Lecture 2

Non-Atomic Routing Games
Wardrop Equilibrium

Roberto Cominetti
Universidad Adolfo Ibáñez

Journées SMAI MODE 2020
Lecture 2: Non-Atomic Routing Games

1. Non-Atomic Routing Games
   - Wardrop equilibrium – Definition
   - Wardrop equilibrium – Characterizations
   - Wardrop equilibrium – Existence & Uniqueness

2. Inefficiency of Equilibria
   - Price-of-Anarchy
   - PoA for highly congested networks
Non-atomic Routing Games
Urban traffic flows under congestion

SANTIAGO
6.000.000 people
11.000.000 daily trips
1.750.000 car trips

Morning peak
500.000 car trips
29.000 OD pairs

2266 nodes / 7636 arcs
Non-atomic routing games

Games with many players become computationally hard. Such situations can be idealized by considering players as a continuum and to focus on the fraction of players that use each strategy.

We illustrate this with routing games on transportation networks.

We are given a graph \((V, E)\) with
- a set of edges \(e \in E\) with continuous non-decreasing costs \(c_e : \mathbb{R}_+ \rightarrow \mathbb{R}_+\)
- a set of OD pairs \(\kappa \in \mathcal{K}\) with corresponding routes \(r \in \mathcal{R}_\kappa \subseteq 2^E\)
- a set of aggregate demands \(d_\kappa \geq 0\) for each \(\kappa \in \mathcal{K}\)
Wardrop equilibrium

- Continuum of players / each one has a negligible impact on congestion.
- Perfectly divisible / aggregate demands $d_\kappa \geq 0$ for each OD pair $\kappa \in \mathcal{K}$.

Let $\mathcal{F}$ be the set of splittings $(y, x)$ of the demands $d_\kappa$ into route-flows $y_r \geq 0$, together with their induced edge-loads $x_e$:

$$d_\kappa = \sum_{r \in \mathcal{R}_\kappa} y_r \quad (\forall \kappa \in \mathcal{K}),$$

$$x_e = \sum_{r \ni e} y_r \quad (\forall e \in E).$$
Wardrop equilibrium

- Continuum of players / each one has a negligible impact on congestion.
- Perfectly divisible / aggregate demands \( d_\kappa \geq 0 \) for each OD pair \( \kappa \in \mathcal{K} \).

Let \( \mathcal{F} \) be the set of splittings \((y, x)\) of the demands \( d_\kappa \) into route-flows \( y_r \geq 0 \), together with their induced edge-loads \( x_e \):

\[
\begin{align*}
  d_\kappa &= \sum_{r \in \mathcal{R}_\kappa} y_r \quad (\forall \kappa \in \mathcal{K}), \\
  x_e &= \sum_{r \ni e} y_r \quad (\forall e \in \mathcal{E}).
\end{align*}
\]

The analog of Nash equilibria for a continuum of players is:

**Definition (Wardrop, 1952)**

A Wardrop equilibrium is a pair \((\hat{y}, \hat{x}) \in \mathcal{F}\) that uses only shortest routes:

\[
(\forall \kappa \in \mathcal{K})(\forall r, r' \in \mathcal{R}_\kappa) \quad \hat{y}_r > 0 \Rightarrow \sum_{e \in r} c_e(\hat{x}_e) \leq \sum_{e \in r'} c_e(\hat{x}_e).
\]
Example: Single OD with 2 identical parallel links

At equilibrium the demand splits 50%-50% : \((\frac{d}{2}, \frac{d}{2})\).
Example (Braess Paradox):

Total demand is $d = 1$.

The upper and lower routes have cost $T_u = x_1 + 1$ and $T_l = 1 + x_2$. Wardrop equilibrium sends $\frac{1}{2}$ on each route with travel time $T_{eq} = 1.5$. 
Example (Braess Paradox):

Total demand is $d = 1$.

The upper and lower routes have cost $T_u = x_1 + 1$ and $T_l = 1 + x_2$. Wardrop equilibrium sends $\frac{1}{2}$ on each route with travel time $T_{eq} = 1.5$.

A central arc $(a, b)$ with cost 0 is added. The new Wardrop equilibrium sends all the flow along the zig-zag path $o-a-b-d$ with travel time $T_{eq} = 2.0$. 
An example with 2 OD pairs

Demands $d_1 = d_2 = 1$

The pair $\kappa_1$ sends all its flow $d_1 = 1$ through the central arc whose cost is always better that the upper route. Given this, $\kappa_2$ sends a traffic 0.8 on the central route until the cost equalizes the lower route which gets a flow of 0.2. The equilibrium cost for both pairs is 1.8.

