \sup_{z_1\in\mathcal{Z}_1(t)} J^-(t,x;z_1,z_2),$$

we have proved

$$\sup_{z_1\in\mathcal{Z}_1(t)} \inf_{z_2\in\mathcal{Z}_2(t)} J^-(t,x;z_1,z_2) = \inf_{z_2\in\mathcal{Z}_2(t)} \sup_{z_1\in\mathcal{Z}_1(t)} J^-(t,x;z_1,z_2) = J(t,x;z_1^*,z_2^*).$$

A similar argument allows to prove

$$\sup_{z_1\in\mathcal{Z}_1(t)} \inf_{z_2\in\mathcal{Z}_2(t)} J^+(t,x;z_1,z_2) = \inf_{z_2\in\mathcal{Z}_2(t)} \sup_{z_1\in\mathcal{Z}_1(t)} J^+(t,x;z_1,z_2) = J(t,x;z_1^*,z_2^*).$$

Hence we have proved that

$$V^-(t,x) = J(t,x,z_1^\star,z_2^\star) = V^+(t,x),$$

which concludes the proof of item 4. □

In [24] and [25], as well an in other papers, the authors formulate (what we call) the Hamadène-Lepeltier-Isaacs condition, as follows.

Hypothesis 3.17. There exists $z_1^\star : [0,T]\times\mathbb{R}^d\times\mathbb{R}^d \to U_1$ and $z_2^\star : [0,T]\times\mathbb{R}^d\times\mathbb{R}^d \to U_2$ Borel such that for all $(s,x,p)\in[0,T]\times\mathbb{R}^d\times\mathbb{R}^d$ we have

$$\begin{aligned} H^0_{CV}(s,x,p,z_1^\star(s,x,p),z_2^\star(s,x,p)) &\ge H^0_{CV}(s,x,p,u_1,z_2^\star(s,x,p)), \qquad \forall u_1\in U_1, \\ H^0_{CV}(s,x,p,z_1^\star(s,x,p),z_2^\star(s,x,p)) &\le H^0_{CV}(s,x,p,z_1^\star(s,x,p),u_2), \qquad \forall u_2\in U_2. \end{aligned} \tag{3.24}$$

We now show that, if σ defined in (1.2), is non-degenerate in the sense of Definition 2.17, then the Hamadène-Lepeltier-Isaacs hypothesis 3.17 implies our Pathwise-Isaacs condition, i.e. Property $\pi(t,x;z_1^\star,z_2^\star)$, see (3.7). The non-degeneracy condition is necessary to apply [39], Theorem 6. to prove that the state equation (1.1), controlled by feedback controls $(z_1^\star,z_2^\star)$, has a strong solution.

Proposition 3.18. We assume Hypothesis 3.1 and the non-degeneracy condition (2.12). We also suppose the existence of v^+ and v^-, quasi-strong solutions of the Bellmann-Isaacs equations (1.6) and (1.7) of class $C^{0,1}$.

If Hypothesis 3.17 holds true then, for all $(t,x)\in[0,T]\times\mathbb{R}^d$, there exists a couple of processes $(\tilde{z}_1^\star,\tilde{z}_2^\star)$ such that Property $\pi(t,x,\tilde{z}_1^\star,\tilde{z}_2^\star)$, defined in (3.7), is verified.

Proof. Let $x \in \mathbb{R}^d$ and $(z_1^\star, z_2^\star)$, the Borel applications defined above. Let us consider below the state equation controlled by the feedback controls $(z_1^\star, z_2^\star)$, where v is equal to v^+ or v^- and $s \in [t, T]$:

$$\begin{cases} dy(s) = f\left(s, y(s), z_1^\star(s, y(s), \partial_x v(s, y(s))), z_2^\star(s, y(s), \partial_x v(s, y(s)))\right) ds + \sigma(s, y(s)) dW_s, \\ y(t) = x, \end{cases} \tag{3.25}$$

The SDE (3.25), whose unknown is y, can be considered as an SDE with Lipschitz non-degenerate diffusion coefficient σ and linear growth measurable drift

$$(t, y) \mapsto f\left(s, y, z_1^\star(s, y, \partial_x v(s, y)), z_2^\star(s, y, \partial_x v(s, y))\right).$$

By [39], Theorem 6., there exists a unique strong solution $y = y(s; t, x)$ to (3.25).