**Exercise:** Find the equilibrium when $d_1 = d_2 = 2$
Wardrop equilibrium – Characterizations
Wardrop equilibrium – Characterizations

Introducing the route costs and minimal times

$$T_r(x) = \sum_{e \in r} c_e(x_e) ; \quad \tau_{\kappa}(x) = \min_{r \in R_{\kappa}} T_r(x).$$

the conditions for Wardrop equilibrium are

$$(\forall \kappa \in \mathcal{K})(\forall r \in R_{\kappa}) \quad y_r > 0 \Rightarrow T_r(x) = \tau_{\kappa}(x).$$

**Theorem** (Beckman-McGuire-Winsten, 1956)

For a feasible flow $(y, x) \in \mathcal{F}$ the following are equivalent:

a) $(y, x)$ is a Wardrop equilibrium

b) $\sum_{r \in R} T_r(x)(y'_r - y_r) \geq 0 \quad \forall (y', x') \in \mathcal{F}$

c) $\sum_{e \in E} c_e(x_e)(x'_e - x_e) \geq 0 \quad \forall (y', x') \in \mathcal{F}$

d) $(y, x)$ is an optimal solution of $\min_{(y, x) \in \mathcal{F}} \sum_{e \in E} \int_0^{x_e} c_e(z) \, dz.$

**Proof:** For simplicity we consider the case of a single OD.
Wardrop equilibrium – Characterization 1

Proposition

A feasible flow \((y, x) \in \mathcal{F}\) is a WE iff

\[
(VI) \quad \sum_{r \in \mathcal{R}} T_r(x)(y'_r - y_r) \geq 0 \quad \forall (y', x') \in \mathcal{F}.
\]
Proposition

A feasible flow \((y, x) \in \mathcal{F}\) is a WE iff

\[
(\forall r \in \mathcal{R}) \quad \sum_{r \in \mathcal{R}} T_r(x)(y'_r - y_r) \geq 0 \quad \forall (y', x') \in \mathcal{F}.
\]

Proof:

(\(\Rightarrow\)) If \((y, x)\) is WE then for all \((y', x') \in \mathcal{F}\) we have

\[
\sum_r T_r(x) y'_r \geq \sum_r \tau(x) y'_r = \tau(x) d = \sum_r \tau(x) y_r = \sum_r T_r(x) y_r.
\]

(\(\Leftarrow\)) Let \((y, x) \in \mathcal{F}\) a solution of \((\forall r \in \mathcal{R}) \sum_{r \in \mathcal{R}} T_r(x)(y'_r - y_r) \geq 0\). If \(y_r > 0\) we may consider the flow \(y'\) identical to \(y\) except for \(y'_r = y_r - \epsilon\) and \(y'_p = y_p + \epsilon\) with \(p \in \mathcal{R}\) a shortest path

\[
\Rightarrow \quad 0 \leq \sum_{q \in \mathcal{R}} T_q(x)(y'_q - y_q) = \epsilon T_p(x) - \epsilon T_r(x)
\]

so that \(T_r(x) \leq T_p(x) = \tau(x)\). Therefore \(y_r > 0 \Rightarrow T_r(x) = \tau(x)\). \(\square\)
Proposition

A feasible flow \( (y, x) \in \mathcal{F} \) is a WE iff

\[
(VI) \quad \sum_{e \in E} c_e(x_e)(x'_e - x_e) \geq 0 \quad \forall (y', x') \in \mathcal{F}.
\]
Proposition

A feasible flow \((y, x) \in \mathcal{F}\) is a WE iff

\[
(VI) \quad \sum_{e \in E} c_e(x_e)(x'_e - x_e) \geq 0 \quad \forall (y', x') \in \mathcal{F}.
\]

Proof: The equivalent form of the \((VI)\) follows from an exchange in the sums

\[
\sum_{r \in \mathcal{R}} T_r(x)(y'_r - y_r) = \sum_{r \in \mathcal{R}} \sum_{e \in r} c_e(x_e)(y'_r - y_r) = \sum_{e \in E} \sum_{r \ni e} c_e(x_e)(y'_r - y_r) = \sum_{e \in E} c_e(x_e)(x'_e - x_e).
\]
Wardrop equilibrium – Characterization 3

**Proposition**

A feasible flow \((y, x) \in \mathcal{F}\) is a WE iff it is an optimal solution of the convex minimization problem

\[
(P) \quad \min_{(y,x) \in \mathcal{F}} \Phi(y, x) = \sum_{e \in E} \int_0^{x_e} c_e(z)dz.
\]

*Remark.* is a continuous analog of Rosenthal’s potential for discrete routing games. In the continuous case equilibria coincide with the minima of the potential.
Wardrop equilibrium – Characterization 3

Proposition

A feasible flow \((y, x) \in \mathcal{F}\) is a WE iff it is an optimal solution of the convex minimization problem

\[
(P) \quad \min_{(y, x) \in \mathcal{F}} \, \Phi(y, x) = \sum_{e \in E} \int_0^{x_e} c_e(z) \, dz.
\]

Proof: Since \(c_e(\cdot)\) is non-decreasing the function \(\Phi(y, x)\) is convex, so that \((y, x) \in \mathcal{F}\) is a minimum iff for all \((y', x') \in \mathcal{F}\) we have

\[
0 \leq \langle \nabla \Phi(y, x), (y', x') - (y, x) \rangle = \sum_{e \in E} c_e(x_e)(x'_e - x_e).
\]

Remark. \(\Phi\) is a continuous analog of Rosenthal’s potential for discrete routing games. In the continuous case equilibria \(\text{coincide}\) with the minima of the potential.
Wardrop equilibrium – Existence & Uniqueness
Theorem

A non-atomic routing game has a Wardop equilibrium. Moreover, if \((y, x)\) and \((y', x')\) are two equilibria then \(c_e(x_e) = c_e(x'_e)\). In particular, if \(c_e(\cdot)\) is strictly increasing then \(x\) is unique.
Wardrop equilibrium – Existence and uniqueness

Theorem

A non-atomic routing game has a Wardrop equilibrium. Moreover, if \((y, x)\) and \((y', x')\) are two equilibria then \(c_e(x_e) = c_e(x'_e)\). In particular, if \(c_e(\cdot)\) is strictly increasing then \(x\) is unique.

Proof: \(\Phi\) is continuous \(\Rightarrow\) its minimum on \(\mathcal{F}\) is attained \(\Rightarrow\) existence of WE.

If \((y, x)\) and \((y', x')\) are two equilibria, using (VI) we get

\[
\sum_{e \in E} c_e(x_e)(x'_e - x_e) \geq 0 \\
\sum_{e \in E} c_e(x'_e)(x_e - x'_e) \geq 0 \\
\sum_{e \in E} (c_e(x_e) - c_e(x'_e))(x'_e - x_e) \geq 0
\]

Since \(c_e(\cdot)\) is non-decreasing each term in the sum is negative so that \((c_e(x_e) - c_e(x'_e))(x'_e - x_e) = 0\) for all \(e \in E\), hence \(c_e(x_e) = c_e(x'_e)\). \(\square\)
Variational Characterization

Wardrop equilibria are the optimal solutions of the convex program

\[
(P) \quad \min_{(y,x) \in \mathcal{F}} \sum_{e \in E} \int_0^{x_e} c_e(z) dz.
\]

- \((P)\) is large scale \(\approx 220 \times 10^6\) variables for Santiago
- Objective function different from the social cost

\[
SC(x) = \sum_{e \in E} x_e c_e(x_e)
\]
Dual Characterization (Fukushima, 1984)

Change of variables: \( x_e \leftrightarrow t_e \)

\[
(D) \quad \min_t \sum_{e \in E} \int_0^{t_e} c_e^{-1}(z) \, dz - \sum_{\kappa \in \mathcal{K}} d_{\kappa} \, \tau_{\kappa}(t) \\
\Phi(t)
\]

strictly convex

\[
\tau_{\kappa}(t) = \min_{r \in R_{\kappa}} \sum_{e \in r} t_e = \text{ODs minimum travel times}
\]

concave, polyhedral

Non-smooth but efficiently computable (Bellman, Dijkstra,...)

\[
\tau_i^{\kappa} = \min_{e \in E_i^+} \{ t_e + \tau_j^{\kappa} \}
\]
Inefficiency of Equilibria – Price-of-Anarchy
Quantifying Inefficiency: Price-of-Anarchy

For non-atomic routing games

Social cost = Total travel time $= \sum_{e \in E} x_e c_e(x_e)$

$$\text{PoA} = \frac{\text{Social Cost of Equilibrium}}{\text{Minimum Social Cost}} \geq 1$$

**Theorem (Roughgarden-Tardos, 2002; Roughgarden, 2003)**

- $\text{PoA} \leq \frac{4}{3}$ for non-atomic routing games with affine costs.
- $\text{PoA} \leq \frac{\sqrt{k+1}}{\sqrt{k+1} - k/(k+1)} \sim O\left(\frac{k}{\log k}\right)$ for polynomials of degree $k$.