Then, for $\omega \in \Omega$, setting $\tilde{x} = y(s; t, x)$, $\tilde{p} = \partial_x v(s; t, y(s; t, x))$, it follows that (3.24) reads as

$$\begin{aligned} &H^0_{CV}(s, \tilde{x}, \tilde{p}, z_1^\star(s, \tilde{x}, \tilde{p}), z_2^\star(s, \tilde{x}, \tilde{p})) \geq H^0_{CV}(s, \tilde{x}, \tilde{p}, u_1, z_2^\star(s, \tilde{x}, \tilde{p})), \qquad \forall u_1 \in U_1, \\ &H^0_{CV}(s, \tilde{x}, \tilde{p}, z_1^\star(s, \tilde{x}, \tilde{p}), z_2^\star(s, \tilde{x}, \tilde{p})) \leq H^0_{CV}(s, \tilde{x}, \tilde{p}, z_1^\star(s, \tilde{x}, \tilde{p}), u_2), \qquad \forall u_2 \in U_2. \end{aligned} \tag{3.26}$$

We rewrite now $\tilde{z}_i^\star(s) = z_i^\star(s, \tilde{x}, \tilde{p}), i = 1, 2$, and we observe that the process $\tilde{z}_i^\star(s) \in \mathcal{Z}_i(t)$ for $i \in \{1, 2\}$. We remark now that the solution $y(s)$ of (3.25), coincides with a solution $y(s; t, x, \tilde{z}_1^\star, \tilde{z}_2^\star)$ of (1.1). At this point we remark that (3.26) corresponds to (3.7) in Property $\pi(t, x; \tilde{z}_1^\star, \tilde{z}_2^\star)$. □

Remark 3.19. Note that in Proposition 3.18 do not intervene the following additional hypotheses in [25]:

1. $(u, v) \mapsto f_1(s, x, u, v)$ and $(u, v) \mapsto l(s, x, u, v)$ are continuous for any fixed (s, x), see Section 4.1, Assumption A;

2. $p \mapsto H^0_{CV}(s, x, p, z_1^\star(s, x, p), z_2^\star(s, x, p))$ is continuous for any fixed $(s, \tilde{x})$, see [25], Assumption A3, item (ii).

Item 1. is necessary to prove the equivalence of the Isaacs condition (Definition 1.3) with Hypothesis 3.17 via a selection theorem, as mentioned in [24] or [33] Proposition 3.1.

Our methodology, alternatively, proves the existence of Nash equilibrium supposing the existence of $C^{0,1}$-quasi-strong solutions of the Bellman-Isaacs PDEs.

4 The case of control theory

As mentioned at the end of the Introduction, the techniques described in previous sections allow also to establish a verification theorem for stochastic control problem, which generalizes Theorem 4.9 of [22]. This is the object of Theorem 4.3, which follows from Corollary 4.2. That corollary is an immediate consequence of Lemma 3.10 and practically extends Lemma 4.10 in [22].

We now describe our optimal control problem. As in Section 1, let us fix a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_s)_{s \in [0,T]}, \mathbb{P})$ satisfying the usual conditions, a finite dimensional Hilbert space, say $\mathbb{R}^d$ that will be the state space, a finite dimensional Hilbert space, say $\mathbb{R}^m$ (the noise space), one compact set $U \subseteq \mathbb{R}^k$ (the control space). We will deal with fixed horizon problem so that we fix $T \in [0, \infty[$ at the beginning and an initial time and state $(t, x) \in [0, T] \times \mathbb{R}^d$. W is a $(\mathcal{F}_s)_{s \in [t,T]}$-$d$-dimensional Brownian motion.