Bounds attained for simple 2-link networks with fine-tuned demands.
PoA and PoS in non-atomic routing games

Note that

\[
\text{Total travel time} = \sum_{e \in E} x_e c_e(x_e) = \sum_{r \in \mathcal{R}} y_r T_r(x) = \sum_{\kappa \in \mathcal{K}} d_\kappa \tau_\kappa(x).
\]

All Wardrop equilibria have the same value of \( c_e(x_e) \)
⇒ the same value of \( T_r(x) \)
⇒ the same minimal times \( \tau_\kappa(x) \)
⇒ social cost is constant on the set of Wardrop equilibria
⇒ PoS=PoA.
PoA and PoS in non-atomic routing games

Note that

\[
\text{Total travel time} = \sum_{e \in E} x_e c_e(x_e) = \sum_{r \in R} y_r T_r(x) = \sum_{\kappa \in K} d_\kappa \tau_\kappa(x).
\]

All Wardrop equilibria have the same value of \( c_e(x_e) \)
⇒ the same value of \( T_r(x) \)
⇒ the same minimal times \( \tau_\kappa(x) \)
⇒ social cost is constant on the set of Wardrop equilibria
⇒ PoS=PoA.

Example. In the Braess paradox, when the central arc is unavailable Wardrop equilibrium splits half and half between with a travel time of 1.5. This coincides with the social optimum that minimizes \( x_1(x_1 + 1) + x_2(x_2 + 1) \) \( \Rightarrow \) PoA=PoS=1.

If we allow the central arc, the new equilibrium sends all the flow on the zig-zag path with travel time 2. The social optimum does not change and the price of anarchy increases to PoA=PoS=\( \frac{4}{3} \).
Example: Pigou network

Let $c : [0, \infty) \to [0, \infty)$ be continuous and increasing and fix $d > 0$.

Wardrop equilibrium is $x = d$ with social cost $d \cdot c(d)$.

Minimum social cost is $\min_{x \in [0,d]} x \cdot c(x) + (d-x) c(d)$.

Hence, PoA on this simple graph can be as large as

$$\alpha(c) = \sup_{d>0} \sup_{x \in [0,d]} \frac{dc(d)}{x \cdot c(x) + (d-x) c(d)} \geq 1.$$

This value allows to bound the PoA on any graph.
PoA in non-atomic routing games

**Theorem (Correa-Schulz-Stier, 2004)**

In a non-atomic routing game on a graph \((N, A)\) with arc costs \(c_e(\cdot)\) we have

\[
\text{PoA} = \text{PoS} \leq \alpha \triangleq \max_{e \in E} \alpha(c_e).
\]

**Proof:** Let \((y, x)\) be a WE and \((\bar{y}, \bar{x})\) a minimizer of \(C(y, x)\). Taking \(d = x_e\) and \(x = \bar{x}_e\) in the expression for the supremum \(\alpha(c_a)\) we get the inequality

\[
x_e c_e(x_e) \leq \alpha [\bar{x}_e c_e(\bar{x}_e) + (x_e - \bar{x}_e)c_e(x_e)]
\]

which added together and in view of VI yield

\[
C(y, x) \leq \alpha \left[ C(\bar{y}, \bar{x}) + \sum_{e \in E} c_e(x_e)(x_e - \bar{x}_e) \right] \leq \alpha C_{\text{min}}.
\]

\(\square\)
PoA in non-atomic routing games

Note that $\alpha(c)$ can be expressed as $\alpha(c) = 1/[1 - \beta(c)]$ where

$$\beta(c) = \sup_{d > 0} \sup_{x \in [0, d]} \frac{x[c(d) - c(x)]}{d c(d)} = \sup \frac{A_1}{A_2}.$$ 