The state equation is

$$\begin{cases} dy(s) = [b(s, y(s)) + f_1(s, y(s), z(s))] \, ds + \sigma(s, y(s))dW_s, \\ y(t) = x, \end{cases} \tag{4.1}$$

where b, σ are defined in (1.2), while $f_1 : [0,T] \times \mathbb{R}^d \times U \mapsto \mathbb{R}^d$. The process $\mathcal{Z}(t) \ni z : [t,T] \times \Omega \mapsto U$ is the control processes, where $\mathcal{Z}(t)$ is the set of *admissible control processes*, that is $(\mathcal{F}_s)_{s\in[t,T]}$-progressively measurable processes taking values in U. We denote the solution to (4.1) with

$$y(s; t, x, z) \text{ or } y(s), \;\; s \in [t, T], x \in \mathbb{R}^d, z \in \mathcal{Z}(t). \tag{4.2}$$

The payoff function is defined as

$$J(t, x; z) = \mathbb{E}(\tilde{J}(t, x; z)), \tag{4.3}$$

provided previous expectation exists (otherwise it will be set to $+\infty$), where

$$\tilde{J}(t, x; z) = \int_t^T l(s, y(s; t, x, z); z(s))ds + g(y(T; t, x, z)), \tag{4.4}$$

adopting very close notations to Section 1. The objective is to minimize the payoff, hence the value function is

$$V(t, x) = \inf_{z\in\mathcal{Z}(t)} J(t, x; z).$$

The problem is now of the type considered in [22].

Definition 4.1. Let $t \in [0,T]$. If there exists a control $z^\star \in \mathcal{Z}(t)$ such that $J(t, x; z^\star) = V(t, x)$ for any $x \in \mathbb{R}^d$, we say that the control $z^\star$ is optimal for the problem (4.1) and (4.3).

The current value Hamiltonian is defined, for $(s, x, p, u) \in [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \times U$, as

$$H_{CV}(s, x, p, u) = \langle b(s, x), p\rangle + \langle f_1(s, x, u), p\rangle + l(s, x; u),$$

and the minimum value Hamiltonian is

$$H(s, x, p) = \inf_{u\in U} H_{CV}(s, x, p, u). \tag{4.5}$$

Since the first term of H_{CV} does not depend on the control z, we define

$$H^0_{CV}(s, x, p, u) = \langle f_1(s, x, u), p\rangle + l(s, x, u)$$

and the corresponding lower value quantity gives

$$H^0(s, x, p) = \inf_{u\in U} H^0_{CV}(s, x, p, u),$$

so that

$$H(s, x, p) = \langle b(s, x), p\rangle + H^0(s, x, p).$$

Defining formally the operator $\mathcal{L}_0$ as

$$\mathcal{L}_0 u(s,x) = \partial_s u(s,x) + \langle b(s,x), \partial_x u(s,x)\rangle + \frac{1}{2} Tr[\sigma^\top(s,x)\partial_{xx}u(s,x)\sigma(s,x)],$$

it is possible to write the HJB equation associated with problem (4.1) and (4.3) as

$$\begin{cases} \mathcal{L}_0 v(s,x) + H^0(s,x,\partial_x v(s,x)) = 0, \\ v(T,x) = g(x). \end{cases} \tag{4.6}$$

We will consider quasi-strong and quasi-strict solutions for the HJB equation as in Definitions 3.7 and 3.6.

The optimal control problem appears here as a particular case of the game theory setting.

Explicitly, if the set U_1 is the singleton $\{u_1\}$, then $\mathcal{Z}_1(t)$ is necessarily the singleton $\{z_1\}$ where $z_1 \equiv u_1$. In this framework, for $t \in [0,T]$, defining $U = U_2$, $\mathcal{Z}(t) = \mathcal{Z}_2(t)$, and omitting u_1 and z_1, the zero-sum differential game (1.1) and (1.3) reduces to the control problem (4.1) and (4.3). In particular, the lower value V^- is the value of the optimal control problem. In Corollary 4.2 and Theorem 4.3, we need a slight reformulation of Hypotheses 3.1 and 3.4. In the game theory setting, f_1 and l are functions of (the four variables) $(s,x,u_1,u_2) \in [t,T]\times\mathbb{R}^d\times U_1\times U_2$, Taking into account what precedes, in the control setting, the above Hypotheses are meant to hold for f_1 and l which are functions of (the three variables) $(s,x,u)\in[t,T]\times\mathbb{R}^d\times U$. Indeed in control there is only one agent trying to minimize the payoff.