If $c(\cdot)$ is affine we have $A_1 \leq \frac{1}{4} A_2$ so that $\beta(c) \leq \frac{1}{4}$. Taking $x = \frac{1}{2} d \to \infty$ we attain asymptotically $\beta(c) = \frac{1}{4}$, and therefore $\alpha(c) = \frac{4}{3}$. 
PoA with polynomial costs

Proposition

For polynomials \( c(x) = a_0 + a_1 x + \cdots + a_k x^k \) with \( a_i \geq 0 \) and \( a_k > 0 \) we have

\[
\alpha(c) = \alpha_k \triangleq \left[ 1 - k(k + 1)^{-(k+1)/k} \right]^{-1} \sim \frac{k}{\ln k}.
\]

<table>
<thead>
<tr>
<th>( k )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha_k )</td>
<td>1.3333</td>
<td>1.6258</td>
<td>1.8956</td>
<td>2.1505</td>
<td>2.3944</td>
<td>2.6297</td>
</tr>
</tbody>
</table>
PoA with polynomial costs

Proposition

For polynomials \( c(x) = a_0 + a_1 x + \cdots + a_k x^k \) with \( a_i \geq 0 \) and \( a_k > 0 \) we have

\[
\alpha(c) = \alpha_k \triangleq \left[ 1 - k(k+1)^{-(k+1)/k} \right]^{-1} \sim \frac{k}{\ln k}.
\]

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha_k )</td>
<td>1.3333</td>
<td>1.6258</td>
<td>1.8956</td>
<td>2.1505</td>
<td>2.3944</td>
<td>2.6297</td>
</tr>
</tbody>
</table>

Proof: Note that \( \beta(c) = \sup_{d > 0} \sup_{x \in [0, d]} \frac{x}{d} \left[ 1 - \frac{c(x)}{c(d)} \right] \). From \( a_i \geq 0 \) we have that \( c(x)/x^k \) is decreasing so that \( c(x)/x^k \geq c(d)/d^k \) and then

\[
\beta(c) \leq \sup_{d > 0} \sup_{x \in [0, d]} \frac{x}{d} \left[ 1 - \left( \frac{x}{d} \right)^k \right] = \sup_{z \in [0,1]} z(1 - z^k)
\]

which is attained at \( z^* = (k+1)^{-1/k} \). Hence \( \beta(c) \leq k(k+1)^{-(k+1)/k} \) and therefore \( \alpha(c) \leq \alpha_k \). This bound is tight: take \( x = z^* d \) with \( d \to \infty \). □
Empirical observation (Youn et al. 2008, O’Hare et al. 2016,...)

In practice PoA is usually close to 1 both under high and low traffic, with fluctuations in the intermediate regime.
Is it always true?

- Is it always the case that PoA=1 when the demand is small, and it goes back to one as the demand grows to ∞?
- Is it at least true for single OD networks?
- Is it at least true for parallel networks?
- Is it true for convex and smooth costs?
PoA may oscillate and remain bounded away from 1 even for simple networks with smooth strongly convex costs:

\[
c_1(x) = (1 + \frac{1}{2} \sin(\log x)) x^2
\]

\[
c_2(x) = x^2
\]

\[
c_3(x) = (1 + \frac{1}{2} \cos(\log x)) x^2
\]
...but eventually yes!

**Definition (Karamata, 1930)**

A function $c : [0, \infty) \rightarrow (0, \infty)$ is called *regularly varying* iff

$$(\forall x > 0) \text{ there exists } \lim_{t \to \infty} \frac{c(tx)}{c(t)} \in (0, \infty)$$

**Remark:** The limit is necessarily of the form $x^\beta$ for some $\beta \geq 0$.
...but eventually yes!

**Definition (Karamata, 1930)**

A function $c : [0, \infty) \to (0, \infty)$ is called **regularly varying** iff

$$(\forall x > 0) \text{ there exists } \lim_{t \to \infty} \frac{c(tx)}{c(t)} \in (0, \infty)$$

**Remark:** The limit is necessarily of the form $x^\beta$ for some $\beta \geq 0$

- This class relevant in probability, large deviations, number theory.
- Examples: polynomials, logarithmic/poly-log functions,…

**Theorem (Colini-C-Mertikopoulos-Scarsini, 2016, 2017)**

- **Regularly varying costs:** $\text{PoA} \to 1$ **in the high congestion regime**.
- **Polynomial costs:** $\text{PoA} \to 1$ **plus sharp convergence rates**.