The Corollary 4.2 below is a direct consequence of Lemma 3.10.

Corollary 4.2. We assume Hypotheses 3.1 and 3.4. We also suppose the existence of a function v such that $v^- = v^+ = v$ satisfies Hypothesis 3.8, where (1.6) (resp. (1.7)) is replaced by (4.6). Then, $\forall (t,x)\in[0,T]\times\mathbb{R}^d$, $\forall z\in\mathcal{Z}(t)$, setting $y(r) = y(r;t,x,z)$, as in (4.2), we have

$$\begin{aligned}\tilde{J}(t,x;z) = v(t,x) + \int_t^T \Big(H^0_{CV}(r,y(r),\partial_x v(r,y(r),z(r)) - H^0(r,y(r),\partial_x v(r,y(r)))\Big)\,dr \\ + M_T,\end{aligned}$$

where M is a (square-integrable) martingale and $\tilde{J}$ is defined in (4.4).

Proof. We consider the stochastic game where the first player is frozen and so U_1 is a singleton. In this case $H^{0,-}$ and $H^{0,+}$ respectively in (1.8) and (1.9) coincide with H^0 so that the Bellman-Isaacs PDEs (1.6) and (1.7) are identical to (4.6). Consequently $v^- := v$ (resp. of course $v^+ = v$) is the quasi-strong solutions of (1.7) (resp. (1.6)). The result follows from Lemma 3.10 recalling that $\mathcal{Z}_1(t)$ is constituted by constant processes, and taking $z_2 = z$.

□

We now state a more general Verification Theorem than Theorem 4.9 of [22].

Theorem 4.3. Assume that Hypotheses 3.1 and 3.4 hold. We also suppose the existence of a function v such that $(v^- = v^+ =)v$ satisfies Hypothesis 3.8. Let $t\in[0,T], x\in\mathbb{R}^d$. Then we have the following.

1. For any $z\in\mathcal{Z}(t)$ the functional (4.3) is well-defined. In particular the random variable $\tilde{J}(t,x,z):\Omega\mapsto\mathbb{R}$ defined in (4.4) is lower-semiintegrable.

2. $v(t,x)\le V(t,x)$.

3. If $z^\star \in \mathcal{Z}(t)$ satisfies (setting $y(r) = y(r; t, x, z^\star)$, as in (4.2))

$$H^0(r, y(r), \partial_x v(r, y(r))) = H^0_{CV}(r, y(r), \partial_x v(r, y(r)), z^\star(r)), \tag{4.7}$$

for a.e. $(r, x) \in [t, T] \times \mathbb{R}^d$, $\mathbb{P}$-a.s., then $z^\star$ is optimal in the sense of Definition 4.1. Moreover $v(t, x) = V(t, x)$ and $V(t, x)$ is finite.

Proof. Applying Corollary 4.2 for a $z \in \mathcal{Z}(t)$, we obtain

$$\begin{aligned} \tilde{J}(t, x, z) = v(t, x) \quad &+ \quad \textstyle\int_t^T \left(H^0_{CV}(r, y(r; x, z(r)), \partial_x v(r, y(r; x, z(r)), z(r))\right. \\ &- \quad H^0(r, y(r; x, z(r)), \partial_x v(r, y(r; x, z(r))))\left.\right) dr + M_T(z), \end{aligned} \tag{4.8}$$

where $M_T(z)$ is a (square-integrable) martingale.

Obviously, the integral in (4.8) is always greater or equal than zero, hence,

$$\tilde{J}(t, x, z) \geq v(t, x) + M_T(z).$$

Taking expectation, it follows that for a generic $z \in \mathcal{Z}(t)$

$$J(t, x; z) \geq v(t, x)$$

and therefore item 1. follows. Then, taking the infimum over z allows to prove item 2.

Concerning item 3., if $z^\star$ satisfies (4.7), the integral in (4.8) vanishes and so $v(t, x) = J(t, x; z^\star)$. Consequently $v(t, x) = V(t, x)$ and the result is finally proved.

□

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